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Improved hydrodynamics from the AdS/CFT duality

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We generalize (linearized) relativistic hydrodynamics by including all order gradient expansion of the energy-momentum tensor, parametrized by four momenta-dependent transport coefficients, one of which is the usual shear viscosity. We then apply the AdS/CFT duality for $\mathcal{N} = 4$ SUSY in order to compute the retarded correlators of the energy-momentum tensor. From these correlators we determine a large set of transport coefficients of third- and fourth-order hydrodynamics. We find that higher order terms have a tendency to reduce the effect of viscosity.

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I. INTRODUCTION

The relativistic heavy ion collider (RHIC) produces hadronic matter with temperatures ranging between the initial $T_i \sim 2T_c$ and the final (or freeze-out) value $T_f \sim T_c/2$ [1,2], where $T_c \approx 170$ MeV is the QCD critical temperature. It has been shown [3–5] that the flows (radial, elliptic) associated with the plasma expansion are well and consistently described by near-ideal relativistic hydrodynamics, with freeze-out implemented via hadronic cascades. Since elliptic flow is dominated by the early times of the fireball expansion, when $T > T_c$ and matter is in the so-called quark-gluon plasma (QGP) phase [6], this lead to the conclusion [7,8] that QGP is a ‘‘perfect liquid’’, presumably because it is actually in a new—strongly coupled—regime of QCD. Those views were discussed in detail and eventually accepted in the 2004 ‘‘white papers’’ of all four experimental collaborations [9]. The full understanding of QCD dynamics at strong coupling, even in the deconfined phase, remains a challenge and one usually appeals to either lattice simulations or phenomenological models. While the lattice is considered a reliable source for QCD thermodynamics, it usually fails to provide accurate data on transport coefficients. Thus, in order to understand transport properties of QCD, at the moment we have to appeal to various microscopic models (for recent reviews on the strongly coupled QGP see e.g. [10,11]).

Starting from 2004 RHIC experiments have discovered and studied phenomena known as the ‘‘cone’’ and the ‘‘ridge’’, associated with the propagation of the energy deposited by quenched hard jets (for the description of the phenomena and recent data see Refs. [12–14] and references therein). The former was associated with conical hydrodynamic flows induced by fast particles propagating through the medium [15]. This has in turn initiated studies, within AdS/CFT, of such processes induced by a steadily moving heavy quark, see [16–19]. Although these studies did not appeal to any hydro description, their results were found to be in very good agreement with (even unimproved) hydrodynamics.

Another structure, known as the ‘‘soft ridge’’ has been observed in two-particle correlations: its origin is attributed to initial state fluctuations in the colliding nuclei. For experimental data and phenomenological discussion see the talks at the BNL dedicated workshop [20]. Although in this paper we will not discuss the phenomenology of those objects, we nevertheless stress that they provide the strongest motivation for a detailed study of small perturbations on top of (hydrodynamically expanding) matter.

One obvious step in understanding these perturbations is to study them using *linearized* hydrodynamics, assuming their amplitude to be small. On the other hand, these objects start their evolution at much smaller scales compared to nuclear (or fireball) radii. For example initial state fluctuations are believed to be given by the ‘‘saturation scale’’ $1/Q_s$, which is only 0.2 fm, about 30 times smaller than the fireball as a whole. Therefore, the evolution of small perturbations includes much larger spatial gradients, and in order to treat them better one would naturally try to improve the accuracy of hydrodynamics, including *higher order derivative terms*. Since the latter appear with many new transport coefficients, the usual phenomenological approach which derives viscosity from the data would hardly be possible. Instead, some self-consistent approach is needed to calculate as many of them as necessary.

Such model of choice for the present study is $\mathcal{N} = 4$ SUSY at large N_c. Via the celebrated AdS/CFT correspondence [21] this gauge theory at strong coupling admits a dual description in terms of weakly coupled gravity in $AdS_5 \otimes S_5$ space. The finite temperature version of this field theory is dual to the AdS-Schwarzschild black hole (brane). The laws for Schwarzschild black hole thermodynamics imply that the entropy density is proportional to the area of the horizon [22].[1] The equilibrium pressure is $P = \pi^2 N_c^2 T^4/8$, while the energy density is $\epsilon = 3P$ due to the conformal symmetry of the microscopic theory.

[1]For theories involving higher order curvature corrections to the Einsteinian gravity, the relation between area of the horizon and entropy is invalid [23].

Refs. [24] pioneered the study of transport coefficients via dual description. For a static plasma and in the limit of large 't Hooft coupling $\lambda \gg 1$, the ratio of shear viscosity to entropy is independent of the coupling and is in fact remarkably small

$$\frac{\eta_0}{s} = \frac{1}{4\pi}. \tag{1.1}$$

Furthermore, Refs. [24] conjectured that this value for the ratio is a universal lower bound, valid for all physical systems in nature. While AdS/CFT leads Eq. (1.1), it does not provide any explanation of this result from the gauge theory side. So far, no microscopic mechanism for low viscosity has been established for QCD, though a promising proposal can be found in [25].

It is rather difficult to extract viscosity from experimental data precisely, because it is small and its effects are $O(10\%)$ or so, comparable to other uncertainties. The phenomenological studies of RHIC data (such as in Ref [26]) typically focus on the elliptic flow dependence on centrality or transverse momentum, $v_2(b, p_t)$. The optimal η_0/s for these fits occurs at a value of the order of the suggested minimum, although deviations from it by a factor two or so are still possible. Another argument to support very low viscosity comes from discussions of the overall entropy production, such as in Ref. [27]. Those works suggested that there is a tension between the total entropy (measured by the observed multiplicity of produced hadrons) and the very short thermalization time (initial time for hydro evolution), unless the viscosity over entropy ratio is pushed down, maybe even below the bound. The third (more indirect) argument for low viscosity is the survival till freeze-out of the ''cones'' and ''ridges'', suggesting smallness of dissipative effects. Therefore all these approaches indicate a very small viscosity value.

The hydrodynamic representation of the energy-momentum tensor is

$$\langle T^{\mu\nu}\rangle = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \Pi^{\mu\nu}, \tag{1.2}$$

where the average is taken over the thermal bath. While at some microscopic scale l the system is assumed to be locally at thermal equilibrium, at some macroscopic scale $L \gg l$ the local fluid velocity field u is a function of space-time coordinates. The ''tensor of dissipations'' $\Pi^{\mu\nu}$, added to the ideal-fluid part, represents all the deviations from the equilibrium state induced by such a flow field. In the long wavelength limit $L \gg l$, these fluctuations can be expanded in terms of gradients of the velocity field, or in powers of l/L. The first order Navier-Stocks (NS) hydrodynamics retains only the first gradient[2]

$$\Pi^{\mu\nu} \sim \eta_0 \nabla^\mu u^\nu. \tag{1.3}$$

In this work we will discuss higher order gradients, which will provide certain corrections to the first order viscosity term when gradients grow. These corrections are relevant for smaller size objects in the plasma or to earlier time of hydro evolution.

The high order gradient expansion generically includes two types of terms: (i) nonlinear terms in the velocity field (like $(\nabla u)^2$) and (ii) linear terms with multiple gradient operators acting on a single velocity field (like $\nabla\nabla u$). These two types of terms are controlled by two different parameters. The nonlinearities are important when the field amplitude is large. However, even for small amplitude waves, one can get large contributions from the linear terms when the momenta associated with the wave are large.

Recently, second order hydrodynamics (next-to-NS) attracted significant attention [1,2]. The main reason is that NS hydrodynamics is known to have causality problems. The acausal effects create numerical instabilities when solving hydrodynamic equations. The problem originates from the fact that NS equations imply instantaneous response to any perturbation introduced in the system. In order to circumvent this problem, one may introduce a relaxation time. It explicitly appears as a new transport coefficient when the gradient expansion is extended to second order:

$$\Pi^{\mu\nu} \sim \eta_0[1 - \tau(u\nabla)]\nabla^\mu u^\nu \rightarrow \eta_0[1 + i\tau\omega](k^\mu u^\nu). \tag{1.4}$$

In fact, in order to restore causality it is not sufficient to include second gradients only: all order gradients need to be resummed. A very popular resummation scheme is due to Israel and Stewart (IS) [28]. It essentially generalizes viscosity to an ω-dependent but k-independent (complex) function

$$\eta^{IS}(\omega) = \frac{\eta_0}{1 - i\omega\tau}. \tag{1.5}$$

Equation (1.5) can be viewed as a Pade-like resummation of (1.4). The relaxation time τ provides a scale for exponential relaxation. The position of the pole below the real axis and the ''good'' falling off asymptotic behavior make the model causal. In other words, as a function of complex ω, the viscosity is analytic in the upper half plane. In coordinate space this simple pole corresponds to a memory function with exponential falloff:

$$\Pi^{\mu\nu}(x,t) \sim \frac{\eta_0}{\tau}\int_0^t dt' e^{-(t-t')/\tau}\nabla^\mu u^\nu(x,t'). \tag{1.6}$$

In this paper we will be studying all order gradient expansion in the linear approximation. Instead of introducing new transport coefficients at each new order, we will be thinking of viscosity and other transport coefficients as frequency and momentum-dependent functions. We will

[2]Throughout this paper we will be considering conformal theory only, for which there is only shear viscosity since the bulk viscosity is zero.

be working in the framework of $\mathcal{N} = 4$ SUSY. For the rest of this paper we set all dimensionfull units to be related with the temperature, $2\pi T = 1$. Among our results, we will show that the IS resummation, although well-known and used, is still simplistic model for high order gradient terms, which is even qualitatively inconsistent with AdS/CFT results. Not only it misses important nonlinear terms already at second order [29,30], but (as we will show below) it is also incorrect in the linear approximation starting from the third order.

More generally, we will find that higher order terms do have a tendency to cancel (or reduce) the effect of NS viscosity. In particular, in our earlier paper [31], we argued that the extremely low viscosity suggested by Refs. [26,27] may essentially be some ''effective viscosity'', which includes these high order gradient terms. The real systems probed in RHIC collisions have finite gradients and the inclusion of their effects may demand going beyond NS approximation. In [31] we attempted to extract a momentum-dependent viscosity from the imaginary part of the sound dispersion curve. Our main observation was that the effective viscosity as probed at finite momenta turns out to be smaller compared to the value at the origin. Motivated by [27], we discussed in [31] the implications of a momentum-dependent viscosity on the entropy production for Bjorken expansion [32]. We discovered that the inclusion of momentum-dependence made it possible to push the hydrodynamic description a bit further into earlier times of the collisions, with the entropy production due to viscous hydro stabilized at around 20% of the total entropy produced in the collision. The conclusion is that the account for a momentum-dependent viscosity reduces the sensitivity to thermalization time. Now, with the result reported below, our previous approach [31] based on the sound dispersion curve looks rather naive (for a much more elaborated study of hydrodynamic theory as an effective theory for the lowest modes see Ref. [33]). In general, we will see that the sound dispersion curve does not contain enough information to define the ''generalized viscosity'' function. Nevertheless, we qualitatively captured at least the right trend: full second order hydro with all nonlinear terms included has the same trend towards reducing the entropy production [29,34].

Our goal in this paper is to put the idea of a momentum-dependent viscosity on a more solid ground compared to our naive treatment in [31]. In the present analysis we will be focusing on the retarded correlators of the stress tensor. The correlators contain information not only about the positions of the poles but also about their residues. The complete information on the correlators is equivalent to the knowledge of the energy-momentum tensor in the linearized approximation.

In a conformal theory in four dimensions, there are only three independent correlators of the energy-momentum tensor. These are correlators in the sound (G^S), shear (G^D), and scalar (G^T) channels. AdS/CFT correspondence provides a tool to compute these correlators by solving certain linearized gravity equations in the background of the AdS-Schwarzschild black hole [35–37]. These equations essentially describe graviton's propagation from the AdS boundary, where the field theory is defined, to the horizon of the black hole. Absorptive boundary conditions are imposed there. Dissipation takes place at the horizon while there is no dissipation in the bulk of AdS. However, the bulk curvature acts as a nonlinear medium, which provides a source for complicated dispersion. It is this dispersion which, by means of the duality, is mapped into momenta-dependent transport coefficients.

Our strategy is to first write a most generic hydrolike representation of the energy-momentum tensor $T^{\mu\nu}$, in terms of the fluid velocity field u. We find that, generically, there are four structures (or operators involving derivatives of u or the metric g) which can occur in $T^{\mu\nu}$ and are consistent with all symmetries. Each structure enters with a coefficient which is momentum-dependent. These are the generalized transport coefficient we are looking for. One of them is associated with the shear viscosity, while the remaining three encode responses of the system to external (4d) gravity perturbations. We call them gravitational susceptibilities of the fluid (GSF). The operators which are multiplied by the GSFs involve the Weyl tensor of the metric and vanish in the flat Minkowski space.

We then proceed by using this hydrolike representation of $T^{\mu\nu}$ in order to compute its correlators in the three channels introduced above. We then attempt to determine the momentum-dependent transport coefficients from the matching to the functions G^S, G^D, and G^T computed directly from the bulk gravity side.

Our program ran into a problem, which we were not able to resolve completely: there are in fact four independent transport functions to be extracted from three equations. Despite the fact that we could not determine the entire functions, we were able to get them to quite high order in the perturbative expansion at small momenta. In particular, we found the shear viscosity function to fifth order in the gradient expansion. This involves several new transport coefficients, most of which are obtained numerically.

The conceptual problem mentioned above, prevented us from computing shear viscosity in the whole kinematic region of arbitrary frequency and momentum. Instead, we build a model similar to IS which utilizes the information about the new transport coefficients and preserves the causality condition. We propose this model for phenomenological studies of hydrodynamics at RHIC, but any application of this model is left beyond the scope of this paper.

The paper is organized in the following way. In Sec. II we present the general setup for computing the retarded correlators from the bulk gravity and from the generalized hydro on the boundary. Section III presents some results. A phenomenological model for generalized viscosity is proposed in Sec. IV. Our conclusions are summarized in Sec. V. Two Appendices supplementing Sec. II provide details of some analytical computations.

II. GENERALITIES

The retarded correlators of two energy-momentum tensors are defined as follows

$$G^{\mu\nu\alpha\beta}(k,\omega) = -i\int_0^\infty dt \int d^3x e^{-i\omega t+ikx} \times \langle[T^{\mu\nu}(x,t), T^{\alpha\beta}(0)]\rangle. \qquad (2.1)$$

Here the average is over the equilibrated thermal bath. For conformally invariant plasma with traceless $T^{\mu\nu}$, there are only three independent correlators $G^T \equiv G^{xyxy}$ (tensor), $G^D \equiv G^{txtx}$ (shear), and $G^S \equiv G^{tztz}$ (sound) with the vector k pointing in the z-direction. All other correlators are related to these three either by rotational symmetry or by the equations of motion.

A. Life in the bulk: Retarded correlators from gravity

In this subsection we closely follow the setup and results of Ref. [38]. From the bulk gravity side, in order to compute the retarded correlators at nonzero temperature one has to solve certain wave equations (one for each symmetry channel). These equations describe propagation of the corresponding metric perturbations (gravitons) in the AdS-Schwarzschild BH background of the dual description. The differential equations are of the form

$$\frac{d^2}{dr^2}Z_a(r) + p_a(r)\frac{d}{dr}Z_a(r) + q_a(r)Z_a(r) = 0, \qquad (2.2)$$

where the coefficients $p_a(r)$, $q_a(r)$ depend on the frequency ω and momentum k, and $a = T, D, S$ labels the three symmetry channels. The coefficient functions are given by the following expressions.

(i) The scalar channel

$$p_T(r) = -\frac{1+r^2}{rf}, \qquad q_T(r) = \frac{\omega^2 - k^2 f}{rf^2}, \qquad (2.3)$$

where $f = 1 - r^2$. The function f is inherited from the AdS-BH metric.

(ii) The shear channel

$$p_D(r) = \frac{(\omega^2 - k^2 f)f + 2r^2\omega^2}{rf(k^2 f - \omega^2)}, \quad q_D(r) = \frac{\omega^2 - k^2 f}{rf^2}. \qquad (2.4)$$

(iii) The sound channel

$$p_S(r) = -\frac{3\omega^2(1+r^2) + k^2(2r^2 - 3r^4 - 3)}{rf(3\omega^2 + k^2(r^2-3))}, \qquad q_S(r) = \frac{3\omega^4 + k^4(3 - 4r^2 + r^4) + k^2(4r^2\omega^2 - 6\omega^2 - 4r^3 f)}{rf^2(3\omega^2 + k^2(r^2-3))}. \qquad (2.5)$$

The fifth dimension coordinate r ranges from 0 to 1, where $r = 0$ corresponds to the boundary of the asymptotically AdS space, and $r = 1$ corresponds to the event horizon of the background metric.

The information about the retarded correlation functions is encoded in the solutions to Eq. (2.2), which satisfy the incoming wave condition at the horizon $Z_a(r \to 1) \sim \exp[-i\omega/2]$. At $r = 0$ the solution can be written as a linear combination of two independent local solutions,

$$Z_a(r) = \mathcal{A}_a Z_a^I(r) + \mathcal{B}_a Z_a^{II}(r), \qquad (2.6)$$

Here Z_a^I is irregular in the origin while Z_a^{II} is a regular solution.

The prescription to compute the correlators G follows from the Minkowski formulation of the AdS/CFT correspondence and amounts to computing the ratio between the two coefficients in the expansion (2.6)

$$\tilde{G}^a(\omega,k) = -8P\frac{\mathcal{B}_a(\omega,k)}{\mathcal{A}_a(\omega,k)}. \qquad (2.7)$$

For the three symmetry channels the correlators G are related to $\tilde{G}$,

$$G^{xyxy} = \frac{1}{2}\tilde{G}^T; \qquad G^{txtx} = \frac{1}{2}\frac{k^2}{\omega^2 - k^2}\tilde{G}^D. \qquad (2.8)$$

For the sound channel the relation is a bit more involved and includes a contribution from contact terms [39]

$$G^{tttt} = \frac{1}{2}\left[\frac{4}{3}\frac{k^4}{(\omega^2-k^2)^2}\tilde{G}^S + \frac{1}{12} \times \frac{29k^4 - 30k^2\omega^2 + 9\omega^4}{(k^2-\omega^2)^2}\right]. \qquad (2.9)$$

Equation (2.2) has real coefficients which are even functions of frequency. In other words this equation propagates waves without any dissipation. The dissipation (time irreversal) effects are introduced by the boundary conditions at the horizon. However, the AdS-BH metric acts as a nonlinear medium for the propagating graviton. The nonlinear dependences on frequency and momenta which appear in (2.2) are to be mapped onto highly nontrivial momenta dependence of the transport coefficient functions.

Let us make a technical remark on numerical solution. The equations for the shear and sound channels have

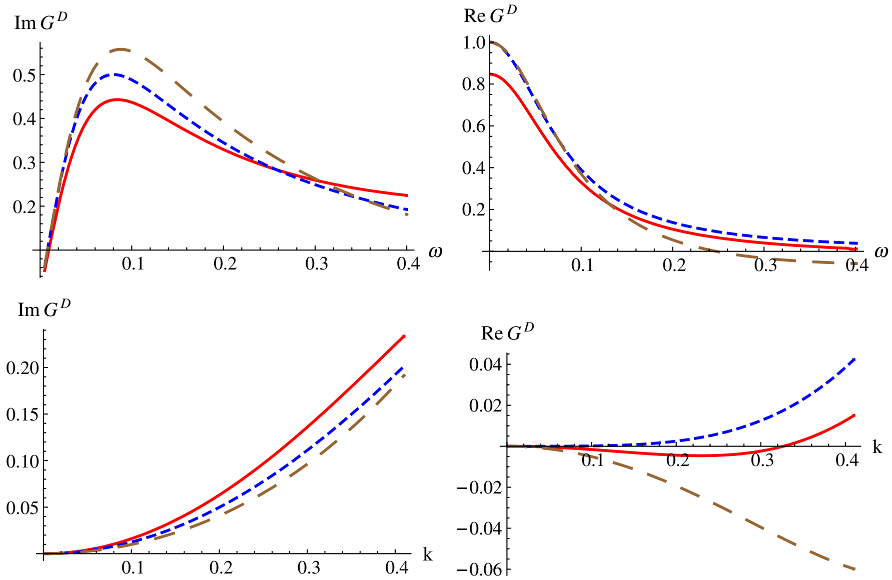

FIG. 1 (color online). Shear channel: top $k = 0.4$; bottom $\omega = 0.4$. Solid line corresponds to the AdS/CFT correlator. Short dashes display the NS hydrodynamics while long dashes show the IS hydrodynamics.

singular points inside the bulk $r = [0, 1]$. For the shear channel it appears for $\omega < k$ at $r_0 = \sqrt{1 - \omega^2/k^2}$ and for the sound it is at $r_0 = \sqrt{3(1 - \omega^2/k^2)}$ (condition that r_0 is inside the bulk). It would be interesting to understand if these points have any special physical role. To ensure that there is no instability caused by these singularities, we split our numerical solution into two intervals $[0, r_0]$ and $[r_0, 1]$ and matched the solutions at the singular points using analytic solutions in the vicinity of r_0.

The correlators computed from the gravity side agree with the field theory correlators up to a constant [35]. In particular, in the sound channel the relation between the correlators G^{tttt} and G^{tztz} is

$$\omega^2(G^{tttt} + \epsilon) = k^2(G^{tztz} + P). \tag{2.10}$$

The analytical expansion for the correlators at small momenta can be found in Appendix A. For the shear and sound channels we show some numerical results alongside the corresponding curves for the NS and IS hydrodynamics on Figs. 1 and 2.

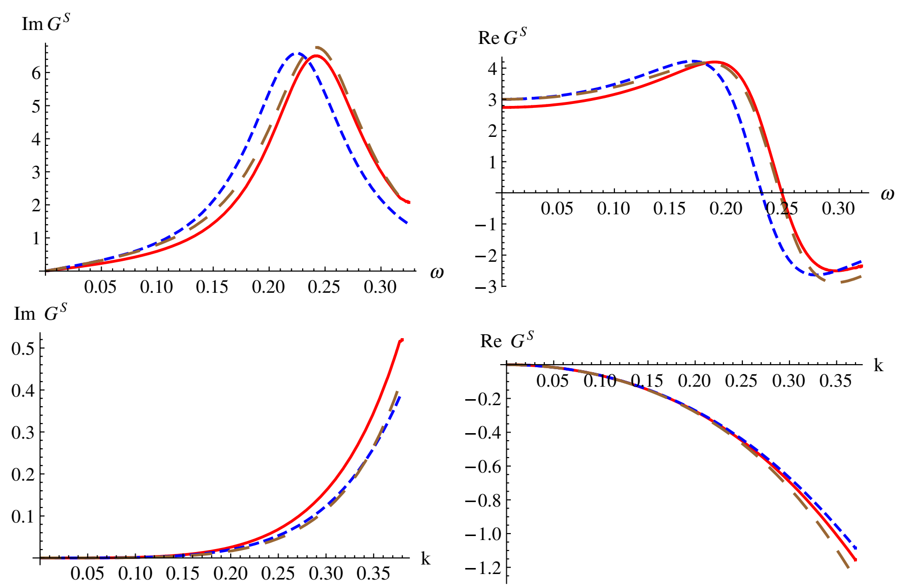

FIG. 2 (color online). Sound channel: top $k = 0.4$; bottom $\omega = 0.4$. Solid line corresponds to the AdS/CFT correlator. Short dashes display the NS hydrodynamics while long dashes show the IS hydrodynamics.

B. Life on the boundary: All order hydrodynamics

The thermal field theory on the four dimensional boundary is defined by means of the generating functional

$$Z[h] = \int D\phi e^{S_0[\phi]+\int d^4x h_{\mu\nu}T^{\mu\nu}}, \tag{2.11}$$

where ϕ collectively denotes all fields of the theory, S_0 is the flat metric action and $h_{\mu\nu}$ is an external perturbation of the Minkowski space.

The expectation value of the energy-momentum tensor at nonvanishing external field $h_{\mu\nu}$ is

$$\langle T^{\mu\nu}\rangle_{\rm cl}^{h} = \frac{\delta \ln Z}{\delta h} = \langle T^{\mu\nu}\rangle_{\rm cl}^{h=0} + h_{\alpha\beta}\tilde{G}^{\alpha\beta\mu\nu}. \tag{2.12}$$

Within the linear response theory we keep terms linear in h only. The correlators $\tilde{G}^{\alpha\beta\mu\nu}$ differ from the retarded correlators $G^{\alpha\beta\mu\nu}$ by constant contact terms [35].

We use Eq. (2.12) to define hydrodynamic variables. Here $\langle T^{\mu\nu}\rangle_{\rm cl}^{h=0}$ corresponds to the thermal equilibrium. The equilibrium energy density is

$$\epsilon_0 \equiv \langle T^{00}\rangle_{\rm cl}^{h=0}. \tag{2.13}$$

The external perturbation $h_{\mu\nu}$ shifts the theory from its thermal equilibrium. The out-of- equilibrium energy density is

$$\epsilon \equiv \langle T^{00}\rangle_{\rm cl}^{h} = \epsilon_0 + h_{\alpha\beta}\tilde{G}^{\alpha\beta 00}. \tag{2.14}$$

We can also define fluid's three-velocity v^i

$$(\epsilon_0 + P_0)v^i \equiv \langle T^{0i}\rangle_{\rm cl}^{h} = h_{\alpha\beta}\tilde{G}^{\alpha\beta 0i} \tag{2.15}$$

and fluid‘s 4-velocity $u^\mu = (\sqrt{1+v^2}, v)$ satisfying $u^2 = -1$.

The action (2.11) is required to be invariant under the local Weyl transformation (see the extensive discussion in Ref. [29])

$$g_{\mu\nu} \to e^{-2\Omega(x,t)}g_{\mu\nu}. \tag{2.16}$$

The invariance of the action implies that $T^{\mu\nu}$ and the velocity field u transform homogeneously

$$T^{\mu\nu} \to e^{6\Omega(x,t)}T^{\mu\nu}; \qquad u^\mu \to e^{\Omega(x,t)}u^\mu. \tag{2.17}$$

In our construction below we will be imposing the Weyl invariance. To this goal we will employ the Weyl tensor $C^\lambda_{\mu\nu\alpha}$

$$C^\lambda_{\mu\nu\alpha} = R^\lambda{}_{\mu\nu\alpha} - \frac{1}{2}(g^\lambda_\nu R_{\mu\alpha} - g^\lambda_\alpha R_{\mu\nu} - g_{\mu\nu}R^\lambda{}_\alpha + g_{\mu\alpha}R^\lambda{}_\nu) + \frac{1}{6}R(g^\lambda_\nu g_{\mu\alpha} - g^\lambda_\alpha g_{\mu\nu}),$$

which is constructed to be invariant under this transformation. Here $R^\lambda{}_{\mu\nu\alpha}$, $R^\lambda{}_\alpha$ and R stand for the Riemann, Ricci tensors and the scalar curvature.

The hydro representation of the energy-momentum tensor

$$\langle T^{\mu\nu}\rangle_{\rm cl}^{h} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \Pi^{\langle\mu\nu\rangle}. \tag{2.18}$$

Here for any tensor $\Pi^{\mu\nu}$ we define its traceless and symmetric component (following the notations of Ref [29])

$$\Pi^{\langle\mu\nu\rangle} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\mu\beta}(\Pi_{\alpha\beta} + \Pi_{\beta\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\Pi_{\alpha\beta} \tag{2.19}$$

with the projector

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \tag{2.20}$$

which is commonly introduced to ensure transversity of the tensor of dissipations Π:

$$u_\mu \Pi^{\mu\nu} = 0. \tag{2.21}$$

This transversity is equivalent to the condition of no dissipation in the fluid‘s rest frame. The tracelessness of $T^{\mu\nu}$ implies that $\Pi^{\langle\mu\nu\rangle}$ is also traceless.[3] The metric g is the full metric, namely, the Minkowski metric perturbed by h.

The energy-momentum conservation leads to equations of motion for the fluid:

$$\nabla_\mu\langle T^{\mu\nu}\rangle = 0. \tag{2.22}$$

Here ∇_μ stands for covariant derivative with respect to the metric g.

The tensor $\Pi^{\mu\nu}$ is considered to have all order gradient expansion. Within the linearized approximation discussed above, and constrained by the Lorentz and Weyl symmetries, there are four independent structures (operators) one can write down[4]

$$\Pi^{\mu\nu} = -2\eta\nabla^\mu u^\nu + 2\kappa u_\alpha u_\beta C^{\mu\alpha\nu\beta} + \rho(u_\alpha\nabla_\beta + u_\beta\nabla_\alpha)C^{\mu\alpha\nu\beta} + \xi\nabla_\alpha\nabla_\beta C^{\mu\alpha\nu\beta}. \tag{2.23}$$

By representing $\Pi^{\mu\nu}$ in the form (2.23) we essentially postulate a constitutive relation between $\langle T^{ij}\rangle$ and v^i. This structure implies that the fluid can be perturbed either by inducing some velocity perturbation or by shaking the metric. These perturbations are not fully independent and can be related by the equations of motion: the gravity perturbations create perturbations of velocity (see Appendix B).

[3] We ignore the Weyl anomaly since it is nonlinear in the metric perturbations

[4] To our understanding, there are no more structures one could possibly add to the expansion (2.23). The only tensors with more than four Lorentz indices and linear in h are the ones obtained by applying covariant derivatives to the Weyl tensor.

Each of the four transport coefficient functions η, κ, ρ and ξ are considered to be functions of the Lorentz scalar operators ∇^2 and $(u\nabla)$

$$\eta = \eta[\nabla^2, (u\nabla)]; \qquad \kappa = \kappa[\nabla^2, (u\nabla)]; \\ \rho = \rho[\nabla^2, (u\nabla)]; \qquad \xi = \xi[\nabla^2, (u\nabla)]; \tag{2.24}$$

In momentum space representation (adequate for our framework of linear approximation) these functions will depend on $i\omega$ and k^2: $\nabla^2 \to \omega^2 - k^2$ and $(u\nabla) \to -i\omega$.

The first term generalizes the usual shear viscosity coefficient η_0 defined at zero frequency and momentum. It also contains the relaxation time term of second order hydrodynamics. The other terms (GSFs) are due to metric perturbations, absent in Minkowski space. However, as was pointed out in Ref. [29], these terms contribute directly to two-point functions of stress tensors, as computed from the bulk gravity side. Also the "κ" term has been first introduced in Ref. [29].[5]

From the Minkowski perspective, the physical role of κ, ρ and ξ is not obvious. It is well-known that the correlators of $T^{\mu\nu}$ contain not only "thermal" physics but in addition get contaminated by the vacuum or zero temperature contributions due to pair production (this is because the underlying microscopic theory is a quantum field theory). However, a naive subtraction of $T = 0$ contributions leads to sign alternating results for imaginary parts of the correlators [38,40,41] which cannot be identified with true thermal spectral functions. This suggests a presence of interference terms between "vacuum" and thermal amplitudes.

It is tempting to identify the viscosity term with pure hydrodynamic (thermal) physics associated with the matter flow, and the GSFs with the nonhydrodynamic or nonmatter effects and the interference thereof. This conjecture is nicely supported by the ξ term, which at first glance does not depend on the fluid's velocity and temperature at all.[6] Consequently, when looking at the correlators, we will find that in all three channels the contributions due to the ξ term could be naturally identified with the vacuum ($T = 0$) effects. The spectral functions computed from the viscosity terms only are positive definite, as they should.

We would like to comment on the Weyl invariance and nonlinear completions. When introducing the all order (linearized) hydrodynamics (2.23) we presented the tensor $\Pi^{\mu\nu}$ as transforming homogeneously under the Weyl transformation (2.16). It is obvious, however, that $\Pi^{\mu\nu}$, as it appears in (2.23), does not have this property. This is because, while the tensor $C^{\mu\alpha\nu\beta}$ is Weyl invariant, its derivatives are not. Furthermore, higher order derivatives put as arguments of the transport coefficient functions also destroy the desired transformation properties. The correct statement is that the Weyl invariance is recovered up to nonlinear terms, which by themselves are of no interest to us in this paper.[7]

It is then a legitimate and interesting question to ask if for any higher order derivative term there exists a nonlinear completion needed to restore the right transformation property under the Weyl transformation. Can it happen that some of the higher order derivatives both in the viscosity term and the GSF terms cannot be completed to meet the requirement of the Weyl invariance and should be forbidden (similarly to the fate of the bulk viscosity term)? The answer is negative and there is no additional selection principle based on the Weyl symmetry. For any number of derivatives there exist a nonlinear completion with the formal construction given in Ref. [43]. It is based on the fact that, instead of the covariant derivative ∇^μ, one can introduce an even longer derivative D^μ involving the Weyl connection constructed from the field u itself. Any number of these derivatives acting on $C^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}$ leaves a Weyl-invariant result. This procedure generates nonlinear terms, which are of no interest to us in this paper. For our purposes it is sufficient to know about their existence. Note, however, that the procedure of Ref. [43] can be used to reconstruct these nonlinear terms from the higher order linear terms discussed below. That would certainly provide more insight on hydrodynamics at order three and higher.

If we were not to impose the Weyl invariance, we would introduce another shear viscosity term in the expansion (2.23),

$$\eta_2 \nabla^\nu \nabla^\mu \nabla^\alpha u_\alpha.$$

This term would normally contribute to the sound channel starting from third order hydrodynamics. We would like to argue that this term is in fact forbidden by Weyl invariance. As was explained above, in order to comply with Weyl invariance the correct prescription is to use long derivatives D^α instead of ∇^α. However, the long derivative D^α has the property $D^\alpha u_\alpha = 0$, which eliminates the η_2 term.

The hydro ansatz (2.23) can be probed by small gravity perturbations. Using linear response theory we can then compute the retarded correlators in the three symmetry channels (the computation is presented in Appendix B).

(i) The scalar:

$$G^T(k, w) = -i\omega\eta - \kappa\frac{1}{2}(w^2 + k^2) - \rho\frac{i\omega}{2}(w^2 - k^2) + \xi\frac{1}{4}(\omega^2 - k^2)^2 \tag{2.25}$$

[5] In [29] κ was introduced as constant.

[6] Up to nonlinear terms it actually coincides with the stress-energy tensor of the conformal gravity [42].

[7] For the ξ term with constant ξ there exists a well-known nonlinear completion (see e.g. [42]): under the Weyl transformation the tensor $\nabla_\alpha \nabla_\beta C^{\mu\alpha\nu\beta} - 1/2 C^{\mu\alpha\nu\beta} R_{\alpha\beta}$ transforms homogeneously.

(ii) The shear:

$$G^D(k, w) = (\epsilon + P)\frac{\bar{\eta}k^2 - i\bar{\kappa}\omega k^2/2 - \bar{\rho}k^2(k^2 - 2\omega^2)/4 + i\bar{\xi}\omega k^2(\omega^2 - k^2)/4}{-i\omega + \bar{\eta}k^2} \quad (2.26)$$

(iii) The sound:

$$G^S(k, w) = (\epsilon + P)\frac{k^2 - 4i\bar{\eta}\omega k^2 - 2\bar{\kappa}\omega^2 k^2 - 2i\bar{\rho}\omega^3 k^2 + \bar{\xi}\omega^4 k^2}{k^2 - 3\omega^2 - 4i\bar{\eta}\omega k^2} \quad (2.27)$$

with

$$\bar{\eta} \equiv \eta/(\epsilon + P); \qquad \bar{\kappa} \equiv \kappa/(\epsilon + P);$$
$$\bar{\rho} \equiv \rho/(\epsilon + P); \qquad \bar{\xi} \equiv \xi/(\epsilon + P). \quad (2.28)$$

Note that when $k = 0$ the SO(3) symmetry of the space is restored. Modulo trivial rescaling we do indeed observe that the three correlators G^T, G^D, and G^S all coincide:

$$G^T|_{k\to 0} = -\frac{\omega^2}{k^2}G^D|_{k\to 0} = -3/4\frac{\omega^2}{k^2}G^S|_{k\to 0} - (\epsilon + P)/4.$$

At large frequencies $w \gg 1$, the temperature effects should be negligible and the correlators G are expected to coincide with the correlators computed in the vacuum:

$$G^T(\omega, k)_{T=0} = (\epsilon + P)(\omega^2 - k^2)^2 \ln(k^2 - \omega^2);$$
$$G^D(\omega, k)_{T=0} = -(\epsilon + P)k^2(\omega^2 - k^2)\ln(k^2 - \omega^2);$$
$$G^S(\omega, k)_{T=0} = -(\epsilon + P)(4/3)k^2(\omega^2 - k^2)\ln(k^2 - \omega^2). \quad (2.29)$$

The asymptotics (2.29) is indeed observed in the correlators computed from the bulk gravity (see the previous section). What is interesting to note is that the behavior (2.29) is naturally identified with the ξ terms in the correlators, suggesting $\xi \sim \ln(k^2 - \omega^2)$ at asymptotically large ω. It is then tempting to identify the ξ terms as responsible for the contribution to the correlators of the nonhydro pair creation effects, while the κ and ρ terms could be regarded as interference contributions between the "vacuum" and "hydro" physics. Within such interpretation it is natural to identify η as purely hydrodynamical effects associated with the matter flow. Thus if one is interested in pure thermal/hydrodynamic correlators, one first has to determine η as functions of momenta and then compute the correlators with the GSFs set to zero.

Despite this nice interpretation of ξ as the pure vacuum term, all GSF terms in fact fully mix when considered as functions of momenta. If we consider ($\omega \to 0$, $k \to \infty$) asymptotics, all correlators tend to behave proportional to $k^4 \ln k^2$. From this behavior we can learn about the asymptotic behavior of the GSFs themselves

$$\kappa \sim k^2 \ln k^2, \qquad \rho \sim \sqrt{k^2}\ln k^2, \qquad \xi \sim \ln k^2. \quad (2.30)$$

III. WHEN THE BULK MEETS THE BOUNDARY: RESULTS

There should be one to one correspondence between linearized $T^{\mu\nu}$ and the full set of its correlators. Our program is to equate the expressions (2.25), (2.26), and (2.27) for the correlators to the correlators computed from the bulk gravity. The goal is to invert these equations in order to determine the four transport coefficient functions. We have got an apparent problem as we end up having only three equations for four unknown functions. This system does not seem to have a unique solution. Despite our failure to simultaneously determine all transport coefficient functions, we are able to extract them perturbatively in the long-wave limit approximation.

In the near long-wave limit all of the coefficient functions are expandable in power series[8]

$$\eta = \eta_0(1 + i\eta_{0,1}\omega + \eta_{2,0}k^2 + \eta_{0,2}w^2 + i\eta_{2,1}\omega k^2 + i\eta_{0,3}\omega^3 + \eta_{4,0}k^4 + \eta_{2,2}\omega^2 k^2 + \eta_{0,4}\omega^4 + \cdots);$$
$$\kappa = \kappa_0(1 + i\kappa_{0,1}\omega + \kappa_{2,0}k^2 + \kappa_{0,2}w^2 + i\kappa_{2,1}\omega k^2 + i\kappa_{0,3}\omega^3 + \cdots);$$
$$\rho = \rho_0(1 + i\rho_{0,1}\omega + \rho_{2,0}k^2 + \rho_{0,2}w^2 + \cdots)$$
$$\xi = \xi_0(1 + i\xi_{0,1}\omega + \cdots) \quad (3.1)$$

Here we explicitly list all terms up to fifth order. The third order coefficients are determined (practically all) analytically. The other coefficients are extracted numerically. We achieved a good accuracy with the forth order coefficients while the rest have large errors.

[8]We believe this expansion has a finite radius of convergence, The radius of convergence is given by the first singularity, which coincides with the first quasinormal mode of the scalar channel.

$$\eta_0 = (\epsilon + P)/2; \qquad \tau \equiv \eta_{0,1} = 2 - \ln 2;$$
$$\eta_{2,0} = -1/2; \qquad \kappa_0 = 2\eta_0; \tag{3.2}$$
$$\kappa_{0,1} = 5/2 - 2\ln 2; \qquad \rho_0 = 4\eta_0.$$

The viscosity η_0 is of course just (1.1). The coefficient $\eta_{0,1}$ is the relaxation time, which within the AdS/CFT approach was first addressed in Ref. [44]. It was correctly determined in Refs. [29,30] and later in [45]. In [29] it was found by looking at the first correction to speed of sound. $\eta_{0,1}$ can be consistently deduced from any of the three correlators. κ_0 was found also in [29] by matching the k^2 term in G^T. Independently and consistently, it can be also found from the shear and sound channels (the ωk^2 term in the numerator of G^D and the w^2k^2 term in the numerator of the function G^S).

The coefficient $\eta_{2,0}$ appears at third order hydro, which was left beyond the scopes of [29]. However, this coefficient could be easily read off from the analysis of Ref. [29], in particular, from the k^4 correction to the diffusive pole in the shear channel. The result is consistent with the k^4 term in the numerator of G^D. The coefficient ρ_0 is deduced from the $\omega = 0$ limit of the function G^D. Finally we analytically extracted the coefficient $\kappa_{0,1}$. This comes from matching the coefficients of the ωk^2 in the scalar channel.

The remaining coefficients were found numerically. Let consider the coefficient $\eta_{0,2}$ as an example of our numerical procedure. We were able to get a very accurate fit of the coefficient in front of the ω^3 term in the expansion of the correlator G^T. This coefficient is then trivially related to $\eta_{0,2}$ and κ_0, $\kappa_{0,1}$, ρ_0, the latter being all previously determined. The result is

$$\eta_{0,2} \simeq -1.379 \pm 0.001 \simeq -\frac{3}{2} + \frac{\ln^2 2}{4} \tag{3.3}$$

where the last expression is our guess for the analytic expression. The error in Eq. (3.3), as well as other errors quoted below, reflect our confidence in the results provided.

Despite the fact that we were not able to find a method to extract four unknown coefficient functions from three equations, there seems to be a recurrent procedure, which make this task possible, at least perturbatively near the long-wave limit. The coefficient $\kappa_{2,0}$ can be obtained from the $\omega = 0$ limit of the sound correlator G^S. Once this one is known, the $\omega = 0$ limit of G^T reveals the coefficient ξ_0, etc.

Below we present our numerical results. 4th order hydro

$$\eta_{2,1} = -2.275 \pm 0.005; \qquad \eta_{0,3} = -0.082 \pm 0.003 \tag{3.4}$$

5th order hydro

$$\eta_{4,0} = 0.565 \pm 0.005; \qquad \eta_{0,4} = 2.9 \pm 0.1; \qquad \eta_{2,2} = 1.1 \pm 0.2; \tag{3.5}$$

The GSF's coefficients

$$\begin{aligned} &\kappa_{2,0} = -1.6 \pm 0.05; \qquad \kappa_{0,2} = 0.04 \pm 0.01; \\ &\kappa_{0,3} = -1.95 \pm 0.05; \qquad \kappa_{2,1} = -1.6 \pm 0.2; \\ &\rho_{0,1} = 0.92 \pm 0.01; \qquad \rho_{0,2} = -0.68 \pm 0.04; \\ &\rho_{2,0} = -0.755 \pm 0.005; \qquad \xi_0 = -2.6 \pm 0.1; \\ &\xi_{0,1} = -1.1 \pm 0.2. \end{aligned} \tag{3.6}$$

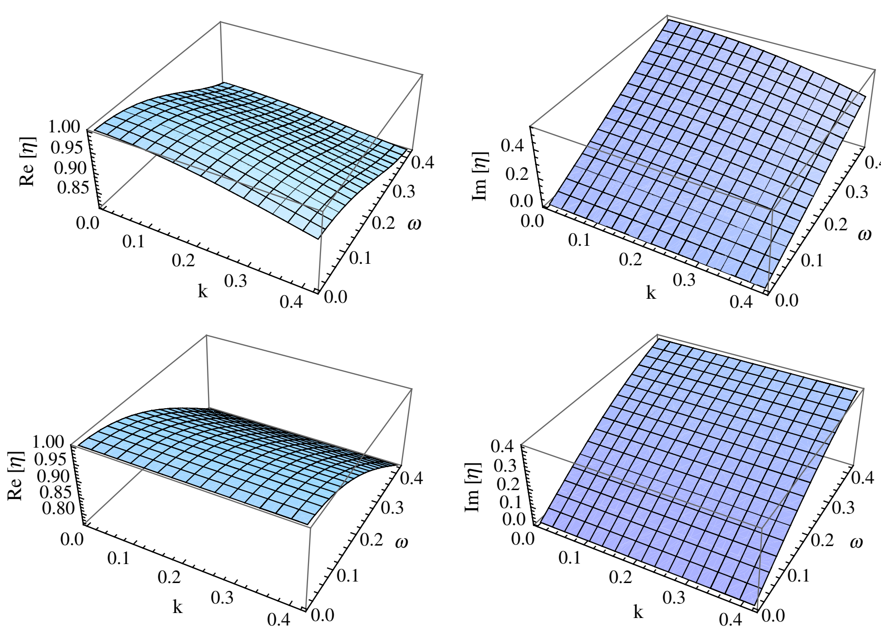

FIG. 3 (color online). Viscosity function (divided by η_0): top AdS/CFT; bottom IS.

To summarize our knowledge of viscosity function η, we plot it and compare to the IS one (Fig. 3). The NS value is, of course, $\eta = \eta_0$. For ω, $k \leq 0.4$ we can expect up to 15% correction due to momenta-dependence of the viscosity function.

On quasinormal modes and analytic structure of the viscosity function

The quasinormal modes are poles of the retarded correlators. They have been analyzed in all three channels in [39]. We would like to argue that entire information about quasinormal modes is coded in the viscosity function η, while the GSF do not have any poles. If this were not true, we would observe appearance of identical quasinormal modes in all three channels, which is not the case at least for a number of low lying modes.

$$\eta(k^2, w) = \sum_{n=0}^{\infty} \frac{\eta_n(k^2, w)}{\omega - \omega_n(k^2)}. \tag{3.7}$$

We further argue that ω_n coincide with the quasinormal modes of the scalar channel (poles of G^T), which have been analyzed in the past (see the table below). At $k^2 = 0$ they can be computed quasiclassically for large n (in fact quasiclassics works well down to $n = 2$) [46]

$$\omega_n(k^2 = 0) \simeq \omega_0 + n(\pm 1 - i). \tag{3.8}$$

No analytical expression for nonzero k is known.

The quasinormal modes of the shear and sound channels are obtained from the following dispersion relations.

$$\begin{aligned} -i\omega + \eta(k^2, \omega)k^2 = 0; \\ -3\omega^2 + k^2 - 4i\eta(k^2, \omega)\omega k^2 = 0 \end{aligned} \tag{3.9}$$

As well-known, these dispersion relations admit hydrodynamic modes as lowest modes in the spectrum. Higher modes will appear as distorted spectrum ω_n. Furthermore, the higher the mode the less distortion should be present. In other words, the spectra of all three channels will become degenerate for high modes. This tendency is clearly observed in the following table copied from Ref. [39] ($k^2 = 1$)

	Scalar channel		Shear channel		Sound channel	
n	$\Re e\omega_n$	$\Im m\omega_n$	$\Re e\omega_n$	$\Im m\omega_n$	$\Re e\omega_n$	$\Im m\omega_n$
1	±1.954331	−1.267327	±1.759116	−1.291594	±1.733511	−1.343008
2	±2.880263	−2.297957	±2.733081	−2.330405	±2.705540	−2.357062
3	±3.836632	−3.314907	±3.715933	−3.345343	±3.689392	−3.363863
4	±4.807392	−4.325871	±4.703643	−4.353487	±4.678736	−4.367981
5	±5.786182	−5.333622	±5.694472	−5.358205	±5.671091	−5.370784

Finally we would like to note that from the behavior of the sound dispersion curve [39] one can deduce the following asymptotic behavior of the viscosity function

$$\eta(k^2 \sim \omega^2 \to \infty) \to \frac{i}{2\omega}, \tag{3.10}$$

which supports our understanding that it is a falling function at large momenta.

IV. MODEL FOR IMPROVED CAUSAL HYDRODYNAMICS

Though in this paper we do not pursuit any practical applications, we would like to propose an improved and causal hydrodynamics for future use by hydro practitioners.

While we were not able to achieve our prime goal, of deducing the viscosity function in full range of frequency and momentum, we were able to get several new coefficients for the small momenta expansion. Below we present a resummation scheme similar to IS, which is an ansatz aimed at providing a good model for the entire viscosity function. The model is constructed with the requirement of causality built in.

Causality implies that the imaginary part of the poles is always negative and the function vanishes at infinite frequencies. This is equivalent to the validity of the dispersion relation:

$$\eta(k^2, \omega) = \int \frac{d\omega'}{2\pi i} \frac{\Re e\eta(k^2, \omega')}{\omega' - \omega}. \tag{4.1}$$

In addition, in order to relate the viscosity function to the thermal spectral functions, we require that both real and imaginary parts of it remain positive for all values of momentum and frequency.

Similarly to the IS model, we take a Pade-like resummation ansatz which reproduces all low momentum coefficients in the expansion.

$$\eta_{\mathrm{model}^1} = \eta_0 \sum_{i=1}^{3} \frac{d_i}{a_i + b_i k^2 - iw}. \tag{4.2}$$

This ansatz has three pure imaginary poles and it reproduces exactly eight first coefficients in the expansion (3.1).

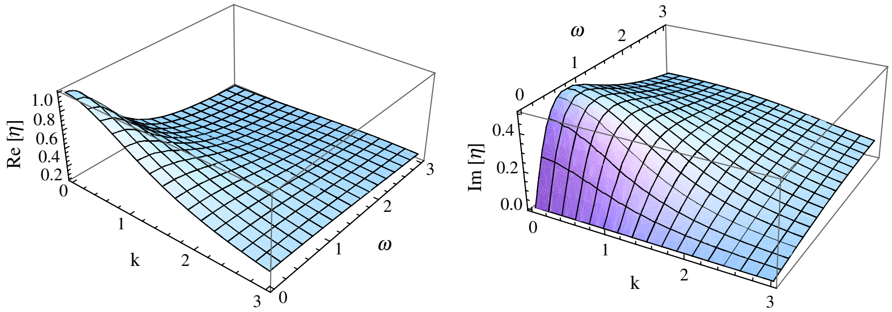

FIG. 4 (color online). Viscosity function (divided by η_0): the model.

$$d_1 = 0.736, \qquad a_1 = 0.72731, \qquad b_1 = 0.3263$$
$$d_2 = 2.1, \qquad a_2 = 0.10618, \qquad b_2 = 0.3042,$$
$$d_3 = -2.1016, \qquad a_3 = 0.10620, \qquad b_3 = 0.3038.$$

The resummed viscosity function is plotted in Fig. 4. This model could be further improved by accounting for the asymptotic behavior (3.10) as well as for information about quasinormal modes of the scalar channel. The second and third poles practically cancel each other. Despite the fact that it does not accurately reproduce the expansion, it turns out to be a very good approximation to retain only one pole, similarly to IS but with three-momentum dependence.

$$\eta_{\text{model}^2} = \frac{\eta_0}{1 - \eta_{2,0}k^2 - iw\eta_{0,1}}. \tag{4.3}$$

Within about 10% accuracy (and in some regions with much better one) the second model is equivalent to the first one. Since the entire effect of momenta-dependence is not expected to be very large, the second model should be more than sufficient for any phenomenological applications. We note that the group velocity for the sound mode computed within this model is always smaller than one, confirming causality of the model. The viscosity function can be Fourier transformed into the memory function

$$D(x, t) = \int d\omega d^3k e^{-i\omega t + ikx}\eta(k^2, \omega), \tag{4.4}$$

which leads to the following expression for the dissipation tensor Π:

$$\Pi^{\mu\nu} = -2\int_0^t dt' \int d^3x' D(x - x', t - t')\nabla'^{\mu}u^{\nu}(x', t'). \tag{4.5}$$

Performing the Fourier transform explicitly we obtain

$$D_{\text{model}^2}(x, t) = \int d\omega d^3k e^{-i\omega t + ikx}\eta_{\text{model}^2}(k^2, \omega) = \frac{1}{2\sqrt{2}}\frac{\eta_0}{\eta_{0,1}}\left(\frac{-\eta_{0,1}}{\eta_{2,0}t}\right)^{3/2} e^{-t/\eta_{0,1}} e^{x^2\eta_{0,1}/(\eta_{2,0}t)}. \tag{4.6}$$

We remind the reader that $\eta_{2,0}$ is negative.

V. SUMMARY AND DISCUSSION

In this paper we initiated a study of all order velocity gradient expansion of linearized relativistic hydrodynamics near equilibrium. The research was carried out within the $\mathcal{N} = 4$ SYM theory at large N_c. More specifically, we parametrized the energy-momentum tensor of the theory in terms of four momenta-dependent functions. These functions generalize the notion of the usual constant transport coefficients, such as viscosity, into momenta-dependent ones. We then attempted to determine all four functions based on the information on retarded correlators of two stress tensors. The latter were computed via the AdS/CFT prescription for computing retarded correlators from bulk gravity waves.

Out of four transport coefficient functions, which we introduced in (2.23), η appears as a coefficient of the operator constructed from velocity gradients and is a generalization of shear viscosity. The remaining three coefficients (GSFs) arise as coefficients of four-dimensional metric perturbations and appear in front of three operators involving the curvature Weyl tensor.

In this paper we extended the previous knowledge of hydrodynamic transport coefficients at first and second order to some higher order coefficients. We were able to find only those, which contribute to linearized hydrodynamics. We gave analytic values for two coefficients of the third order hydro. We provided very accurate numerical estimates for two coefficients of the forth order and one of the fifth. In addition, we introduced and determined several new coefficients associated with the GSFs.

To illustrate the effect of the higher order terms in the viscosity function, we compute the sound dispersion curve by solving (perturbatively) Eq. (3.9) to the order k^6:

$$\omega_{\rm AdS} = \pm\frac{k}{\sqrt{3}}\left(1+\left(\frac{1}{2}-\frac{\ln[2]}{3}\right)k^2 - 0.088k^4\right) - i\frac{k^2}{3}\left(1-\frac{k^2}{12}(4-8\ln 2+\ln^2 2) - 0.15k^4\right) \quad (5.1)$$

As we have already emphasized in Ref. [31], the sound width gets negative corrections from higher order terms. This is in sharp contrast to the IS model, which leads to a qualitatively opposite effect, with the correction being positive

$$\omega_{\rm IS} = \pm\frac{k}{\sqrt{3}}\left(1+\left(\frac{1}{2}-\frac{\ln[2]}{3}\right)k^2\right) - i\frac{k^2}{3}\left(1+\frac{k^2}{3}\ln 2(2-\ln 2)\right). \quad (5.2)$$

Based on the new information about higher order terms in the expansion of the viscosity function, we have proposed an improved causal IS-like (single pole) hydrodynamics, which we hope can be used by hydro practitioners. Compared to IS, this model emphasizes the importance of the space-momentum dependence of the viscosity function. It leads to qualitatively different predictions, as seen from the width of the sound pole. On the basis of this example, we cautiously suggest that the results based on the IS theory might be in fact less reliable than it was previously thought. We also propose to exploit our improved model for nonlinear phenomena (such as Bjorken expansion and elliptic flow) even though such applications have no theoretical justification.

Admittedly, the problem we had set up was not yet fully solved by the present paper. Using a perturbative procedure, we found several new higher order (constant) transport coefficients, either analytically or numerically. Nevertheless, a generic problem remains to be solved: the correlators we used as our input seemed not to be sufficient to determine the four transport coefficient functions parametrizing all relevant kinematic structures. While it is possible to follow the iterative approach used by us to determine more coefficients in the expansion of the transport functions, we have no proof that this procedure will actually work at *all* higher orders with unique results. It is not excluded that some additional inputs (apart of the correlators) are required in order to solve the problem in full.

As an alternative approach to the problem, one may switch from solving the bulk equations for gravity waves—the basis for computing the correlators—to a membrane paradigm-type approach based on vibrations and translations of the horizon, as it is done in Ref. [30] and the following papers [34,47]. This approach provides a quite general procedure to derive the next order derivative terms for boundary hydrodynamics. In this approach, the boundary energy-momentum tensor with appropriate gradient corrections is obtained through the usual holographic renormalization procedure, with the bulk solution reflecting perturbation of the near-horizon "membrane". If the boundary metric is not taken as flat Minkowski (as it was done in [30]), but rather as a slightly perturbed one, the method of [30] would reveal the GSFs alongside the viscosity function. Furthermore, the approach of [30] has a potential to determine not only linear but also nonlinear terms, the latter being beyond the scope of our present paper. Here we obviously mean third and higher order hydrodynamics. We have not pursued this direction, but believe it is worth studying it as it is important to learn about the gradient structure as a way to understand the nonequilibrium effects in plasma.

An important general problem is a separation between the hydrodynamic (thermal) physics associated with the matter flow and the vacuum (zero temperature) effects associated with the pair production, as both contribute to the retarded correlators. We hope that we proposed the right approach to it, by identifying the different roles played by the viscosity function and the GSFs. While the former is purely "hydrodynamical", the latter includes nonthermal physics and interference. This separation of roles is very plausible, supported by the results at hand, but it was not proven by us in general. We have argued in the text that the pole structure of the correlators is entirely included in the viscosity function, while the GSFs have no poles. The overall role of the GSFs is somewhat unclear. On the one hand, they are formally introduced in (2.23) as a response of the fluid to external gravitational shakings. On the other hand, from the analysis of the correlators we identify the GSFs as being responsible for flat space nonhydrodynamic (nonthermal) effects associated with pair production of the underlying microscopic field theory. The metric perturbations in effect mimic nonhydro physics.

Are the GSFs relevant for RHIC experiment? We believe the answer is "no". We had to deal with them only because the correlators used for our analysis contain both thermal physics and vacuum effects (such as pair creation). One, of course, could propose another type of experiment, in which plasma would be exposed to a real gravitational wave. In this type of experiment, the GSFs would determine the physical response of the fluid.

The so-called contact (or Schwinger) terms are QFT phenomena originating in UV. One could suspect that the GSFs originate from those. However, if this were the case, the only effect they would produce is to shift the correlators by finite order polynomials in momenta. This is not the case, however. From the explicit expressions for the correlators (2.26) and (2.27) one can see that the numerator terms involving the GSFs cannot cancel the corresponding poles. Thus they include more physics than just the contact terms part of which is not coming from the UV.

Last but not least, it remains to be seen how relevant the effects of momenta-dependent viscosity are for realistic applications of relativistic hydrodynamics to heavy ion collisions. Our previous paper [31] argues that they might be quite substantial at the early times of the collision. As we explained in the Introduction, recently the phenomenology shifted to the ‘‘fate of small initial state fluctuations’’, related with the conical structure and ‘‘ridges’’. Although we have not applied our results, it is clear that such flows would be sensitive to higher gradients, as the size of those fluctuations is an order of magnitude smaller than the nuclear size associated with radial and elliptic flows studied before. In agreement with our proposal [31], the viscosity function (its real part) is a decreasing function both of frequency and momenta. This behavior might be the reason behind the low viscosity observed at RHIC. It may also explain the exceptionally good survival of various hydrodynamic flows, particularly the sound waves.

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APPENDIX A: ANALYTIC EXPANSIONS OF CORRELATORS

In this Appendix we present analytic expansions of retarded correlators at small frequency and momentum, as computed from the bulk gravity. The expansions are obtained following Appendix of Ref. [29] where a perturbative approach to solving Eq. (2.2) is set. We reproduce and extend their results to include some of higher order terms.

1. Scalar channel

$$\begin{aligned} A &= 1 + i\frac{\ln 2}{2}\omega + \ln 2\left(\frac{3\ln 2}{8} - 1\right)\omega^2 + \ln 2 k^2 - \frac{\ln^2 2}{2}k^4 \\ &\quad - i\frac{\ln^2 2}{2}\omega k^2 + \left(\frac{5}{4} - \frac{\ln^3 2}{6}\right)k^6 \ldots \\ B &= \frac{1}{2}k^2 + i\frac{1}{2}\omega + \left(\frac{3}{4} - \frac{\ln 2}{2}\right)k^4 - \left(\frac{1}{2} - \frac{\ln 2}{4}\right)\omega^2 \\ &\quad - i\frac{\ln 2}{4}\omega k^2 + \frac{\ln 2}{4}(3 - \ln 2)k^6 \ldots \end{aligned} \tag{A1}$$

For the retarded correlator G^T we obtain

$$\begin{aligned} \frac{1}{(\epsilon + P)}G^T &= -\frac{B}{A} \\ &= -\frac{1}{2}k^2 - i\frac{1}{2}\omega - \frac{1}{2}(\ln 2 - 1)\omega^2 \\ &\quad - \frac{1}{4}(3 - 4\ln 2)k^4 + i\ln 2\omega k^2 - \ln^2 2 k^6 \ldots \end{aligned} \tag{A2}$$

2. Shear channel

$$\begin{aligned} A &= \omega + i\frac{1}{2}k^2 + i\frac{1}{4}k^4 + \frac{\ln 2}{4}wk^2 + i\frac{\ln 2}{2}\omega^2 \ldots; \\ B &= \frac{i}{2}(k^2 - \omega^2)\left(1 + i\frac{2 - \ln 2}{2}\omega - \frac{1}{2}k^2 \ldots\right) \end{aligned} \tag{A3}$$

The correlator reads

$$\frac{1}{(\epsilon + P)}G^D = -\frac{k^2}{\omega^2 - k^2}\frac{B}{A} = \frac{ik^2/2[1 + i(2 - \ln 2)\omega - k^2/2\ldots] + \omega k^2/2 + \cdots}{\omega + ik^2/2[1 + i(2 - \ln 2)\omega - k^2/2 + \cdots]}. \tag{A4}$$

3. Sound channel

$$A = 8\left[9\omega^2 - 3k^2 + i6\omega k^2 + (\ln 2 - 4)k^4 + 9\ln 2\left(\frac{\ln 2}{2} - 1\right)\omega^4 + 3\ln 2\left(2 - \frac{\ln 2}{2}\right)\omega^2 k^2\right]$$
$$B = 3[18\omega^2 - 30k^2 + i12\omega k^2 + 2(5\ln 2 - 12)k^4 + 3\ln 2(12 - 5\ln 2)\omega^2 k^2 + 9\ln 2(\ln 2 - 2)\omega^4 \ldots] \quad \text{(A5)}$$

The analytically controlled part of the sound correlator

$$\frac{1}{(\epsilon + P)} G^S = \left(\frac{4}{3}\frac{k^4}{(\omega^2 - k^2)^2}\frac{B}{A} + \frac{1}{12}\frac{29k^4 - 30k^2\omega^2 + 9\omega^4}{(k^2 - \omega^2)^2} - \frac{3}{4}\right)\frac{\omega^2}{k^2} + 1$$
$$= \frac{-k^2 + i2[1 - i\omega(\ln 2 - 2) + \cdots]\omega k^2 + 2\omega^2 k^2 \ldots}{3\omega^2 - k^2 + i2\omega k^2[1 - i\omega(\ln 2 - 2) + \cdots]}. \quad \text{(A6)}$$

Here we used Eq. (2.10).

APPENDIX B: CORRELATORS FROM GENERALIZED HYDRODYNAMICS

In this Appendix we compute the retarded correlators from the hydrodynamic ansatz (2.23) using 4d metric perturbations. The nonperturbed space has the Minkowski metric $g^{\mu\nu} = diag\{-1, 1, 1, 1\}$.

1. Scalar channel

The perturbation is $h \equiv h_{xy}(z, t)$. The fluid remains at rest. We first compute Christoffels coefficients and Riemann tensor

$$\Gamma^t_{xy} = \Gamma^x_{ty} = \Gamma^y_{tx} = \frac{1}{3}\dot{h};$$
$$\Gamma^x_{zy} = \Gamma^y_{zx} = -\Gamma^z_{xy} = \frac{1}{2}h'$$
$$R^t_{xty} = R^x_{tty} = R^y_{ttx} = \frac{1}{2}\ddot{h}; \quad \text{(B1)}$$
$$R^x_{zzy} = R^y_{zzx} = -R^z_{xzy} = \frac{1}{2}h'';$$
$$R^t_{xzy} = R^x_{tzy} = R^y_{tzx} = -R^z_{xzy} = R^x_{zty} = R^y_{ztx} = \frac{1}{2}\dot{h}'$$

The only nonzero component of the Ricci tensor is R_{xy} while the scalar curvature is zero

$$R_{xy} = \frac{1}{2}(\ddot{h} - h'') \qquad R = 0. \quad \text{(B2)}$$

The relevant nonzero components of the Weyl tensor are

$$C_{txty} = -C_{txyt} = C_{xtyt} = C_{ytxt} = C_{xzyz} = C_{yzxz}$$
$$= -\frac{1}{4}(\ddot{h} + h'') \quad \text{(B3)}$$
$$C_{xzyt} = C_{yzxt} = C_{xtyz} = C_{ytxz} = -\frac{1}{2}\dot{h}'.$$

The xy component of the stress tensor reads

$$\langle T^{xy}\rangle = -Ph - \eta\dot{h} - \frac{1}{2}\kappa[\ddot{h} + h''] + \rho\frac{1}{2}[\dddot{h} - \dot{h}'']$$
$$- \xi\frac{1}{4}[\ddddot{h} - 2\ddot{h}'' + h''''']. \quad \text{(B4)}$$

In momentum space this becomes

$$\langle T^{xy}\rangle = -\left[P - i\omega\eta - \kappa\frac{1}{2}(w^2 + k^2) - \rho\frac{i\omega}{2}(w^2 - k^2) + \xi\frac{1}{4}(\omega^2 - k^2)^2\right] \quad h(k, w). \quad \text{(B5)}$$

From Eq. (B5) one can read off the correlator $G^T = G^{xyxy}$

$$\tilde{G}^{xyxy}(k, w) = P - i\omega\eta - \kappa\frac{1}{2}(w^2 + k^2) - \rho\frac{i\omega}{2}(w^2 - k^2) + \xi\frac{1}{4}(\omega^2 - k^2)^2. \quad \text{(B6)}$$

The retarded correlator $G^{xyxy} = \tilde{G}^{xyxy} - P$. The transverse static susceptibility χ^T is momenta dependent and is given by the functions κ and ξ:

$$\chi^T(k) = -\kappa(k, 0)k^2/2 + \xi(k, 0)k^4/4. \quad \text{(B7)}$$

2. Shear channel

The perturbation $h \equiv h_{tx}(z, t)$. The fluid's four velocities is $u^\mu = (1, v, 0, 0)$ and $u_\mu = (-1, v + h, 0, 0)$ The Christoffels coefficients and the Riemann tensor are

$$\Gamma^x_{tz} = -\Gamma^t_{xz} = -\Gamma^z_{xt} = \frac{1}{2}h'; \qquad \Gamma^x_{tt} = \dot{h}$$
$$R^t_{txtz} = -R^t_{xzt} = R^x_{ttz} = -R^x_{tzt} = -R^z_{txt} = -\frac{1}{2}\dot{h}'$$
$$R^z_{xzt} = -R^z_{xtz} = R^x_{ztz} = -R^z_{txz} = -\frac{1}{2}h''. \quad \text{(B8)}$$

The nonzero Ricci components and curvature are

$$R_{xz} = -\frac{1}{2}\dot{h}'; \qquad R_{xz} = -\frac{1}{2}h''; \qquad R = 0. \quad \text{(B9)}$$

The relevant nonzero Weyl components read

$$C_{xtzt} = C_{ztxt} = \frac{1}{4}\dot{h}'; \qquad C_{xzzt} = C_{ztxz} = \frac{1}{4}h''. \quad \text{(B10)}$$

The components of the stress tensor

$$\langle T^{tt}\rangle = \epsilon; \qquad T^{tx} = (\epsilon + P)v + Ph$$
$$\langle T^{xz}\rangle = -\eta v' + \kappa\frac{1}{2}\dot{h}' + \rho\frac{1}{4}(h''' - 2\ddot{h}') + \xi\frac{1}{4}(\dddot{h}' - \dot{h}'''). \quad \text{(B11)}$$

Equations of motion relate the metric perturbation h to the induced three-velocity v:

$$\partial_t\langle T^{tx}\rangle = (\epsilon + P)(\dot{v} + \dot{h});$$
$$\partial_z\langle T^{zx}\rangle = -\eta v'' + \kappa\frac{1}{2}\dot{h}'' + \rho\frac{1}{4}(h'''' - 2\ddot{h}'' + \xi\frac{1}{4}(\dddot{h}''h - \dot{h}'''') \quad \text{(B12)}$$

which leads to the relation (in momentum space)

$$v = h\frac{i\omega - i\bar{\kappa}\omega k^2/2 - \bar{\rho}k^2(k^2 - 2\omega^2)/4 + i\bar{\xi}\omega k^2(\omega^2 - k^2)/4}{-i\omega + \bar{\eta}k^2}. \quad \text{(B13)}$$

Substituting this relation back into the expression for $\langle T^{tx}\rangle$ we can read off the correlator $G^D = G^{txtx}$

$$\tilde{G}^{txtx} = (\epsilon + P)\frac{\bar{\eta}k^2 - i\bar{\kappa}\omega k^2/2 - \bar{\rho}k^2(k^2 - 2\omega^2)/4 + i\bar{\xi}\omega k^2(\omega^2 - k^2)/4}{-i\omega + \bar{\eta}k^2} - \epsilon \quad \text{(B14)}$$

Note the appearance of the extra terms proportional to the GSFs in the numerator, whereas in the normal diffusion scenario the residue is usually given by the viscosity only.

The retarded correlator $G^{txtx} = \tilde{G}^{txtx} + \epsilon$. The shear static susceptibility χ^D:

$$\chi^D(k) = (\epsilon + P)\left[1 - \frac{\bar{\rho}(k, 0)}{4\bar{\eta}(k, 0)}k^2\right]. \quad \text{(B15)}$$

3. Sound channel

The perturbation which generates sound is $h \equiv h_{tz}(z, t)$. The fluid‘s four velocities is $u^\mu = (1, 0, 0, v)$ and $u_\mu = (-1, 0, 0, v + h)$.

Christoffels and Riemann are

$$\Gamma^t_{zz} = -h'; \qquad \Gamma^z_{tt} = \dot{h};$$
$$R^z_{tztz} = -R^z_{ttz} = R^t_{zzt} = -R^t_{ztz}. \quad \text{(B16)}$$

Contrary to the cases of tensor and shear perturbations, the sound perturbation has a nonvanishing scalar curvature.

$$R_{zz} = -R_{tt} = -\dot{h}'; \qquad R = -2\dot{h}'; \qquad C_{ztzt} = \frac{1}{3}\dot{h}'. \quad \text{(B17)}$$

The relevant components of the stress tensor

$$\langle T^{tt}\rangle = \epsilon; \qquad \langle T^{tz}\rangle = (\epsilon + P)v + Ph;$$
$$\langle T^{zz}\rangle = P - \eta\frac{4}{3}v' + \kappa\frac{2}{3}\dot{h}' - \rho\frac{2}{3}\ddot{h}' + \xi\frac{1}{3}\dddot{h}'. \quad \text{(B18)}$$

Equations of motion can be solved for v relating it to the perturbation h

$$v = h\frac{3\omega^2 - 2\bar{\kappa}\omega^2k^2 - 2i\bar{\rho}\omega^3k^2 + \bar{\xi}\omega^4k^2}{k^2 - 3\omega^2 - 4i\bar{\eta}\omega k^2}. \quad \text{(B19)}$$

Substituting v back into the expression for T^{tz} we can read off the correlator $G^S = G^{tztz}$

$$\tilde{G}^{tztz} = (\epsilon + P) \times \frac{k^2 - 4i\bar{\eta}\omega k^2 - 2\bar{\kappa}\omega^2k^2 - 2i\bar{\rho}\omega^3k^2 + \bar{\xi}\omega^4k^2}{k^2 - 3\omega^2 - 4i\bar{\eta}\omega k^2} - \epsilon. \quad \text{(B20)}$$

The retarded correlator $G^{tztz} = \tilde{G}^{tztz} + \epsilon$.

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