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Holography and the speed of sound at high temperatures

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We show that in a general class of strongly interacting theories at high temperatures the speed of sound approaches the conformal value $c_s^2 = 1/3$ universally from *below*. This class includes theories holographically dual to a theory of gravity coupled to a single scalar field, representing the operator of the scale anomaly.

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I. INTRODUCTION

The discovery of the holographic correspondence between string and gauge theories [1–3] has led to a number of applications to modelling properties of quark-gluon plasma of QCD in the regime relevant to heavy-ion collision experiments (see, e.g., Ref. [4] for a guide and references).

Although the precise form of the theory holographically dual to QCD is not known, many features of QCD can be implemented on the string theory side of the correspondence. Alternatively, one can pursue a bottom-up approach, by constructing an effective theory, or a model, with a *minimal* set of operators needed to describe the relevant physics. In this paper, we consider a general class of such models, describing the physics of scaling violation in QCD. In Refs. [5–7] such theories have been considered with the aim to model, or mimic, the equation of state of QCD known numerically from the lattice calculations. This has been achieved by tuning the potential $V(\phi)$ of the scalar field in the model.

The main goal of this paper is to establish properties of the equation of state $p(\epsilon)$, which are *universal* with respect to the choice of the potential $V(\phi)$. An obvious universal property is that the conformal symmetry is gradually restored, and thus $p(\epsilon) \to \epsilon/3$, as the temperature $T \to \infty$. This is natural since the temperature becomes the only relevant scale. It is a welcomed feature of the models since the same phenomenon occurs in QCD.

Here, we shall show that the *deviation* of the speed of sound $c_s^2 = dp/d\epsilon$ from the conformal limit $c_s^2 = 1/3$ is universally *negative* in such holographic models, at least to the leading order in inverse temperature.

At the outset, we must point out that $c_s^2 = 1/3$ is by no means a universal upper bound on the sound velocity. A counterexample is given by the speed of sound in QCD at large isospin chemical potential [8], where c_s can approach the speed of light in a certain limit. Nevertheless, a number of string theory examples [9–11] of holographically dual theories do indeed consistently show $c_s^2 \leq 1/3$.

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In QCD, Monte Carlo lattice calculations [12] show that this inequality is fulfilled. In the regime $T \gg \Lambda_{\mathrm{QCD}}$, this inequality follows from asymptotic freedom $\beta(\alpha) < 0$. For a pure N_c-color Yang-Mills theory at high T, for example, [13]

$$c_s^2 \simeq 1/3 + 5N_c/(36\pi)\beta(\alpha) < 1/3. \tag{1}$$

How general is the inequality $c_s^2 < 1/3$ and what are the prerequisites for it to hold? For example, does it hold in regimes where the theory is close to being conformal, but is still strongly interacting, as is apparently the case for QCD in the range of $T \sim (2\text{–}3)T_c$?

To investigate this question, we consider a class of theories, or models, which possess a holographically dual description. The minimal set of operators that we need to consider consists of the stress-energy tensor $T^{\mu\nu}$ and a scalar operator $\mathcal{O}$ necessary to provide a nonzero right-hand side of the scale (trace) anomaly equation. As an example, in QCD, the role of such an operator is played by $\mathrm{Tr}F^2$: $\theta = \beta(\alpha)/(8\pi\alpha^2)\,\mathrm{Tr}F^2$.

At this point it is helpful to express the energy density ϵ and pressure p in terms of the heat function (enthalpy) $w = \epsilon + p$ and the trace of the stress-energy tensor $\theta = \epsilon - 3p$:

$$\epsilon = (3w + \theta)/4; \qquad p = (w - \theta)/4. \tag{2}$$

Then

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{1 - d\theta/dw}{3 + d\theta/dw}. \tag{3}$$

In conformal theories $\theta \equiv 0$ and $c_s^2 = 1/3$. The inequality $c_s^2 < 1/3$ is equivalent to $d\theta/dw > 0$. This inequality is somewhat reminiscent, but not equivalent, to the inequality $d(pT^{-4})/dT = \theta/T^5 > 0$ conjectured in Ref. [14] and the weaker constraint $\int dT\theta/T^5 > 0$ proposed in Refs. [15,16].

Since the enthalpy w appears to play here the role of a more natural thermal parameter than T, it is helpful to keep in mind that w is a monotonous function of T: $dw/dT = c_v + s > 0$.

II. THE THEORY

The class of the five-dimensional (5D) holographic theories that we consider contains a scalar field ϕ, holographically dual to the (scaling violating) operator $\mathcal{O}$, coupled to the metric $g_{\mu\nu}$ (dual to $T^{\mu\nu}$). The fields are governed by the action

$$S_5 = \frac{1}{2\kappa^2}\left(\int_M d^5x\sqrt{-g}\left(R - V(\phi) - \frac{1}{2}(\partial\phi)^2\right) - 2\int_{\partial M} d^4x\sqrt{-\gamma}K\right), \tag{4}$$

where R is the Ricci scalar, g is the determinant of the metric, γ is the determinant of the induced metric on the UV boundary ∂M, K is the extrinsic curvature on ∂M, and κ^2 is the 5D Einstein gravitational constant. The value of κ^2 is inversely proportional to the number of the degrees of freedom in the dual four-dimensional theory, e.g., N_c^2 in a gauge theory with large number of colors N_c. The smallness of κ^2 (i.e., the largeness of the number of colors) controls the semiclassical approximation which we use. The last term in the action is the Gibbons-Hawking term, which removes the boundary terms arising upon integration by parts of the terms in R linear in second derivatives of the metric [17]. This boundary term does not affect classical equations of motion, but is essential for evaluating variations of the action with respect to the boundary values of the fields.

We assume that the potential $V(\phi)$ has an extremum at $\phi = 0$. The value of $V(0) = 2\Lambda = -12$, where Λ is the cosmological constant necessary to achieve proper asymptotics of the metric near the boundary.

It should be noted that we are going to treat the holographic theory with a method technically different from the one used in either Refs. [5,6] or Ref. [7]. Our approach is complementary to the existing ones and has its own advantages, which might go beyond the specific application which we consider in this paper.

According to the holographic correspondence, the correlation functions of the dual four-dimensional (4D) theory are equal to the variations of the 5D action under the changes of the boundary conditions on the 5D fields. We shall first determine the extremum of the action, and then consider first-order variations, which we can relate to energy density and pressure. The most general metric (up to general coordinate transformations) possessing three-dimensional Euclidean isometry is given by

$$ds^2 = \frac{1}{z^2}(-f(z)dt^2 + d\vec{x}^2) + e^{2B(z)}\frac{dz^2}{z^2 f(z)}. \tag{5}$$

The functions $B(z)$ and $f(z)$ depend on the holographic coordinate z and will be determined by extremizing the action. The boundary ∂M is located at $z = \varepsilon$ with ε acting as an UV regulator. In relations involving only physical quantities ε can be taken to 0.

Substituting the metric of Eq. (5) into the equations of motion, we find

$$\dot{B} = -\tfrac{1}{6}\dot{\phi}^2, \tag{6}$$

$$\ddot{f} = (4 + \dot{B})\dot{f}, \tag{7}$$

$$-6\dot{f} + f(24 - \dot{\phi}^2) + 2e^{2B}V(\phi) = 0, \tag{8}$$

$$\ddot{\phi}f + \dot{\phi}(\dot{f} - f(4 + \dot{B})) - e^{2B}V'(\phi) = 0, \tag{9}$$

where a dot denotes a $\log z$ derivative, e.g., $\dot{\phi} = zd\phi/dz$, while $V' = dV/d\phi$. The holographic correspondence provides the boundary conditions at the UV boundary $z = \varepsilon$. The boundary condition on ϕ,

$$\phi(\varepsilon) = c\varepsilon^{\Delta_-}, \tag{10}$$

corresponds to introducing a term $c\mathcal{O}$ into the action of the 4D theory dual to the theory in Eq. (4), where c is the source of the scalar operator $\mathcal{O}$, dual to the field ϕ. The dimensions of the source and the operator are given by

$$[c] = \Delta_-; \qquad [\mathcal{O}] = \Delta_+; \qquad \Delta_+ + \Delta_- = 4. \tag{11}$$

The metric must approach the Lorentz invariant form at the UV boundary, hence,

$$f(\varepsilon) = 1. \tag{12}$$

Equation (7) can be integrated once to give

$$\dot{f} = -wz^4e^B. \tag{13}$$

The integration constant w must be positive if the metric is to possess a horizon $f(z_H) = 0$ at some value of z_H.

There are no more independent boundary conditions. The boundary value of B is determined by Eq. (8), which is algebraic in B. The role of the second boundary condition for Eq. (9) is played by the requirement that ϕ is finite at the horizon, $z = z_H$, which is a regular singular point of the second order differential Eq. (9).

Thus, we find a two-parameter family of solutions. These parameters are c from Eq. (10) and w from Eq. (13). Different members of this family are related by rescaling $z \to z/\lambda$, $w \to \lambda^4 w$, $c \to \lambda^{\Delta_-}c$, which represents scale invariance inherent in the action given by Eq. (4).

Near the $z = \varepsilon \to 0$ boundary, $\phi \to 0$, $B \to 0$, $\dot{B} \to 0$, and $\dot{f} \to 0$. Equation (9) for ϕ can be linearized and the asymptotic behavior of ϕ near the boundary can be determined easily:

$$\phi(z) \to (c - d\varepsilon^{\Delta_+ - \Delta_-})z^{\Delta_-} + dz^{\Delta_+} + \ldots, \tag{14}$$

where the curvature of the potential $V''(0) \equiv m^2$ determines the indices $\Delta_\pm = 2 \pm \sqrt{4 + m^2}$. The coefficient of the first term is related to c by Eq. (10). The coefficient d of the second linearly independent solution should be determined by the finiteness condition at the horizon.

By calculating the derivative of the 5D action with respect to c and matching it, by holographic correspondence, to the expectation value $\langle\mathcal{O}\rangle$, one finds [18]

$$\langle\mathcal{O}\rangle = -\partial S_5/\partial c = -d(\Delta_+ - \Delta_-). \tag{15}$$

III. EQUATION OF STATE

Rather than following existing approaches to calculating the speed of sound, based on the relation $c_s^2 = d\log T/d\log s$, [5,6] or looking at the poles of the two-point function for the sound channel [19,20], we shall directly calculate *one*-point functions $\langle T^{tt}\rangle$ and $\langle T^{xx}\rangle$ and use the relation $c_s^2 = dp/d\epsilon$. As we shall see, this quickly yields some closed-form results, not evident in the alternative approaches.

In order to calculate the one-point function, such as $\langle T^{tt}\rangle$, we observe that the generalization of the 4D theory to a curved 4D background with metric $h_{\mu\nu}$ corresponds, holographically, to imposing the following boundary condition on the bulk 5D metric

$$g_{\mu\nu}(\varepsilon) = h_{\mu\nu}\varepsilon^{-2}. \tag{16}$$

Holographic correspondence then gives us

$$\langle T^{\mu\nu}\rangle = 2\delta S_5/\delta h_{\mu\nu}. \tag{17}$$

Variation of the boundary condition causes variations of the metric in the bulk, as it follows the equations of motion. But since we are varying around the extremum of the action, the only contribution to the first-order variation comes from the boundary. This contribution consists of the variation of the boundary terms, which appear during integration by parts while deriving equations of motion for, e.g., g_{tt}, as well as the variation of the Gibbons-Hawking term. Altogether this gives

$$\langle T^{tt}\rangle = -\frac{6}{\varepsilon^4}e^{-B(\varepsilon)}; \qquad \langle T^{xx}\rangle = w - \langle T^{tt}\rangle. \tag{18}$$

Both of these quantities are divergent as $\varepsilon \to 0$. This represents the familiar vacuum energy divergence in the quantum field theory. A simple vacuum subtraction

$$\epsilon = \langle T^{tt}\rangle - \langle T^{tt}\rangle_{T=0}, \qquad p = \langle T^{xx}\rangle - \langle T^{xx}\rangle_{T=0}, \tag{19}$$

takes care of this, and we find for the energy density and the pressure, after solving Eq. (8) for B at $z = \varepsilon \to 0$ with ϕ from Eq. (14):

$$\begin{aligned}\epsilon &= (3w - cd\Delta_-(\Delta_+ - \Delta_-))/4,\\ p &= (w + cd\Delta_-(\Delta_+ - \Delta_-))/4.\end{aligned} \tag{20}$$

We can now conclude that the integration constant introduced in Eq. (13) is the enthalpy, $w = \epsilon + p$, of the corresponding 4D field theory. We also observe that the scale anomaly

$$\theta = \epsilon - 3p = -cd\Delta_-(\Delta_+ - \Delta_-) = \Delta_- c\langle\mathcal{O}\rangle \tag{21}$$

is related to the expectation value of the operator $\mathcal{O}$ [using Eq. (15)]. This relation is easy to derive directly on the field theory side by observing that the violation of the scale invariance comes from the dimensionful parameter c. Naturally, it is proportional to the dimension of c, Δ_-.

From Eq. (20), the speed of sound can be expressed now in terms of the derivative $\partial d/\partial w$ at fixed c and dimensions $\Delta_\pm$:

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{1 + c\Delta_-(\Delta_+ - \Delta_-)(\partial d/\partial w)}{3 - c\Delta_-(\Delta_+ - \Delta_-)(\partial d/\partial w)}. \tag{22}$$

It should be emphasized that this equation is an *exact* expression valid for *all* temperatures.

IV. HIGH TEMPERATURE

The high-temperature limit is equivalent to the small c limit, since c is the only other dimensionful parameter in the theory, and high-temperature expansion is controlled by the dimensionless parameter c/T^{Δ_-}, or $cw^{-\Delta_-/4}$. In this limit,

$$c_s^2 = 1/3 + (4/9)c\Delta_-(\Delta_+ - \Delta_-)(\partial d/\partial w) + \ldots. \tag{23}$$

We shall now show that, even though generally the dependence of d on w can be only found numerically, in the high-temperature (i.e., high w) limit an analytic expression can be found for arbitrary potential $V(\phi)$.

One can begin by observing that at large w the function f varies very rapidly according to Eq. (13). This means one can neglect variations of the function B between the boundary $z = \varepsilon$ and the horizon $z = z_H$, since z_H becomes small (as $w^{-1/4}$). Since on the boundary $B = 0$ (up to terms of order $\varepsilon^{2\Delta_-}$, negligible here, according to Eq. (8)), we find from Eq. (13)

$$f(z) = 1 - wz^4/4. \tag{24}$$

Another consequence is that ϕ, which is small at $z = \varepsilon$, remains small up to z_H ($\phi \sim cz_H^{\Delta_-} \sim c/T^{\Delta_-} \ll 1$), and the linearized approximation to Eq. (9) is valid not only near the boundary, but all the way to the horizon. With $B = 0$ and f from Eq. (24) we obtain

$$\left(1 - \frac{1}{4}wz^4\right)\phi'' - \left(\frac{3}{z} + \frac{wz^4}{4}\right)\phi' - \frac{m^2}{z^2}\phi = 0. \tag{25}$$

Equation (25) can be solved analytically

$$\begin{aligned}\phi(z) &= cz^{\Delta_-}{}_2F_1(\Delta_-/4, \Delta_-/4, \Delta_-/2, wz^4/4)\\ &\quad + dz^{\Delta_+}{}_2F_1(\Delta_+/4, \Delta_+/4, \Delta_+/2, wz^4/4),\end{aligned} \tag{26}$$

where the coefficients follow the notations of Eq. (14) (up to terms $\mathcal{O}(\varepsilon^{\Delta_+ - \Delta_-})$, here negligible). Both linearly independent solutions are logarithmically divergent at the horizon $z = z_H$, where $wz_H^4/4 = 1$. The condition $|\phi(z_H)| < \infty$ requires us to select the linear combination in which these divergences cancel. This fixes d in terms of

c:

$$d = -cw^{(\Delta_+ - \Delta_-)/4} D(\Delta_-), \tag{27}$$

where the function $D(\Delta_-) = 1/D(\Delta_+)$ is given by

$$D(\Delta_-) = \frac{\pi 2^{\Delta_-}}{2 - \Delta_-} \cot(\pi\Delta_-/4) \frac{\Gamma(\Delta_-/2)^2}{\Gamma(\Delta_-/4)^4}. \tag{28}$$

Substituting Eq. (27) into Eq. (22) we find our main result

$$c_s^2 = \frac{1}{3} - \frac{1}{9} c^2 \Delta_- (\Delta_+ - \Delta_-)^2 w^{-\Delta_-/2} D(\Delta_-) + \ldots. \tag{29}$$

It is clear that the correction term is *negative* for all values of $0 < \Delta_- < 2$ (i.e., $2 < \Delta_+ < 4$). As expected, the correction vanishes with the power of w dictated by the dimension of the operator $\mathcal{O}$.

V. DISCUSSION

When this work was completed, the authors learned about a similar result [21], obtained using the relation $c_s^2 = d\log T/d\log s$. In Ref. [21] the speed of sound is expressed in terms of the value ϕ_H of the scalar field at the horizon (since both s and T are calculated at the horizon) $c_s^2 = 1/3 - C\phi_H^2 + \ldots$. The coefficient C is then given by an integral from the boundary to the horizon of the square of a hypergeometric function, which is manifestly positive. We have verified that, upon evaluating the integral, the result of Ref. [21] coincides with Eq. (29). For completeness of comparison we evaluate ϕ_H explicitly by using Eq. (26) and applying relation Eq. (27)

$$\phi_H = cw^{\Delta/4-1} 2^{-\Delta/2} (2\Delta - 4) \frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2)}, \tag{30}$$

where $\Delta = \Delta_+$. Using Eq. (29) we then find

$$c_s^2 = \frac{1}{3} - \frac{1}{18\pi} (4 - \Delta)(4 - 2\Delta) \tan(\pi\Delta/4) \phi_H^2 + \ldots. \tag{31}$$

As we have pointed out already, and as Eq. (30) demonstrates explicitly, in the high-temperature (large w) limit the value of ϕ remains small everywhere between the boundary and the horizon. This is the origin of the universality we find. The speed of sound near the high T limit depends only on the behavior of the scalar potential near the origin, i.e., specifically, on $V''(0) = m^2 = \Delta(\Delta - 4)$.

One can interpret and further generalize our results in the following way. In the high-temperature limit, Eq. (21) becomes $\theta = \Delta_- c^2 \chi_{\mathcal{O}} + \ldots$. The susceptibility $\chi_{\mathcal{O}} \equiv \partial\langle\mathcal{O}\rangle/\partial c$ can be related to the effective potential for $\langle\mathcal{O}\rangle$, defined as the Legendre transform of the generating functional $W(c)$ (in holography $W = -S_5$), i.e., $\Gamma(\langle\mathcal{O}\rangle) \equiv W(c) + c\langle\mathcal{O}\rangle$, as $\chi_{\mathcal{O}} = 1/\Gamma''(0)$. Stability implies $\Gamma''(0) > 0$, and thus $\chi_{\mathcal{O}} > 0$. Consequently, $\theta > 0$, as conjectured in [14]. Holographic models confirm these expectations according to Eqs. (15) and (27).

We also find that $d\theta/dw > 0$, which requires $d\chi_{\mathcal{O}}/dw > 0$. That means that the curvature $\Gamma''(0) = \chi_{\mathcal{O}}^{-1}$ *decreases* with temperature. This behavior is unusual, if one recalls that in a weakly coupled $\lambda\phi^4$ scalar theory, the leading perturbative temperature correction to the curvature ($\sim \lambda T^2$) increases with temperature (e.g., the restoration of a broken symmetry is a well-known manifestation of this). The opposite behavior of the curvature of the effective potential $\Gamma(\langle\mathcal{O}\rangle)$ can be understood by counting dimensions $[\chi_{\mathcal{O}}^{-1}] = \Delta_- - \Delta_+ < 0$, which means $\chi_{\mathcal{O}}^{-1} \sim T^{\Delta_- - \Delta_+}$ and decreases with T.

In conclusion, we have shown that in a quite general class of gravity dual theories with a single scalar operator representing the scale anomaly the speed of sound always approaches the conformal value $c_s^2 = 1/3$ from *below*.

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] S. S. Gubser and A. Karch, arXiv:0901.0935.
[5] S. S. Gubser and A. Nellore, Phys. Rev. D **78**, 086007 (2008).
[6] S. S. Gubser, A. Nellore, S. S. Pufu, and F. D. Rocha, Phys. Rev. Lett. **101**, 131601 (2008).
[7] U. Gursoy, E. Kiritsis, L. Mazzanti, and F. Nitti, J. High Energy Phys. 05 (2009) 033.
[8] D. T. Son and M. A. Stephanov, Yad. Fiz. **64**, 899 (2001) [Phys. At. Nucl. **64**, 834 (2001)].
[9] A. Buchel and J. T. Liu, J. High Energy Phys. 11 (2003) 031.
[10] P. Benincasa, A. Buchel, and A. O. Starinets, Nucl. Phys. **B733**, 160 (2006).
[11] A. Buchel, Nucl. Phys.**B708**, 451 (2005).
[12] F. Karsch, Proc. Sci., POD07 (2007) 026.

[13] J. I. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1993).
[14] J. D. Bjorken, Phys. Rev. D **27**, 140 (1983).
[15] T. Appelquist, A. G. Cohen, and M. Schmaltz, Phys. Rev. D **60**, 045003 (1999).
[16] T. Appelquist, A. G. Cohen, M. Schmaltz, and R. Shrock, Phys. Lett. B **459**, 235 (1999).
[17] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[18] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).
[19] P. K. Kovtun and A. O. Starinets, Phys. Rev. D **72**, 086009 (2005).
[20] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. 12 (2002) 054.
[21] A. Cherman, T. D. Cohen, and A. Nelore, arXiv:0905.0903.