Bound on the speed of sound from holography

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We show that the squared speed of sound v_s^2 is bounded from above at high temperatures by the conformal value of $1/3$ in a class of strongly coupled four-dimensional field theories, given some mild technical assumptions. This class consists of field theories that have gravity duals sourced by a single-scalar field. There are no known examples to date of field theories with gravity duals for which v_s^2 exceeds $1/3$ in energetically favored configurations. We conjecture that $v_s^2 = 1/3$ represents an upper bound for a broad class of four-dimensional theories.

I. INTRODUCTION

The gauge/gravity duality, which relates gauge theories to string theories in higher-dimensional spaces [1], has been used in recent years to shed light on the properties of plasmas described by strongly coupled, large N_c gauge theories. When a field theory that has a string dual is in the large N_c and strong coupling limits, the string dual generally reduces to a classical supergravity theory. This allows one to get information on strongly coupled quantum gauge theories by doing classical calculations in higher-dimensional "holographic" gravitational theories. For instance, transport coefficients, which are normally theoretically inaccessible in strongly coupled systems, can be calculated in theories with gravity duals.

Unfortunately, there are no known gravity duals to the gauge theories that are currently used to describe nature. In particular, there is no known gravity dual to QCD, which for $N_c = 3$ describes the strongly coupled quark-gluon plasma under exploration at RHIC. The gauge/gravity duality therefore makes no quantitative predictions for phenomenologically interesting theories. However, one might hope to learn some qualitative lessons by searching for quantities that do not depend sensitively on the details of any particular gravity dual. With luck, insights gleaned from such "universal" properties may extend to theories with no known duals. A striking example of a universal quantity is the ratio η/s of shear viscosity to entropy density. This takes the value $1/4\pi$ in all theories with gravity duals (i.e., certain field theories that are in the large N_c and strong 't Hooft coupling limits) [2]. The phenomenological implications of the universality of η/s in theories with gravity duals remain unclear [3]. Moreover, the deviations from η/s as one moves away from the supergravity

limit do not appear to be universal, as they can either increase or decrease η/s [4]. Nevertheless, the search for universal behaviors in holographic theories remains important both for theoretical reasons and for its possible phenomenological relevance.

In this paper, we show that in the simplest class of nonconformal four-dimensional (4D) theories with gravity duals, the squared speed of sound v_s^2 is always bounded from above by $1/3$ at sufficiently high temperatures. (We work in natural units with $c = \hbar = k_B = 1$ throughout, and we focus on 4D field theories for definiteness.) Of course, v_s is not universal in the same sense as η/s; in general it depends on the temperature T, the chemical potential μ, and other details of a system. However, as we will show, this dependence takes a certain universal form at high T in the class of theories that we consider. Furthermore, since $v_s^2 > 1/3$ has to date never been observed in energetically stable configurations of theories with gravity duals, it is tempting to speculate that $v_s^2 = 1/3$ is an upper bound in a broad class of strongly coupled gauge theories.

II. SINGLE-SCALAR SYSTEMS

The squared speed of sound can be written as $v_s^2 = \partial p / \partial \epsilon$, where p is the pressure of a system and ϵ is its energy density. For systems at zero chemical potential, v_s^2 can also be written in terms of the entropy density s as

$$v_s^2 = \frac{d \log T}{d \log s}. \tag{1}$$

In a 4D conformal field theory (at finite T), the entropy density s behaves as $s \sim T^3$, and $v_s^2 = 1/3$. However, in nonconformal field theories, v_s^2 has a nontrivial dependence on the temperature and other properties of the theories. The simplest class of gravity duals that describe nonconformal strongly coupled 4D field theories at large N_c and zero chemical potential is the so-called "single-

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scalar model'' [5–7] with the action

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right]. \quad (2)$$

Above, $\kappa_5^2/(8\pi)$ is the five-dimensional (5D) gravitational constant, ϕ is a real scalar field, and $V(\phi)$ is a smooth potential that is symmetric about an extremum at $\phi = 0$. Different potentials correspond to different dual gauge theories.

We study systems at finite temperature with translational invariance in the $(t, \vec{x})$ directions and $SO(3)$ invariance in the $\vec{x}$ directions. The most general metric ansatz consistent with these symmetries is

$$ds^2 = a^2(-h dt^2 + d\vec{x}^2) + \frac{dr^2}{b^2 h}, \quad (3)$$

where a, b, and h are functions of the holographic coordinate r only, and $\phi = \phi(r)$. A black hole horizon occurs at $r = r_h$, where h has a simple zero. a, b, and ϕ are all regular at $r = r_h$. For this system, s and T can be read off in the usual way

$$s = \frac{2\pi}{\kappa_5^2} |a(r_h)|^3, \qquad T = \frac{|a(r_h)b(r_h)h'(r_h)|}{4\pi}. \quad (4)$$

We consider theories that become approximately conformal deep in the UV. This corresponds to assuming that the gravity dual approaches 5D anti-de Sitter space (AdS_5) as $|a| \to \infty$. To obtain asymptotically AdS_5 geometries in the UV as $\phi \to 0$, $V(\phi)$ must behave as

$$\lim_{r\to\infty} V(\phi) = -\frac{12}{L^2} + \frac{1}{2L^2} \Delta(\Delta - 4)\phi^2 + \mathcal{O}(\phi^4), \quad (5)$$

with L the AdS curvature radius, $\Delta(\Delta - 4)/L^2$ the squared mass m^2 of the scalar, and Δ the scaling dimension of the gauge theory operator $\mathcal{O}_\phi$ dual to the scalar field ϕ [1]. We restrict our attention to relevant $\mathcal{O}_\phi$, for which $\Delta < 4$. Since stable backgrounds must satisfy the Breitenlohner-Freedman bound $m^2L^2 > -4$ [8], we focus on $2 < \Delta < 4$ in our analysis.

A potential with just the constant term $-12/L^2$ would have an AdS_5-Schwarzschild black hole as a solution to the equations of motion with a conformal field theory (CFT) at finite T living on the AdS_5 boundary. Thus, a 4D gauge theory that has a description in terms of a 5D single-scalar gravity dual is just a CFT deformed by the addition of a relevant operator. As illustrated in Refs. [5,6], the thermodynamics of these single-scalar systems is quite rich: Many equations of state $v_s(T)$ are possible, and the system can undergo first- or second-order phase transitions, depending on the form of $V(\phi)$.

We now demonstrate that in all models of this class, $v_s^2 \leq 1/3$ at high T. As our derivation will self-consistently show, small $\phi_H \equiv \phi(r_h)$ corresponds to asymptotically high T [9], and at high T these models are sensitive only to the universal part of $V(\phi)$ given in Eq. (5). By high T,

we mean temperatures that are large compared to all other scales in the problem. Since these systems are approximately conformal in the UV, we expect that at very high T, the background geometry will approximate an AdS-Schwarzschild black hole. Thus, v_s^2 should approach the conformal value of $1/3$ as $T \to \infty$. However, the sign of the first nonzero correction in a high-temperature expansion of v_s^2 is less obvious.

To determine the sign of this correction, we solve the equations of motion resulting from Eq. (2) perturbatively around the AdS-Schwarzschild black hole solution. Working to second order in ϕ_H, it is possible to obtain a closed-form expression for the backreaction of the scalar field on the geometry, which then allows us to evaluate s and T and find v_s^2 using Eq. (1). We sketch the derivation in an appendix and simply present the result here:

$$v_s^2(\phi_H) = 1/3 - C(\Delta)\phi_H^2 + \mathcal{O}(\phi_H^3),$$

$$\text{where } C(\Delta) = \frac{1}{576}(\Delta - 4)^2 \Delta \left[16 + (\Delta - 4)\Delta \right.$$

$$\left. \times \int_1^\infty ds\, {}_2F_1(2 - \Delta/4; 1 + \Delta/4; 2; 1 - s)^2 \right]$$

$$= \frac{1}{18\pi}(4 - \Delta)(4 - 2\Delta)\tan(\pi\Delta/4), \quad (6)$$

and ${}_2F_1$ is a hypergeometric function. The simplified form in the last line can be obtained by standard identities. Since $C(\Delta)$ is positive, $v_s^2 \leq 1/3$. This result was also obtained by different methods in Ref. [10].

Using the methods discussed in the Appendix, it is not hard to show that

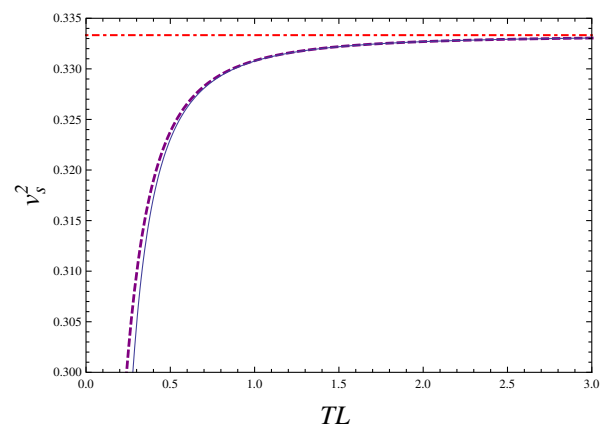$$\phi_H = (\pi L T)^{\Delta - 4} \frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2 - 1)} \quad (7)$$

FIG. 1 (color online). Plot of the high-temperature approximation to v_s^2 in Eqs. (6) and (7) for $\Delta = 3$ (solid line) versus a numerical solution (dashed line) for v_s^2 found using the methods of Ref. [5]. The numerical solution is for $V(\phi) = -\frac{12}{L^2} \times \cosh(\frac{1}{2}\phi)$, which corresponds to $\Delta = 3$. The sound bound $v_s^2 = 1/3$ is shown as a horizontal dotted-dashed line.

plus corrections that go to zero as $T \to \infty$. As promised, small ϕ_H corresponds to high temperatures. In Fig. 1, we demonstrate that the closed-form result for v_s^2 in Eqs. (6) and (7) matches a numerical solution for v_s^2 at large T.

III. SPEED OF SOUND IN OTHER MODELS

The result that we have obtained above for single-scalar holographic models fits into a pattern. All other calculations of v_s^2 in theories with gravity duals in the literature to date have found $v_s^2 \leq 1/3$ at high temperatures, at least in energetically stable systems. We review some representative results from the literature below.

First, we note that single-scalar models with potentials that do not fall into the class that we considered above [11] also obey $v_s^2 \leq 1/3$ at high temperatures.

The speed of sound has been calculated to be $1/5$ [12] in the finite-temperature generalization of the Sakai-Sugimoto model [13], which is dual to a strongly coupled field theory that contains fundamental matter [14]. It has also been shown that $v_s^2 \leq 1/3$ at high T in the $D3/D7$ system [15].

The speed of sound at high temperatures has also been computed in a 4D cascading gauge theory [16]. This strongly coupled theory has the effective gauge group $SU(K) \times SU(K + P)$ at high T, where $P \ll K$, and v_s^2 was calculated using a gravity dual to be

$$v_s^2 = \frac{1}{3} - \frac{4}{9} \frac{P^2}{K} + \mathcal{O}\left(\frac{P^4}{K^2}\right). \tag{8}$$

At high T, v_s^2 has also been calculated in $\mathcal{N} = 2^*$ gauge theory [17]. This theory is 4D $\mathcal{N} = 4$ super Yang-Mills theory at finite T, which is deformed by turning on small masses for the bosons and fermions in two of the $\mathcal{N} = 1$ chiral multiplets that are part of the $\mathcal{N} = 4$ gauge theory. To leading order in the deformations, it was found that

$$v_s^2 = \frac{1}{3} - \frac{2[\Gamma(3/4)]^4 m_f^2}{9\pi^4 T^2} + \mathcal{O}(m_b, m_f/T^4). \tag{9}$$

IV. SOUND BOUND CONJECTURE

From the examples above, it appears that $v_s^2 \leq 1/3$ at high temperatures in a much broader class of systems than just the single-scalar models, for which we were able to show a universal bound on the speed of sound at high temperatures. It is not clear just how broad the class of theories that satisfy the bound is. Motivated by the examples above, we conjecture that $v_s^2 \leq 1/3$ at high temperatures in all 4D theories with gravity duals, at least for energetically stable systems [18] at zero chemical potential.

It is not implausible that the bound holds more broadly. In all of the examples in the literature so far where v_s has been calculated in stable systems away from the high-T limit, v_s^2 remains less than $1/3$ [5,20]. Thus, the sound

bound $v_s^2 \leq 1/3$ may continue to hold away from the high-T limit for some broad class of theories.

It is a challenge for future work to determine the class of systems for which such a sound bound may apply. Note that the sound bound cannot apply to *all* field theories in nature. For instance, in QCD at zero temperature and nonzero isospin chemical potential μ_I, the speed of sound can be accurately calculated in chiral perturbation theory provided one works in the regime $\Lambda \gg \mu_I, m_\pi$ (where Λ is a typical hadronic scale and m_π is the pion mass) [21]. In the nontrivial phase with $\mu_I > m_\pi$,

$$v_s^2 = \frac{\mu_I^2 - m_\pi^2}{\mu_I^2 + 3m_\pi^2}, \tag{10}$$

so $v_s^2 \to 1$ as $m_\pi^2 \to 0$. Similar behavior was seen earlier in more *ad hoc* models [22,23].

The counterexample above shows that the sound bound cannot generally apply to all systems with chemical potentials. It is not yet known whether the bound might be violated in holographic systems at finite chemical potential. Ref. [24] showed that $v_s^2 \leq 1/3$ in a $D3/D7$ holographic model at finite isospin chemical potential where $\mu_I < m_\pi$, and it would be interesting to investigate v_s^2 when $\mu_I > m_\pi$ in systems with gravity duals.

There are some heuristic field-theoretic arguments that suggest that $v_s^2 \leq 1/3$ at high temperatures for systems at zero chemical potential. For example [25], suppose we write the entropy in terms of the number of effective degrees of freedom $N_{\rm eff}(T)$, so that $s = \frac{16\pi^2}{45} N_{\rm eff}(T) T^3$. Then we have $v_s^{-2} = 3 + T N_{\rm eff}'(T)/N_{\rm eff}(T)$. In asymptotically free theories, one expects that at high temperatures $N_{\rm eff}'(T) \geq 0$, with $N_{\rm eff}'(T) \to 0$ at large T. This makes it plausible that in such theories, which remain well defined in the UV, the speed of sound will approach $1/3$ from below. In fact, this is what happens in QCD at zero chemical potential [26].

Of course, the heuristic argument above only applies cleanly to theories that are weakly coupled at high temperatures, so it does not apply directly to theories with gravity duals. It was conjectured in Ref. [27] that for strongly coupled theories, $N_{\rm eff}(T)$ in the UV must be greater than in the IR. However, this by itself does not imply the sound bound unless $N_{\rm eff}(T)$ grows monotonically, which is not always the case.

V. CONCLUSIONS

We have shown that $v_s^2 \leq 1/3$ at high temperatures in theories with single-scalar gravity duals. Our techniques can be extended to show that $v_s^2 \leq 1/3$ in systems with multiple scalars and also to compute transport coefficients other than v_s^2, as will be discussed in a companion paper [28]. It would be interesting to study other strongly coupled systems with gravity duals to determine what class of systems satisfies the sound bound of $v_s^2 \leq 1/3$, and to

see if there are classes of systems where such a bound can apply beyond just the high-temperature limit. To investigate whether there is a class of field theories without gravity duals for which such a bound might apply, it would be useful to explore the $1/N_c$ and $1/\lambda$ corrections to v_s^2 in gauge/string duality.

APPENDIX

Consider the action Eq. (2) with $V(\phi)$ as in Eq. (5) and the metric ansatz Eq. (3). There are three independent equations of motion: the scalar equation and the *tt* and *rr* components of Einstein's equations. Let us work in the gauge $a = r$ so that $r \rightarrow \infty$ at the boundary. When ϕ vanishes identically, the general solution to the equations of motion is AdS$_5$-Schwarzschild. When ϕ is everywhere small, it assumes the profile

$$\phi(r) = \phi_0 {}_2F_1(1 - \Delta/4; \Delta/4; 1; 1 - r^4/r_h^4), \quad (A1)$$

where ϕ_0 is an integration constant that measures the smallness of the scalar. Equation (A1) is one solution to the linearized scalar equation of motion in AdS$_5$-Schwarzschild. We have discarded a second solution that is not regular at $r = r_h$, the location of the black hole horizon.

Now imagine fixing the entropy density s of AdS$_5$-Schwarzschild at some large s_0 in the high-temperature regime. We need to find the temperature that corresponds to s_0 when ϕ is turned on. To this end, we pursue perturbation expansions of the metric and the scalar in powers of ϕ_0. Working up to $\mathcal{O}(\phi_0^n)$ is sufficient for computing corrections to v_s^2 up to $\mathcal{O}(\phi_0^n)$. The metric backreacts on the scalar at odd orders in ϕ_0, and the scalar backreacts on the metric at even orders in ϕ_0. Meanwhile, the dual CFT Lagrangian has been deformed by the addition of the relevant term $\Lambda^{4-\Delta}\mathcal{O}_\phi$. Here, Λ is a new energy scale that also appears in the leading behavior of the scalar at the boundary

$$\phi \approx (\Lambda L)^{4-\Delta} r^{\Delta-4}. \quad (A2)$$

We must keep Λ fixed when computing equations of state, so we set $\Lambda L = 1$. Comparing the asymptotic form of Eq. (A1) as $r \rightarrow \infty$ with Eq. (A2) then gives rise to a relationship connecting r_h and ϕ_0:

$$r_h^{\Delta-4}\Gamma(\Delta/4)^2 = \phi_0\Gamma(\Delta/2 - 1). \quad (A3)$$

Fixing Λ thus imposes a consistency condition on the expansion. Small ϕ_0 necessarily corresponds to large r_h, which means that the exact single-scalar background only approaches AdS$_5$-Schwarzschild in the "conformal" high-temperature limit. The four boundary conditions we impose at each order in ϕ_0: (1) maintain $\Lambda L = 1$ and hence also Eq. (A3); (2) retain the horizon location $r = r_h$ so that s remains at s_0; (3) ensure that the solution is regular at $r = r_h$; and (4) preserve the boundary asymptotics $h \rightarrow 1$ and $b \rightarrow r/L$. We do not present our results for b and h here because they are cumbersome, but we note that $\phi_H = \phi_0$ up to $\mathcal{O}(\phi_0^2)$. v_s^2 is computed straightforwardly to $\mathcal{O}(\phi_H^2)$ by plugging the $\mathcal{O}(\phi_0^2)$ results for b and h into Eqs. (4), eliminating r_h using Eq. (A3), and subsequently applying the formula Eq. (1).

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