Mass-dependent β-function

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(Received 10 August 2009; published 29 September 2009)

Threshold effects related to fermion masses are considered for an all-order β-function based on a background field momentum subtraction scheme. Far away from all thresholds, the suggested β-function reduces to the conjectured all-order form inspired by the Novikov–Shifman–Vainshtein–Zakharov β-function of $\mathcal{N} = 1$ supersymmetric gauge theories with a fixed integer number of fermion flavors. At (formally) infinite masses the corresponding pure Yang–Mills β-function is recovered. We discuss applications to the phase diagram of non-Abelian field theories.

I. INTRODUCTION

The β-function of non-Abelian gauge field theories is scheme independent (universal) up to two loops. This statement holds if a fixed number of active matter species is considered. Regarding the renormalization group running of the coupling for a fixed number of flavors is legitimate as long as the energy scales under consideration are either far above or below the fermion masses, that is, the relevant thresholds. From the viewpoint of renormalization theory, it is legal to use the fixed flavor schemes even close or across the thresholds, although the freezing out (By default, we evolve from higher to lower energy scales.) of heavy flavors is ignored. Sometimes this effect is incorporated by piecing together β-functions for different integer numbers of flavors exactly at the mass of the flavors which become inactive. While this procedure might appear somewhat *ad hoc*, it features already a crucial aspect of the threshold effect, the scale dependence of the β-function coefficients. They are no longer scheme independent in the above sense. The switching also involves a blending between different schemes. It is plausible that the transition between different numbers of active flavors at a threshold should happen gradually. Therefore, we base the following investigation on a background field momentum subtraction scheme [1], which features naturally a smooth switching. Momentum subtraction schemes [2–6] respect the decoupling theorem [7], but, in general, spoil the Slavnov–Taylor identities. The latter shortcoming is cured by the use of the background field method [8]. Alternatively, we could have used the physical charge approach of Ref. [9]. The outcome, however, is qualitatively and quantitatively close to that in the background field momentum subtraction scheme (See especially Fig. 2 in [9].) and there the expressions can be handled analytically.

An all-order β-function without threshold effects has been conjectured in [10]. It is inspired by the Novikov–Shifman–Vainshtein–Zakharov β-function of $\mathcal{N} = 1$ supersymmetric gauge theories [11]. The conjectured β-function possesses the correct limits for exactly known

cases, like super Yang–Mills theory [11] or planar equivalence in a large-N_c limit [12]. At two-loop order it coincides with the universal β-function.

We base our suggestion for a mass-dependent all-orders β-function on the postulate that, far away from all thresholds, it reduces to the just mentioned all-order β-function for a fixed number of flavors. At two-loop order it is to be identical with the β-function in background field momentum subtraction scheme [1]. From the thus obtained mass-dependent β-function we find that threshold effects are felt 2 orders of magnitude away from the mass of the fermion. (See Fig. 5.)

One field of research where the mass dependence of the β-function is of importance is the conformal window of non-Abelian gauge field theories. In the picture laid out in [13], the interplay between the matter content of a theory and chiral symmetry breaking gives rise to different phases as follows (See Fig. 1.): For no or only little matter the (dominant) antiscreening of the non-Abelian gauge bosons inhibits the occurrence of an infrared fixed point (A). This is the case for quantum chromodynamics. For slightly more matter a perturbative Caswell–Banks–Zaks [14] fixed point develops. When arguing based on the two-loop β-function, this amounts to a change of the sign of the second coefficient (B). This fixed point, however, need not be realized, as, for an insufficient matter content, the value of the coupling may suffice to trigger chiral symmetry breaking. In that case, the fermions acquire a dynamical mass and decouple, at least in parts, from the dynamics. The effective number of flavors is reduced and the antiscreening dominates again (C). Only above a certain amount of matter, the fixed point is reached before chiral symmetry breaking sets in. The coupling constant freezes (D). Just before this, the quasiconformal case is to be found, where the fixed point is almost realized (E). There the β-function is small for a value of the coupling α close to its critical value for chiral symmetry breaking. Consequently, the coupling stays almost constant (it walks) for a large interval of scales before chiral symmetry breaking is triggered and the coupling constant starts running again. Beyond a given amount of matter the theory loses
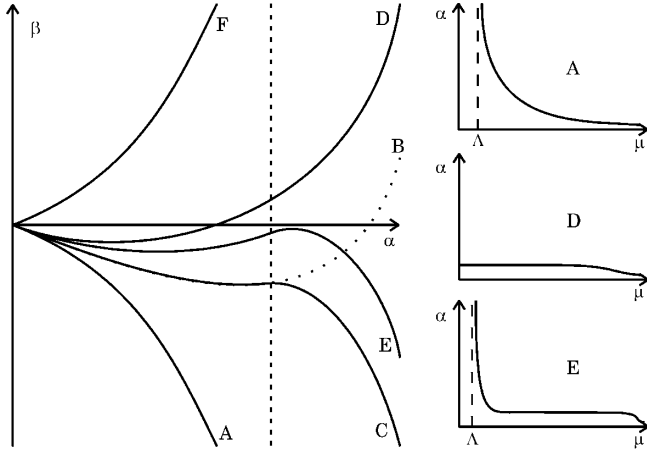
FIG. 1.   behavior of the β-function as a function of the coupling α and of the coupling as a function of the energy scale μ, depending on the matter content of the theory. (A) No or little matter; (B) existence of a perturbative Caswell–Banks–Zaks fixed point; (C) actual shape due to chiral symmetry breaking; (D) realized fixed point; (E) quasiconformal case; (F) loss of asymptotic freedom. The dashed line in the plot on the left-hand side indicates the critical values of the coupling for chiral symmetry breaking.

asymptotic freedom. The first coefficient of the β-function changes sign. This discussion indicates why the acquisition of (more) mass by the fermions is a crucial building block for the understanding of the dynamics of the theory.

The motivation for carrying out the investigation in [13] was the determination of walking, that is, quasiconformal [15] technicolor [16] models, which are consistent with available electroweak precision data [17,18]. {Walking technicolor models possess a rich collider phenomenology for the LHC [19] as well as dark matter candidates [20] and are also interesting models for studies in the AdS/CFT framework [21].} In technicolor models, chiral symmetry breaking among fermions (techniquarks) added to the standard model without Higgs sector simultaneously breaks the electroweak symmetry dynamically. Technicolor hence provides the masses for the weak gauge bosons. The most intensely researched way to also render massive the standard model fermions is extended technicolor [22]. Apart from this primary purpose, extended technicolor may serve to make extra Nambu–Goldstone-modes (Three of the potentially more modes arising from the chiral symmetry breaking are absorbed as the longitudinal modes of the weak gauge bosons.) sufficiently massive, and or stabilize the vacuum alignment [23,24]. The effects from extended technicolor are already in place before the chiral condensate of technicolor forms. Therefore, as shall be explained below, some of the effects connected to the stabilization of the vacuum alignment and the Nambu–Goldstone masses are similar in nature to an explicit mass term in the sense of the electroweakly induced quark masses relative to the chiral dynamics of quantum chromodynamics.

The just described occurrence of walking is due to inherently nonperturbative effects from chiral symmetry breaking. In the context of the mass-dependent β-function discussed here, we would like to point out that there is also the possibility to see walking from an interplay of more perturbative effects: Let us think of an asymptotically free gauge theory, which would have an infrared fixed point for massless fermions. "Hard" fermion masses in the sense of, for example, the electroweakly induced quark masses as seen by quantum chromodynamics lead to a reduction of the screening due to the fermions. If this occurs at the correct energy scale (Here, the occurrence of walking depends on the initial condition for the coupling constant.), the infrared fixed point is never reached, but a situation analogous to the one depicted in Fig. 1(e) can arise.

The paper is organized as follows. In Sec. II, we derive our mass-dependent all-order β-function. Section III contains the application to the question of the determination of the conformal window. The subsection under Sec. III studies the quantitative influence of fermion masses on the (quasi)conformal window. Section IV summarizes the results. The appendix contains plots of the (quasi)conformal windows for gauge field theories with fermions transforming under various representations of $SU(N_c)$, $Sp(2N_c)$, and $SO(N_c)$ gauge groups.

II. A MASS-DEPENDENT β-FUNCTION

The β-function describes the change of the gauge coupling g of non-Abelian gauge field theories with the energy scale μ. It is subtraction scheme independent up to two loops,

$$\beta(g) = -\frac{\beta_0}{(4\pi)^2} g^3 - \frac{\beta_1}{(4\pi)^4} g^5 - \dots, \qquad (1)$$

$$\beta_0 = \tfrac{11}{3} C_2(G) - \tfrac{4}{3} T(R) N_f, \qquad (2)$$

$$\beta_1 = \tfrac{34}{3} C_2(G)^2 - \tfrac{20}{3} C_2(G) T(R) N_f - 4 C_2(R) T(R) N_f. \quad (3)$$

In the background field momentum subtraction scheme, however, the first two β-function coefficients are different [1],

$$\beta_0 \mapsto \bar{\beta}_0 = \frac{11}{3} C_2(G) - \frac{4}{3} T(R) \sum_{j=1}^{N_f} b_0(x_j), \qquad (4)$$

$$\beta_1 \mapsto \bar{\beta}_1 = \frac{34}{3} C_2(G)^2 - T(R) \sum_{j=1}^{N_f} b_1(x_j). \qquad (5)$$

Here, $x_j = -\mu^2/(4m_j^2)$, m_j is the mass of the fermion flavor j. Further,

$$b_0(x) = 1 + 3[1 - G(x)]/(2x), \qquad (6)$$

(which coincides with the result from the physical charge approach [25]) where $G(x) = (2y \ln y)/(y^2 - 1)$ as well as

$y = (\sqrt{1-1/x} - 1)/(\sqrt{1-1/x} + 1)$ and

$$b_1(x) = \frac{16(1-x^2)C_2(R) + (1 + 8x^2)C_2(G)}{6x^2(1-x)}\sigma(x) - \frac{2}{3x^2}(C_2(G) - 2C_2(R))I(x) + \frac{2}{3x}\tilde{I}_3^{(4)}(x)C_2(G)$$

$$+ [(1 + 3x - 10x^2 + 12x^3)C_2(G) - 3(3 - 3x - 4x^2 + 8x^3)C_2(R)]\frac{4}{3x}G(x)^2 - [(147 - 4x - 100x^2 + 8x^3)C_2(G)$$

$$+ 168(1-x)C_2(R) + 6(9 + 4x)\ln(-4x)C_2(G)]\frac{1}{9x}G(x) + [(99 + 62x)C_2(G) + 12(11 + 3x)C_2(R)$$

$$+ 2(27 + 24x - 2x^2)\ln(-4x)C_2(G)]\frac{1}{9x}, \tag{7}$$

$$\sigma(x) = \{2\,\mathrm{Li}_2(-y) + \mathrm{Li}_2(y) + [\ln(1-y) + 2\ln(1+y) \\ - (3/4)\ln y]\ln y\}(1 - y^2)/y \tag{8}$$

$$I(x) = 6[\zeta_3 + 4\,\mathrm{Li}_3(-y) + 2\,\mathrm{Li}_3(y)] - 8[2\,\mathrm{Li}_2(-y) \\ + \mathrm{Li}_2(y)]\ln y - 2[2\ln(1+y) + \ln(1-y)]\ln^2 y, \tag{9}$$

$$\tilde{I}_3^{(4)}(x) = 6\zeta_3 - 6\,\mathrm{Li}_3(y) + 6\,\mathrm{Li}_2(y)\ln y + 2\ln(1-y)\ln^2 y. \tag{10}$$

Here, $\zeta_3 = \zeta(3) = 1.2020569\ldots$ stands for Apéry's constant, that is, Riemann's zeta function evaluated at 3. $\mathrm{Li}_n(z) = \sum_{j=1}^{\infty}\frac{z^j}{j^n}$ is the polylogarithm.

In the background field method, the β-function is obtained from the background field renormalization constant Z_A according to

$$\mu^2\frac{d}{d\mu^2}\frac{\alpha}{4\pi} = \lim_{\epsilon\to 0}\alpha\mu\frac{d}{d\mu}\ln Z_A. \tag{11}$$

Here, $\alpha = g/(4\pi)$ and ϵ emanates from dimensional regularization, $d = 4 - 2\epsilon$. (Complications arising from the renormalization of the gauge parameter can be avoided by adopting background field Landau gauge, but the physical result does not depend on this choice.) Consequently, only background field propagator diagrams have to be evaluated, that is, diagrams with two external couplings to the external field and no other external legs. Z_A is obtained from,

$$Z_A = \frac{1 + \Pi_0(Q^2, \mu^2, \{m_j^2\})}{1 + \Pi(Q^2, \mu^2, \{m_j^2\})}, \tag{12}$$

where Π_0 is the bare and Π the renormalized self-energy amplitude. The renormalized masses $\{m_j\}$ of the fermions be defined as poles of the corresponding propagators. In the momentum subtraction scheme, the renormalized self-energy amplitude is fixed by the condition

$$\Pi(Q^2, \mu^2, \{m_j^2\})|_{Q^2=\mu^2} = 0. \tag{13}$$

After computing the renormalized self-energy amplitude to two-loop order, the corresponding β-function can be extracted by differentiation according to Eq. (11). In this context, $I(x)$ and $\tilde{I}_3^{(4)}(x)$ are master integrals appearing in the computation of Π.

Let us consider N_f mass degenerate flavors, all transforming under the same representation of the gauge group. Then the modifications of $\bar{\beta}_0$ and $\bar{\beta}_1$ relative to β_0 and β_1 can be collected in "numbers of active flavors"

$$N_{f,0}/N_f = b_0(x), \tag{14}$$

$$N_{f,1}/N_f = b_1(x)/[(20/3)C_2(G) + 4C_2(R)]. \tag{15}$$

Both go to N_f for massless fermions and to zero for very massive fermions, thereby satisfying the decoupling theorem [7]. The concept of number of active flavors can even be given a gauge invariant meaning [9], which makes such numbers observable, in principle. This, however, raises all the more the question why there should be different numbers of active flavors in each term of the β-function. Furthermore, $N_{f,0}$ interpolates monotonously between the two limiting cases N_f and zero, which would be expected from a number of active flavors, while $N_{f,1}$ does not. The latter is, in general, not even positive for all values of x. This suggests that the interpretation as active number of flavors is more appropriate for $N_{f,0}$ than it is for $N_{f,1}$. We shall revisit this point below.

The apparent contradiction to the statement about scheme independence of the two-loop β-function arises from the fact that the background field momentum subtraction scheme is not a fixed subtraction scheme in the above sense, but can be seen as a smooth interpolation between subtraction schemes as a function of the scale [5]. The transition takes place when the scale closes in on the mass of a flavor, which then freezes out gradually. We can also think about the other way to incorporate the change of the number of active flavors with the energy scale, that is, the matching of running couplings for different integer numbers of flavors. The matching is performed exactly where the scale meets the mass of a given flavor. In this sense, also the two-loop β-function coefficients are functions of the scale and do not coincide with the subtraction scheme independent expressions for a fixed number of flavors.

Based on the argument of universality in [10], an all-order β-function was conjectured,

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \frac{\beta_0 - \frac{2}{3} T(R) N_f \gamma(g^2)}{1 - \frac{g^2}{8\pi^2} C_2(G)(1 + 2\frac{\beta_0'}{\beta_0})}, \qquad (16)$$

where $\beta_0' = C_2(G) - T(R)N_f$. γ stands for the anomalous dimension of the fermion mass operator, $\gamma(g^2) = (3/2)C_2(R)g^2/(4\pi^2) + O(g^4)$. An expansion to two-loop order reproduces the universal two-loop coefficients. Many expression would achieve this, but, as shown in [10], (16) is consistent with limiting cases in which the β-function is known exactly like super Yang–Mills theory [11] or planar equivalence in a large-N_c limit [12].

Our objective is to find a mass-dependent all-order β-function that when expanded to second order reproduced the two-loop β-function in the background field momentum subtraction scheme. (If we chose a different mass-dependent two-loop β-function as starting point the resulting all-order β-function would be accordingly different.) Further, in the massless limit the mass-dependent all-order β-function is to coincide with the mass-independent all-order β-function. This ascertains that the exactly known supersymmetric results can be reproduced. Likewise, in the ultramassive limit it is to coincide with the pure Yang–Mills version of the mass-independent all-order β-function. This implies also that all terms involving the Casimir $C_2(R)$ have to be absorbed in the term involving the anomalous dimension.

We propose the following mass-dependent β-function,

$$\bar{\beta}(g) = -\frac{g^3}{(4\pi)^2} \frac{\bar{\beta}_0 - \frac{2}{3} T(R) \sum_{j=1}^{N_f} \bar{\gamma}(x_j)}{1 - \frac{g^2}{8\pi^2} C_2(G)(1 + 2\frac{\bar{\beta}_0'}{\bar{\beta}_0})}, \qquad (17)$$

where

$$\bar{\gamma}(x_j) = \frac{3}{2} \frac{g^2}{4\pi^2} \frac{1}{4} b_1(x_j)|_{C_2(G)\to 0} \qquad (18)$$

and

$$\bar{\beta}_0' = C_2(G) + T(R) \sum_{j=1}^{N_f} \left[\frac{2}{3} b_0(x_j) - \frac{1}{4} \frac{b_1(x_j)|_{C_2(R)\to 0}}{C_2(G)} \right].$$

It fulfils all of the aforementioned requirements: Taking all masses m_j to zero reproduces Eq. (16). In fact, all barred, that is, mass-dependent quantities $\bar{\beta}_0$, $\bar{\beta}_1$, $\bar{\beta}_0'$ and especially, $\bar{\gamma}$ go to their unbarred, that is, massless counterparts separately. $\bar{\gamma}(x_j)/\bar{\gamma}(x_j \to -\infty)$ has the same limits, 0 for $x \to 0$ and 1 for $x \to -\infty$, as found from the mass-dependent anomalous dimension $\gamma_m = \gamma_{m=0}[1 + \ln(1 - 4x)/(4x)]$ given in [3]. Expanding to two-loop order reproduces Eqs. (4) and (5). [This last feature would also be achieved if a factor $1 + O(g^2)$ was introduced in $\bar{\gamma}$ and/or $\bar{\beta}_0'$. (The same is already true in the massless case.) Its effect would, however, only make an appearance at third order and could, hence, be absorbed in a change of the renormalization scheme. Likewise, if one wanted to accommodate a particular three-loop term, one could include

such a factor and adjust the denominator accordingly, which would amount to a change of scheme.]

Like Eq. (16), Eq. (17) can be generalized to allow for flavors which transform under different representations of the gauge group. In this case it reads

$$\bar{\beta}(g) = -\frac{g^3}{(4\pi)^2} \frac{\bar{\beta}_0 - \frac{2}{3} \sum_{j=1}^{N_f} T(R_j) \bar{\gamma}(x_j)}{1 - \frac{g^2}{8\pi^2} C_2(G)(1 + \frac{\bar{\beta}_0'}{\bar{\beta}_0})}, \qquad (19)$$

where

$$\bar{\beta}_0' = C_2(G) + \sum_{j=1}^{N_f} T(R_j) \left[\frac{2}{3} b_0(x_j) - \frac{1}{4} \frac{b_1(x_j)|_{C_2(R)\to 0}}{C_2(G)} \right] \qquad (20)$$

and R_j is the representation of flavor j.

Through Eq. (17), the coupling g now depends on the anomalous dimension γ in a twofold way: As in Eq. (16), there is a direct dependence in the numerator. Here additionally, the β-function depends on the masses, which via the renormalization group equations

$$\mu \frac{d}{d\mu} g(\mu) = +\beta[g(\mu), \{m_k(\mu)/\mu\}], \qquad (21)$$

$$\mu \frac{d}{d\mu} m_j(\mu) = -\gamma[g(\mu), \{m_k(\mu)/\mu\}], \qquad (22)$$

are functionals of the anomalous dimension.

III. IMPACT ON THE CONFORMAL WINDOW

Here, predominantly for the sake of notational simplicity, we concentrate on walking technicolor models with two techniflavors, the generalization to more flavors being straightforward. (The most intensely studied technicolor models, minimal and next-to-minimal walking technicolor [17,18,26], feature two techniflavors.)

Models in which the techniquarks do not transform under a (pseudo)real representation of the technicolor gauge group have the essential $SU(2)_L \times SU(2)_R$ flavor symmetry. When it breaks to $SU(2)_V$, the electroweak $SU(2)_L \times U(1)_Y$ breaks to $U(1)_{em}$, regardless of the embedding [23]. The breaking leads to three pseudo-Nambu–Goldstone-modes, which become the longitudinal degrees of freedom of the weak gauge bosons.

Models with techniquarks in (pseudo)real representations of the technicolor gauge group have an $SU(4)$ unbroken flavor symmetry. For real representations it breaks to $SO(4)$, which leaves behind six Nambu–Goldstone-modes beyond the 3 degrees of freedom, which are absorbed as longitudinal degrees of freedom of the weak gauge bosons. They have to be sufficiently massive, as these potentially lightest states of a technicolor theory have not been detected to date. It turns out that electroweak radiative corrections can take them outside the direct ex-

clusion limit for technimesonic pseudoscalars [24], which is slightly above the mass of the weak gauge bosons [27]. On top of that, these extra Nambu–Goldstone-modes carry nonzero technibaryon number and hence, as opposed to technimesons, they can only be produced in pairs. The positive electroweak contributions to the squared masses of the Nambu–Goldstone-modes also stabilize the embedding of the electroweak gauge group, which leads to its correct breaking [23]. It could have been embedded in such a way that it remains unbroken, but the latter embedding would be destabilized by electroweak radiative corrections.

This is different in the pseudoreal case. There, the breaking goes to $Sp(4)$, which yields two uneaten Nambu-Goldstone-modes. They receive negative contributions to their squared masses from electroweak radiative corrections. Thus, the embedding that leads to the correct breaking of the electroweak gauge group is destabilized. It must be stabilized by an additional mechanism, which is usually incorporated with the extended technicolor sector. In fact, looking at the non-(pseudo)real case, all it takes to provide suitable masses to the weak gauge bosons is an $SU(2)_L \times SU(2)_R \to SU(2)_V$ breaking pattern. Hence, above the technicolor scale not the full $SU(4)$ symmetry has to be preserved, but only the $SU(2)_L \times SU(2)_R$.

Looking, for the moment, exclusively at the two-flavor technicolor sector neither with electroweak interactions nor extended technicolor effects, it consists of four Weyl fermions, U_L, D_L, U_R, and D_R. Representing the right-handed fields such that they transform as left-handed fields, $-i\sigma^2 U_R^*$ and $-i\sigma^2 D_R^*$, the $SU(4)$ flavor symmetry becomes apparent. The assignment of the left and right fields is dictated by the coupling to the electroweak interactions. From the viewpoint of the flavor symmetries, the mass terms

$$\mathcal{L}_m = m(\bar{U}_L U_R + \bar{U}_R U_L + \bar{D}_L D_R + \bar{D}_R D_L) \quad (23)$$

and

$$\mathcal{L}_\lambda = \lambda(\bar{U}_L D_L + \bar{D}_L U_L + \bar{U}_R D_R + \bar{D}_R U_R) \quad (24)$$

are equivalent: Imagine, for example, that the four Weyl flavors are arranged in a column vector. Then the respective mass terms can be expressed by contraction with the matrices

$$\begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{pmatrix}, \quad (25)$$

which are linked by an $SU(4)$ transformation. $\mathbb{O}$ and $\mathbb{1}$ are 2×2 zero and unit matrices, respectively.

The first mass term above, however, breaks the electroweak symmetry, while the second does not; it breaks the $SU(4)$ to $SO(4) \simeq SU(2)_L \times SU(2)_R$. Consequently, it contributes to the masses of those Nambu–Goldstone-modes, which link left- with right-fields, that is, the modes with nonzero technibaryon number. Apart from a direct

application to dynamical electroweak symmetry breaking, for which the electroweak symmetry must be unbroken before the chiral condensate is formed, the investigation of the impact of an explicit mass term of $\mathcal{L}_m$-type is interesting *per se* as well as for quantum chromodynamics, and natural for a study in the framework of lattice field theory.

For techniquarks in a pseudoreal representation of the technicolor gauge group, terms which break the $SU(4)$ flavor symmetry to $SU(2)_L \times SU(2)_R$ are needed to stabilize the vacuum. The motivation for studying additional mass terms of $\mathcal{L}_\lambda$-type for real representations (and yet another motivation for studying it for pseudoreal representations) is to control the amount of walking or even avoid conformality of an otherwise promising candidate [28]. (In fact, it appears to be interesting to compare the implications from our mass-dependent all-order β-function with results from lattice studies, which is an active field in the context of walking technicolor theories [28,29].) For this purpose, we do not even have to increase the value of the mass parameter λ beyond its value when used to make the extra Nambu–Goldstone-modes sufficiently massive: The extended technicolor-induced term is of dynamical origin. Thus, arguing based on a Gell-Mann–Oakes–Renner relation,

$$m_\pi^2 f_\pi^2 = 2\lambda \langle Q\bar{Q} \rangle, \quad (26)$$

where $m_\pi = O$ (few 0.1 TeV) is the extended technicolor contribution to the mass of the corresponding pions, $f_\pi = O$ (few TeV) their decay constant and $\langle Q\bar{Q} \rangle = O(\text{TeV}^3)$ the related techniquark condensate, results in λ being of $O(\text{TeV})$ as well. This is where also the technicolor scale is situated. Hence, an interference of related threshold effects with the technicolor phase transition appears natural. Thus, the extended technicolor might affect the technicolor phase transition.

On the other hand, the dynamically generated techniquark mass in the $\mathcal{L}_m$ channel can be estimated to be [30]

$$\Sigma(0) \approx 2\pi F_\pi / \sqrt{d_R}, \quad (27)$$

where for technicolor models with two techniflavors $F_\pi = \Lambda_{\text{ew}} = 246$ GeV. d_R is the dimension of the representation of the technicolor gauge group with respect to which the techniquarks transform. Hence, $\Sigma(0)$ is also of $O(\text{TeV})$ and it seems likely that also the critical value of the coupling α^* is influenced by the presence of $\mathcal{L}_\lambda$.

As another example consider partially (electroweakly) gauged technicolor [17]. In partially gauged technicolor, be it with matter in a single or different simultaneously present representations of the technicolor gauge group, only some of the techniquarks are gauged under the electroweak gauge group. This is usually done to alleviate constraints from electroweak precision data, in particular, from the oblique parameters. At the same time it allows to bring the theory close to the (quasi)conformal window. The part of the techniquarks that is not gauged under the

electroweak can be given masses without breaking the electroweak symmetry. Depending on the flavor symmetry among the electroweakly gauged techniquarks of a partially gauged technicolor model, specific mass terms that leave the electroweak symmetry intact might be possible in this sector as well, as was already explained above.

The conformal window

In [13], the lower bound of the conformal window in the plane spanned by the number of colors of an $SU(N)$ theory and the number of flavors was determined by equating the coupling at the Caswell–Banks–Zaks fixed point with the critical coupling [31] for the formation of a chiral condensate as obtained within the ladder-rainbow-approximation to the Dyson–Schwinger–equations. Solving for the number of flavors as a function of the number of colors yields the dotted curves in Fig. 2, for four different representations; from top to bottom, the fundamental, the two-index antisymmetric, the two-index symmetric and the adjoint. Asymptotic freedom is lost above the solid curves.

In [10], the conjectured β-function (16) was used to determine the lower bound of the conformal window. To
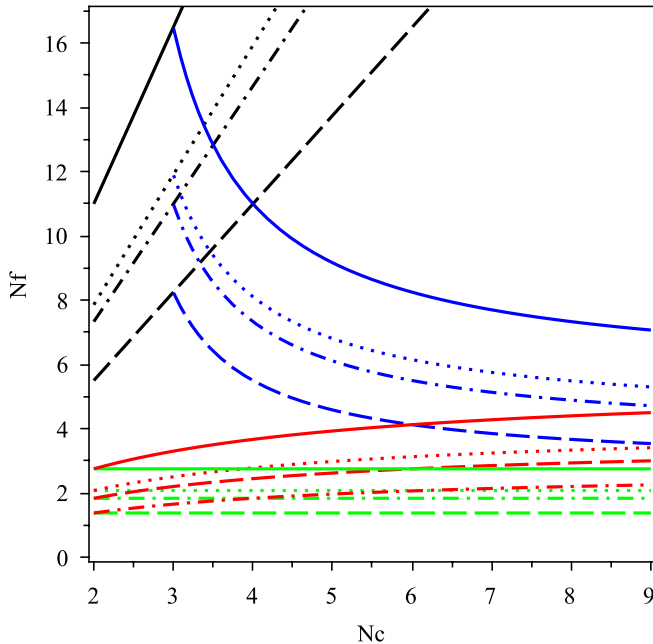

FIG. 2 (color online). Massless fermions: Conformal window for $SU(N)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising line), two-index antisymmetric (blue; curved, falling line), two-index symmetric (red; curved, rising line) or adjoint (green; straight, horizontal line) representation of the gauge group. Above the solid curves asymptotic freedom is lost. The dotted lines show the lower bound for the conformal window according to the rainbow-ladder-approximation to the Dyson–Schwinger-equations. The lower bounds for the conformal window are depicted by the dash-dotted ($\gamma = 1$) and the dashed ($\gamma = 2$; minimal lower bound) curves.

this end, it was equated to zero and the value of the anomalous dimension γ was kept fixed. While the ladder-rainbow-approximation yields a critical value of one, the only theoretically hard upper bound on γ arises from the requirement of unitarity of the gauge field theory and is two. (This follows from the fact that in a conformal field theory the dimension $3 - \gamma$ of all nontrivial spinless operators and thus, that of the chiral condensate must be larger or equal to unity [32].) The lower bound for the conformal window must, hence, not lie below the thus obtained curve. (There are also indications from duality arguments [33] for choosing the maximum value for γ.) Again, after solving for the number of flavors as a function of the number of colors, this approach gives rise to the dash-dotted curves ($\gamma = 1$) and dashed curves ($\gamma = 2$), respectively, in Fig. 2. This approach has also been applied to gauge groups other than $SU(N)$ and to multiple representations [34,35]. Interestingly, this approach leads to a universal formula $1 = \kappa 2N_f T(R)/C_2(G)$ for the lower bound of the conformal window [36]. In [36], this relation was found in the framework of the world line formalism, and $\kappa \approx 1/4$ was determined from matching to SQCD. From Eq. (16), one finds $\kappa = (2 + \gamma)/11$. (A combination of the two results would yield $\gamma \approx 3/4$.) Such a relation was also found in [37].

Here, we study the influence of threshold effects due to the fermion masses based on the β-function (17) by fixing $\bar{\gamma}$ to the two benchmark values from the massless study. For vanishing mass, we find the same results as before. For nonzero values of the mass, the "lower bound of the conformal window" is moved towards a larger number of flavors. In the presence of massive fermions this term ("...") is somewhat abused, here. Clearly, in a theory, where all fermions have a nonzero mass, however tiny it may be, they freeze out for scales far enough below this mass, and we are effectively left with a pure Yang–Mills theory, where the antiscreening from the gluons is unchallenged. Hence, in a theory in which all flavors are massive, there is strictly speaking no conformal window. Therefore, what above was called "lower bound of the conformal window" in the massive case is the phenomenologically decisive minimal number of flavors above which the coupling develops a plateau (walks). Consequently, we should talk of a "quasiconformal window," which in the limit $m \to 0$ coincides with the conformal window. In the massless case, the walking setups were to be found slightly below the lower edge of the conformal window. (There, above the edge, the theory does evolve into the fixed point.) In the massive case, in a setup slightly below the modified bound, there will also be at least some walking. In that case, the amount of walking, that is, the range of scales of quasiconformal behavior, is determined by a direct interplay of the freezing out of the flavors due to the explicit mass and the onset of chiral symmetry breaking. (Independent of walking, a similar interplay between quark

mass effects and chiral symmetry breaking exists in quantum chromodynamics for the strange quark.) If the number of flavors is really below this bound, it never comes near the fixed point and does not show any walking. If the number of flavors is above this bound (and the theory is still asymptotically free) the theory approaches the fixed point very closely and stays in its vicinity until the flavors start decoupling gradually. Once the flavors are decoupled sufficiently, the coupling starts running again. This means also that in such a setting, the position of the low-energy end of the plateau is not determined by the initial conditions for the renormalization group evolution, but by the value of the fermion masses. As examples we present the quasiconformal windows for $\gamma = 1$ and $x = -4$ ($\mu = 4m$) in Fig. 3 and for $\gamma = 2$ and $x = -4$ in Fig. 4. For comparison, the massless case is also always shown. (We concentrate here on equal masses for all fermions. The phenomenology becomes richer, once we allow for different masses, which, on the other hand, would also break more symmetries.)

At this point, let us revisit briefly the notion of "number of active flavors." The determination of the conformal window relies on setting the β-function (17) equal to

zero. This amounts to finding the zero of its numerator. For fixed $\bar{\gamma}$, the latter depends only on $\bar{\beta}_0$ and not $\bar{\beta}_1$. Hence, the number of active flavors that counts, at least for this way of determining the quasiconformal window, is $N_{f,0}$. This aspect together with the fact that $N_{f,1}$ is not monotonously interpolating between the limiting cases indicates that $N_{f,1}$ or $\bar{\beta}_1$, and actually probably all the $N_{f,j}$ or $\bar{\beta}_j$ with $j \geq 1$, encode something else than a mere modification of the number of flavors.

In Fig. 5, in order to show the influence of the mass of the fermions, we plot the critical number of flavors as a function of $x = -\mu^2/(4m^2)$ for an $SU(2)$ field theory with two adjoint Dirac flavors, that is, the core of minimal walking technicolor. First of all, one can see that the switching zone spans 4 orders of magnitude. In other words, threshold effects are felt for energy scales, which are a hundred times bigger than the mass of the fermions. Further, taking the mass to infinity (x to zero) does not lead to the zero flavor result because γ is kept fixed. As for the range of the mass effect, for $\gamma = 1$, an x of $O(-10^1)$ lifts the lower bound to the value obtained from the ladder-rainbow-approximation. A value for x slightly below -1,
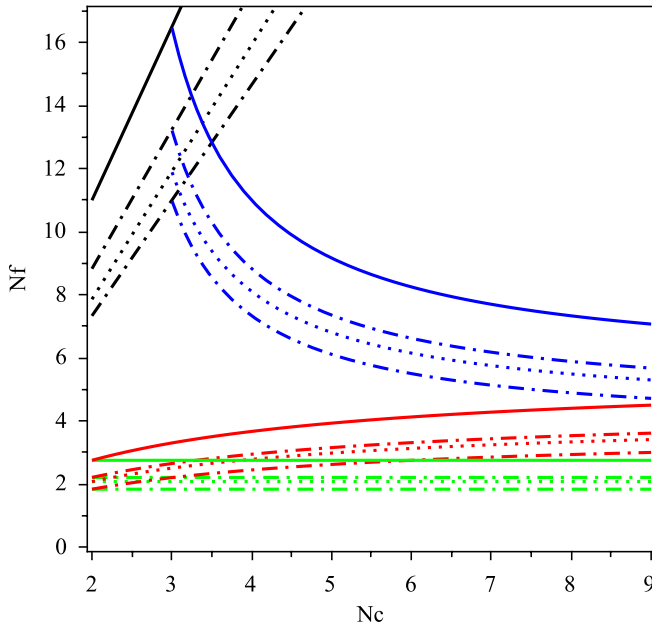

FIG. 3 (color online). Massive fermions, $\gamma = 1$: Quasiconformal window for $SU(N)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising line), two-index antisymmetric (blue; curved, falling line), two-index symmetric (red; curved, rising line) or adjoint (green; straight, horizontal line) representation of the gauge group. Above the solid curves asymptotic freedom is lost. The dotted lines show the lower bound for the conformal window according to the rainbow-ladder-approximation to the Dyson–Schwinger-equations. For massless fermions the lower bound is depicted by the lower dash-dotted curve; for fermions with $x = -4$ ($\mu = 4m$), by the upper dash-dotted curve.
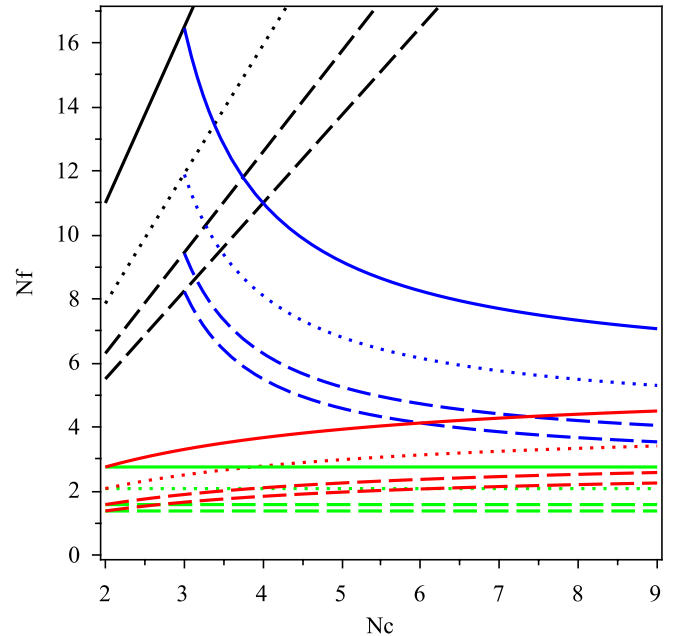
FIG. 4 (color online). Massive fermions, $\gamma = 2$: Quasiconformal window for $SU(N)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising line), two-index antisymmetric (blue; curved, falling line), two-index symmetric (red) or adjoint (green; straight, horizontal line) representation of the gauge group. Above the solid curves asymptotic freedom is lost. The dotted lines show the lower bound for the conformal window according to the rainbow-ladder-approximation to the Dyson–Schwinger-equations. For massless fermions the lower bound is depicted by the lower dashed curve; for fermions with $x = -4$ ($\mu = 4m$), by the upper dashed curve.
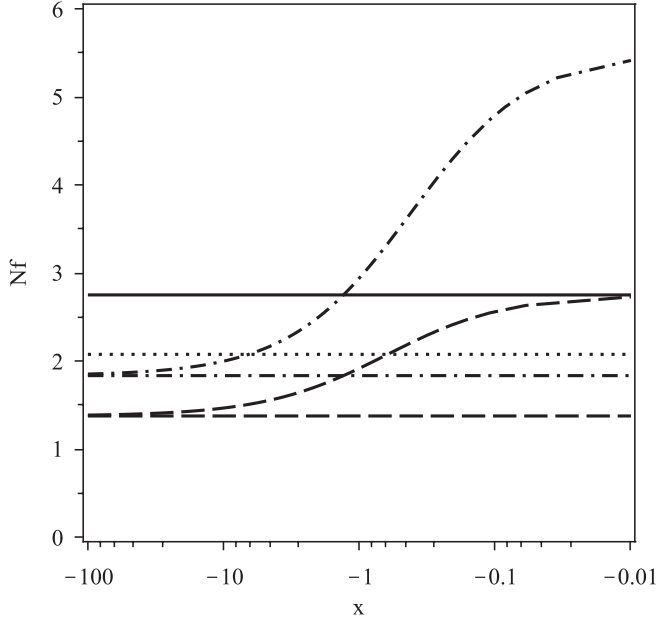
FIG. 5. Lower bounds for the (quasi)conformal window, two flavors, adjoint representation, $SU(2)$: ladder-rainbow-approximation (dotted line); $\gamma = 1$ massless (horizontal straight line, dash-dotted), massive (curve, dash-dotted); $\gamma = 2$ massless (horizontal straight line, dashed), massive (curve, dashed). Above the solid line asymptotic freedom is lost. The zone in which the flavors are gradually switched off, spans 4 orders of magnitude.

that is, $2m \lesssim \mu$, closes the quasiconformal window altogether, as asymptotic freedom is lost, before quasiconformality is reached. For $\gamma = 2$, the ladder-rainbow-value is only reached for x slightly above -1, that is, $2m \gtrsim \mu$; the quasiconformal window closes for infinite fermion masses, $x \rightarrow 0$.

In principle, the quasiconformal windows, shown in Figs. 3 and 4 for $SU(N_c)$ gauge theories, have to be

replotted for every value of x. Therefore, in the appendix, we have attempted to give a universal representation that permits to read off N_f for the lower bound of the quasiconformal window for $SU(N_c)$ (Fig. 7), $Sp(2N_c)$ (Fig. 8), and $SO(N_c)$ (Fig. 9) gauge theories, from single plots. Figure 6 explains the procedure.

IV. SUMMARY

We propose a mass-dependent all-order β-function (17), which combines the ideas that, in [10], led to the supersymmetry inspired mass-independent all-order β-function (16) with results for the β-function coefficients in the mass-dependent background field momentum subtraction scheme. (Formally, it is legal to use mass-independent subtraction schemes. They, however, neglect completely the influence of the constituent masses on the evolution of the gauge coupling constant.) For vanishing fermion masses, the mass-independent all-order β-function (16) is recovered from the mass-dependent all-order β-function (17). We use the β-function (17) to study the impact of explicit fermion masses on the lower bound of the (quasi)conformal window of the corresponding field theory: Effects of the fermion mass can be sensed at scales more than a hundred times bigger than the fermion mass; (See Fig. 5.) the lower bound of the (quasi)conformal window can be moved to a considerably higher number of flavors; in fact, it can be closed up altogether. (See Figs. 3–5.) This means that the corresponding theory does not show quasiconformal behavior for any number of flavors below the value for which asymptotic freedom is lost. In the investigation of dynamical electroweak symmetry breaking by walking technicolor theories, our approach may be useful for studying effects of extended technicolor interactions, which may be primarily required to stabilize the vacuum and/or render additional Nambu–Goldstone-modes sufficiently massive. (These effects interfere with the chiral
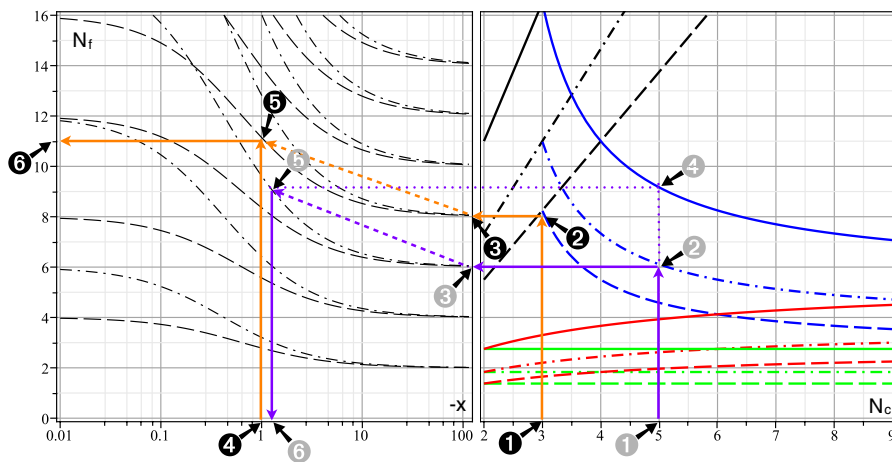

FIG. 6 (color online). Examples for how to use Figs. 7–9 on the example of the (quasi)conformal window for $SU(N_c)$ theories. More explanations are given at the end of the body of the text of the appendix.
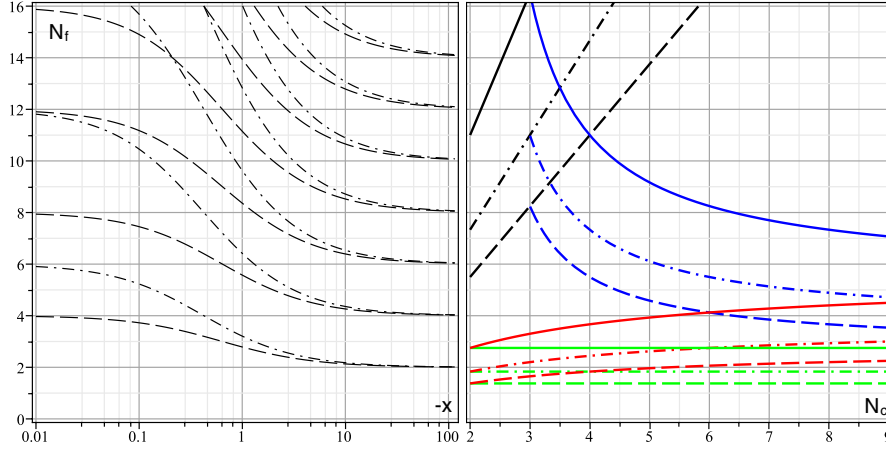
FIG. 7 (color online). (Quasi) conformal window for $SU(N_c)$ theories. *Right-hand side*: Conformal window for $SU(N_c)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising line), two-index antisymmetric (blue; curved, falling line), two-index symmetric (red; curved, rising line) or adjoint (green; straight, horizontal line) representation of the gauge group. Above the solid curves asymptotic freedom is lost. The lower bounds for the conformal window are depicted by the dash-dotted ($\gamma = 1$) and the dashed ($\gamma = 2$; minimal lower bound) curves. *Left-hand side*: Effective rescaling of the N_f axis due to massive fermions for $\gamma = 1$ (dash-dotted line) and $\gamma = 2$ (dashed line).
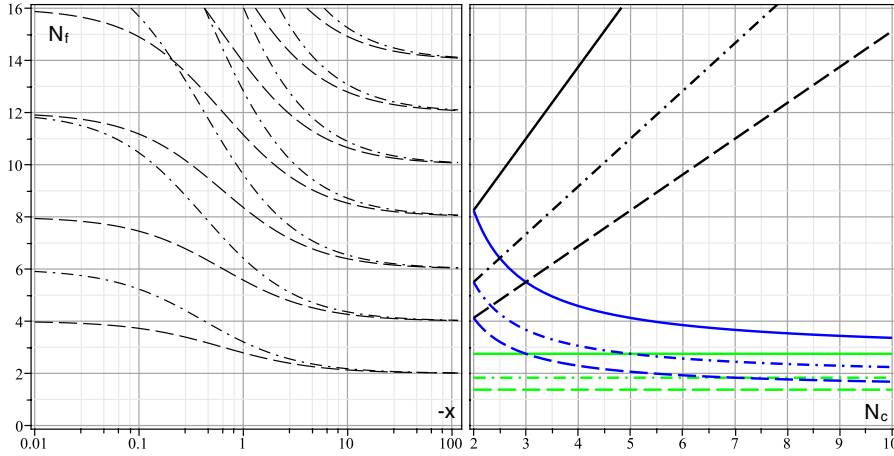

FIG. 8 (color online). (Quasi) conformal window for $Sp(2N_c)$ theories. *Right-hand side*: Conformal window for $Sp(2N_c)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising line), two-index antisymmetric (blue; curved, falling line) or adjoint (green; straight, horizontal line) representation of the gauge group. Above the solid curves asymptotic freedom is lost. The lower bounds for the conformal window are depicted by the dash-dotted ($\gamma = 1$) and the dashed ($\gamma = 2$; minimal lower bound) curves. *Left-hand side*: Effective rescaling of the N_f axis due to massive fermions for $\gamma = 1$ (dash-dotted line) and $\gamma = 2$ (dashed line).
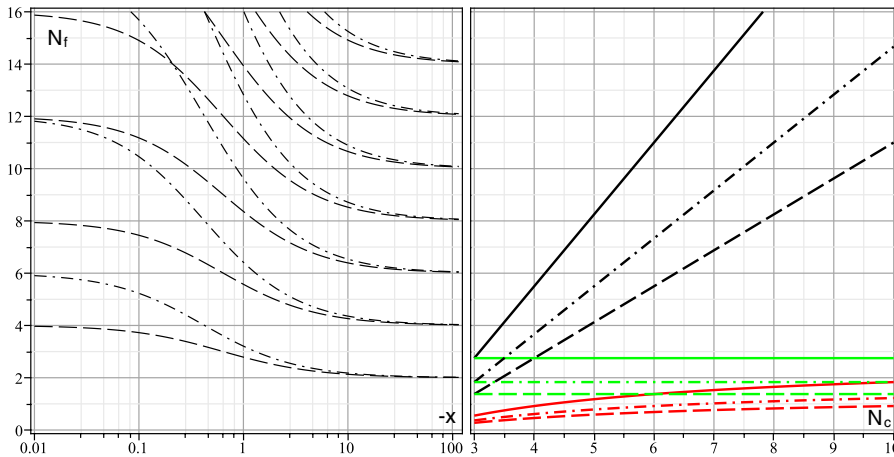

FIG. 9 (color online). (Quasi) conformal window for $SO(N_c)$ theories. *Right-hand side*: Conformal window for $SO(N_c)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising line), two-index symmetric (red; curved, rising line) or adjoint (green; straight, horizontal line) representation of the gauge group. Above the solid curves asymptotic freedom is lost. The lower bounds for the conformal window are depicted by the dash-dotted ($\gamma = 1$) and the dashed ($\gamma = 2$; minimal lower bound) curves. *Left-hand side*: Effective rescaling of the N_f axis due to massive fermions for $\gamma = 1$ (dash-dotted line) and $\gamma = 2$ (dashed line).

dynamics of technicolor in a similar way as electroweakly induced masses interfere with the chiral dynamics of quantum chromodynamics.) Additionally, it might be extended technicolor effects that make a theory quasiconformal, which from its bare technicolor structure would be completely conformal and hence, not suited for breaking the electroweak symmetry dynamically. It would be interesting to address the issue of mass dependence, in particular, and of other extended technicolor effects, in general, in lattice simulations and to contrast them to the present investigation.

ACKNOWLEDGMENTS

APPENDIX: QUASICONFORMAL WINDOWS

The critical number of flavors obtained by setting the mass-dependent β-function equal to zero at a fixed value for $\bar{\gamma}$ is given by,

$$N_f = \frac{11}{2} \frac{C_2(G)}{T(R)} [\bar{\gamma} + 2b_0(x)]^{-1}. \quad \text{(A1)}$$

The modification due to the mass of the fermion is, hence, universal in the sense that it neither depends on the gauge group nor the representation. The latter is encoded in the fraction $C_2(G)/T(R)$, while the mass effect are contained in $b_0(x)$.

We are here making use of this universality by giving an illustration for the quasiconformal windows for theories constructed from the fundamental, 2-index antisymmetric, 2-index symmetric or adjoint representation of an $SU(N_c)$ (Fig. 7), $Sp(2N_c)$ (Fig. 8), or $SO(N_c)$ (Fig. 9) gauge group. {The $m = 0$ case was first treated in [13] (SU), [34] (Sp, SO)}. Said representation consists of split figures; their left half depicts the quasiconformal windows for vanishing mass and their right half the modification due to a nonzero mass, for both benchmark values, $\bar{\gamma} = 1$ and $\bar{\gamma} = 2$, respectively.

Figure 6 explains how to use the depiction:

On the right-hand side, we have the conformal window for $SU(N_c)$ gauge theories with fermions in the (from top to bottom) fundamental (black; straight, rising), two-index antisymmetric (blue; curved, falling), two-index symmetric (red; curved, rising) or adjoint (green; straight, horizontal) representation of the gauge group [here $SU(N_c)$]. Above the solid curves, asymptotic freedom is lost. The lower bounds for the conformal window are depicted by the dash-dotted ($\gamma = 1$) and the dashed ($\gamma = 2$; minimal lower bound) curves.

The left-hand side shows the effective rescaling of the N_f axis due to massive fermions for $\gamma = 1$ (dash-dotted) and $\gamma = 2$ (dashed). Note that the curves signalling the loss of asymptotic freedom are not rescaled.

The black numbers mark the first example, in which we read off the lower bound of the quasiconformal window for $x = -1$ for fermions in the fundamental representation of $SU(3)$ with the criterion $\gamma = 2$: (1) In the plot on the right-hand side, pick $N_c = 3$; (2) go up to the $\gamma = 2$ curve for fermions in the fundamental representation, which yields $N_f \approx 8$ for the lower bound of the massless conformal window; (3) find the $N_f = 8$ curve in the plot on the left-hand side; (4) pick the $-x = 1$ line; (5) find the crossing between the curve and the line; (6) read off the result on the axis; here, $N_f \approx 11$.

The grey numbers indicate the second example, in which the value of $-x$ is to be determined for which the quasi-conformal window closes up completely for fermions in the 2-index antisymmetric representation of $SU(5)$ and using the criterion $\gamma = 1$: (1) In the plot on the right-hand side, pick $N_c = 5$; (2) go to the $\gamma = 1$ curve for fermions in the 2-index antisymmetric representation, which yields $N_f \approx 6$ for the lower bound of the massless conformal window; (3) find the $N_f = 6$ curve in the plot on the left-hand side; (4) determine the number of flavors for which asymptotic freedom is lost from the plot on the right-hand side, which is $N_f \gtrsim 9$; (5) find where the curve in the plot on the left-hand side reaches this value; (6) go down to the $-x$ axis, which yields $-x \gtrsim 1$.

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