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Effects of a fundamental mass term in two-dimensional super Yang-Mills theory

Uwe Trittmann

Department of Physics & Astronomy, Otterbein College, Westerville, Ohio 43081, USA

Stephen S. Pinsky

Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA

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We show that adding a vacuum expectation value to a gauge field left over from a dimensional reduction of three-dimensional pure supersymmetric Yang-Mills theory generates mass terms for the fundamental fields in the two-dimensional theory while supersymmetry stays intact. This is similar to the adjoint mass term that is generated by a Chern-Simons term in this theory. We study the spectrum of the two-dimensional theory as a function of the vacuum expectation value and of the Chern-Simons coupling. Apart from some symmetry issues a straightforward picture arises. We show that at least one massless state exists if the Chern-Simons coupling vanishes. The numerical spectrum separates into (almost) massless and very heavy states as the Chern-Simons coupling grows. We present evidence that the gap survives the continuum limit. We display structure functions and other properties of some of the bound states.

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I. INTRODUCTION

With the LHC experiments at CERN to begin data taking shortly, it may soon be clear whether supersymmetry is realized in Nature. Notwithstanding the experimental verdict, supersymmetry provides solutions to profound questions in particle physics [1,2], and, as a symmetry, it is useful to simplify calculations. We use it within the framework of supersymmetric discretized light-cone quantization (SDLCQ) to solve quantum field theories. SDLCQ comes with a set of strengths and weaknesses documented in the literature [3,4]. In particular, SDLCQ is a Hamiltonian approach and practically limited to theories with enough supersymmetry to render them finite. It is primarily a numerical approach, and as such it is cheaper to consider theories in lower dimensions. The SDLCQ Hamiltonian is manifestly invariant under supersymmetry which is hard to achieve in conventional lattice gauge theory because of the asymmetric treatment of bosons and fermions, although progress is being made [5].

In a line of publications, we have been deciphering the properties of bound states of theories that share features with QCD or are interesting in their own right. Starting from a two-dimensional pure super Yang-Mills (SYM) theory [6,7], we have been adding fundamental matter to emulate quarks [8,9], a Chern-Simons (CS) term to simulate effective gluon masses [10], and tackled higher dimensional theories [11,12]. As a natural extension to previous work, we set out to construct a mass term for the fundamental particles in the present note. Of course, supersymmetry itself prevents the use of ordinary mass terms, but one does not have to think too hard to fix this problem. Inspired by work of Myers *et al.* [13] on $\mathcal{N} = 2$ SYM in four dimensions, we might try to add a vacuum expectation value (VEV) to a gauge field left over from the dimensional reduction of a higher-dimensional version of the theory. Shifting the field by its VEV should then produce fundamental mass terms invariant under supersymmetry. It turns out that the simplest scenario suffices: we can start with a three-dimensional $\mathcal{N} = 1$ SYM theory, reduce it to two dimensions, have the transverse gauge field acquire a VEV, and produce the desired fundamental mass term in the dimensionally reduced theory.

In the present note, we work out the details and study the ensuing spectrum of bound states as a function both of the VEV (“quark mass”) and the CS coupling (“gluon mass”). Though the focus is on the effects of the VEV-induced mass terms, it is natural to include a CS term, too. We find that the theory containing both terms is not invariant under any of the customary discrete symmetries. However, mass differences between nominal parity partners are tiny due to a small symmetry-breaking term. Apart from this glitch, exploring the model is straightforward and yields few surprises. This is good news since stable continuum results can be extracted and a theory with an interesting mass spectrum emerges. If no CS term is present, massless states exist, otherwise the lightest states remain massive. Masses tend to decrease with growing resolution, but even at finite K some states become massless at special values of the VEV, if the CS coupling vanishes. As the couplings grow, few nearly massless states are clearly separated from the bulk of heavy states. While it is hard to show directly that this feature survives the continuum limit, it is likely to be true at substantial CS coupling judging from the strong coupling limit and numerical evidence. We present the theory in the next section, derive some analytical results, display numerical results, and conclude.

II. SUPER YANG-MILLS THEORY IN TWO DIMENSIONS

A two-dimensional super Yang-Mills theory with a Chern-Simons term is generated conveniently by dimensionally reducing its three-dimensional pendant. The action of $\mathcal{N}=1$ supersymmetric gauge theory in three dimensions coupled to fundamental matter with a Chern-Simons term is

$$S_{2+1} = S_{\rm SYM} + S_{\rm fund} + S_{\rm CS}, \tag{1}$$

with

$$S_{\rm SYM} = \int d^3x\, {\rm Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\Lambda}\Gamma^\mu D_\mu \Lambda\Big), \tag{2}$$

$$S_{\rm fund} = \int d^3x (D_\mu \xi^\dagger D^\mu \xi + i\bar{\Psi}D_\mu\Gamma^\mu\Psi - g[\bar{\Psi}\Lambda\xi + \xi^\dagger\bar{\Lambda}\Psi]), \tag{3}$$

$$S_{\rm CS} = \int d^3x \frac{\hat{\kappa}}{2}\Big(\epsilon^{\mu\nu\lambda}\Big(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}gA_\mu A_\nu A_\lambda\Big) + 2\bar{\Lambda}\Lambda\Big). \tag{4}$$

The gauge part, $S_{\rm SYM}$, of the action describes a system of gauge bosons $(A_\mu)_{ab}$ and their superpartners, the Majorana fermions Λ_{ab} with color indices $a, b = 1, \ldots, N_c$, transforming under the adjoint representation of $SU(N_c)$. The matter content of the theory consists of a complex scalar ξ_a, and a Dirac fermion Ψ_a, both transforming under the fundamental representation of the gauge group. In matrix notation the covariant derivatives and the gauge field strength are defined as usual

$$\begin{aligned} D_\mu\xi &= \partial_\mu\xi + igA_\mu\xi, \qquad D_\mu\xi^\dagger = \partial_\mu\xi^\dagger - ig\xi^\dagger A_\mu,\\ D_\mu\Psi &= \partial_\mu\Psi + igA_\mu\Psi, \qquad D_\mu\Psi^\dagger = \partial_\mu\Psi^\dagger - ig\Psi^\dagger A_\mu,\\ D_\mu\Lambda &= \partial_\mu\Lambda + ig[A_\mu, \Lambda],\\ F_{\mu\nu} &= \partial_{[\mu}A_{\nu]} + ig[A_\mu, A_\nu]. \end{aligned} \tag{5}$$

The action (1) is invariant under supersymmetry transformations parametrized by a constant two-component Majorana spinor $\varepsilon \equiv (\varepsilon_1, \varepsilon_2)^{\rm T}$; $\bar{\varepsilon} \equiv \varepsilon^{\rm T}\Gamma^0$:

$$\begin{aligned} \delta A_\mu &= \tfrac{i}{2}\bar{\varepsilon}\Gamma_\mu\Lambda, \qquad \delta\Lambda = \tfrac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\varepsilon,\\ \delta\xi &= \tfrac{i}{2}\bar{\varepsilon}\Psi, \qquad \delta\xi^\dagger = -\tfrac{i}{2}\bar{\Psi}\varepsilon,\\ \delta\Psi &= -\tfrac{1}{2}\Gamma^\mu\varepsilon D_\mu\xi, \qquad \delta\bar{\Psi} = -\tfrac{1}{2}D_\mu\xi^\dagger\bar{\varepsilon}\Gamma^\mu, \end{aligned} \tag{6}$$

where $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$. Using standard Noether techniques, we can determine the conserved current density $\mathcal{J}^\mu = N^\mu + K^\mu$, consisting of the familiar Noether (on shell) current density N^μ, and K^μ, related to the change of the Lagrangian under a supersymmetry transformation and having the form of a space-time divergence. We dimensionally reduce the theory to two dimensions by omitting all transverse derivatives, $\partial_\perp(\ldots) = \partial^\perp(\ldots) \equiv 0$. Note that $A^\perp$ will remain part of the two-dimensional theory.

At this point it is useful to transcribe to light-cone coordinates, $x^\pm = (x^0 \pm x^1)/\sqrt{2}$. We calculate the supercharge Q^α (a two-component spinor in two dimensions) by integrating the plus-component of the supercurrent $\mathcal{J}^+$ over all space, i.e. over x^- and $x^\perp$. The latter yields a constant factor which can be reabsorbed in a rescaling of the fields. Using light-cone coordinates allows us to express the supercharge in terms of the *physical* fields by imposing the light-cone gauge condition $A^+ = A_- = 0$. Since the other component of the gauge field can be eliminated by a constraint equation, we are left with the adjoint $A^\perp_{ab}$ and the fundamental scalar ξ_a as physical bosonic fields, whereas the (left-moving) physical fermionic fields λ_{ab} and ψ_a are components of the spinors appearing in the action (1)

$$\Lambda = \begin{pmatrix}\lambda\\ \tilde{\lambda}\end{pmatrix}, \qquad \Psi = \begin{pmatrix}\psi\\ \tilde{\psi}\end{pmatrix}. \tag{7}$$

We have used the imaginary (Majorana) representation

$$\Gamma^0 = \sigma_2, \qquad \Gamma^1 = i\sigma_1, \qquad \Gamma^\perp = i\sigma_3, \tag{8}$$

to render the Majorana spinor field real, $\Lambda^\dagger = \Lambda^T$. The supercharge components are labeled $Q = (Q^+, Q^-)^T$, to reflect their relation to the Lorentz generators via the superalgebra in its $\mathcal{N} = (1,1)$ form

$$\{Q^+, Q^+\} = 2\sqrt{2}P^+, \qquad \{Q^-, Q^-\} = 2\sqrt{2}P^-, \qquad \{Q^+, Q^-\} = 0. \tag{9}$$

The two-dimensional supercharge reads

$$Q^- = Q^-_{\rm SYM} + Q^-_{\rm fund} + Q^-_{\rm CS}, \tag{10}$$

where

$$\begin{aligned} Q^-_{\rm SYM} &= ig\sqrt{2}\int dx^-\, {\rm Tr}\Big[\Big(-[A^\perp, \partial_- A^\perp] + \frac{i}{\sqrt{2}}\{\lambda, \lambda\}\Big)\frac{1}{\partial_-}\lambda\Big],\\ Q^-_{\rm fund} &= ig\sqrt{2}\int dx^-\Big({\rm Tr}\Big[(\partial_-\xi\xi^\dagger - \xi\partial_-\xi^\dagger + \sqrt{2}i\psi\psi^\dagger)\frac{1}{\partial_-}\lambda\Big] + i\xi^\dagger A^\perp\psi + i\psi^\dagger A^\perp\xi\Big),\\ Q^-_{\rm CS} &= \hat{\kappa}\sqrt{2}\int dx^-\, {\rm Tr}(\lambda A^\perp). \end{aligned}$$

To generate a mass term, we will assume[1] that the gauge field $A^{\perp}_{ab}$ acquires a vacuum expectation value

$$\hat{v}_{ab} := \langle A^{\perp}_{ab} \rangle = \hat{v}\delta_{ab}.$$

Shifting the field by its VEV, and expressing the theory in terms of the new field

$$(A^{\perp}_{ab})' = A^{\perp}_{ab} - \langle A^{\perp}_{ab} \rangle, \tag{11}$$

will yield extra terms in the supercharge, which can be interpreted as mass terms for the fundamental particles of the theory. The only part of the supercharge that is affected by the shift of the perpendicular gauge field is $Q^-_{\rm fund}$, since the color-neutral, constant VEV appears in a derivative in $Q^-_{\rm SYM}$, and in a trace in $Q^-_{\rm CS}$. The effect of the shift, Eq. (11), is on the last two terms of $Q^-_{\rm fund}$, giving rise to an extra operator in the supercharge

$$Q^-_{\rm XS} = -g\hat{v}\sqrt{2}\int dx^-(\xi^\dagger\psi + \psi^\dagger\xi). \tag{12}$$

At this point, we employ the framework of SDLCQ (see, e.g. [14]) to obtain the mode decomposition of the supercharge listed in the Appendix that will allow us to evaluate the theory on a computer. In particular, we quantize by imposing the canonical commutation relations

$$\begin{aligned}
[A^{\perp}_{ab}(0,x^-), \partial_- A^{\perp}_{cd}(0,y^-)] &= i\delta_{ad}\delta_{bc}\delta(x^- - y^-),\\
\{\lambda_{ab}(0,x^-), \lambda_{cd}(0,y^-)\} &= \sqrt{2}\delta_{ad}\delta_{bc}\delta(x^- - y^-),\\
[\xi_a(0,x^-), \partial_-\xi_b(0,y^-)] &= i\delta_{ab}\delta(x^- - y^-),\\
\{\psi_a(0,x^-), \psi^\dagger_b(0,y^-)\} &= \sqrt{2}\delta_{ab}\delta(x^- - y^-).
\end{aligned} \tag{13}$$

The compactification of the theory on a on a lightlike circle $(-L < x^- < L)$ leads to discrete momentum modes defined via

$$A^{\perp}_{ab}(0,x^-) = \frac{1}{\sqrt{4\pi}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(A_{ab}(n)e^{-in\pi x^-/L} + A^\dagger_{ba}(n)e^{in\pi x^-/L}), \tag{14}$$

$$\lambda_{ab}(0,x^-) = \frac{1}{2^{1/4}\sqrt{2L}}\sum_{n=1}^{\infty}(B_{ab}(n)e^{-in\pi x^-/L} + B^\dagger_{ba}(n)e^{in\pi x^-/L}), \tag{15}$$

$$\xi_a(0,x^-) = \frac{1}{\sqrt{4\pi}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(C_a(n)e^{-in\pi x^-/L} + \tilde{C}^\dagger_a(n)e^{in\pi x^-/L}), \tag{16}$$

$$\psi_a(0,x^-) = \frac{1}{2^{1/4}\sqrt{2L}}\sum_{n=1}^{\infty}(D_a(n)e^{-in\pi x^-/L} + \tilde{D}^\dagger_a(n)e^{in\pi x^-/L}). \tag{17}$$

Normalization is chosen such that the commutation relations (13) in terms of creation and annihilation operators of are cast into their customary form, namely,

$$[A_{ab}(n), A^\dagger_{cd}(n')] = \{B_{ab}(n), B^\dagger_{cd}(n')\} = \left(\delta_{ad}\delta_{bc} - \frac{1}{N}\delta_{ab}\delta_{cd}\right)\delta_{nn'}, \tag{18}$$

$$[C_a(n), C^\dagger_b(n')] = [\tilde{C}_a(n), \tilde{C}^\dagger_b(n')] = \{D_a(n), D^\dagger_b(n')\} = \{\tilde{D}_a(n), \tilde{D}^\dagger_b(n')\} = \delta_{ab}\delta_{nn'}.$$

The extra part of the supercharge becomes

$$Q^-_{\rm XS} = -\frac{g\hat{v}}{2^{1/4}}\sqrt{\frac{L}{\pi}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(C^\dagger_a(n)D_a(n) + \tilde{C}^\dagger_a(n)\tilde{D}_a(n) + D^\dagger_a(n)C_a(n) + \tilde{D}^\dagger_a(n)\tilde{C}_a(n)).$$

The operators of $Q^-_{\rm XS}$ induce extra terms in the Hamiltonian

$$\begin{aligned}
P^-_{\rm XS} &= \frac{1}{2\sqrt{2}}\{Q^-_{\rm XS}, Q^-_{\rm XS}\}\\
&= \frac{g^2\hat{v}^2L}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}(D^\dagger_a(n)D_a(n) + \tilde{D}^\dagger_a(n)\tilde{D}_a(n) + C^\dagger_a(n)C_a(n) + \tilde{C}^\dagger_a(n)\tilde{C}_a(n)),
\end{aligned} \tag{19}$$

which are *bona fide* mass terms with correct dimensions, since the VEV is dimensionless, and g has dimension of mass in the two-dimensional theory. Note that there are additional induced terms, e.g. $\{Q_{\rm SYM}, Q_{\rm XS}\}$. To obtain the spectrum of the theory one has to solve the matrix eigenvalue problem

$$2P^+P^-|n\rangle = M_n^2|n\rangle, \tag{20}$$

which will yield the mass (squared) eigenvalues M_n^2, and the eigenfunctions of the bound states of the theory, parametrized by the harmonic resolution K induced by the compactification and related to the light-cone momentum $P^+ = \frac{\pi}{L}K$. When generating matrix elements it becomes convenient to use the rescaled parameters $v = \hat{v}/\sqrt{N_c}$ and $\kappa = \hat{\kappa}/\sqrt{N_c}$, because the effective gauge coupling is $g\sqrt{N_c}$. We will refer to v and κ as the VEV and the Chern-Simons coupling in the remainder of the paper.

III. SYMMETRIES

The theory is marginally invariant under two discrete symmetries, in the sense that different parts of the supercharge will respect different symmetries. Parity acts on the annihilation operators introduced in Eqs. (14)–(17) as fol-

[1]Introducing the VEV earlier would have required a modification of the super transformation of the fundamental fermion, entailing Ψ/A mixing. It must, however, lead to the same supercharge and Hamiltonian.

TABLE I. Transformation properties of the parts of the supercharge under parity $\mathcal{P}$ and orientation reversal $\mathcal{O}$ as defined in the text.

$\mathcal{A}$	$\mathcal{P}\mathcal{A}$	$\mathcal{O}\mathcal{A}$
$Q^-_{\rm SYM}$	$+Q^-_{\rm SYM}$	$+Q^-_{\rm SYM}$
$Q^-_{\rm fund}$	$+Q^-_{\rm fund}$	$+Q^-_{\rm fund}$
$Q^-_{\rm CS}$	$-Q^-_{\rm CS}$	$+Q^-_{\rm CS}$
$Q^-_{\rm XS}$	$-Q^-_{\rm XS}$	$-Q^-_{\rm XS}$

lows:

$$\mathcal{P}\colon A_{ab} \to -A_{ab}, \qquad B_{ab} \to B_{ab}, \qquad C_a \to -C_a, \qquad \tilde{C}_a \to -\tilde{C}_a, \qquad D_a \to D_a, \qquad \tilde{D}_a \to \tilde{D}_a. \tag{21}$$

Note that this is the light-cone analogue of parity, and as such the transformations might not be intuitively clear. The Hamiltonian P^- commutes with the parity operator only in the absence of both the VEV-induced mass terms and the Chern-Simons term, Eq. (A3); the latter mixes parity-odd with parity-even adjoints. Mass eigenvalues are degenerate under parity, but not under the $\mathcal{O}$ symmetry [15] reversing the orientation of a string of partons

$$\mathcal{O}\colon A_{ab} \to -A_{ba}, \qquad B_{ab} \to -B_{ba}, \qquad C_a \to \tilde{C}_a, \qquad D_a \to -\tilde{D}_a. \tag{22}$$

Adding the mass terms, $Q^-_{\rm XS}$, to the supercharge without the Chern-Simons term destroys both symmetries, yet the combination $\mathcal{P}\mathcal{O}$ is intact as inspection shows, with doubly degenerate eigenvalues. No symmetry remains if both Chern-Simons and mass terms are present, although mass differences between nominal parity partners are extremely small. For a summary of the symmetry properties of the parts of the supercharge, see Table I. Obviously, symmetry is restored as $\kappa \to \infty$ ($\mathcal{O}$ is a good symmetry), and as $v \to \infty$ ($\mathcal{P}\mathcal{O}$ good).

IV. ANALYTICAL RESULTS

For $K = 3$ we can solve the matrix Eq. (20) in closed form in the absence of a Chern-Simons term. We can then use the discrete symmetry $\mathcal{P}\mathcal{O}$ to reduce the number of bosonic and fermionic states to four, say in the $\mathcal{P}\mathcal{O}$ even sector, with states

$$\begin{aligned}
|1\rangle_{b+} &= \frac{1}{\sqrt{2}N_c}\,\mathrm{Tr}[\tilde{C}^\dagger(2)C^\dagger(1) + \tilde{C}^\dagger(1)C^\dagger(2)]]|0\rangle,\\
|2\rangle_{b+} &= \frac{1}{\sqrt{2}N_c}\,\mathrm{Tr}[\tilde{D}^\dagger(2)D^\dagger(1) - \tilde{D}^\dagger(1)D^\dagger(2)]]|0\rangle,\\
|3\rangle_{b+} &= \frac{1}{N_c}\,\mathrm{Tr}[\tilde{C}^\dagger(1)A^\dagger(1)C^\dagger(1)]]|0\rangle,\\
|4\rangle_{b+} &= \frac{1}{\sqrt{2}N_c}\,\mathrm{Tr}[\tilde{C}^\dagger(1)B^\dagger(1)D^\dagger(1) + \tilde{D}^\dagger(1)B^\dagger(1)C^\dagger(1)]]|0\rangle,\\
|1\rangle_{f+} &= \frac{1}{\sqrt{2}N_c}\,\mathrm{Tr}[\tilde{C}^\dagger(2)D^\dagger(1) + \tilde{D}^\dagger(1)C^\dagger(2)]]|0\rangle,\\
|2\rangle_{f+} &= \frac{1}{\sqrt{2}N_c}\,\mathrm{Tr}[\tilde{C}^\dagger(1)D^\dagger(2) + \tilde{D}^\dagger(2)C^\dagger(1)]]|0\rangle,\\
|3\rangle_{f+} &= \frac{1}{\sqrt{2}N_c}\,\mathrm{Tr}[\tilde{C}^\dagger(1)A^\dagger(1)D^\dagger(1) + \tilde{D}^\dagger(1)A^\dagger(1)C^\dagger(1)]]|0\rangle,\\
|4\rangle_{f+} &= \frac{1}{N_c}\,\mathrm{Tr}[\tilde{D}^\dagger(1)B^\dagger(1)D^\dagger(1)]]|0\rangle.
\end{aligned}$$

In this basis the supercharge matrix reads

$$(Q_{b+}) = {}_{f+}\langle m|Q^-|n\rangle_{b+} = -\frac{ig\sqrt{N_c L}}{2^{1/4}\pi}\begin{pmatrix} \sqrt{\pi}v & \sqrt{\frac{\pi}{2}}v & \frac{1}{2\sqrt{2}} & 0\\ \sqrt{\frac{\pi}{2}}v & -\sqrt{\pi}v & \frac{1}{2} & i\sqrt{2}\\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2\pi}v & 0\\ i\frac{3}{2\sqrt{2}} & -i & 0 & \sqrt{2\pi}v \end{pmatrix}.$$

We note that $Q^-_{\rm SYM}|n\rangle_{b+} = 0$ for $K = 3$, and that the eigenvalues of the two $\mathcal{PO}$ sectors are degenerate. The solutions of the characteristic polynomial of $\mathcal{Q}_{b+}(\mathcal{Q}_{b+})^\dagger$

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

with

$$a_1 = -5 - 7\pi v^2, \qquad a_2 = \frac{451}{64} + 25\pi v^2 + \frac{73}{4}\pi^2 v^4,$$
$$a_3 = -\frac{255}{128} - \frac{615}{32}\pi v^2 - \frac{165}{4}\pi^2 v^4 - 21\pi^3 v^6,$$
$$a_4 = 9\pi^4\Big(v^2 - \frac{\sqrt{33}-5}{8\pi}\Big)^2\Big(v^2 + \frac{\sqrt{33}+5}{8\pi}\Big)^2,$$

are related to the eigenvalues M_n^2 of the mass (squared) operator in the bosonic $\mathcal{PO}$-even sector

$$\mathcal{M}^2|_{b+} = 2P^+P^-|_{b+} = \sqrt{2}P^+\mathcal{Q}_{b+}(\mathcal{Q}_{b+})^\dagger,$$

by letting $M_n^2 = \sqrt{2}\frac{\pi K}{L}(\frac{g\sqrt{N_c L}}{2^{1/4}\pi})^2\lambda$.

Solving this generic quartic equation is intricate. For our purposes it suffices to determine at which values of the VEV massless eigenstates exist. For $\lambda = 0$ to be a solution, the constant term a_4 has to vanish, which is obviously the case for $v = ((\sqrt{33}-5)/8\pi)^{1/2} \approx 0.172$. The theory of quartic equations furthermore asserts that λ becomes unity when $1 + \sum_{i=1}^4 a_i = 0$, which gives rise to another quartic equation in v^2.

What type of spectrum do we expect for extreme choices of the parameters v and κ? As $v \to \infty$ at fixed κ, we have $P^- \to P^-_{\rm XS}$, and obtain a free spectrum of states with one fundamental parton of mass $m = \pi$ in units $g^2\hat{v}^2/\pi$ at each end of a chain of adjoints. Clearly, the lowest mass is, in the same units,

$$M^2_{{\rm lowest}, v\to\infty} = K\Big(\frac{\pi}{K/2} + \frac{\pi}{K/2}\Big) = 4\pi, \qquad (23)$$

and the highest mass (e.g. of two fundamentals with smallest possible momenta linked by $K-2$ adjoints) is $M^2_{\rm highest} = 2\pi K$. We note that their degeneracies are vastly different. There are 4(8) states of lowest mass at even(odd) K, but $8\cdot 3^{K-3}$ of mass $2\pi K$. The former states are part of a set of $4(K-1)$ states with no adjoint partons which have generally the smallest masses in the spectrum. According to Eq. (23), the smallest mass is obtained when two fundamentals split the total momentum, i.e. when both have large momentum. Obviously, adding more (adjoint) partons increases the mass. The same reasoning leads to a similar free particle spectrum for large κ at fixed v. Here, the $4(K-1)$ states with no adjoint partons will be massless, and the lightest massive states will have a mass (squared) of $\frac{K\pi}{K-2}$ in units $\hat{\kappa}^2/\pi$.

It seems clear that light states at large v or κ will remain light as these parameters decrease. We thus conclude that the lighter states in the spectrum, regardless of the values of v or κ, will be the ones in which the fundamental partons have large momenta, i.e. states with a minimal number of partons. Interaction between states of different parton number will increase the average number of partons in these states, but not dramatically.

V. NUMERICAL RESULTS

A. Masses as functions of the physical parameters

We plot the spectrum of the theory without CS term as a function of the VEV, Fig. 1, and see that the most prominent feature is a quadratic rise of masses (squared) with v. At small v we find $4(K-1)$ light states, four of which (one in each symmetry sector) become exactly[2] massless at various values of v. How do bound-state masses decrease as their constituents increase in mass? The change of mass at small v is roughly[3] $\delta M_n^2 = \langle n|2P^+P^-_{\rm XS}|n\rangle$. It is clear from the form of the Hamiltonian, Eq. (19), that a decrease in bound-state mass is possible even as v grows for states having overlap with certain basis states containing two fermionic fundamentals due to fermionic statistics. As we saw in Sec. IV, the lightest states will be short, having a large overlap with these special basis states. In fact, this seems to be the very reason why they are light. Close inspection of Fig. 1 shows that a few of the more massive states see their mass decreasing over some v range as well, which is, of course, not contradicting our finding.

From the analytical considerations in Sec. IV it is clear that the lowest $4(K-1)$ states are special in that they do not contain adjoint partons at large v. The quadratic nature of the interaction terms suggests to interpret the pattern of masses $M_n^2(v)$ in Fig. 1 as an overlay of $K-1$ parabolas with centers shifted in v, and individual masses distorted by eigenvalue repulsion. Indeed, for even (odd) K we see that at $\frac{K}{2}$ ($\frac{K-1}{2}$) VEV values a massless state is present. At fixed K, all masses eventually rise with v, because in the tug-of-war between the VEV and the effects of the admixture of fermionic "mass-reducing" basis states, the former must win. On the other hand, if K grows, so does the number of these basis states, keeping the effects of the growing VEV in check over a larger region. As more states come on line at higher K, the parabolas will add up to straighter lines at lower and lower mass, and we speculate that in the continuum limit a countable-infinite number of massless states emerges, all being linear combinations of infinitely many two-parton states. This hypothesis is sup-

[2]We showed in Sec. IV analytically that one of the states in each sector becomes massless for a certain value of v. Judging from numerical evidence, we can assume that this remains true at higher K.

[3]Since at $K = 3$ we have decreasing masses but $Q^-_{\rm SYM} = 0$, $Q^-_{\rm XS}$ alone is responsible for the decrease; likely also at higher K.

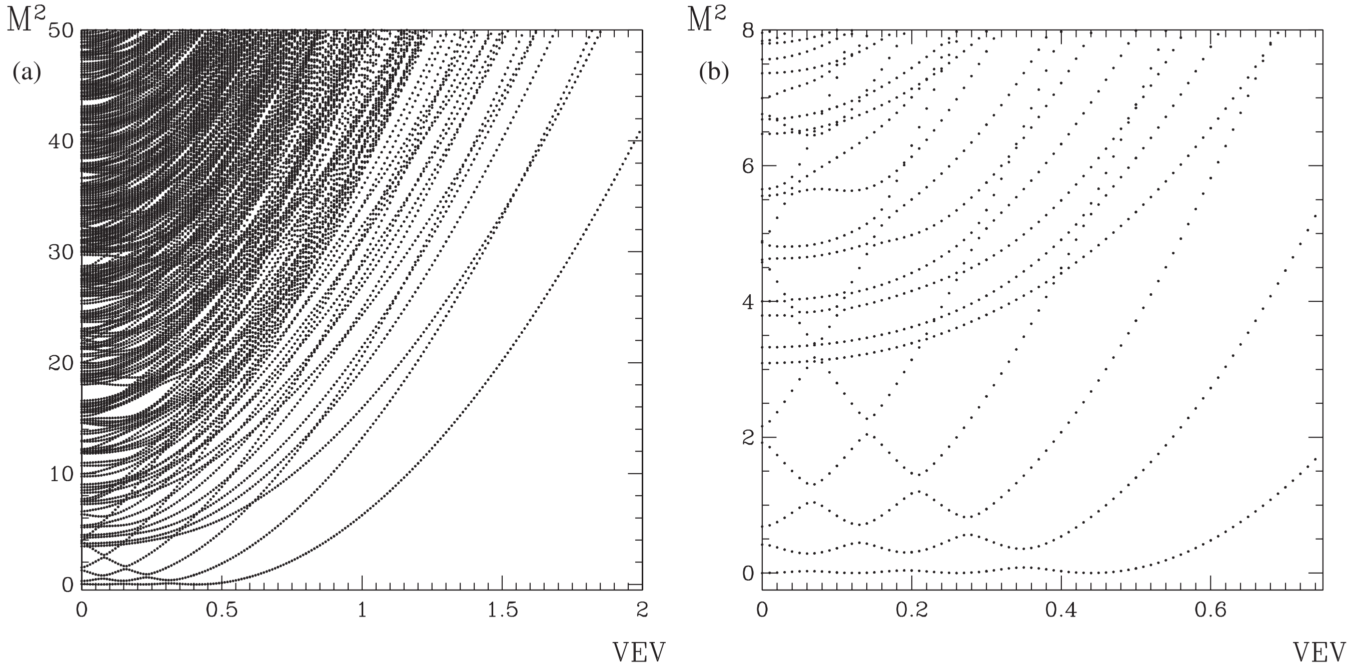

FIG. 1. The spectrum as a function of the VEV at $\kappa = 0$ in the bosonic $\mathcal{PO}$-even sector: (a) overview at $K = 7$ (left); and (b) detailed view of the lightest states at $K = 8$ (right). Masses (squared) are in units $g^2 N_c/\pi$.

ported by the behavior of average number of partons $\langle n \rangle$ of the lowest states. We saw in Sec. IV that $\langle n \rangle = 2$ for $v \to \infty$. Moreover, Fig. 2(a), displaying the average parton numbers of the ten lightest states at $K = 7$ as a function of the VEV, shows a trend towards short states already for intermediate values of v. It seems thus that at large K we could have a separation between (almost) massless states and heavy states. We come back to this issue in the next subsection.

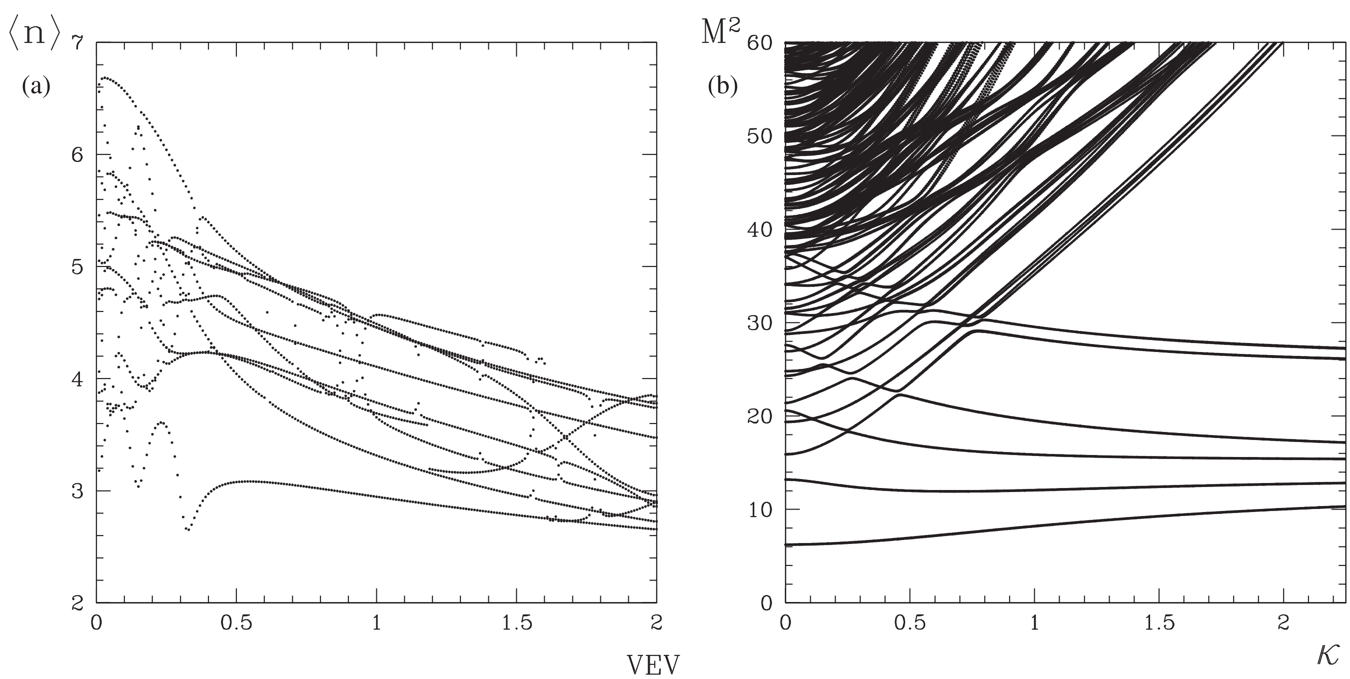

FIG. 2. Figure showing (a) the average parton number of the ten lightest states as a function of the VEV at $\kappa = 0$ and $K = 7$ (right), and (b) the bosonic spectrum with masses (squared) in units $g^2 N_c/\pi$ as a function of the Chern-Simons coupling κ in units g at $v = 1$ and $K = 7$.

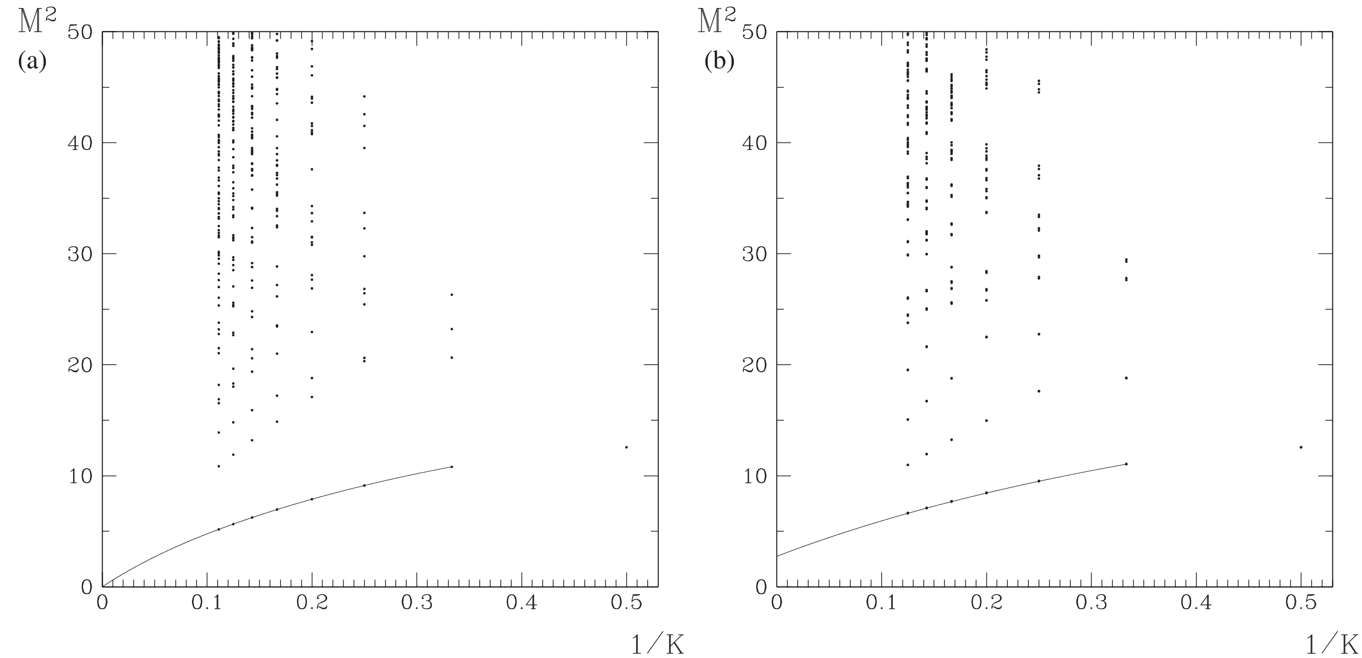

FIG. 3. Figure showing (a) the bosonic spectrum as a function of the inverse harmonic resolution: at $v = 1$ (left); and (b) the spectrum at $\kappa = g/\sqrt{\pi}$, $v = 1$ as a function of the inverse resolution (right). Masses (squared) are in units $g^2 N_c/\pi$.

The effect of adding a Chern-Simons term on the spectrum is as anticipated, see Fig. 2(b). The adjoints become massive, and the only states remaining light as κ grows are the $4(K-1)$ two-parton states without adjoints. Note that each "line" $M_n^2(\kappa)$ in the plot is actually a double line of two almost degenerate mass eigenvalues. The reason for the approximate degeneracy is that the states (at least the light ones) are largely devoid of partons subject to the symmetry-breaking CS term. Remarkably, the symmetry-breaking is very small for all states. The main effect of the CS term is to lift the masses of all bound states. As κ grows, the spectrum at fixed resolution K separates into light and very heavy states.

B. The continuum limit

While we are able to present evidence that an infinite set of massless states exists for all values of the VEV, we can demonstrate decisively that a massless state exists at a specific VEV. We do this by plotting the masses as a function of the inverse harmonic resolution $1/K$ at fixed VEV, say $v = 1$, in Fig. 3(a) and extrapolating to the continuum limit. A fit of seven data points to a polynomial of fifth degree, included in Fig. 3(a), yields a continuum mass (squared) of $M^2_{\text{lowest},v=1,\kappa=0}(K \to \infty) = 0.0015 \pm 0.2257$ in units $g^2 N_c/\pi$. We attributed a systematic error to this value by performing a fit to a polynomial of fourth and sixth degree, respectively, and taking the larger difference between these extrapolations and the value above as the uncertainty. The continuum mass is thus consistent with zero.

On the other hand, there is no massless state in the continuum limit if a Chern-Simons term is present. Plotting the spectrum at $v = 1$ and $\kappa = g/\sqrt{\pi}$ as a function of $1/K$ in Fig. 3(b), we see that a fit to a polynomial of fourth degree in $1/K$ to the masses of the lightest states (six data points) suggests that no massless states exist when $\kappa \neq 0$, since $M^2_{\text{lowest},v=1,\kappa=1}(K \to \infty) = 2.74 \pm 0.30$ in units $g^2 N_c/\pi$, where we estimated the systematic error as described above.

Finally, we need to settle the question whether a prominent feature of the spectra as a function of the physical parameters, namely, the gap between light and heavy states, persists as the unphysical parameter K is removed in the continuum limit. Figures 4(a) and 4(b) reveal that the gap survives the limit if the Chern-Simons coupling is substantial, whereas in its absence, the gap seems to collapse, at least at small v. Both fitting functions are polynomials of third degree. The estimated errors are large enough at $\kappa = 0$ to prevent us from concluding that the heavier states remain massive. We were unable to perform the analysis at substantially different parameters because we could not label states unambiguously, i.e. decide whether they belong to the light or the heavy states.

C. Structure functions

It is interesting to look at the wave functions of the bound states. In the full theory (massive fundamental and

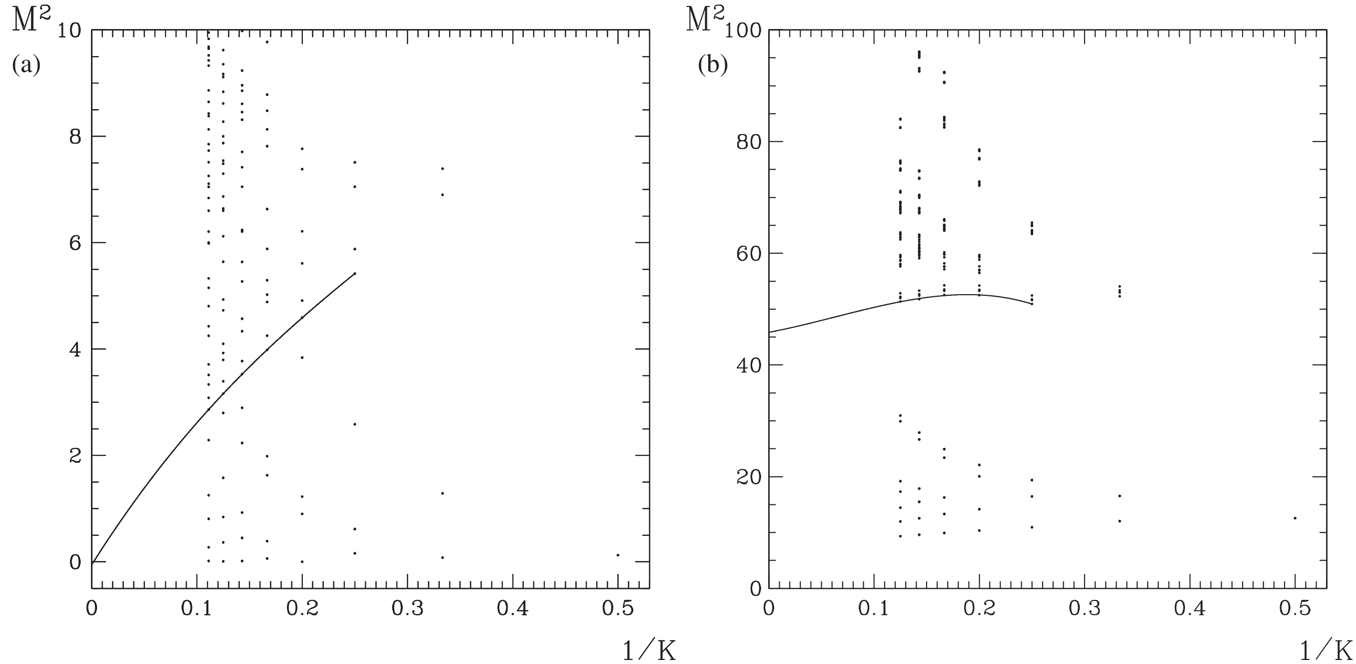

FIG. 4. The spectrum as a function of the inverse resolution: (a) at $\upsilon = 0.1$, $\kappa = 0$; and (b) at $\upsilon = 1$, $\kappa = 3g/\sqrt{\pi}$. Masses (squared) are in units g^2N_c/π.

adjoint partons) the amount of information encoded in them is too large to be useful. Instead, we display the expectation values of various particle number operators in the lightest two states in Table II as a function of K. It is obvious that symmetry breaking is exceedingly small and that it does not grow significantly with K. The two lightest states have almost identical properties. The fact that the average parton number is creeping up as K grows is caused by the increase of the number of adjoint bosons in the states. Both average fundamental boson and fermion number are roughly one, implying that the state consists largely of two-parton basis states, roughly half of which have two bosons, the other half having two fermions. Some of the latter have the effect of lowering the bound-state mass, producing the light mass we observe.

TABLE II. Properties of the two lightest bosonic bound states at $\upsilon = 1$ and $\kappa = g/\sqrt{\pi}$. Listed are the average numbers of adjoint bosons (aB), fundamental bosons (fB), adjoint (aF), and fundamental fermions (fF).

K	M^2	$\langle n \rangle$	$\langle n_{\mathrm{aB}} \rangle$	$\langle n_{\mathrm{fB}} \rangle$	$\langle n_{\mathrm{aF}} \rangle$	$\langle n_{\mathrm{fF}} \rangle$
3	11.0413	2.2630	0.2260	1.0045	0.0370	0.9955
	11.0686	2.2627	0.2486	0.9862	0.0140	1.0138
4	9.5070	2.4420	0.3803	1.0118	0.0617	0.9882
	9.5413	2.4429	0.4105	0.9709	0.0324	1.0291
5	8.4447	2.5676	0.4884	1.0340	0.0792	0.9660
	8.4814	2.5699	0.5262	0.9439	0.0436	1.0561
6	7.6705	2.6595	0.5669	1.0506	0.0926	0.9494
	7.7075	2.6626	0.6097	0.9242	0.0529	1.0758
7	7.0823	2.7294	0.6258	1.0646	0.1036	0.9354
	7.1188	2.7330	0.6721	0.9081	0.0609	1.0919
8	6.6194	2.7844	0.6715	1.0771	0.1129	0.9230
	6.6550	2.7882	0.7203	0.8942	0.0680	1.1058

We define the discrete version of the customary structure functions at harmonic resolution K as

$$g_A(n) = \sum_{q=2}^{K} \sum_{n_1,\ldots,n_q=1}^{K-q} \delta\left(\sum_{i=1}^{q} n_i - K\right) \times \sum_{l=1}^{q} \delta_n^{n_l} \delta_{A_l}^{A} |\psi(n_1,\ldots,n_q)|^2.$$

They are normalized such that the summation over the argument (n) yields the average number of type A partons in a state. The possible types are adjoint bosons (aB), adjoint fermions (aF), fundamental bosons (fB), and fundamental fermions (fF). What do we expect? From Sec. IV we know that the lightest states are short, and to minimize mass, their fundamental partons should gobble up as much of the total momentum as possible while splitting it evenly. Therefore, we anticipate the structure functions to be peaked around longitudinal momentum fraction $x = 0.5$ for fundamentals, and around $x = 1/K$ for adjoints. Furthermore, odds are that there is only one adjoint parton, and we have a preference for states with two fundamental fermions since they can lower the bound-state mass. Hence, in the bosonic sector, we should find more adjoint bosons than fermions in the light bound states. We see from Fig. 5 that our expectations are largely met for the lightest

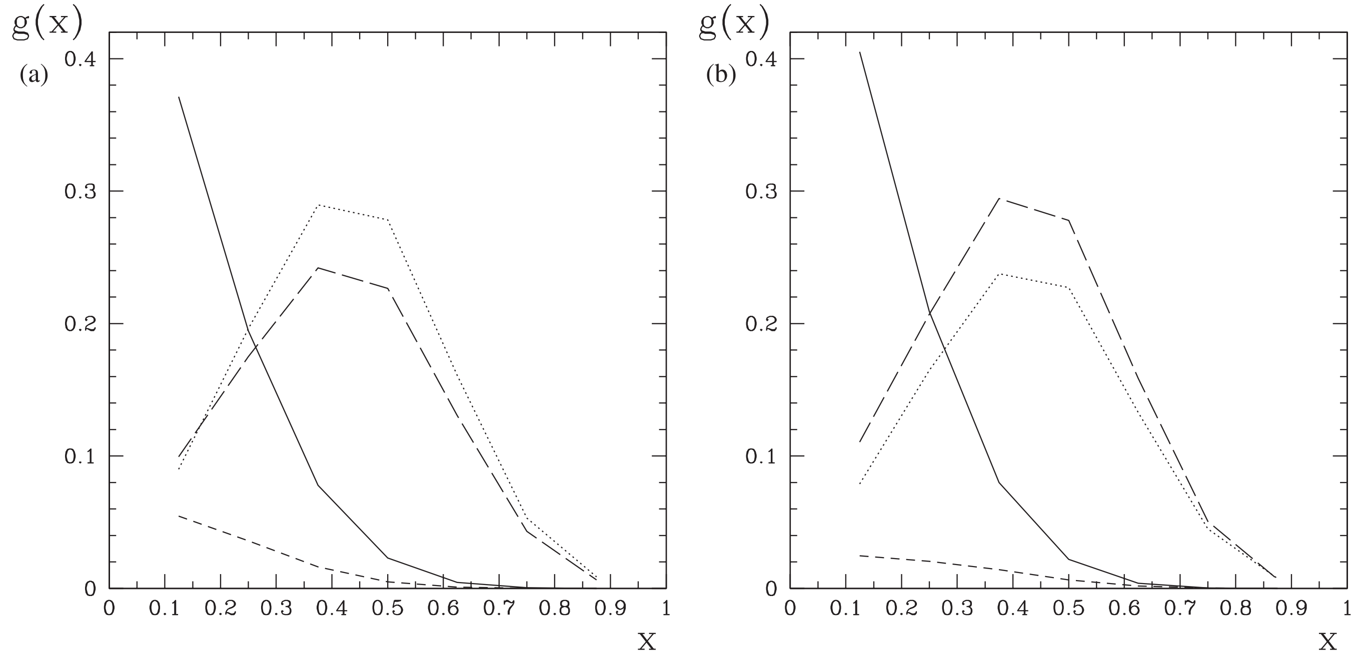

FIG. 5. Structure functions of the lowest two bosonic states as a function of the longitudinal momentum fraction $x = n/K$ at $v = 1$, $\kappa = g/\sqrt{\pi}$, and $K = 8$: (a) lowest state, and (b) second lowest state. Solid lines: g_{aB}. Long-dashed lines: g_{fF}. Short-dashed lines: g_{aF}. Dotted lines: g_{fB}.

two states. Obviously, the two lightest states are very similar in mass due to the small breaking of the $\mathcal{PO}$ symmetry, and they also have very similar eigenfunctions as evident from their structure functions. Apart from the flip in importance of g_{fF} with g_{fB}, there is a minor reduction of g_{aF} towards smaller x in the heavier state. The lighter state has a slightly smaller number of fundamental fermions which is somewhat surprising. Since only some of the two-fundamental-fermion basis states can lower the bound-state mass, this is not in conflict with our previous conclusions, however.

VI. DISCUSSION

We generated mass terms for the fundamental fields in a two-dimensional SYM theory by adding a vacuum expectation value to the perpendicular gauge boson left over from a dimensional reduction of the associated three-dimensional theory. Supersymmetry stays intact this way. In earlier work we had fabricated an analogous mass term for the adjoints of the theory by adding a Chern-Simons term. As a function of the new parameter $v := \langle A^{\perp} \rangle$, the majority of the states will see their masses increase with v. The lightest state, however, experiences masslessness for several values of v. We presented evidence for our hypothesis that in the continuum limit an infinite number of massless states will be present in this theory for a large range of v values. As the main difference between the theories with and without a Chern-Simons term, we showed that in the latter case the lightest state is massless, whereas no massless state exists if κ is substantial. Although we did not push for extreme numerical precision, we feel that we have a safe handle on the continuum limit, meaning that the properties of individual states show little variation with the harmonic resolution.

In the sense that we were able to add mass terms to all species of this generic supersymmetric gauge theory, and to study their effects on the spectrum, this work concludes our exploration of two-dimensional SYM. It would be interesting to extend our investigations to higher dimensions, but the computational effort likely will be prohibitively high if we want to strive for the same quality in terms of convergence control.

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APPENDIX: THE SDLCQ SUPERCHARGE IN MODE DECOMPOSITION

For completeness we list the four parts of the supercharge

$$Q^- = Q^-_{\rm SYM} + Q^-_{\rm fund} + Q^-_{\rm CS} + Q^-_{\rm XS}.$$

Note that the supercharge is a Hermitian operator. Its parts are

$$Q^-_{\rm SYM} = -\frac{ig\sqrt{L}}{2^{1/4}\pi} \sum_{n_1,n_2,n_3=1}^{\infty} \delta_{n_1+n_2,n_3}\bigg\{+\frac{1}{2\sqrt{n_1n_2}}\frac{n_1-n_2}{n_3}[A^\dagger_{ac}(n_1)A^\dagger_{cb}(n_2)B_{ab}(n_3) - B^\dagger_{ab}(n_3)A_{ac}(n_1)A_{cb}(n_2)]$$
$$+\frac{1}{2\sqrt{n_1n_3}}\frac{n_1+n_3}{n_2}[A^\dagger_{ab}(n_3)A_{ac}(n_1)B_{cb}(n_2) - A^\dagger_{ac}(n_1)B^\dagger_{ca}(n_2)A_{ab}(n_3)]$$
$$+\frac{1}{2\sqrt{n_2n_3}}\frac{n_2+n_3}{n_1}[B^\dagger_{ac}(n_1)A^\dagger_{cb}(n_2)A_{ab}(n_3) - A^\dagger_{ab}(n_3)B_{ac}(n_1)A_{cb}(n_2)]$$
$$+\left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}\right)[B^\dagger_{ac}(n_1)B^\dagger_{cb}(n_2)B_{ab}(n_3) + B^\dagger_{ab}(n_3)B_{ac}(n_1)B_{cb}(n_2)]\bigg\}. \tag{A1}$$

$$Q^-_{\rm fund} = -\frac{ig\sqrt{L}}{2^{1/4}\pi} \sum_{n_1,n_2,n_3=1}^{\infty}\bigg\{\frac{(n_2+n_3)}{n_1\sqrt{2n_2n_3}}(\tilde{C}^\dagger_i(n_3)\tilde{C}_j(n_2)B_{ji}(n_1) - \tilde{C}^\dagger_a(n_2)B^\dagger_{ab}(n_1)\tilde{C}_b(n_3) + B^\dagger_{ba}(n_1)C^\dagger_a(n_2)C_b(n_3)$$
$$- C^\dagger_a(n_3)B_{ab}(n_1)C_b(n_2)) + \frac{1}{n_1}(\tilde{D}^\dagger_a(n_2)B^\dagger_{ab}(n_1)\tilde{D}_b(n_3) + \tilde{D}^\dagger_a(n_3)\tilde{D}_b(n_2)B_{ba}(n_1) + B^\dagger_{ab}(n_1)D^\dagger_b(n_2)D_a(n_3)$$
$$+ D^\dagger_a(n_3)B_{ab}(n_1)D_b(n_2)) - \frac{i}{2\sqrt{n_2n_3}}(C^\dagger_a(n_3)A_{ab}(n_2)D_b(n_1) + A^\dagger_{ab}(n_2)D^\dagger_b(n_1)C_a(n_3) + \tilde{D}^\dagger_b(n_1)A^\dagger_{ba}(n_2)\tilde{C}_a(n_3)$$
$$+ \tilde{C}^\dagger_a(n_3)\tilde{D}_b(n_1)A_{ba}(n_2)) - \frac{i}{2\sqrt{n_1n_2}}(A^\dagger_{ba}(n_2)C^\dagger_a(n_1)D_b(n_3) + D^\dagger_b(n_3)A_{ba}(n_2)C_a(n_1) + \tilde{D}^\dagger_b(n_3)\tilde{C}_a(n_1)A_{ab}(n_2)$$
$$+ \tilde{C}^\dagger_a(n_1)A^\dagger_{ab}(n_2)\tilde{D}_b(n_3))\bigg\}\delta_{n_3,n_1+n_2}. \tag{A2}$$

The Chern-Simons term is

$$Q^-_{\rm CS} = -\frac{ig\sqrt{L}}{2^{1/4}\pi}\left(i\sqrt{\pi}\frac{\hat{\kappa}}{g}\right)\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(A^\dagger_{ab}(n)B_{ab}(n) + B^\dagger_{ab}(n)A_{ab}(n)). \tag{A3}$$

Finally, the extra terms induced by shifting the gauge field by its VEV are

$$Q^-_{\rm XS} = -\frac{ig\sqrt{L}}{2^{1/4}\pi}(-i\sqrt{\pi}\hat{v})\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\times(C^\dagger_a(n)D_a(n) + \tilde{C}^\dagger_a(n)\tilde{D}_a(n) + D^\dagger_a(n)C_a(n) + \tilde{D}^\dagger_a(n)\tilde{C}_a(n)). \tag{A4}$$

The common factor $\frac{g\sqrt{L}}{2^{1/4}\pi}$ is dropped in numerical calculations to obtain dimensionless matrix elements. In the mass squared operator $\mathcal{M}^2 = 2P^+P^-$ the compactification length L cancels, due to $P^+ = K\pi/L$. Its eigenvalues carry units of g^2N_c/π, since N_c creeps in via the parton number changing interactions, and is absorbed in a rescaling of the VEV and Chern-Simons couplings in two-body operators: $\hat{v} = v\sqrt{N_c}$, $\hat{\kappa} = \kappa\sqrt{N_c}$.

[1] K. A. Olive, S. Rudaz, and M. A. Shifman, *Proceedings of the International Symposium Celebrating 30 Years Of Supersymmetry* (North-Holland, Amsterdam, 2001).

[2] S. P. Martin, arXiv:hep-ph/9709356.

[3] S. J. Brodsky, H. C. Pauli, and S. S. Pinsky, Phys. Rep. **301**, 299 (1998).

[4] O. Lunin and S. Pinsky, AIP Conf. Proc. **494**, 140 (1999).

[5] D. B. Kaplan, Eur. Phys. J. Special Topics **152**, 89 (2007).

[6] Y. Matsumura, N. Sakai, and T. Sakai, Phys. Rev. D **52**, 2446 (1995).

[7] J. R. Hiller, M. Harada, S. S. Pinsky, N. Salwen, and U. Trittmann, Phys. Rev. D **71**, 085008 (2005).

[8] O. Lunin and S. Pinsky, Phys. Rev. D **63**, 045019 (2001).

[9] J. R. Hiller, S. S. Pinsky, and U. Trittmann, Phys. Rev. D **67**, 115005 (2003); Nucl. Phys. **B661**, 99 (2003).

[10] J. R. Hiller, S. S. Pinsky, and U. Trittmann, Phys. Rev. Lett. **89**, 181602 (2002); Phys. Rev. D **65**, 085046 (2002).

[11] P. Haney, J. R. Hiller, O. Lunin, S. Pinsky, and U. Trittmann, Phys. Rev. D **62**, 075002 (2000).

[12] J. R. Hiller, S. S. Pinsky, and U. Trittmann, Phys. Rev. D **66**, 125015 (2002); Phys. Lett. B **541**, 396 (2002).

[13] R. C. Myers (private communication).see also H. Nastase, arXiv:hep-th/0305069.

[14] J. R. Hiller, S. Pinsky, Y. Proestos, N. Salwen, and U. Trittmann, Phys. Rev. D **76**, 045008 (2007).

[15] D. Kutasov, Nucl. Phys. **B414**, 33 (1994).