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Froissart bound for inelastic cross sections

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We prove that while the total cross section is bounded by $(\pi/m_\pi^2)\ln^2 s$, where s is the square of the c.m. energy and m_π the mass of the pion, the total inelastic cross section is bounded by $(1/4)(\pi/m_\pi^2)\ln^2 s$, which is 4 times smaller. We discuss the implications of this result on the total cross section itself.

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The Froissart bound [1], proved later from local massive field theory and unitarity [2], is generally written as

$$\sigma_T < \frac{\pi}{m_\pi^2}\ln^2 s, \tag{1}$$

where s is the square of the c.m. energy and m_π the pion mass.

The constant in front of $\ln^2 s$ was obtained by Lukaszuk and myself [3]. Many of my friends, especially Peter Landshoff, complained that this constant was much too large. It is true that some fits of the proton-proton and proton-antiproton cross sections [4] indicate the possible presence of a $\ln s$ square term with, however, a much smaller coefficient, about 500 times smaller. Joachim Kupsch, Shasanka Roy, David Atkinson, Porter Johnson, and myself are planning to try to improve this constant by taking into account analyticity and unitarity, including *elastic* unitarity in the elastic region. To date, there is no example of amplitude satisfying these requirements. Atkinson [5] has produced amplitudes satisfying all requirements but where $\sigma_T \propto \ln^{-3} s$.

If we want to undertake such a program a preliminary requirement is to start on a well-defined basis. It was recognized long ago that the Froissart bound is nonlocal by Common [6] and Yndurain [7]; see also [8,9]. Namely, one has, in fact,

$$s^N \int_s^{s+1/s^N} \sigma_T(s')ds' < C_N \ln^2 s. \tag{2}$$

The constant C_N, however, depends on N. The narrower is the interval, the larger is C_N. This comes from the fact that the basic ingredient of the Froissart bound is the convergence of the integral

$$\int_{s_0}^{\infty} \frac{A_s(s,t)}{s^3} ds < \infty, \tag{3}$$

for $0 < t \le 4m_\pi^2$ (sometimes only $0 < t < 4m_\pi^2$, strictly), where A_s is the absorptive part of the scattering amplitude, and t the square of the momentum transfer

$$t = 2k^2(\cos\theta - 1), \qquad s = \left(\sqrt{M_A^2 + k^2} + \sqrt{M_B^2 + k^2}\right)^2. \tag{4}$$

We have explained in [9] and will show elsewhere [10] that, if one wants to preserve the value of the constant in (1), the average should be taken on a large interval, for instance,

$$\bar{\sigma}_T(s) = \frac{1}{s}\int_s^{2s} \sigma_T(s')ds' < \frac{\pi}{m_\pi^2}\ln^2 s + A\ln s + B, \tag{5}$$

where A and B are determined by low energy parameters in the t channel.

Here we want to report something different and seeming naively obvious, namely, that for the inelastic cross section σ_I,

$$\sigma_I < \frac{\pi}{4m_\pi^2}\ln^2 s. \tag{6}$$

The bound is 4 times smaller than the one on the total cross section.

If there was a strictly sharp cutoff in the partial wave distribution, this would indeed be obvious, because if the scattering amplitude $F(s,t)$ is given by

$$F(s,t) = \frac{\sqrt{s}}{2k}\sum_\ell (2\ell+1) f_\ell(s) P_\ell\left(1 + \frac{t}{2k^2}\right), \tag{7}$$

then

$$\sigma_T = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\mathrm{Im} f_\ell(s), \tag{8}$$

and

$$\sigma_I = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)(\mathrm{Im} f_\ell(s) - |f_\ell(s)|^2). \tag{9}$$

Hence

$$\sigma_I < \frac{4\pi}{k^2}\sum_\ell (2\ell+1)(\mathrm{Im} f_\ell - (\mathrm{Im} f_\ell)^2). \tag{10}$$

So while

$$0 \le \mathrm{Im} f_\ell \le 1, \tag{11}$$

$$0 \le \mathrm{Im} f_\ell - (\mathrm{Im} f_\ell)^2 \le 1/4. \tag{12}$$

However, there is no sharp cutoff in the partial wave distribution and it is not the same distribution which maximizes σ_T and σ_I for a given absorptive part:

$$A_s = \mathrm{Im}F, \quad \text{for } t < 4m_\pi^2. \tag{13}$$

Here, for simplicity, we shall not use the average given by (5), and make the traditional assumption that A_s is a continuous function of s for fixed $t < 4m_\pi^2$. Then, from (3), we have

$$0 < A_s(s,t) < \frac{s^2}{\ln s}, \tag{14}$$

on a set of values of s of asymptotic density unity.

We recall the method to get the bound on σ_T total. One tries to maximize

$$\sigma_T \propto \sum_\ell (2\ell+1)\mathrm{Im} f_\ell, \tag{15}$$

for a given A_s, with $x = 1 + t/(2k^2)$

$$A_s(s,t) = \sum_\ell (2\ell+1)\mathrm{Im} f_\ell P_\ell(x), \tag{16}$$

neglecting the deviation of $\sqrt{s}/(2k)$ from unity.

It is known that the optimal distribution is

$$\begin{aligned} \mathrm{Im}\, f_\ell &= 1, \quad \text{for } 0 \le \ell \le L_T, \\ \mathrm{Im} f_{L_T+1} &= \epsilon, \qquad 0 \le \epsilon \le 1. \end{aligned} \tag{17}$$

Then we have from (16)

$$P'_{L_T}(x) + P'_{L_T+1}(x) < \frac{s^2}{\ln s}. \tag{18}$$

Using standard bounds on Legendre polynomials one gets

$$L_T(A_s) \le \frac{k}{\sqrt{t}} \ln s. \tag{19}$$

The Froissart bound follows from that:

$$\sigma_T \le \frac{4\pi}{t} \ln^2 s, \tag{20}$$

giving (1) for $t = 4m_\pi^2$. A recent new derivation of this result has been proposed [11].

If, on the other hand, we want to maximize σ_I, where

$$\sigma_I = \sum_\ell (2\ell+1)(\mathrm{Im} f_\ell - (\mathrm{Im} f_\ell)^2), \tag{21}$$

we find that the optimal distribution for given A_s is (see Appendix A)

$$\mathrm{Im}\, f_\ell = \tfrac{1}{2}[1 - P_\ell(x)/P_{\bar{L}}(x)] \tag{22}$$

for $0 \le \ell \le L_I$, with

$$L_I < \bar{L} < L_I + 1. \tag{23}$$

It is obvious that

$$L_I(A_s) > L_T(2A_s) > L_T(A_s). \tag{24}$$

Starting from (22) one can get a closed expression for A_s:

$$A_s = \frac{1}{2}\bigg[P'_{L_I}(x) + P'_{L_I+1}(x) - \frac{(L_I+1)^2 P^2_{L_I}(x) - (x^2-1){P'_{L_I}}^2}{P_{\bar{L}(x)}}\bigg]. \tag{25}$$

In fact, we shall not use this expression. However, since

$$L_T(s^2/\ln s) \simeq \frac{k}{\sqrt{t}} \ln s, \tag{26}$$

$L_I > (k/\sqrt{t}) \ln s$. The sum

$$\frac{1}{2}\sum_0^{L_I} (2\ell+1)\bigg[1 - P_\ell\bigg(1 + \frac{t}{2k^2}\bigg)\bigg/ P_{\bar{L}}\bigg(1 + \frac{t}{2k^2}\bigg)\bigg] P_\ell(x), \tag{27}$$

can split into $\sum_{\ell=0}^{L_I-\Delta} + \sum_{\ell=L_I-\Delta}^{L_I}$. We choose

$$\Delta = \lambda k. \tag{28}$$

For $\ell < L_I - \Delta$,

$$\frac{P_\ell(x)}{P_{\bar{L}}(x)} \le \frac{P_\ell(x)}{P_{L_I}(x)} < \frac{P_{L_I-\Delta}(x)}{P_{L_I}(x)}. \tag{29}$$

We prove, in Appendix B that

$$\frac{P_{L_I-\Delta}(x)}{P_{L_I}(x)} < 4\exp(-\Delta\sqrt{x-1}), \tag{30}$$

(this is a *very crude bound*, but sufficient for our purpose).

Hence, with the choice (28), we get

$$\begin{aligned} &\frac{1}{2}\sum_{\ell=0}^{L_I-\lambda k} (2\ell+1)[1 - P_\ell(x)/P_{\bar{L}}(x)]P_\ell(x) \\ &\quad > \frac{1}{2}[1 - 4\exp(-\lambda\sqrt{t/2})] \sum_{\ell=0}^{L_I-\lambda k} (2\ell+1)P_\ell(x). \end{aligned} \tag{31}$$

So taking $\lambda = \sqrt{2/t}\ln 8$ and $t < 2k^2$, we get

$$\sum_{\ell=0}^{L_I-\lambda k} (2\ell+1)P_\ell(x) < \frac{4s^2}{\ln s}. \tag{32}$$

Hence we are back to the same problem as for σ_T, except for a change of scale, and we get

$$L_I - \Delta < \frac{k}{\sqrt{t}}(\ln s + C). \tag{33}$$

Now, from (28),

$$L_I < \frac{k}{\sqrt{t}}(\ln s + C'), \tag{34}$$

so that

$$\sigma_I < \sum_{\ell=0}^{L_I} (2\ell+1)\frac{1}{4} = \frac{k^2 \ln^2 s}{4t}, \tag{35}$$

and

$$\sigma_I < \frac{\pi}{t}[\ln^2 s + \mathcal{O}(\ln s)], \tag{36}$$

and, if $t = 4m_\pi^2$,

$$\sigma_I < \frac{\pi}{4m_\pi^2}\ln^2 s. \tag{37}$$

There remains of course the fact that (37) holds only on a set of asymptotic density unity if A_s is a continuous function of s for fixed t. The scale in s cannot be fixed, as was the case for the total cross section. As we said before, the only thing we know is that the integral (3) converges for $0 \le t < 4m_\pi^2$, sometimes also for $t = 4m_\pi^2$. For σ_I, one would like to have the analog of (5), but, so far, we have not been able to get it. Another way out is to assume that, beyond a certain energy, A_s is monotonous. The case where it is monotonous decreasing is uninteresting, and so we take A_s to be monotonous increasing. If

$$I(t) = \int_{s_0}^{\infty} \frac{A_s(s,t)}{s^3} ds, \quad \text{then } A_s(s,t) < 2s^2 I(t). \tag{38}$$

Then, all constants can be fixed in the bounds on σ_T and σ_I, and the scale problem is removed. Further, if $I(t)$ goes to infinity as t approaches $4m_\pi^2$, we know that $I(t)$ behaves like a negative power of $(4m_\pi^2 - t)$. By taking $t = 4m_\pi^2 - 1/\ln s$, one can manage to prove that (1) and (32) still hold, with corrective terms of the order of $\ln s \ln(\ln s)$. It is a matter of taste to decide if this monotonicity assumption is acceptable. Here we shall not give detailed calculations, because we hope to find the analog of (5) for the inelastic cross section, and to get the best possible estimates without any artificial assumption.

This ends the rigorous part of this paper. Now comes the fact that most theoreticians believe that the worse that can happen at high energies is that the elastic cross section reaches half of the total cross section, which corresponds to an expanding black disk. This is the case in the model of Chou and Yang [12], and in the model of Cheng and Wu [13], later developed by Bourrely, Soffer, and Wu [14], and also in general considerations by Van Hove [15] who introduced what became known as the "overlap function" which is

$$\sum_\ell (2\ell+1)[\mathrm{Im} f_\ell - (\mathrm{Im} f_\ell)^2] P_\ell(\cos\theta), \tag{39}$$

which represents the overlap between inelastic final states produced by two two-body states corresponding to different directions. Here Van Hove neglects the real part of the elastic amplitude. From

$$o_\ell = \mathrm{Im} f_\ell - (\mathrm{Im} f_\ell)^2, \tag{40}$$

one gets

$$\mathrm{Im}\, f_\ell = \frac{1 \pm \sqrt{1 - 4o_\ell}}{2}, \tag{41}$$

For large ℓ one has to choose the *minus* sign, and Van Hove argues that by continuity, or, better analyticity in ℓ, one has to keep the minus sign down to $\ell = 0$, which means that $\mathrm{Im} f_\ell$ is less than $1/2$. However, not everybody agrees with this. See, for instance, the talk of Troshin in La Londe-les-Maures [16]. In his view, the scattering amplitude becomes dominantly elastic in the high energy limit. To say the least, this seems to me extremely unlikely and, therefore, I tend to believe that we have

$$\sigma_T < \frac{1}{2}\frac{\pi}{m_\pi^2}\ln^2 s. \tag{42}$$

Certainly, this is not enough to satisfy Landshoof, but it represents, nevertheless, progress.

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APPENDIX A

Calling $\mathrm{Im} f_\ell = y_\ell$, we try to maximize

$$\sigma_I = \sum_\ell (2\ell+1)(y_\ell - y_\ell^2), \tag{A1}$$

for given

$$A_s = \sum_\ell (2\ell+1) y_\ell P_\ell(x), \qquad x > 1. \tag{A2}$$

We start with a heuristic variational argument. We have

$$\delta A_s = 0 = \sum_\ell (2\ell+1)\delta y_\ell P_\ell, \tag{A3}$$

$$\delta \sigma_I = 0 = \sum_\ell (2\ell+1)\delta y_\ell (1 - 2y_\ell). \tag{A4}$$

Hence, using a Lagrange multiplyer:

$$y_\ell = \tfrac{1}{2}[1 - cP_\ell(x)]. \tag{A5}$$

This is, in fact, the correct answer. We shall prove it.

Assume that $\{y_\ell\}$ is the maximizing distribution. Consider only two terms, y_ℓ and y_L. Another distribution contains $y_\ell + \Delta y_\ell$ and $y_L + \Delta y_L$. A_s is fixed. Hence,

$$(2\ell+1)\Delta_\ell P_\ell + (2L+1)\Delta_L P_L = 0. \tag{A6}$$

On the other hand,

$$\Delta\sigma_I = (2\ell+1)\Delta_\ell(1-2y_\ell) + (2L+1)\Delta_L(1-2y_L) - (2\ell+1)\Delta_\ell^2 - (2L+1)\Delta_L^2. \tag{A7}$$

If we choose

$$1 - 2y_\ell = cP_\ell, \qquad 1 - 2y_L = cP_L, \tag{A8}$$

we get, from (A6)

$$\Delta\sigma_I = -(2\ell + 1)\Delta_\ell^2 - (2L + 1)\Delta_L^2 < 0. \tag{A9}$$

Hence the choice (A8) maximizes σ_I. Therefore, we take

$$y_\ell = \tfrac{1}{2}[1 - cP_\ell(x)]. \tag{A10}$$

Now, what is c? If the sum is

$$A_s = \frac{1}{2}\sum_{\ell=0}^{\ell=L_I}(2\ell + 1)P_\ell(x)[1 - cP_\ell(x)], \tag{A11}$$

obviously $0 < c < 1/P_{L_I}$ because $0 < \mathrm{Im} f_\ell < 1$. But it is not possible for c to be less than $1/P_{L_I+1}$, because we could apply our previous reasoning to the last two partial waves, the last one being zero. This would lead to changing c. So

$$\frac{1}{P_{L_I}} < c < \frac{1}{P_{L_I+1}}. \tag{A12}$$

Now we give, for completeness, in the case where $c = 1/P_{L_I}$ exactly, the complete expression for A_s, even though we do not use it. From Gradshtein and Ryzhik [17], we get

$$\sum_{\ell=0}^{L}(2\ell + 1)P_\ell^2 = (L + 1)(P'_{L+1}P_L - P'_L P_{L+1}), \tag{A13}$$

and so

$$A_s = \tfrac{1}{2}[P'_L + P'_{L+1} - (L + 1)(P'_{L+1} - P'_L P_{L+1}/P_L)]. \tag{A14}$$

Notice that A_s vanishes for $x = 1$. It is possible to get an expression with $x - 1$ explicitly factored out, using the Legendre differential equation and recursive relations.

APPENDIX B

We derive an upper bound on

$$P_{L-\Delta}(x)/P_L(x), \qquad x > 1. \tag{B1}$$

From

$$P_\ell = \frac{1}{\pi}\int_0^\pi (x + \sqrt{x^2 - 1}\cos\phi)^\ell d\phi, \tag{B2}$$

we get, using the Minkowsky-Hölder inequality, for $x > 1$,

$$P_{L-\Delta}(x) < [P_L(x)]^{(L-\Delta)/L}, \tag{B3}$$

so

$$\frac{P_{L-\Delta}(x)}{P_L(x)} < \frac{1}{[P_L(x)]^{\Delta/L}}. \tag{B4}$$

Now, we need a lower bound for P_ℓ. A very crude lower bound is enough: cutting the integral (B2) at $\phi = \pi/4$, we get

$$P_\ell(x) > \frac{1}{4}\left[x + \sqrt{\frac{x^2 - 1}{2}}\right]^\ell, \tag{B5}$$

and, for $1 < x < 7$,

$$P_\ell(x) > \tfrac{1}{4}\exp(\ell\sqrt{x - 1}). \tag{B6}$$

Since $x = 1 + t/(2k^2)$ and $t < 4m_\pi^2$, this corresponds to $k > 0.4m_\pi$, ridiculously small in these high energy considerations.

One could do much better than that. For instance, Roy [18] quotes an unpublished optimal result of mine:

$$P_\ell(x) > \frac{2N!}{(N!)^2}\left(\frac{N + 1}{2(2N + 1)}\right)^N\left(x + \frac{N}{N + 1}\sqrt{x^2 - 1}\right)^\ell, \tag{B7}$$

for $N = 1, 2, 3, 4, \ldots$. For instance,

$$P_\ell(x) > \frac{54}{100}\left(x + \frac{2}{3}\sqrt{x^2 - 1}\right)^\ell. \tag{B8}$$

If we take $\ell = 2$ and $\ell = 3$, we see that this bound is saturated for $x \to \infty$. However, these refinements are not really needed for our purpose. Inequality (B6) is enough.

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