Spectral sum rules for the quark-gluon plasma

P. Romatschke and D. T. Son

Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA

(Received 15 April 2009; published 21 September 2009)

We derive sum rules involving the spectral density of the stress-energy tensor in $\mathcal{N} = 4$ super-Yang-Mills theory and pure Yang-Mills theory. The sum rules come from the hydrodynamic behavior at small momenta and the conformal (in the case of $\mathcal{N} = 4$ super-Yang-Mills theory) or asymptotically free (as for the pure Yang-Mills theory) behavior at large momenta. These sum rules may help constrain quark-gluon plasma transport coefficients obtained from lattice QCD.

DOI: 10.1103/PhysRevD.80.065021 PACS numbers: 12.38.Mh, 11.25.Hf, 11.10.Wx

I. INTRODUCTION

Recently, much interest has been concentrated on the transport properties of the strongly coupled quark-gluon plasma (QGP) created at the Relativistic Heavy Ion Collider (RHIC) [1–4]. Attempts have been made to extract these coefficients from the lattice [5–9]. These calculations rely on the reconstruction of the real-time spectral function from Euclidean (imaginary-time) correlation functions, which for numerical data is an ill-defined procedure unless extra assumptions are made. In practice, the reconstruction amounts to postulating a form of the spectral density, and then fitting the parameters of the ansatz using lattice data.

Clearly, it would be of great help if some constraints on the spectral density can be derived. For nonrelativistic fluids, there exist sum rules (for example, the f-sum rule) that constrain the spectral densities [10]. One may wonder if such sum rules exist in a relativistic theory. Some progress has been made in this direction: for instance, Kharzeev and Tuchin [11] (and later Karsch, Kharzeev, and Tuchin [12]) wrote down a sum rule for the spectral density of the trace of the stress-energy tensor. The slope of this spectral density at zero frequency is the bulk viscosity ζ. Although the sum rule does not fix the form of the spectral density, with some assumptions the authors of Refs. [11,12] argued that the bulk viscosity becomes large near the QCD phase transition. (As we shall see below, the precise form of our sum rule in the bulk channel is slightly different from that of Kharzeev and Tuchin, but some features of the latter remain intact. We point out that the difference stems from a subtle noncommutativity of limits.) Several sum rules are also argued to hold for weakly coupled relativistic theories by Teaney [13].

In this paper, we derive certain sum rules for the spectral density in hot gauge theories. We start with the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, which is a prototype of the strongly coupled QGP, and derive the following spectral sum rule:

$$\frac{2}{5}\epsilon = \frac{2}{\pi}\int \frac{d\omega}{\omega}[\rho(\omega) - \rho_{T=0}(\omega)], \tag{1}$$

where $\rho = -\mathrm{Im}G_R(\omega)$, and G_R is the retarded propagator of the T^{xy} component of the stress-energy tensor. Here $\rho_{T=0}(\omega)$ is the spectral density at zero temperature, and ϵ is the finite-temperature energy density. Besides $\mathcal{N} = 4$ SYM theory at infinite 't Hooft coupling, Eq. (1) is valid for any theory whose gravitational dual has purely Einstein gravity. It is also valid for $\mathcal{N} = 4$ SYM theory at any nonzero coupling.[1]

Another sum rule relates a linear combination of second-order hydrodynamic coefficients with the spectral function,

$$\eta\tau_\pi - \frac{1}{2}\kappa = \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega^3}[\rho(\omega) - \rho_{T=0}(\omega) - \eta\omega], \tag{2}$$

where η is the shear viscosity, and τ_π (the relaxation time) and κ are defined in Ref. [14].

We then move to the bulk sector of QCD and show the following sum rule:

$$3(\epsilon + P)(1 - 3c_s^2) - 4(\epsilon - 3P) = \frac{2}{\pi}\int \frac{d\omega}{\omega}[\rho^{\mathrm{bulk}}(\omega) - \rho^{\mathrm{bulk}}_{T=0}(\omega)], \tag{3}$$

where c_s is the speed of sound and $\rho^{\mathrm{bulk}}(\omega)$ is the spectral density for the trace of the stress-energy tensor T^μ_μ. Equation (3) is similar to, but different from, the sum rule suggested by Kharzeev and Tuchin, and by Karsch, Kharzeev, and Tuchin. We shall show that the sum rule is a consequence of the hydrodynamic behavior of the QGP at large distances, and of asymptotic freedom at small distances.

The structure of our paper is as follows. In Sec. II we remind the reader how spectral sum rules can be derived. In Sec. III we derive two sum rules for the spectral function in the shear channel. One sum rule relates the total energy density with the spectral density, and another sum rule relates a linear combination of second-order hydrodynamic coefficients with the same spectral density. We verify both sum rules by numerically computing the spectral integral.

[1] If the coupling is vanishing, the stress-energy tensor is not the only dimension four operator anymore, and one has to consider its individual parts from scalars, vectors, etc.

We also comment on the possible form of the shear sum rule for pure Yang-Mills theory. In Sec. IV we turn to pure Yang-Mills theory and derive a sum rule for the bulk channel.

II. KRAMERS-KRONIG RELATION

For definiteness, consider the retarded correlator of the T^{xy} component of the stress-energy tensor in, say, the $\mathcal{N}=4$ SYM theory. The spectral function is defined to coincide, up to a sign, to the imaginary part of the retarded Green function of T^{xy},

$$\rho(\omega,\mathbf{q}) = -\mathrm{Im}G_R(\omega,\mathbf{q}), \tag{4}$$

where we assume $\mathbf{q}$ to be along the z direction. Since G_R is the Fourier transform of a real function (recall that G_R determines a linear response), we have

$$G_R(-\omega,\mathbf{q}) = G_R^*(\omega,-\mathbf{q}). \tag{5}$$

Let us consider the function $f_{\mathbf{q}}(Z)$, defined so that $f_{\mathbf{q}}(\omega^2) = G_R(\omega,\mathbf{q})$, which has a cut from $Z=0$ to $Z=\infty$.

We can write down a Kramers-Kronig relation for $f_{\mathbf{q}}(Z)$. Pretend for a moment that $f_{\mathbf{q}}(Z)\to 0$ as $Z\to\infty$. Taking the integral of $f_{\mathbf{q}}(Z)/(Z+\alpha^2)$ over the contour in Fig. 1, we find

$$\frac{1}{2\pi i}\oint dZ\frac{f_{\mathbf{q}}(Z)}{Z+\alpha^2} = f_{\mathbf{q}}(-\alpha^2), \tag{6}$$

if $f_{\mathbf{q}}(Z)$ does not have singularities except for the positive real semiaxis. In a relativistic field theory, typically G_R diverges as $\omega\to\infty$. For example, the T^{xy} correlator $G_R(\omega)$ grows like $\omega^4\ln\omega$ at large ω (see, e.g., [15]). One can subtract this zero-temperature piece $G_R^{T=0}(\omega)$ and we denote $\delta G_R = G_R - G_R^{T=0}$. But, as will be shown, there remains a constant piece which also needs to be subtracted. Therefore, we define

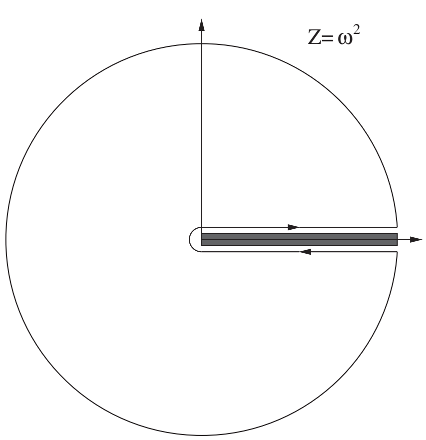

FIG. 1. Integration contour for the integral (6).

$$f_{\mathbf{q}}(\omega^2) = \delta G_R(\omega,\mathbf{q}) - \delta G_R^\infty, \qquad \delta G_R^\infty = \lim_{\omega\to+i\infty}\delta G_R(\omega,\mathbf{q}), \tag{7}$$

so that the Kramers-Kronig relation (6) for this function $f_{\mathbf{q}}(Z)$ is valid.

In a conformal, large-N_c theory,[2] the asymptotics of $G_R(\omega)$ at small ω and k is known to second-order from hydrodynamics [14],

$$G_R(\omega,q) = G_R(0) - i\eta\omega + \left(\eta\tau_\pi - \frac{1}{2}\kappa\right)\omega^2 - \frac{1}{2}\kappa q^2 + \cdots. \tag{8}$$

The constant $G_R(0)$ may depend on the way the correlator is defined. When the correlator is defined through the response to metric perturbations, one finds $G_R(0)=P$ [14]. For this particular correlator, $f_q(Z)$ does not have singularities outside the positive real semiaxis and the contour integral may be deformed to give

$$\begin{aligned} f_q(-\alpha^2) &= \delta G_R(0) - \delta G_R^\infty + \eta\alpha + \left(\eta\tau_\pi - \frac{\kappa}{2}\right)\alpha^2 - \frac{1}{2}\kappa q^2 + \cdots \\ &= -\frac{2}{\pi}\int_0^\infty d\omega\frac{\omega}{\omega^2+\alpha^2}\delta\rho(\omega,q), \end{aligned} \tag{9}$$

where $\delta\rho(\omega,q) = \rho(\omega,q) - \rho_{T=0}(\omega,q)$.

Setting $\alpha=0$ in this formula, we find

$$\begin{aligned} -f_q(0) &= \delta G_R^\infty - \delta G_R(0) + \frac{1}{2}\kappa q^2 + \mathcal{O}(q^4) \\ &= \frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\delta\rho(\omega,q), \end{aligned} \tag{10}$$

which for $q=0$ will become Eq. (1). Subtracting the α-independent part, the Kramers-Kronig relation becomes

$$\eta\alpha + \left(\eta\tau_\pi - \frac{1}{2}\kappa\right)\alpha^2 + \cdots = \frac{2\alpha^2}{\pi}\int_0^\infty\frac{d\omega}{\omega}\frac{\delta\rho(\omega,q)}{\omega^2+\alpha^2}. \tag{11}$$

From this we derive another sum rule,

$$\eta\tau_\pi - \frac{1}{2}\kappa + \mathcal{O}(q^2) = \frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega^3}[\delta\rho(\omega,q) - \eta\omega], \tag{12}$$

which for $q=0$ is Eq. (2). Note that the coefficient κ enters this equation, which is interesting because in the hydrodynamic equations κ couples only to curvature tensors [14] and hence drops out for flat space. Equation (12) suggests that κ can be determined from flat-space physics. In fact, κ can already be obtained from Euclidean correla-

[2] Outside the large-N_c limit, there can be nonanalytic terms (such as $\omega^{3/2}$) present in $G_R(\omega)$ [16].

tors at $\omega = 0$ and small q. For weakly coupled $SU(N_c)$ gauge theory, we find a nonzero value for κ at the lowest order of perturbation theory (see Appendix A). Curiously, κ divided by the entropy density only differs by a factor of about two between strongly coupled $\mathcal{N} = 4$ SYM and free $SU(N_c)$ gauge theory.

A note on the definition of the correlators

It is clear from the previous discussion that, in order to derive the sum rule in a particular theory, one should use the same definition for the correlation function in the ultraviolet (UV) ($\omega \to \infty$) and infrared (IR) ($\omega \to 0$). In this paper, we define the correlators through the partition function $\mathcal{Z}$ in curve spacetime. The one- and two-point functions are given as the first and second derivatives of $\ln\mathcal{Z}$ with respect to the metric. In Euclidean signature, one has

$$\delta \ln\mathcal{Z} = \frac{1}{2}\int dx \langle T^{\mu\nu}(x)\rangle \delta g_{\mu\nu}(x) + \frac{1}{8} \times \int dx dy \langle T^{\mu\nu}(x) T^{\rho\sigma}(y)\rangle \delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y) + \cdots \tag{13}$$

In other words,

$$\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle = 4\frac{\delta^2 \ln\mathcal{Z}}{\delta g_{\rho\sigma}(y)\delta g_{\mu\nu}(x)}\bigg|_{g_{\alpha\beta}=\delta_{\alpha\beta}} = 2\frac{\delta}{\delta g_{\rho\sigma}(y)}\langle\sqrt{g}T^{\mu\nu}(x)\rangle\bigg|_{g_{\alpha\beta}=\delta_{\alpha\beta}}. \tag{14}$$

An alternative definition of the correlator is through the path integral in flat space,

$$\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle' = \frac{1}{\mathcal{Z}}\int \mathcal{D}A e^{-S_E} T^{\mu\nu}(x)T^{\rho\sigma}(y), \tag{15}$$

where A represents all fields in the theory and S_E is the Euclidean action. The two correlators differ by a contact term[3]

$$\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle = \langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle' - 4\left\langle\frac{\delta^2 S_E}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)}\right\rangle. \tag{16}$$

Analogously, one can define the Minkowski-space correlation functions. The retarded Green function is found from the linear response,

$$\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle_R = -2\frac{\delta}{\delta g_{\rho\sigma}(y)}\langle\sqrt{-g}T^{\mu\nu}(x)\rangle|_{g_{\alpha\beta}=\eta_{\alpha\beta}}. \tag{17}$$

The advantage of using the correlator defined through Eqs. (13) and (17) is that we know this correlator at low momenta through hydrodynamics. Indeed, using the hydrodynamic equations one can establish how a system responds to external gravitational perturbations, and then use Eq. (14) to find the correlation functions (see, e.g., [17,18]). In addition, this is the most natural definition that comes out of AdS/CFT correspondence. Note that on the lattice, so far what is normally measured is (15). However, this difference does not matter as far as the Kramers-Kronig relation (9) is concerned, because subtracting δG_R^∞ from $\delta G_R(0)$ makes the contact term drop out from $f_q(0)$.

III. SHEAR SUM RULES IN $\mathcal{N} = 4$ SYM AND PURE YANG-MILLS THEORY

A. AdS/CFT calculation of $f_q(0)$

For the case of large 't Hooft coupling, properties of $\mathcal{N} = 4$ SYM can be calculated using the AdS/CFT duality [19–21]. In particular, it is known how to calculate finite-temperature correlators $G_R(\omega, \mathbf{q})$ in AdS/CFT [17]. To find the $\langle T^{xy}T^{xy}\rangle$ correlator, we solve the equation of motion for the xy component of the metric, which is essentially the equation for a minimally coupled scalar,

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z\phi) - g^{\mu\nu}k_\mu k_\nu\phi = 0, \tag{18}$$

where $\mu = 0\ldots3$ indexes the usual four field theory dimensions and g is the determinant of the (five-dimensional) metric. We denote the fifth dimension by z (not to be confused with the spatial direction in the previous section), where $z = 0$ corresponds to the four-dimensional boundary of AdS_5 space. Finite-temperature correlators in AdS/CFT can be studied by considering the metric of a static black hole in the bulk. The location of the event horizon z_H of the black hole is related to its Hawking temperature, $z_H = (\pi T)^{-1}$. Being interested in δG_R at large imaginary $\omega \gg T$, we can restrict ourselves to the region of anti–de Sitter (AdS) space very close to the boundary.

For convenience, we shall use here the metric in Fefferman-Graham coordinates, which has the following form [22] near the boundary at $z = 0$:

$$ds^2 = R^2\frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2} + \frac{R^2z^2}{4z_H^4}(3dt^2 + d\vec{x}^2) + \mathcal{O}(z^4), \tag{19}$$

where R is the scale set by the AdS radius. In these coordinates, Eq. (18) becomes

[3]A contact term is a term in $\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle$ that is proportional to $\delta^4(x-y)$ or its derivatives (corresponding to a constant or polynomial in momentum space). For theories which do not have derivatives of the metric in the action (such as Yang-Mills theory), the contact terms can only be constants.

$$\phi'' - \frac{3}{z}\phi' + \left(1 + \frac{3z^4}{4z_H^4}\right)\omega^2\phi - \left(1 - \frac{z^4}{4z_H^4}\right)\mathbf{q}^2\phi = 0. \tag{20}$$

We will put $\mathbf{q} = \mathbf{0}$ from now on, and consider Euclidean momentum $\omega^2 = -Q^2$. In the deep Euclidean region $Q^2 \to +\infty$, most of the interesting dynamics happen near the boundary, so one can solve Eq. (20) iteratively in inverse powers of z_H. We expand the solution as

$$\phi = \phi_0 + \phi_1 + \cdots, \tag{21}$$

where $\phi_0 = \frac{1}{2}(Qz)^2 K_2(Qz)$ is obtained by sending $z_H \to \infty$ in Eq. (20) and demanding regularity at $z \to \infty$ (see also [13]). The first correction ϕ_1 satisfies

$$\phi_1'' - \frac{3}{z}\phi_1' - Q^2\phi_1 = j \equiv 3Q^2\frac{z^4}{4z_H^4}\phi_0. \tag{22}$$

The solution to this equation is formally given by Green's function,

$$\phi_1(z) = \int dz' G(z, z') j(z'), \tag{23}$$

where $G(z, z')$ can be constructed from the known solutions to the homogeneous Eq. (22),

$$f_1(z) = (Qz)^2 K_2(Qz), \qquad f_2(z) = (Qz)^2 I_2(Qz), \tag{24}$$

as

$$G(z, z') = -\frac{1}{W[f_1, f_2](z')}[f_1(z)f_2(z')\theta(z - z') + f_2(z)f_1(z')\theta(z' - z)], \tag{25}$$

where the Wronskian $W[f_1, f_2] = f_1 f_2' - f_1' f_2$ evaluates to $W[f_1, f_2](z') = Q^4 z'^3$.

Evaluating the integral (25) using

$$\int_0^\infty dz z^5 K_2^2(z) = \frac{32}{5}, \tag{26}$$

we then find the small z asymptotics of ϕ_1 to be given by

$$\phi_1(z) = -\frac{3}{10}\frac{z^4}{z_H^4}. \tag{27}$$

Recalling that the correlation function is given by [15]

$$G_R(\omega) = -\frac{N_c^2}{8\pi^2}\lim_{z\to 0}\frac{\phi'(z)\phi(z)}{z^3}, \tag{28}$$

which at $T = 0$ reproduces the well-known result,

$$G_R^{T=0}(\omega) = -\frac{N_c^2}{8\pi^2}\frac{\phi_0'(z)}{z^3}\bigg|_{z\to 0} = \frac{N_c^2}{32\pi^2}Q^4 \ln Q. \tag{29}$$

The first correction due to temperature is

$$\lim_{\omega\to i\infty}\delta G_R(\omega)|_{\phi_1} = -\frac{N_c^2}{8\pi^2}\frac{\phi_1'(z)}{z^3} = \frac{3}{20\pi^2}\frac{N_c^2}{z_H^4} = \frac{3\pi^2}{20}N_c^2 T^4 = \frac{2}{5}\epsilon. \tag{30}$$

There are also contributions to the correlators from the boundary terms in the action (i.e., $\int d^4x\sqrt{-\gamma}$, cf. [23]) but one can check that they contribute the same amount at any ω, and they are the only contribution at $\omega = 0$ (the contact terms). Thus, in $\mathcal{N} = 4$ SYM theory one has $f_{q=0}(0) = -\frac{2}{5}\epsilon$ and hence the sum rule (10) becomes

$$\frac{2}{5}\epsilon = \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega}[\rho(\omega) - \rho_{T=0}(\omega)]. \tag{31}$$

We will call this sum rule the "shear sum rule" as the slope of $\rho(\omega)$ at $\omega = 0$ is the shear viscosity. It is clear from our derivation that this shear sum rule holds in any theory with an Einstein gravitational dual.

B. Rederivation of $f_q(0)$ in $\mathcal{N} = 4$ SYM from OPE

Within $\mathcal{N} = 4$ SYM theory the sum rule can be derived without relying on gravity. We start with the operator product expansion (OPE) of the stress-energy tensor [24],

$$T_{\mu\nu}(x)T_{\rho\sigma}(0) \sim C_T\frac{\mathcal{I}_{\mu\nu,\rho\sigma}(x)}{x^8} + \hat{A}_{\mu\nu\rho\sigma\alpha\beta}(x)T_{\alpha\beta}(0) + \cdots \tag{32}$$

Here $\hat{A}$ contains various Lorentz structures, all scaling as x^{-4}, and is given explicitly in Ref. [24] in terms of three constants a, b, and c. In a thermal ensemble the second term averages to a constant contribution to the correlator. Setting $\mu = \rho = x$, $\nu = \sigma = y$, and performing Fourier transform, we get[4]

$$\delta G_R(\omega)|_{\omega\to i\infty} = -\frac{18(a + b)}{14a - 2b - 5c}P. \tag{33}$$

In $\mathcal{N} = 4$ SYM theory the coefficients a, b, and c are given by [25]

$$a = -\frac{16}{9\pi^6}(N_c^2 - 1), \qquad b = -\frac{17}{9\pi^6}(N_c^2 - 1), \qquad c = -\frac{92}{9\pi^6}(N_c^2 - 1), \tag{34}$$

and hence

$$\delta G_R(\omega)|_{\omega\to i\infty} = \frac{11}{5}P. \tag{35}$$

[4] In Ref. [24] the correlator is defined as the second derivative of the partition function with respect to the *upper* components of the metric, while we differentiate $\ln Z$ with respect to the lower components. This difference, together with the sign change by going from Euclidean to retarded propagator, has been taken into account in Eq. (33).

On the other hand, the shift of G_R when $\omega \to 0$ can be found from hydrodynamics, which predicts $\delta G_R(0) = P$ [14,17]. Therefore we find $f_{q=0}(0) = -\frac{6}{5}P$, which corresponds to (1) since for a conformal field theory $P = \frac{1}{3}\epsilon$.

A remark is in order. It is known that the constants a, b, and c are independent of the coupling in $\mathcal{N} = 4$ SYM theory: in fact, their value can be found from one-loop calculations [25]. Therefore, the sum rule is valid for any nonzero value of the coupling.

C. Calculation of $f_q(0)$ in pure Yang-Mills theory

In pure Yang-Mills theory, the UV behavior is that of a weakly coupled field theory. The leading terms in the OPE are the same as for free fields. The coefficients a, b, c can be found from the general formulas [24]

$$a = \frac{1}{27\pi^6} n_\phi - \frac{2}{\pi^6} n_v, \tag{36a}$$

$$b = -\frac{4}{27\pi^6} n_\phi - \frac{1}{2\pi^6} n_f, \tag{36b}$$

$$c = -\frac{1}{27\pi^6} n_\phi - \frac{1}{\pi^6} n_f - \frac{8}{\pi^6} n_v, \tag{36c}$$

where n_s, n_f, and n_v are the number of real scalars, Dirac fermions, and gauge fields in the theory.

For pure Yang-Mills theory, by repeating the calculations in Sec. III B, we find $f_{q=0}(0) = -2P$ (this can be checked directly by computing the relevant Feynman diagram; see Appendix B). However, there is an additional subtlety here that was not present in the previous section because the OPE of two components of the stress-energy tensor may involve terms like $\alpha_s F^2$, where α_s is the strong coupling constant. Though formally higher order in α_s, these terms average to $\epsilon - 3P$ which is a constant independent of the scale x in Eq. (32). Therefore, we can tentatively write a sum rule for pure Yang-Mills theory,

$$\frac{\epsilon + P}{2} + C(\epsilon - 3P) = \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega}[\rho(\omega) - \rho_{T=0}(\omega)]. \tag{37}$$

where the constant C is left to be determined by a more accurate calculation. (However, naively applying the results of Refs. [26] about the absence of the leading-order gluon condensate contribution in the tensor glueball channel would imply that $C = 0$).

In the large-N_c limit, where second-order hydrodynamic coefficients are well defined, the sum rule (12) is valid (except that one has to use a proper definition of κ for nonconformal theories [27]).

D. Numerical verification of sum rules in AdS/CFT

In AdS/CFT, the spectral function $\rho(\omega)$ can be calculated numerically for arbitrary frequency/momenta from the solution to the mode equation (18). For convenience, we adopt a metric and coordinates such that Eq. (18) becomes [15]

$$\phi'' - \frac{1+u^2}{uf(u)}\phi' + \frac{\mathrm{w}^2 - \mathrm{q}^2 f(u)}{uf^2(u)}\phi = 0, \tag{38}$$

where $u = z^2/z_H^2$, $f(u) = 1 - u^2$, $\mathrm{w} = \omega/(2\pi T)$, and $\mathrm{q} = q/(2\pi T)$. Again, we set the spatial momentum $\mathrm{q} = 0$ in the following, although the method described below can also handle nonvanishing momenta.

We follow the algorithm by Teaney [13], which is outlined here for completeness, fixing some typos in Ref. [13].[5] Equation (38) is recast in a system of coupled first-order equations by introducing $\pi = \phi'$. Discretizing derivatives as $\phi' = (\phi(u + \delta u) - \phi(u))/\delta u$, Eq. (38) can be explicitly integrated forward from a point close to the boundary $u = u_0$. If π is taken to be defined at half-integer step sizes $\pi(u) = \pi((n + 1/2)\delta u)$ and ϕ at integer step sizes $\phi(u) = \pi(n\delta u)$, then the resulting algorithm is second-order accurate in δu ("leapfrog algorithm"), suggesting numerical stability. To start the algorithm, initial conditions for ϕ and π at $u = u_0$, $u = u_0 + \frac{1}{2}\delta u$, respectively, need to be specified. For u_0 sufficiently close to the boundary $u = 0$, Eq. (38) may be solved analytically, yielding the pair of solutions

$$\Phi_1(u) = u^2\left[1 - \frac{\mathrm{w}^2}{3}u + \cdots\right],$$
$$\Phi_2(u) = -\frac{\mathrm{w}^4}{2}\log(u)\Phi_1(u) + 1 + \mathrm{w}^2 u - \frac{2}{9}\mathrm{w}^6 u^3 + \cdots, \tag{39}$$

where the u^2 term in $\Phi_2(u)$ is arbitrary (it can be any multiple of $\Phi_1(u)$) and was set to zero in accordance with the convention by Kovtun and Starinets [28]. The analytic result for $\Phi_1(u)$ (and its derivative) is used as initial conditions for $\phi(u_0)$, $\pi(u_0 + \frac{1}{2}\delta u)$, which can then be integrated forward to give a numerical solution $\phi_1(u_1)$ with u_1 close to the horizon $u = 1$ (the same procedure for $\Phi_2(u)$ gives $\phi_2(u_1)$). The physically interesting solution for $\phi(u)$ is the one that corresponds to an incoming wave at the horizon, $\phi(u) \sim (1-u)^{-i\mathrm{w}/2}$. Solving Eq. (38) analytically close to $u = 1$ one finds for the incoming wave solution

$$\phi^{\mathrm{inc}}(u) = (1-u)^{-i\mathrm{w}/2}\left[1 - (1-u)\frac{2i\mathrm{w}^3 + 3\mathrm{w}^2 - i\mathrm{w}}{4(1+\mathrm{w}^2)} - (1-u)^2\frac{\mathrm{w}(4\mathrm{w}^3 + 7i\mathrm{w}^2 - 2\mathrm{w} + 4i)}{32(\mathrm{w}^2 + 3i\mathrm{w} - 2)} + \cdots\right]. \tag{40}$$

The real solutions $\phi_1(u)$ and $\phi_2(u)$ are linear combinations of the incoming and outgoing wave solutions,

[5] A version of the C++ code will be made available at http://hep.itp.tuwien.ac.at/~paulrom

$$\phi_1(u) = A(\mathrm{w})\phi^{\mathrm{inc}}(u) + B(\mathrm{w})\bar{\phi}^{\mathrm{inc}}(u),$$
$$\phi_2(u) = C(\mathrm{w})\phi^{\mathrm{inc}}(u) + D(\mathrm{w})\bar{\phi}^{\mathrm{inc}}(u), \qquad (41)$$

where $\bar{\phi}^{\mathrm{inc}}$ denotes the complex conjugate of ϕ^{inc}. For given w, the complex constants A, B, C, D are, e.g., calculated from the numerical solution of $\phi_{1,2}$ at $u = u_1$ and $u = u_1 - \delta u$ and the analytic solution (40) for ϕ^{inc} close to the boundary. As a consequence, one can construct a numerical solution to Eq. (38) with incoming wave boundary conditions by

$$\phi^{\mathrm{inc,num}}(u) = \frac{D}{AD - BC}\phi_1(u) - \frac{B}{AD - BC}\phi_2(u), \quad (42)$$

which can be normalized to $\phi(0) = 1$ by realizing $\Phi_1(0) = 0$, $\Phi_2(0) = 1$, so that close to the boundary

$$\phi^{\mathrm{inc,norm}}(u_0) = -\frac{D(\mathrm{w})}{B(\mathrm{w})}\Phi_1(u) + \Phi_2(u). \qquad (43)$$

In practice, we found the choices $u_0 = 10^{-6}$, $u_1 = 0.999$ to give acceptable numerical accuracy. Once the normalized solution to the mode equation is known, the retarded correlator is obtained from (28)

$$G_R(\omega, 0) = -\frac{\pi^2 N_c^2 T^4}{4} \lim_{u_0 \to 0} \frac{\partial_u \phi^{\mathrm{inc,norm}}(u_0)}{u_0}. \qquad (44)$$

In particular, using the analytical results for $\Phi_{1,2}$ one finds for the spectral function

$$\delta\rho(\omega, 0) = -4P\frac{D(\mathrm{w})}{B(\mathrm{w})} - 2\pi P \mathrm{w}^4, \qquad (45)$$

where $P = \frac{\pi^2}{8} N_c^2 T^4$ in strongly coupled $\mathcal{N} = 4$ SYM. The numerical result for the spectral function is shown in Fig. 2. As can be seen from this figure, $\delta\rho(\omega, 0)$ first increases as a function of ω, reaching a maximum at around $\mathrm{w} = 0.45$, then decreases strongly and oscillates around zero with an amplitude that decays quasiexponentially. Figure 2(b) shows the spectral function where the leading hydrodynamic behavior $\eta\omega$ has been subtracted. As can be seen from this figure, for small frequencies the spectral function seems to behave as

$$\frac{\delta\rho(\omega, 0)}{\eta\omega} = 1 + a_0 \mathrm{w}^2 + a_1 \mathrm{w}^3 + \mathcal{O}(\mathrm{w}^4), \qquad (46)$$

where numerically we determine $a_0 \simeq 1.72$, $a_1 \simeq -3.0$.

With the value $\frac{1}{8P} f_{q=0}(0) = \frac{3}{20}$ calculated above, the first sum rule (1) would imply the identity

$$\frac{3}{20} = 0.15 \overset{?}{=} \frac{1}{\pi}\int_0^\infty \frac{d\mathrm{w}}{\mathrm{w}}\left(-\frac{D(\mathrm{w})}{B(\mathrm{w})} - \frac{\pi}{2}\mathrm{w}^4\right)$$
$$\simeq \frac{1}{\pi}\int_0^{\mathrm{w}_{\max}} \frac{d\mathrm{w}}{\mathrm{w}}\left(-\frac{D(\mathrm{w})}{B(\mathrm{w})} - \frac{\pi}{2}\mathrm{w}^4\right) = 0.150\,000\,8(44), \qquad (47)$$

which we can confirm up to five-digit accuracy when choosing $w_{\max} = 6$ in practice. For the second sum rule (2), $\eta\tau_\pi - \frac{1}{2}\kappa = 2P\frac{1-\log(2)}{(2\pi T)^2}$ from Ref. [14] implies

$$1 - \log(2) \simeq 0.306\,853 \overset{?}{=} \frac{4}{\pi}\int_0^\infty \frac{d\mathrm{w}}{\mathrm{w}^3}\left(-\frac{D(\mathrm{w})}{B(\mathrm{w})} - \frac{\pi}{2}\mathrm{w}^4 - \frac{\mathrm{w}}{2}\right)$$
$$\simeq -\frac{2}{\pi \mathrm{w}_{\max}} + \frac{4}{\pi}\int_0^{\mathrm{w}_{\max}} \frac{d\mathrm{w}}{\mathrm{w}^3}\left(-\frac{D(\mathrm{w})}{B(\mathrm{w})} - \frac{\pi}{2}\mathrm{w}^4 - \frac{\mathrm{w}}{2}\right)$$
$$= 0.306\,86(2), \qquad (48)$$

indicating that the numerical result matches with four-digit accuracy. While it is possible to improve the numerical accuracy further, we take this agreement of at least one part in 10^{-4} between the analytical and numerical results as an indication that for $\mathcal{N} = 4$ SYM, the sum rules (1) and (2) are correct.

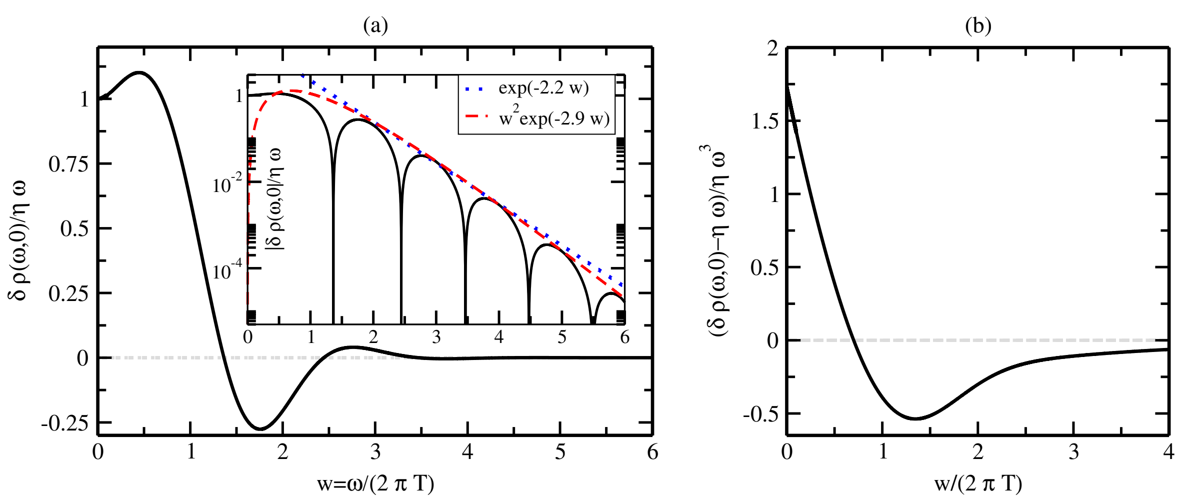

FIG. 2 (color online). Numerical results for the spectral function $\delta\rho(\omega, 0)$ for strongly coupled $\mathcal{N} = 4$ SYM. The results shown correspond to the integrand of the sum rules (1) and (2), respectively. The inset in (a) demonstrates the near-exponential drop in the amplitude of $\delta\rho(\omega, 0)$. Horizontal dashed lines are visual aids.

IV. THE BULK SUM RULE IN QCD

In this section we revisit the sum rule satisfied by the imaginary part of the correlation function of the trace of the stress-energy tensor T^{μ}_{μ}. As the spectral density in this channel is related to the bulk viscosity, this sum rule will be called the "bulk sum rule." We show that this sum rule indeed exists, but its form is slightly different from the one given in Refs. [11,12].

In this section we shall be concerned with metric perturbations of the following form:

$$g_{\mu\nu} = \eta_{\mu\nu} e^{2\Omega}, \tag{49}$$

or $\delta g_{\mu\nu} = \eta_{\mu\nu}(e^{-2\Omega} - 1)$, with $\Omega \ll 1$. For these perturbations, the partition function expansion defined in Eq. (13) can be explicitly given as

$$\begin{aligned}\delta \ln Z = &\int dx \eta_{\mu\nu}(x)\langle T^{\mu\nu}(x)\rangle[-\Omega(x) + \Omega^2(x))] \\ &+ \frac{1}{2}\int dx dy \eta_{\mu\nu}(x)\eta_{\rho\sigma}(y)\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle \\ &\times \Omega(x)\Omega(y).\end{aligned} \tag{50}$$

We then define the correlators of $\theta(x)$ as follows:

$$\begin{aligned}\langle\theta(x)\rangle &\equiv -\frac{\delta \ln Z}{\delta\Omega(x)}\bigg|_{\Omega=0} = \sqrt{-g}g_{\mu\nu}\langle T^{\mu\nu}(x)\rangle|_{\Omega=0} \\ &= \eta_{\mu\nu}(x)\langle T^{\mu\nu}(x)\rangle,\end{aligned} \tag{51}$$

$$\begin{aligned}\langle\theta(x)\theta(y)\rangle &\equiv \frac{\delta^2 \ln Z}{\delta\Omega(x)\delta\Omega(y)}\bigg|_{\Omega=0} \\ &= \eta_{\mu\nu}(x)\eta_{\rho\sigma}(y)\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle \\ &\quad + 2\delta(x-y)\eta_{\mu\nu}(x)\langle T^{\mu\nu}(x)\rangle \\ &\equiv -\frac{\delta}{\delta\Omega(x)}\langle\theta(y)\rangle\bigg|_{\Omega=0},\end{aligned} \tag{52}$$

where we recall that in conformal field theories $\langle\theta(x)\rangle = \langle\theta(x)\theta(y)\rangle = 0$. As a consequence of the definition (51), the correlator $\langle\theta(x)\theta(y)\rangle$ differs from $\eta_{\mu\nu}\eta_{\rho\sigma}\langle T^{\mu\nu}T^{\rho\sigma}\rangle$ by a contact term, which in its turn differs from $\langle T^{\mu}_{\mu}T^{\nu}_{\nu}\rangle'$ by a contact term. Our subsequent calculations are simplest when using the correlator $\langle\theta(x)\theta(y)\rangle$ defined in this fashion.

Consider the pure Yang-Mills theory. We need to know how to couple Yang-Mills to an external metric perturbation of the form (49). This is done through changing the bare coupling $g_s^2 = 4\pi\alpha_s$, so that it is dependent upon the metric. In particular, the Euclidean action of pure Yang-Mills becomes

$$S_E = \int dx \frac{1}{4g_s^2(\Lambda e^{\Omega})}F^2_{\mu\nu}, \tag{53}$$

where we have rescaled the gauge fields so that the field strength tensor is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu, \tag{54}$$

and f^{abc} are the $SU(N)$ structure constants. From this we find

$$\frac{\delta S_E}{\delta\Omega} = \beta(g_s)\frac{\partial S}{\partial g_s} = -\frac{\beta(g_s)}{2g_s^3}F^2_{\mu\nu}, \tag{55}$$

where $\beta(g_s) = \Lambda\partial_\Lambda g_s$ is the beta function. As a consequence, one has

$$\langle\theta(x)\rangle = -\frac{\beta(g_s)}{2g_s^3}\langle F^2_{\mu\nu}(x)\rangle, \tag{56}$$

$$\begin{aligned}\langle\theta(x)\theta(y)\rangle = &\left(\frac{\beta(g_s)}{2g_s^3}\right)^2\langle F^2(x)F^2(y)\rangle + \beta(g_s)\frac{\partial}{\partial g_s}\left(\frac{\beta(g_s)}{2g_s^3}\right) \\ &\times\langle F^2\rangle\delta(x-y).\end{aligned} \tag{57}$$

If we are interested in computing $G_R(\omega)$, it is most convenient to choose $\Lambda \sim \omega$, so the two terms in Eq. (57) can be evaluated perturbatively without large logarithms. In the weak coupling regime

$$\beta(g_s) = -b_0 g_s^3 - b_1 g_s^5 + \cdots, \tag{58}$$

so the first term in Eq. (57) is proportional to $g_s^4(\omega)T^4$, while the second term is proportional to $g_s^4(\omega)(\epsilon - 3P)$. When $\omega \to \infty$, the correlation function vanishes because of asymptotic freedom.

Now let us compute $\langle\theta\theta\rangle_R$ at small frequencies. For that we need to find the response of the system of an external metric perturbation with $\Omega = \Omega(t)$ varying slowly with t. Since the perturbation is spatially homogeneous, we expect the fluid to remain at rest ($u^0 = e^{\Omega}$, $u^i = 0$), but the temperature will have time dependence: $T = T(t)$. It is more convenient to work with the entropy density s instead of T. When metric perturbations are slow, entropy is conserved. The solution to the equation for entropy conservation, $\nabla_\mu(su^\mu) = 0$, is

$$s = e^{3\Omega}s_0, \tag{59}$$

and therefore

$$\frac{\partial}{\partial\Omega} = 3s\frac{\partial}{\partial s}. \tag{60}$$

This result may now be directly applied to the definition of the correlation function, and hence we find

$$\begin{aligned}\langle\theta\theta\rangle_R(\omega \to 0, \mathbf{q} = 0) &= \frac{\partial}{\partial\Omega}\langle\sqrt{-g}T^{\mu}_{\mu}\rangle \\ &= -\left(3s\frac{\partial}{\partial s} - 4\right)(\epsilon - 3P).\end{aligned} \tag{61}$$

Now let us derive the spectral sum rule. Introducing the spectral function $\rho^{\rm bulk}(\omega)$ in the bulk channel, we find

$$\left(3s\frac{\partial}{\partial s} - 4\right)(\epsilon - 3P) = \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega}\delta\rho^{\rm bulk}(\omega), \tag{62}$$

where ϵ and P are now the thermal parts of the energy and pressure (with the divergent vacuum contributions subtracted out). This, we argue, is the correct version of the sum rule by Karsch, Kharzeev, and Tuchin [11,12]. The right-hand side can be transformed into Eq. (3) by using the thermodynamic relations $d\epsilon = Tds$ and $dP/d\epsilon = c_s^2$. Note that Eq. (62) does not coincide with Refs. [11,12], which had $T\partial/\partial T$ instead of $3s\partial/\partial s$. The issue is the noncommutativity of the $q \to 0$ and $\omega \to 0$ limits in the bulk channel (see Appendix C). The correct expression for the right-hand side follows directly from entropy conservation in hydrodynamics.

In the weak coupling limit of high-temperature gauge theory, the pressure is given by [29,30]

$$P = T^4(A + g_s^2 B + \mathcal{O}(g_s^3)), \tag{63}$$

where A, B are constants that are unimportant for the following discussion. Calculating the trace anomaly from $\epsilon = T\frac{dP}{dT} - P$ one finds

$$\epsilon - 3P = 2BT^4 g_s \beta(g_s) \sim \mathcal{O}(g_s^4 T^4), \tag{64}$$

which implies [31]

$$\left(3s\frac{\partial}{\partial s} - 4\right)(\epsilon - 3P) \sim \mathcal{O}(g_s^6 T^4), \tag{65}$$

where $c_s^{-2} = 3 + \frac{2B}{A} g\beta(g_s)$ was used. On the other hand, the integral over the spectral function gives a contribution $\mathcal{O}(g_s^4 T^4)$ at low frequency [31]. In order for our sum rule (62) to hold for weakly coupled QCD, the $\mathcal{O}(g_s^4 T^4)$ contribution must be canceled to leading-order by the high frequency tail in the spectral function. There are indications that this is indeed what is happening in weakly coupled QCD [32].

V. CONCLUSION

In this paper we have written down several sum rules involving the spectral functions in hot gauge theories. The sum rules can be checked in $\mathcal{N} = 4$ SYM theory using gauge/gravity duality. The bulk sum rule for QCD was also derived. We still have some uncertainty in the shear sum rule in QCD, but hopefully this will be resolved in the future.

Some conclusions may be drawn from our work. First, note that the left-hand side (LHS) of the sum rule (2) is positive in strongly coupled $\mathcal{N} = 4$ SYM because $\eta\tau_\pi > \frac{1}{2}\kappa$. This implies that the spectral function $\delta\rho$ in the shear channel must be larger than $\eta\omega$ for some frequencies, or otherwise the integral would not be positive. This feature of the spectral function is clearly seen on Fig. 2. This requirement is not satisfied by the simplest Lorentzian ansatz for the spectral function, $\delta\rho(\omega) \sim \frac{\eta\omega}{\omega^2+\Gamma^2}$. A similar argument can be made for weakly coupled QCD in the large N_c limit, because there $\tau_\pi \sim 6\eta/(sT)$ [33] and $\eta/s \sim \alpha_s^{-2}\ln\alpha_s^{-1/2}$ [34], while $\kappa \sim T^2$ (see Appendix A),[6] so one expects $\eta\tau_\pi > \frac{1}{2}\kappa$. Therefore, to satisfy both shear sum rules in QCD, an ansatz more sophisticated than the simplest Lorentzian ansatz $\delta\rho(\omega) \sim \frac{\eta\omega}{\omega^2+\Gamma^2}$ is needed.

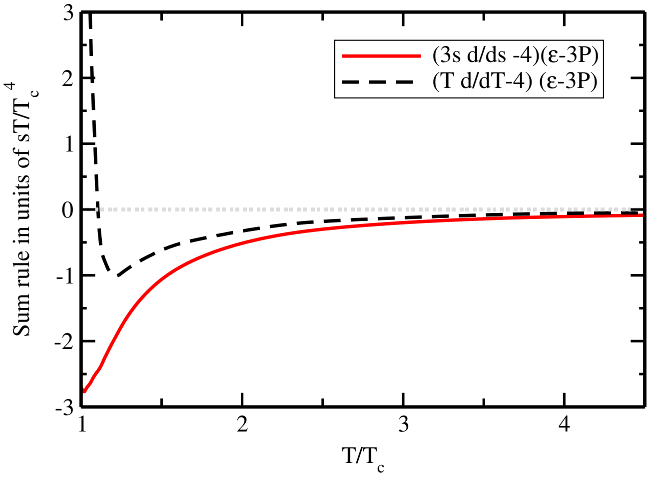

FIG. 3 (color online). The LHS of the bulk sum rule [Eqs. (3) and (62)] (solid line), as evaluated from $SU(3)$ lattice data [36]. The result is negative for all temperatures shown, and should not be associated with the value of the bulk viscosity in $SU(3)$. For comparison, the result from evaluating Eq. (C7) (corresponding to the sum rule by Kharzeev and Tuchin [11]) is shown (dashed lines). The horizontal dotted line is a visual aid.

Moreover, the LHS of our sum rule (3) for the bulk sector can be evaluated using lattice results for the thermodynamics [36]. The result turns out to be negative for all temperatures above the deconfinement transition (see Fig. 3). Figure 3 demonstrates that the sum rule (3) cannot be directly used to extract information about the value of the bulk viscosity in QCD, unless additional phenomenological assumptions are made, for example, as in Refs. [11,12]. In our language, Refs. [11,12] assume $\rho_{T=0}(\omega)$ to contain a "nonperturbative" part (associated with the phenomenological gluon condensate), that—once subtracted—is offsetting the negative LHS of the bulk sum rule (3).

ACKNOWLEDGMENTS

While this work was being completed, we became aware of related work by Simon Caron-Huot [32]. We thank L. Yaffe for discussions, G. Moore and S. Caron-Huot for comments on the manuscript, J. Engels for providing the lattice QCD data used in Fig. 3, and S. Caron-Huot for correcting an error in our earlier treatment of the bulk sum rule and for giving us a preview of Ref. [32]. This work is supported, in part, by U.S. DOE Grant No. DE-FG02-00ER41132.

[6]For nonconformal theories, corrections to the definition of κ should be suppressed by an additional power of α_s [35].

APPENDIX A: THE COEFFICIENT κ

Here we calculate directly the Euclidean correlator

$$G(x_1, x_2) = \langle T_{xy}(x_1) T_{xy}(x_2)\rangle_T \tag{A1}$$

for a free $SU(N_c)$ gauge theory at finite temperature T. We have

$$G(x_1, x_2) = \langle(\partial_x A_a^\alpha - \partial_\alpha A_a^x)(\partial_\alpha A_a^y - \partial_y A_a^\alpha) \times (\partial_x A_b^\beta - \partial_\beta A_b^x)(\partial_\beta A_b^y - \partial_y A_b^\beta)\rangle_T, \tag{A2}$$

where each of the building blocks is a correlator of the form

$$C_{i_1 i_2 j_1 j_2 l_1 l_2 m_1 m_2}(x_1, x_2) = \langle \partial_{i_1} A_a^{i_2} \partial_{j_1} A_a^{j_2} \partial_{l_1} A_b^{l_2} \partial_{m_1} A_b^{m_2}\rangle_T i^4 \times \sum\!\!\!\!\!\!\int_{P_1,P_2,P_3,P_4} e^{i(P_1+P_2)\cdot x_1 + i(P_3+P_4)\cdot x_2} \times P_1^{i_1} P_2^{j_1} P_3^{l_1} P_4^{m_1} \langle A_a^{i_2}(P_1) A_a^{j_2}(P_2) \times A_b^{l_2}(P_3) A_b^{m_2}(P_4)\rangle_T, \tag{A3}$$

and $P = (\mathbf{p}, p_4) = (\mathbf{p}, 2\pi T n)$, $\sum\!\!\!\!\!\!\int_P = T\sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3}$. We will be interested in the Fourier transform of these correlators,

$$\begin{aligned}C_{i_1 i_2 j_1 j_2 l_1 l_2 m_1 m_2}(\mathbf{q}, q_4) &= \int_0^\beta d\tau \int d^3\mathbf{x} e^{-iQ\cdot x} C_{i_1 i_2 j_1 j_2 l_1 l_2 m_1 m_2}(x, 0)\\ &= \sum\!\!\!\!\!\!\int_{P_2,P_3,P_4} (Q - P_2)^{i_1} P_2^{j_1} P_3^{l_1} P_4^{m_1} [\langle A_a^{i_2}(Q-P_2) A_b^{m_2}(P_4)\rangle_T\\ &\times \langle A_a^{j_2}(P_2) A_b^{l_2}(P_3)\rangle_T + \langle A_a^{i_2}(Q-P_2) A_b^{l_2}(P_3)\rangle_T\\ &\times \langle A_a^{j_2}(P_2) A_b^{m_2}(P_4)\rangle_T + \langle A_a^{i_2}(Q-P_2) A_a^{j_2}(P_2)\rangle_T\\ &\times \langle A_b^{l_2}(P_3) A_b^{m_2}(P_4)\rangle_T],\end{aligned} \tag{A4}$$

where

$$\langle A_a^\mu(P_1) A_b^\nu(P_2)\rangle_T = \frac{1}{T}\delta_{n_1+n_2,0}(2\pi)^3 \delta(\mathbf{p_1} + \mathbf{p_2}) \Delta_{ab}^{\mu\nu}(P_1), \tag{A5}$$

and $\Delta_{ab}^{\mu\nu}(P_1)$ is the gluon propagator. Note that the last term in Eq. (A4) corresponds to a disconnected diagram; we are only interested in the connected Green's function, so this term will be dropped in the following. Using Feynman gauge $\Delta_{ab}^{\mu\nu}(P_1) = \delta_{ab}\delta^{\mu\nu} P_1^{-2}$, Eq. (A4) simplifies to

$$C_{i_1 i_2 j_1 j_2 l_1 l_2 m_1 m_2}(\mathbf{q}, q_4) = (N_c^2 - 1) \sum\!\!\!\!\!\!\int_K (Q+K)^{-2} K^{-2} \times (Q+K)^{l_1} K^{m_1} [(Q+K)^{i_1} \times \delta^{i_2 l_2} K^{j_1} \delta^{j_2 m_2} + K^{i_1} \delta^{i_2 m_2} \times (Q+K)^{j_1} \delta^{j_2 l_2}], \tag{A6}$$

and hence the Fourier transform of the stress-energy correlator becomes

$$G(Q) = (N_c^2 - 1) \sum\!\!\!\!\!\!\int_K (Q+K)^{-2} K^{-2} \times [4k_x^2 k_y^2 - 2K\cdot(Q+K)(k_x^2 + k_y^2) + K^2 k_x^2 + (Q+K)^2 k_y^2 + (K\cdot(Q+K))^2]. \tag{A7}$$

Since we are interested here in the case for vanishing external frequency ($p_4 = 0$), the thermal sums are readily evaluated. Dropping the vacuum part and using the substitution $\mathbf{k} \to \mathbf{k} - \mathbf{q}$ in part of the integrand one finds

$$\begin{aligned}T\sum_n (Q+K)^{-2} &= \frac{n(k)}{k},\\ T\sum_n (Q+K)^{-2} K^{-2} &= \frac{n(k)}{k}\left[\frac{1}{|\mathbf{k}+\mathbf{q}|^2 - k^2} + \frac{1}{|\mathbf{k}-\mathbf{q}|^2 - k^2}\right],\\ T\sum_n \frac{K\cdot(Q+K)}{(Q+K)^2 K^2} &= \frac{n(k)}{k}\left[\frac{\mathbf{k}\cdot\mathbf{q}}{|\mathbf{k}+\mathbf{q}|^2 - k^2} - \frac{\mathbf{k}\cdot\mathbf{q}}{|\mathbf{k}-\mathbf{q}|^2 - k^2}\right],\\ T\sum_n \frac{(K\cdot(Q+K))^2}{(Q+K)^2 K^2} &= \frac{n(k)}{k}\left[\frac{(\mathbf{k}\cdot\mathbf{q})^2}{|\mathbf{k}+\mathbf{q}|^2 - k^2} + \frac{(\mathbf{k}\cdot\mathbf{q})^2}{|\mathbf{k}-\mathbf{q}|^2 - k^2}\right].\end{aligned} \tag{A8}$$

Expanding the integrand to $\mathcal{O}(\mathbf{q}^2)$, all the remaining integrals can be done analytically and one finds

$$\delta G(0, \mathbf{q}) = \frac{N_c^2 - 1}{36} T^2 q^2 + \mathcal{O}(\mathbf{q}^4), \tag{A9}$$

so that

$$\kappa = \frac{N_c^2 - 1}{18} T^2. \tag{A10}$$

APPENDIX B: THE COEFFICIENT $f_q(0)$ IN WEAKLY COUPLED $SU(N_c)$

Calculation of $G_R^\infty = \lim_{q_4\to\infty} G_R(Q)$ starts similar to the calculation for κ in the previous section, leading to Eq. (A7) for $G_R(Q)$. For $\mathbf{q} = 0$ the sum-integrals become

$$\begin{aligned}T\sum_n (Q+K)^{-2} &= \frac{1+2n(k)}{2k},\\ T\sum_n (Q+K)^{-2} K^{-2} &= \frac{1+2n(k)}{2k}\left[\frac{1}{q_4^2 + 2ikq_4} + \frac{1}{q_4^2 - 2ikq_4}\right],\\ T\sum_n \frac{K\cdot(Q+K)}{(Q+K)^2 K^2} &= \frac{1+2n(k)}{2k}\left[\frac{ikq_4}{q_4^2 + 2ikq_4} - \frac{ikq_4}{q_4^2 - 2ikq_4}\right],\\ T\sum_n \frac{(K\cdot(Q+K))^2}{(Q+K)^2 K^2} &= -\frac{1+2n(k)}{2k} \times \left[\frac{k^2 q_4^2}{q_4^2 + 2ikq_4} + \frac{k^2 q_4^2}{q_4^2 - 2ikq_4}\right],\end{aligned} \tag{B1}$$

where we used $n(iq_4) = n(2\pi i T n) = 1$. Evaluation of the remaining integrals is straightforward and we find

$$\lim_{q_4\to\infty} G(q_4, 0) - G(q_4, 0)_{T=0} = f_{q=0}(0) = -2P, \quad \text{(B2)}$$

where $P = \frac{2}{90}(N_c^2 - 1)\pi^2 T^4$. This result can also be obtained by integrating the result for the spectral function from Ref. [37].

APPENDIX C: NONCOMMUTATIVITY OF THE $\omega \to 0$ AND $q \to 0$ LIMITS OF THE $\langle\theta\theta\rangle$ CORRELATOR

In Sec. IV we have shown that

$$\lim_{\omega\to 0}\lim_{\mathbf{q}\to 0}\langle\theta\theta\rangle_R(\omega, \mathbf{q}) = -\left(3s\frac{\partial}{\partial s} - 4\right)(\epsilon - 3P). \quad \text{(C1)}$$

Using the same method, we now show that

$$\lim_{\mathbf{q}\to 0}\lim_{\omega\to 0}\langle\theta\theta\rangle_R(\omega, \mathbf{q}) = -\left(T\frac{\partial}{\partial T} - 4\right)(\epsilon - 3P). \quad \text{(C2)}$$

This result is consistent with previous results [38] derived through the Euclidean path integral following the method of Ref. [39]. It should be expected: Euclidean correlators are defined with discrete Matsubara frequencies $\omega_E = 2\pi nT$, and the only sensible zero momentum limit in the Matsubara formalism is to set $\omega_E = 0$ first, and then take $\mathbf{q} \to 0$.

We turn on a static metric perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} e^{-2\Omega(\mathbf{x})}. \quad \text{(C3)}$$

When $\Omega(\mathbf{x})$ varies smoothly, one can use hydrodynamics to find out the response. For static perturbations we expect the response will be static. The velocity field is $u^0 = e^{\Omega}$, $u^i = 0$, and the temperature depends on space, $T = T(\mathbf{x})$. Substituting $T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu}$ into the equation $\nabla_\mu T^{\mu\nu} = 0$, we find

$$\partial_i P - (\epsilon + P)\partial_i \Omega = 0. \quad \text{(C4)}$$

Using $dP = sdT$ and $\epsilon + P = Ts$, the solution to this equation is

$$T = T_0 e^{\Omega}. \quad \text{(C5)}$$

The correlator is found from

$$\langle\theta\theta\rangle(\omega = 0, \mathbf{q} \to 0) = \frac{\partial}{\partial\Omega}(\sqrt{-g}T^\mu_\mu) = -\left(T\frac{\partial}{\partial T} - 4\right)(\epsilon - 3P). \quad \text{(C6)}$$

One apparent paradox is that if one writes down the bulk sum rule for any spatial momentum $\mathbf{q} \neq 0$, the integral should be equal to $-\langle\theta\theta\rangle(0, \mathbf{q})$ which is given by (C2) but not (C1):

$$\left(T\frac{\partial}{\partial T} - 4\right)(\epsilon - 3P) = \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega}\delta\rho^{\mathrm{bulk}}(\omega, \mathbf{q}), \qquad \mathbf{q} \neq 0. \quad \text{(C7)}$$

There is no contradiction, however, as the integral in Eq. (C7) is expected to receive a finite contribution from the region $\omega \sim q$, in particular, from the sound-wave peak at $\omega = c_s q$, as $\langle\theta\theta\rangle$ correlator has a sound-wave pole. When $q \to 0$, this region shrinks to zero size, but its contribution remains finite. The contribution from the sound-wave peak can be calculated as follows: to leading order in hydrodynamic fluctuations, $T^{xx} = T^{yy} = T^{zz} = c_s^2 T^{00}$, so

$$G_R(\omega, q)^{\mathrm{bulk}} = (1 - 3c_s^2)^2 \langle T^{00}T^{00}\rangle_R. \quad \text{(C8)}$$

Defining $\langle T^{00}T^{00}\rangle_R = \frac{q^2}{\omega^2}\chi_L(\omega, q) \simeq \frac{\chi_L(\omega,q)}{c_s^2}$ and using Teaney's result for the spectral density corresponding to χ_L [13],

$$\frac{\delta\rho_L(\omega, q)}{\omega} = \frac{\epsilon + P}{2}\left[\frac{\Gamma_s q^2/2}{(\omega - c_s q)^2 + (\Gamma_s q^2/2)^2} + (\omega \to -\omega)\right], \quad \text{(C9)}$$

where $\Gamma_s = (\epsilon + P)^{-1}(\frac{4}{3}\eta + \zeta)$, the integral over the sound-wave pole at positive frequency gives $\lim_{q\to 0}\frac{2}{\pi} \times \int_0^\infty \frac{d\omega}{\omega}\delta\rho_L(\omega, \mathbf{q}) = \epsilon + P$. Therefore, the contribution for the integral over $\delta\rho^{\mathrm{bulk}}$ is precisely the difference between the LHS of Kharzeev-Tuchin's and our sum rule,

$$\left(T\frac{\partial}{\partial T} - 3s\frac{\partial}{\partial s}\right)(\epsilon - 3P) = \frac{(1 - 3c_s^2)^2}{c_s^2}(\epsilon + P). \quad \text{(C10)}$$

On the other hand, in the bulk sum rule (62), we first compute the spectral function $\rho(\omega)$ at any finite, nonzero ω by setting $\mathbf{q} = 0$ in $\rho(\omega, \mathbf{q})$, and then take the spectral integral. The sound-wave contribution does not appear in this integral, which means that our Eq. (62), but not Kharzeev and Tuchin's version, applies. Note that at zero temperature the two limits $\omega \to 0$ and $\mathbf{q} \to 0$ commute.

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