CP^{N-1} models at a Lifshitz point

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We consider CP^{N-1} models in $d+1$ dimensions around Lifshitz fixed points with dynamical critical exponent z, in the large-N expansion. It is shown that these models are asymptotically free and dynamically generate a mass for the CP^{N-1} fields for all $d = z$. We demonstrate that, for $z = d = 2$, the initially nondynamical gauge field acquires kinetic terms in a way similar to usual CP^{N-1} models in $1 + 1$ dimensions. Lorentz invariance emerges generically in the low-energy electrodynamics, with a nontrivial dielectric constant given by the inverse mass gap and a magnetic permeability which has a logarithmic dependence on scale. At a special multicritical point, the low-energy electrodynamics also has $z = 2$, and an essentially singular dependence of the effective action on $B = \epsilon_{ij}\partial_i A_j$.

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Nonlinear sigma models are ubiquitous in a variety of areas in theoretical physics. In this paper we will deal with the CP^{N-1} model [1], whose fields are N component complex vectors $\vec{\phi}(t,x)$ constrained by

$$\vec{\phi}^{\star}\cdot\vec{\phi} = \frac{1}{g^2} \tag{1}$$

and fields which differ by an overall (space-time dependent) phase are identified,

$$\vec{\phi}(t,x) \sim e^{i\theta(t,x)}\vec{\phi}(t,x). \tag{2}$$

The identification is incorporated by introducing a nondynamical $U(1)$ gauge field, A_μ. The conventional, relativistic, action is

$$S = \frac{1}{2}\int dt\int d^d x (D_\mu\vec{\phi})^{\star}(D^\mu\vec{\phi}), \tag{3}$$

where

$$D_\mu \equiv \partial_\mu + iA_\mu. \tag{4}$$

Integrating out A_μ leads to a nonlinear action which involves only the $\vec{\phi}$ fields. As is well known, in $d = 1$ the model (3) is asymptotically free and generates a mass m for the fields $\vec{\phi}$ by dimensional transmutation, as can be explicitly seen in the 't Hooft large-N expansion $N \to \infty$ $g \to 0$ $g^2 N = \lambda =$ fixed [1]. At the same time, the initially nondynamical gauge field acquires a standard kinetic energy term, with a gauge coupling constant given by m^2. This is the simplest example of a dynamical emergence of gauge dynamics.

The $d = 2$ model is interesting for condensed matter applications. In fact, the $O(3)$ nonlinear sigma model with three component unit vector $\hat{n}$ can be rewritten as the CP^1 model via the identification $\bar{\phi}\,\vec{\sigma}\,\phi = \hat{n}$. It is evident that local phase transformations of the ϕ fields do not affect the "gauge-invariant" field $\hat{n}$. Now there is a usual order-disorder transition: In the magnetically ordered phase of $\hat{n}$, the ϕ field is condensed, and the gauge field is gapped out by the Higgs mechanism. However, gauge field dynamics appears in the disordered phase when the ϕ fields become massive. Normally, the gauge fields also become massive and ϕ fields become confined on the paramagnetic side of the transition due to the compactness of the gauge field [2]. However, suppressing the monopoles [3] of the gauge field (which correspond to "hedgehog" configurations of the original $\hat{n}$ fields) leads to a new critical point [4], and a paramagnetic phase with a gapless photon [5]. There are conjectures that such a model with a noncompact gauge field also describes a possible non-Landau, deconfined, critical point [6] between the Neél and bond-ordered phases of the $d = 2$ quantum antiferromagnet.

In this paper we consider UV modifications of these models, which correspond to Lifshitz-like fixed points with a dynamical critical exponent z,

$$S_L = \frac{1}{2}\int dt\int d^d x[(D_0\vec{\phi})^{\star}(D^0\vec{\phi}) + \alpha(D_i\vec{\phi})^{\star}(D^i\vec{\phi}) + |\mathcal{D}^z\vec{\phi}|^2], \tag{5}$$

where the operator $\mathcal{D}^z$ is a sum of $O(d)$ invariant terms containing z factors of the spatial covariant derivative D_i. For example, in $z = 2$,

$$|\mathcal{D}^z\phi|^2 \equiv a(D_iD_j\vec{\phi})^{\star}\cdot(D_iD_j\vec{\phi}) + b(D^2\vec{\phi})^{\star}\cdot(D^2\vec{\phi}) \tag{6}$$

with a, $b \geq 0$ being parameters. For higher z we would have many more terms corresponding to various orderings of the D_i.

At the fixed point $\alpha = 0$, one needs to scale the time and space coordinates as

$$t \to \gamma^z \qquad tx \to \gamma t. \tag{7}$$

Such fixed points, called Lifshitz fixed points, have a

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variety of applications in classical condensed matter systems [7]. They also have a connection to quantum dimer models [8], which are defined with a "kinetic" term which flips dimers on parallel bonds, and a "potential" term which gives an energy to every flippable plaquette. Finally, there is a constraint that every lattice site should have one and only one dimer touching it. Recently, it was realized [9,10] that neutral (ungauged) one-component Lifshitz fixed points describe special points [the Rokhsar-Kivelson (RK) points] of quantum dimer models [8] on bipartite lattices, where the field ϕ is a height variable dual to a bond which may or may not contain a dimer [11,12]. The standard RK point describes the transition between the smooth and rough phases of the height and is multicritical, in the sense that more than one parameter needs to be tuned to attain the fixed point [9,10]. However, it is possible to construct models with enough symmetries such that the fixed point can be obtained as a regular critical point describing, for example, the phase transition between two different types of bond-ordered states in a bilayer honeycomb lattice [13]. While the above are examples of $z = d = 2$ theories of neutral scalars, examples of $z = 2$ gauge theories also occur in condensed matter, in the description of algebraic spin liquids in $d = 3$ [14,15] and topological critical phases in $d = 2$ [16].

Note that the action of Eq. (5) still contains a single time derivative of the gauge fields A_i, though it has higher spatial derivatives. Thus, even though one cannot easily integrate out the A_μ to obtain a pure spin model, the gauge field is nondynamical to begin with. In the spirit of the renormalization group, one expects that the model defined by Eq. (5) for $N = 2$ is in the universality class of a $z = d = 2$ $O(3)$ nonlinear sigma model.

Recently, Lifshitz-type theories have been suggested as UV completions of low-energy Lorentz-invariant theories of gravity and gauge dynamics [17]. This is because theories which are nonrenormalizable at the usual Lorentz-invariant UV fixed points with $z = 1$ can become renormalizable for nontrivial z. The idea that Lorentz symmetry violation can be regarded as UV regulators of field theories has been around for a while. See [18] for a recent discussion and references. In the same spirit, recently such Lifshitz fixed points have been proposed as UV completions of four-fermion theories similar to the Nambu-Jona Lasinio model in 3 + 1 dimensions and discussed as possible candidates for physics at the weak scale [19,20].

We will find that for any choice of $\mathcal{D}^z$ the theory defined by (6) is asymptotically free for all $d = z$ and generates a mass gap, pretty much as the models in [20]—so that the theory is always in a disordered phase. As a result, the gauge field acquire a kinetic term.

This means that the possible emergence of Lorentz invariance at low energies is a little more nontrivial since this has to happen in the gauge as well as in the scalar sector. For $z = d = 2$ we explicitly show that this indeed happens generically, and discuss the possibilities for higher $d = z$.

For special choices of $\mathcal{D}^z$ (the $a = 0$ multicritical point in our case) something even more interesting happens: one obtains a $z = 2$ electrodynamics with a standard $\vec{E}^2$ term, but the leading term in $B = \epsilon_{ij}\partial_i A_j$ is $(\nabla B)^2$. This naively suggests that a constant B costs no energy: however, a more careful calculation reveals that there is a nonanalytic dependence on constant B of the form $B^{3/2}\exp(-\pi m/B)$. Note that the analytic terms in our $z = 2$, $d = 2$ electrodynamics have the same form as the gauge theory descriptions of algebraic spin liquids in $d = 3$ [14,16], but appear to be dual to the gauge description of the transition between two bond-ordered phases [13] or the topological critical phase [16] in $d = 2$ in which it is the $\vec{E}^2$ term which is replaced by $(\epsilon_{ij}\partial_i E_j)^2$.

I. ASYMPTOTIC FREEDOM AND DYNAMICAL MASS GENERATION

We will study the large-N limit of the model (5) with the constraint (1), using standard techniques [21]. The coupling g in the model (5) becomes dimensionless under Lifshitz scaling (7) when $d = z$. This may be seen by dimension counting: t has length dimension z, so that the length dimensions of $\vec{\phi}$ and g are

$$[\vec{\phi}] \sim [L]^{(z-d)/2}, \qquad [g] \sim [L]^{(d-z)/2}. \tag{8}$$

Whether the coupling is marginally relevant or marginally irrelevant at $z = d$ depends on the dynamics. To investigate this we use standard large-N techniques. Imposing the constraint (1) by a Lagrange multiplier field $\chi(t, x)$, we get the action

$$S_L = \frac{1}{2}\int dt \int d^d x\Big[(D_0\vec{\phi})^\star(D^0\vec{\phi}) + \alpha(D_i\vec{\phi})^\star(D^i\vec{\phi}) + |\mathcal{D}^z\vec{\phi}|^2 + \chi(t,x)\Big(|\vec{\phi}|^2(t,x) - \frac{1}{g^2}\Big)\Big]. \tag{9}$$

Integrating out the field $\vec{\phi}$ we get the effective action

$$S_{\rm eff} = N\Big\{\mathrm{Tr}\log[-D_0^2 + (-1)^z(\mathcal{D}^z)^2 + \chi(t,x)] - \frac{1}{2g^2}\int dt d^d x \chi(t,x)\Big\}. \tag{10}$$

At $N = \infty$ the functional integral over $A_\mu(t, x)$ and $\chi(t, x)$ is dominated by the saddle point of (10). We will assume that the saddle point is translationally invariant and rotationally symmetric in the d spatial dimensions with a vanishing gauge field strength. Thus we may set

$\chi(t, x) = \chi_0$ in the saddle point equation.[1] This also means that, as far as the saddle point equation is concerned, all possible terms in $\mathcal{D}^z$ contribute equally:

$$2N \int \frac{dk_0 d^d k}{(2\pi)^{d+1}} \frac{1}{k_0^2 + \alpha \vec{k}^2 + (\vec{k}^2)^z + \chi_0} = \frac{1}{g^2}. \tag{11}$$

For any finite α, this integral is logarithmically divergent for $d = z$ and behaves as $\log\frac{\Lambda^{2z}}{m^2}$, where Λ is a cutoff on the spatial momentum $\vec{k}$. This immediately implies that a solution to the gap equation is

$$m^2 \equiv \chi_0 \sim \Lambda^{2z} \exp\left[-\frac{A}{g^2 N}\right], \tag{12}$$

where A is a *positive* real number. Since m is (to leading order in $1/N$) the physical mass of the $\vec{\phi}$ field (i.e., in a Lorentzian signature this is the lowest value of the energy of a single particle state), it is clear from (12) that the coupling g^2 has to be asymptotically free, with a beta function

$$\Lambda \frac{d}{d\Lambda} g = -\frac{g^3 N}{A}. \tag{13}$$

It is useful to evaluate the integral in (11) for our primary case of interest, $z = d = 2$. For α, $m \ll \Lambda^2$ we get

$$m = 2\Lambda^2 e^{-[(2\pi)/(g^2 N)]} - \frac{\alpha}{2}. \tag{14}$$

The standard Gaussian fixed point corresponds to $\alpha \gg \Lambda^2$ and leads to a linearly divergent answer in this case.

Dynamical mass generation for this model is thus almost exactly identical to that in the four-fermion model of Ref. [20]. The effective action for the gauge field A_μ and the fluctuations of $\chi(t, x)$ has to be now obtained by substituting

$$\chi(t, x) = \chi_0 + \frac{1}{\sqrt{N}} \delta\chi \qquad A_\mu(t, x) \to \frac{1}{\sqrt{N}} A_\mu(t, x) \tag{15}$$

in (10). Clearly, this will generate kinetic terms for $\delta\chi$ and A_μ. The effect of these will be to provide corrections to the leading order propagator of the $\vec{\phi}$ fields which is simply the integrand of (11). Accordingly, the parameter α will be renormalized. If we go off the critical surface containing the Lifshitz fixed point, the renormalized value of α will be nonzero. Clearly, when the spatial momenta are much smaller than $\sqrt{\alpha}$, the propagator of the $\vec{\phi}$ will be dominated by the $\alpha_{\rm ren} k^2$ term. Therefore at low energies when $\alpha \neq 0$, Lorentz invariance is recovered with a speed of light given by $\frac{1}{\sqrt{\alpha}}$.

II. EFFECTIVE ACTION FOR THE GAUGE FIELDS : $d = z = 2$, $\alpha = 0$

Emergence of Lorentz symmetry at low energies in the gauge field sector is more nontrivial, especially when $\alpha = 0$, which is the case we will concentrate on. By gauge invariance, the induced action for the gauge fields must be functionals of the field strengths F_{0i} and F_{ij} and their derivatives. In addition, it must be symmetric under spatial rotations. For a Lorentz symmetry to emerge, this effective action must contain combinations like

$$\epsilon_0 F_{0i}^2 + \frac{1}{\mu_0} F_{ij}^2 \tag{16}$$

with constant ϵ_0, μ_0. In that one can now rescale t, x, A_0, A_i to get a standard Lorentz-invariant form.

The length dimensions of the dielectric constant ϵ_0 and magnetic permeability μ_0 may be easily seen to be

$$[\epsilon_0] \sim [L]^{z-d+2} \qquad [\mu_0] \sim [L]^{d+z-4} \tag{17}$$

so that the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$ has length dimensions

$$[c] \sim [L]^{1-z} \tag{18}$$

as it should.

It is not at all obvious that terms like (16) *have* to emerge at $\alpha = 0$, since the parent theory has $z = 2$. In fact we will show that for special choices of the operator $\mathcal{D}^z$ this will *not* happen. However, we will find that for generic choices of $\mathcal{D}^z$, terms like (16) do appear.

Let us first address this question for $z = d = 2$, using the form (6). For this purpose, it is sufficient to consider the effective action (10) with $\delta\chi = 0$, so that we essentially have

$$S_{\rm eff} = N\{\mathrm{Tr}\log[-D_0^2 + (a+b)(D_i D^i)^2 + a(B(t,x))^2 - ia\epsilon^{ij}(\partial_i B) D_j + m^2] - (B = 0 \text{ term})\}, \tag{19}$$

where we have used the commutation relation

$$[D_i, D_j] = iF_{ij} \tag{20}$$

and for $d = 2$ renamed $F_{12} = B(t, x)$.

[1] In principle, there could be a condensation of the field strength. However, we will soon see in Sec. II A that for $d = z = 2$ the effective action for a constant $B = \epsilon_{ij}\partial_i A_j$ is always larger than the action with $B = 0$. This rules out condensation of B. Our assumption that the field strength vanishes at the saddle point is thus justified only *a posteriori*. We do not have a proof that this continues to hold for all $d = z$, but this appears to be plausible.

A. Constant magnetic field

It is useful to first evaluate this for a constant B. Then the problem in evaluating the effective action reduces to the problem of determining the eigenvalues of the operator

$$H(B) = -D_0^2 + (a+b)(-D_1^2 - D_2^2)^2 + aB^2 \tag{21}$$

which is closely related to the problem of Landau diamagnetism. Let us choose a Landau gauge

$$A_0 = A_1 = 0 \qquad A_2 = Bx^1. \tag{22}$$

Consider the system to be in a large box with size in the time direction T and spatial sizes L_1, L_2. For large enough T, L_1, L_2 the eigenvalue of ∂_0 can be taken to be continuous, which we will call p_0. It is straightforward to see that the eigenvalues of $H(B)$ are

$$\kappa(p_0, n) = p_0^2 + (a+b)B^2(2n+1)^2 + aB^2 \tag{23}$$

with a degeneracy of the level n given by

$$d(n) = \frac{BL_1L_2}{2\pi}. \tag{24}$$

To evaluate the effective action (19) we use the Nambu-Schwinger-de Witt representation,

$$-S_{\rm eff} = N\int_0^\infty \frac{ds}{s} e^{-m^2 s}\,{\rm Tr}\, e^{-sH(B)} - (B=0 \text{ term}). \tag{25}$$

Using (23) and (24) we have

$$\begin{aligned}{\rm Tr}\, e^{-sH(B)} &= e^{-saB^2}\sum_{n=0}^{\infty} T\int_{-\infty}^{\infty}\frac{dp_0}{2\pi} \\ &\quad\times \frac{BL_1L_2}{2\pi} e^{-[sp_0^2 + 4(a+b)B^2 s(n+1/2)^2]} \\ &= \frac{VTB}{8\pi^2}\sqrt{\frac{\pi}{s}} e^{-saB^2}\vartheta_2[0|4iB^2 s(a+b)/\pi],\end{aligned} \tag{26}$$

where $V = L_1L_2$ denotes the spatial volume and $\vartheta_2[w|\tau]$ is a Jacobi theta function. To examine the small B behavior, it is useful to use standard theta function identities to write

$$\begin{aligned}{\rm Tr}\, e^{-sH(B)} &= \frac{VT}{16\pi s\sqrt{a+b}} e^{-saB^2}\vartheta_4[0|i\pi/(4B^2 s(a+b))] \\ &= \frac{VT}{16\pi s\sqrt{a+b}} e^{-saB^2}\sum_{k=-\infty}^{\infty}(-1)^k \\ &\quad\times e^{-\{(\pi^2k^2)/[4sB^2(a+b)]\}}.\end{aligned} \tag{27}$$

The theta function in (27) may be written in a product representation as

$$\begin{aligned}\vartheta_4[0|i\pi/(4B^2 s(a+b))] = \prod_{n=1}^{\infty}&(1 - e^{-\{[(2n-1)\pi^2]/[4B^2 s(a+b)]\}})^2 \\ &\times(1 - e^{-\{(2n\pi^2)/[4B^2 s(a+b)]\}}).\end{aligned} \tag{28}$$

Since a, $b \geq 0$, this immediately shows that

$${\rm Tr}\; e^{-sH(B)} < {\rm Tr}\, e^{-sH(0)} \tag{29}$$

so that

$$S_{\rm eff}(B) > S_{\rm eff}(0) \tag{30}$$

for any m^2. This provides a justification for setting $B = 0$ in the saddle point equation which determines m^2. The result (30) in fact holds for all d with $z = 2$.

The integral over s in (27) can be performed, leading to the effective action[2]

$$\begin{aligned}\frac{S_{\rm eff}(B)}{VT} = &-\sum_{k\neq 0}(-1)^k\frac{Bm}{4\pi^2 k}\sqrt{1+\frac{aB^2}{m^2}}K_1\left(\frac{\pi k m}{B}\sqrt{\frac{1+\frac{aB^2}{m^2}}{a+b}}\right) \\ &-\frac{1}{16\pi\sqrt{a+b}}\int_0^\infty\frac{ds}{s^2}e^{-m^2 s}(e^{-saB^2}-1),\end{aligned} \tag{31}$$

where K_1 denotes a modified Bessel function. In deriving (31) we have noted that the $k = 0$ term in the sum in (27) is the sole contribution when $B = 0$ and subtracted that.

The first term on the right-hand side of (31) has a nonanalytic dependence on B for small B. This follows from the asymptotic behavior of the modified Bessel function. For $B \ll m$ the sum in (31) is dominated by the $k = 1$ term, which leads to

$$\frac{S_{\rm eff}(B) - S_{\rm eff}(0)}{VT} \simeq \frac{B^{3/2}m^{1/2}b^{1/4}}{4\pi^2\sqrt{2}} e^{-[(\pi m)/(B\sqrt{b})]}. \tag{32}$$

The second term contains various powers of B.

Therefore, at the multicritical point $a = 0$, the effective action vanishes for $B = 0$ in a *nonanalytic fashion*.[3]

For any $a \neq 0$ the final form of the effective action begins with a term proportional to B^2, It follows from (25) that the coefficient of this term is divergent [proportional to $\Gamma(0)$]. To understand this, we use a version of dimensional regularization by adding $(d-2)$ extra spatial directions. Now the first line of (26) will be modified to

[2]Note that there is an overall factor of N in the effective action (39). Since we are performing a $1/N$ expansion, the fields have to be rescaled as in (15). The factor of N cancels for the terms which are quadratic in the fields.

[3]We have performed the sum in (27) using a Euler-McLaurin expansion and verified that there are no polynomial terms in B in this case to very high orders.

$$\mathrm{Tr}\ e^{-sH(B)} = e^{-saB^2}\sum_{n=0}^{\infty} T\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}L^{d-2}\int\frac{d^{d-2}p}{(2\pi)^{d-2}} \times \frac{BL_1L_2}{2\pi}e^{-sp_0^2-sp^4-4(a+b)B^2s(n+1/2)^2}. \tag{33}$$

This leads to the following coefficient of B^2 in the effective action (25)

$$a\int_0^{\infty}\frac{ds}{s}e^{-m^2s}s^{-(d-2)/4} = a\frac{\Gamma(\frac{2-d}{4})}{m^{2-d/2}}. \tag{34}$$

The coefficient has length dimension $(2-d)$ as required. In a dimensional regularization scheme we introduce a scale κ with length dimensions -2, and write this coefficient as

$$\frac{1}{\mu_0\kappa^{(2-d)/2}}, \tag{35}$$

where μ_0 is dimensionless. Then as $\epsilon \to 0$, the finite part of $1/\mu_0$ becomes

$$\frac{1}{\mu_0} \sim \log\left(\frac{\kappa}{m}\right). \tag{36}$$

B. General calculation using heat kernel

In this section we will evaluate the effective action using a standard heat kernel method for general fields $F_{0i}(t,x)$ and $F_{ij}(t,x)$. The calculation presented above for constant magnetic field shows that at the special point $a=0$ the effective action has a nonanalytic dependence on B. We would like to determine whether the action contains terms which are analytic in derivatives of B. As shown in the Appendix, the small s expansion of the heat kernel is of the form

$$\mathrm{Tr}\ e^{-sH(E,B)} = \int d^2x dt\sum_{n=1}^{\infty} b_n(x)s^{(n/4)-1} \tag{37}$$

which leads to the effective action

$$S_{\mathrm{eff}} = N\sum_{n=1}^{\infty}\int d^2x dt b_n(x)\frac{\Gamma(\frac{n}{4}-1)}{m^{2[(n/4)-1]}}. \tag{38}$$

As in the previous subsection the term with $n=4$ can be handled via dimensional regularization. As explained in the Appendix, we should therefore replace (38) by

$$S_{\mathrm{eff}} = N\sum_{n=1}^{\infty}\int d^2x dt b_n(x)\frac{\Gamma(\frac{n}{4}-1+\frac{\epsilon}{4})}{m^{2[(n/4)-1+(\epsilon/4)]}}. \tag{39}$$

Let us first evaluate this for $a=0$ and $b=1$ For this case, explicit calculations yield $b_1(t,x)=b_2(t,x)=b_3(t,x)=b_4(t,x)=b_5(t,x)=0$. The leading contribution comes from $b_6(t,x)$. After a fairly long calculation we find that this leads to the effective action

$$S_{\mathrm{eff},a=0} = \frac{1}{12m}\int dt d^2x\left[F_{0i}^2 + \frac{1}{10}(\partial_i B)(\partial^i B)\right]. \tag{40}$$

Higher powers of field strength are suppressed by powers of N. Note that the term we obtained in the previous section for constant B is nonanalytic in B, and formally irrelevant by power counting since its Taylor series is 0. However, in the absence of this nonanalytic term, any constant B costs no energy, leading to a huge ground state degeneracy. Thus, the term $B^{3/2}e^{-\pi m/B}$ is a dangerously irrelevant operator for this special case $a=0$, $b=1$. Clearly Lorentz invariance is not regained at low energies in this case.

Away from this multicritical point $a\neq 0$. As expected from our constant B calculation, we now find that $b_4\neq 0$. It is straightforward to see that

$$b_4(t,x) \sim F^{ij}F_{ij}. \tag{41}$$

Since we are interested in the action at low energies, we should retain only the lowest nontrivial terms with the least number of derivatives. This leads to the low-energy effective action for gauge fields, up to numerical factors

$$S_{\mathrm{eff}} \sim \int d^2x dt\left[\frac{1}{m}F_{0i}F^{0i} + \frac{\Gamma(\frac{\epsilon}{4})}{m^{\epsilon/2}}F_{ij}F^{ij} + \cdots\right], \tag{42}$$

where the ellipses now stand for terms containing more derivatives and/or more powers of the field strength. As in the previous subsection, in the spirit of dimensional regularization this is really

$$S_{\mathrm{eff}} \sim \int d^2x dt\left[\frac{1}{m}F_{0i}F^{0i} + \log\left(\frac{\kappa}{m}\right)F_{ij}F^{ij} + \cdots\right] \tag{43}$$

so that we have μ_0 given in (35) and

$$\epsilon_0 \sim \frac{1}{m}. \tag{44}$$

Since ϵ_0, μ_0 are constants independent of (t,x) one can now rescale t, x, A_0, A_i to get the form

$$S_{\mathrm{eff}} \sim \int d^2x dt[F_{\mu\nu}F^{\mu\nu} + \cdots]. \tag{45}$$

This demonstrates the emergence of approximate low-energy Lorentz symmetry with a scale dependent speed of light. Note that this happens even when the (renormalized) parameter α_{ren} in (5) is tuned to zero. At this Lifshitz point there is no Lorentz symmetry in the scalar sector. For $\alpha_{\mathrm{ren}}\neq 0$ the speed of light in the scalar sector is different from that in the gauge sector.

III. DISCUSSION

It is clear from Sec. I that the coupling g in the sigma model is asymptotically free for all $d=z$. Does this also mean that gauge dynamics emerges in all dimensions? From Eq. (17) we see that the length dimension of ϵ_0 is always 2 for all $z=d$. If such a term appears, one would expect that $\epsilon_0 \sim m^{2/z}$, since from (12) the length dimension of the mass m is z. For $z \geq 3$ it is not clear how such a term can arise in the effective action obtained by integrating out the massive field $\vec{\phi}$. In fact if such an effective action is an expansion of powers of $1/m$ (apart from logs), one would expect that the lowest dimension operator which would appear must have length dimensions $[L]^{z+d}$. In a similar vein, the magnetic permeability would have *positive* length dimensions. Such a term is also unlikely to come from an effective action. It would be interesting to investigate this issue further.

The model studied in this paper gauges the overall $U(1)$ of the symmetry group. One could, instead, consider gauging the entire $U(N)$ group to obtain a non-Abelian gauged sigma model. This model would generate a mass gap in exactly the same fashion—in fact the gap equation is identical. It would be interesting to see the effective action for the non-Abelian gauge fields in this case. It appears to us that the heat kernel expansion calculation is quite similar to ours.

Another interesting direction is to revisit the *linear* rather than the nonlinear sigma model around a Lifshitz fixed point as has been originally considered in classical statistical mechanics. The length dimension of a $(\vec{\phi}\cdot\vec{\phi})^2$ coupling is given by $(d-3z)$, so that this is a relevant operator for $z > d/3$. It would be interesting to explore if there are IR fixed points for $d > 2$ at finite values of the corresponding coupling, similar to $z = 1$, $d = 2$. This could have interesting applications to particle physics. These vector models can be also interesting from the point of view of AdS/CFT correspondence. Lifshitz fixed points have been argued to have dual gravity descriptions [22]. On the other hand, the dual of usual vector models are higher spin gauge fields in usual AdS [23,24]. It would be interesting to see the nature of the gravity duals for these Lifshitz sigma models.

These issues are currently under investigation.

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In this Appendix we calculate the heat kernel using the technique of [25]. This uses the representation

$$\mathrm{Tr}\, e^{-s\mathcal{O}} = \int dt d^2x \int \frac{d\omega d^2k}{(2\pi)^3} e^{-i(\omega t + k\cdot x)} e^{-s\mathcal{O}} e^{i(\omega t + k\cdot x)}. \tag{A1}$$

Using the basic identity

$$e^{-i(\omega t + k\cdot x)} D_\mu e^{i(\omega t + k\cdot x)} = ik_\mu + D_\mu \tag{A2}$$

we have for our case

$$e^{-i(\omega t + k\cdot x)} H(E,B) e^{i(\omega t + k\cdot x)} = \omega^2 + \vec{k}^4 + a_1 + a_2 + a_3 + a_4, \tag{A3}$$

where

$$\begin{aligned} a_1 &= -4ik^2(k^i D_i) \\ a_2 &= -2i\omega D_0 - 4(k^i D_i)^2 - 2k^2(D^i D_i) \\ a_3 &= i[(D^i D_i)(k^j D_j) + (k^j D_j)(D^i D_i) + 2D_i(k^j D_j)D^i] \\ a_4 &= -D_0^2 + D_i D_j D^i D^j. \end{aligned} \tag{A4}$$

Using (A1) and (A3) and rescaling

$$\omega \to \frac{1}{s^{1/2}}\omega \qquad \vec{k} \to \frac{1}{s^{1/4}}\vec{k} \tag{A5}$$

we get

$$\mathrm{Tr}\, e^{-sH(E,B)} = \frac{1}{s}\int dt d^2x \int \frac{d\omega d^2k}{(2\pi)^3} e^{-(\omega^2+k^4)} e^{-G(E,B)}, \tag{A6}$$

where

$$G \equiv s^{1/4}a_1 + s^{1/2}a_2 + s^{3/4}a_3 + s a_4. \tag{A7}$$

The integrals over ω and $\vec{k}$ can be now evaluated, leading to small-s expansion of the heat kernel of the form (37), leading to the form of the effective action (38).

The term with $n = 4$ has to be treated in dimensional regularization. This means that in (A1) we replace $d^2k \to d^dk$, so that after the rescalings (A5), the equation (A6) becomes

$$\mathrm{Tr}\, e^{-sH(E,B)} = \frac{1}{s^{(d+2)/4}} \times \int dt d^2x \int \frac{d\omega d^dk}{(2\pi)^3} e^{-(\omega^2+k^4)} e^{-G(E,B)}. \tag{A8}$$

For $d = 2 - \epsilon$ we will still evaluate the integrals over $\vec{k}$ by replacing $d^dk \to d^2k$ in (A8). This leads to the small s expansion

$$\mathrm{Tr}\, e^{-sH(E,B)} = \int d^2x dt \sum_{n=1}^{\infty} b_n(x) s^{(n/4)-1+(\epsilon/4)}. \tag{A9}$$

This leads to the expression (39).

The integrals over ω and $\vec{k}$ can be performed using basic symmetry properties. Thus, e.g.,

$$\int_{-\infty}^{\infty} d\omega e^{-\omega^2}\omega^{2n+1} = 0 \quad (n \text{ integer})$$
$$\int_{-\infty}^{\infty} d\omega e^{-\omega^2}\omega^2 = \frac{1}{2}\int_{-\infty}^{\infty} d\omega e^{-\omega^2}$$
$$\int_{-\infty}^{\infty} d\omega e^{-\omega^2}\omega^4 = \frac{3}{4}\int_{-\infty}^{\infty} d\omega e^{-\omega^2}$$
$$\cdots \quad \cdots \tag{A10}$$

while

$$\int d^2k k^{i_1}k^{i_2}k^{1_{2n+1}}e^{-k^4} = 0 \quad (n \text{ integer})$$
$$\int d^2k k^i k^j e^{-k^4} = \frac{1}{2\sqrt{\pi}}\delta^{ij}\int d^2k e^{-k^4}$$
$$\int d^2k(\vec{k}\cdot\vec{k})^2 e^{-k^4} = \frac{1}{2}\int d^2k e^{-k^4}$$
$$\int d^2k(\vec{k}\cdot\vec{k})^3 e^{-k^4} = \frac{1}{\sqrt{\pi}}\int d^2k e^{-k^4}$$
$$\int d^2k(\vec{k}\cdot\vec{k})^4 k^i k^j e^{-k^4} = \frac{1}{2\sqrt{\pi}}\delta^{ij}\int d^2k e^{-k^4}$$
$$\cdots \quad \cdots \tag{A11}$$

Using these integrals, it is straightforward to see that terms with $n = 1, 3, 5$ in the sum (A9) vanish since they have odd numbers of ω and/or $\vec{k}$. A short calculation using the explicit expressions in (A10) and (A11) shows that the various terms cancel, leading to $b_2(x, t) = 0$. The first nontrivial term is therefore for $n = 4$. Here, after several cancellations one is left with

$$b_4(t, x) \sim F_{ij}F^{ij} \tag{A12}$$

which basically comes from rewriting the second term in a_4 as

$$\begin{aligned} D_iD_jD^iD^j &= D_iD^2D_i + D_iD_j[D^i, D^j] \\ &= D_iD^2D_i + \tfrac{1}{2}[D_i, D_j][D^i, D^j] \\ &= D_iD^2D_i + \tfrac{1}{2}F_{ij}F^{ij}. \end{aligned} \tag{A13}$$

The next nonzero term comes at $n = 6$. This leads to an electric field term, which arises from

$$\int \frac{d\omega d^d k}{(2\pi)^3} e^{-(\omega^2+k^4)}(a_1a_2a_1a_2) \tag{A14}$$

which clearly includes a term

$$D_iD_0D_iD_0 \tag{A15}$$

and hence to $F_{0i}F^{0i}$. The $n = 6$ term contains other contributions as well. These contain higher derivative terms in the field strength B. Specifically, for the case $a = 0$, $b = 1$, we obtain the term

$$\frac{1}{120m}(\nabla B)^2. \tag{A16}$$

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