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Magnetic bion condensation: A new mechanism of confinement and mass gap in four dimensions

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In recent work, we derived the long-distance confining dynamics of certain QCD-like gauge theories formulated on small $S^1 \times \mathbb{R}^3$ based on symmetries, an index theorem, and Abelian duality. Here, we give the microscopic derivation. The solution reveals a new mechanism of confinement in QCD(adj) in the regime where we have control over both perturbative and nonperturbative aspects. In particular, consider $SU(2)$ QCD(adj) theory with $1 \leq n_f \leq 4$ Majorana fermions, a theory which undergoes gauge symmetry breaking at small S^1. If the magnetic charge of the Bogomol'nyi-Prasad-Sommerfield (BPS) monopole is normalized to unity, we show that confinement occurs due to condensation of objects with magnetic charge 2, not 1. Because of index theorems, we know that such an object cannot be a two identical monopole configuration. Its net topological charge must vanish, and hence it must be topologically indistinguishable from the perturbative vacuum. We construct such non-self-dual topological excitations, the magnetically charged, topologically null molecules of a BPS monopole and $\overline{\text{KK}}$ antimonopole, which we refer to as magnetic bions. An immediate puzzle with this proposal is the apparent Coulomb repulsion between the BPS-$\overline{\text{KK}}$ pair. An attraction which overcomes the Coulomb repulsion between the two is induced by $2n_f$-fermion exchange. Bion condensation is also the mechanism of confinement in $\mathcal{N} = 1$ SYM on the same four-manifold. The $SU(N)$ generalization hints a possible hidden integrability behind nonsupersymmetric QCD of affine Toda type, and allows us to analytically compute the mass gap in the gauge sector. We currently do not know the extension to $\mathbb{R}^4$.

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I. INTRODUCTION

Probably the most important experimental and phenomenological observation about $SU(3)$ QCD is confinement, i.e., the absence of the free colored particles in isolation. Numerical lattice simulations unambiguously establish confinement in pure Yang-Mills theory and QCD. However, to date the analytical success had been limited. For reviews, see [1–3].

The QCD of nature belongs to a subclass of asymptotically free and confining gauge theories without elementary scalars. This class is referred to as vectorlike or QCD-like. This is a sufficiently good reason to warrant the study of the dynamics of such four-dimensional QCD-like theories. In the last two decades, most theoretical efforts were concentrated into the dynamics of supersymmetric theories. It would be fair to say that despite many remarkable results obtained in such theories, their benefit to QCD-like theories is still in its infancy. There is a very good reason for this. On $\mathbb{R}^4$, there only exists one QCD-like supersymmetric theory, the pure $\mathcal{N} = 1$ SYM. All other supersymmetric theories have scalars, and are hence non-QCD-like by definition. In regimes where such theories are solved or understood quantitatively, such as mass deformation of $\mathcal{N} = 2$ SYM down to $\mathcal{N} = 1$ [4], the scalars never decouple from the dynamics. If they are forced to decouple by tuning certain parameters, one usually looses the theoretical control over the theory [5].

Our goal in this paper is more direct and motivated by the following question: Is there any asymptotically free and confining QCD-like theory in $d = 4$ dimensions (with no special properties such as supersymmetry) in which we can understand its nonperturbative aspects exactly, and can derive the long-distance (confining) dynamics starting with microscopic theory?[1]

On $\mathbb{R}^4$, the answer seems to be out of reach currently. However, on locally four-dimensional settings, such as spatial $S^1 \times \mathbb{R}^3$, the answer is yes. In particular, QCD with multiple adjoint representation fermions on small $S^1 \times \mathbb{R}^3$, ($S^1 \times \mathbb{R}^{2,1}$ in Minkowski setting) [8] becomes analytically tractable. Here, it is important that S^1 is not a thermal circle. It is a spatial circle along which fermions are endowed with periodic spin connection, and the resulting QCD-like theory is a zero temperature field theory on a

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[1]Two archetypes of non-QCD-like theory in which the long-distance theory can be derived starting with the microscopic theory are Polyakov's treatment [6] of the Georgi-Glashow model on $\mathbb{R}^3$, a theory which confines, and Nekrasov's derivation [7] of the $\mathcal{N} = 2$ Seiberg-Witten prepotential, a theory which does not confine. The $\mathcal{N} = 1$ mass deformation of the latter also confines. Our goal is to find such quantitatively tractable examples among QCD-like theories.

space with one compact dimension. The benefits of considering this setup are (i) weak coupling (due to asymptotic freedom) and (ii) unbroken spatial center symmetry. The latter is a consequence of the absence of thermal fluctuations and the fact that the quantum fluctuations favor the center symmetric vacuum.

A. A mechanism of confinement by non-self-dual topological excitations

Below, we will briefly outline the results of [8], and address fundamental questions regarding the microscopic origin of confinement and chiral symmetry realization in QCD(adj). Historically, (starting with the mid-1970s), confinement was thought to be related to *self-dual* topological excitations which are solutions to Prasad-Sommerfield–type first-order differential equations. For example, both the Polyakov model [6] and Seiberg-Witten theory are of this type [4]. In particular, the Seiberg-Witten solution may be viewed as a very elegant realization of the mid-1970's dream of 't Hooft, Polyakov, and Mandelstam. The picture of confinement (which appears in a semiclassical regime) of QCD(adj) does not directly fit to earlier ideas regarding the subject. As we will demonstrate below, it is sourced by *non-self-dual*, yet dynamically stable novel topological excitations, that we will refer to as *magnetic bions*. What makes these excitations more elusive than monopole-instantons, monopoles, or instantons is that, magnetic bions, in the sense of topological charge, are indistinguishable from perturbative vacuum. Thus, there was no reason to search for their existence. This is also what makes the current work different from earlier (above-mentioned) proposals of confinement in non-Abelian gauge theories.

Consider the setup of Ref. [8]. In the small S^1 (weak coupling) limit of $SU(2)$ QCD(adj), the holonomy of the spatial Wilson line along the S^1 direction $U(x) = Pe^{i\int dx_4 A_4(x,x_4)}$ may be regarded as a *compact* adjoint Higgs field. This field acquires a nontrivial (center symmetry respecting) vacuum expectation value, $U = \text{Diag}(e^{i\pi/2}, e^{-i\pi/2})$, due to radiatively induced one-loop Coleman-Weinberg potential. The photons and neutral fermions (A_μ, λ^I) parallel to U remains massless to *all* orders in perturbation theory, and all the other modes acquire masses and hence decouple from the infrared dynamics.

Nonperturbatively, there are topologically stable monopole configurations which are a consequence of gauge symmetry breaking. Since the adjoint Higgs field is compact, other than the Bogomol'nyi-Prasad-Sommerfield (BPS) monopole, there is also a KK monopole. The existence of KK monopoles, which are perhaps the most crucial ingredient in our discussion of QCD(adj), was discovered in 1997, independently by Lee and Yi using D-branes in string theory [9] and by Kraan and van Baal by using calorons configurations [10]. The magnetic and topological charges ($\int F$, $\int F\tilde{F}$) of these monopoles are normalized as

$$\begin{aligned} &\text{BPS}: \left(+1, +\frac{1}{2}\right), \qquad \overline{\text{BPS}}: \left(-1, -\frac{1}{2}\right) \\ &\text{KK}: \left(-1, +\frac{1}{2}\right), \qquad \overline{\text{KK}}: \left(+1, -\frac{1}{2}\right) \end{aligned} \tag{1.1}$$

where a bar denotes antimonopoles.

In [8], we constructed the $d = 3$ dimensional long-distance theory for QCD(adj) formulated on $\mathbb{R}^3 \times S^1$ by employing three tools: Abelian duality, symmetries, and index theorem. This strategy is, in essence, similar to the Seiberg-Witten construction of prepotential in $\mathcal{N} = 2$ SYM [4]. The unique Lagrangian to order e^{-2S_0} dictated by these considerations is

$$\begin{aligned} L^{\text{dQCD}} = &\frac{1}{2}(\partial\sigma)^2 - be^{-2S_0}\cos 2\sigma + i\bar{\psi}^I\gamma_\mu\partial_\mu\psi_I \\ &+ ce^{-S_0}\cos\sigma(\det_{I,J}\psi^I\psi^J + \text{c.c.}) \end{aligned} \tag{1.2}$$

where σ and ψ^I denote the dual photon and fermion. Dimensionless coordinates, measured in units of compactification circumference L, are used. A detailed microscopic derivation of this Lagrangian will be given in Sec. II. The mass gap for gauge bosons is manifest in this Lagrangian. The inverse of the mass gap is the characteristic size of the chromoelectric flux tube, hence confinement is also manifest in dual formulation [8].

B. Microscopic derivation

In this work, we will derive the dual Lagrangian (1.2) by summing over all nonperturbative effects. Before doing so, note a simple but important feature of (1.2). It is clear that fermionic interaction terms arise due to the monopole effects. Any monopole carries a net topological charge. If massless fermions are present in the underlying theory, due to the index theorem, a monopole must be associated with $2n_f$ fermion zero-modes of one chirality and an antimonopole leads to $2n_f$ zero-modes of the opposite chirality. Consequently, the terms involving fermion zero-mode insertions are the sum of the monopole operators:

$$\begin{aligned} &\text{BPS}: e^{i\sigma}\det_{I,J}\psi^I\psi^J, \qquad \text{KK}: e^{-i\sigma}\det_{I,J}\psi^I\psi^J, \\ &\overline{\text{BPS}}: e^{-i\sigma}\det_{I,J}\bar{\psi}^I\bar{\psi}^J, \qquad \overline{\text{KK}}: e^{i\sigma}\det_{I,J}\bar{\psi}^I\bar{\psi}^J, \end{aligned} \tag{1.3}$$

where $e^{iq_m\sigma}$ is the pure monopole operator and $q_m = \pm 1$ are magnetic charges of the corresponding (anti)monopole, and $\det_{I,J}\psi^I\psi^J$ are compulsory zero-modes attached to it. Now, let us inspect the bosonic potential. It is

$$V(\sigma) \sim \cos 2\sigma \sim e^{i2\sigma} + e^{-2i\sigma}. \tag{1.4}$$

Because of the index theorem, a bosonic potential cannot arise due to objects which carry a nonvanishing index. Such objects, by construction, must have fermion zero-mode insertions, and cannot appear in the bosonic potential. It is easy to check that the functional integral $Z = \int D\sigma e^{-\int d^3x[1/2(\partial\sigma)^2 - be^{-2S_0}\cos 2\sigma]}$ is equivalent to a plasma of magnetically charged particles with long-range Coulomb interaction,

$$V(r) = \frac{2(\pm 2)}{4\pi r}, \tag{1.5}$$

where charges are twice the one of the monopoles. In other words, the Debye phenomena (which renders the dual photon massive) is induced not due to excitations with magnetic and topological charge $(\pm 1, \pm \frac{1}{2})$, but rather with charges $(\pm 2, 0)$. Clearly, these are not elementary monopoles. The first question we want to answer is, what are these objects?

A fuller discussion of all pairs and their roles will be given in Sec. II B. For now, let us observe that only a bound state of BPS monopole, and KK antimonopole, BPS$\overline{\text{KK}}$, and its antiparticle can induce the bosonic potential. Such an object has the correct quantum numbers $(1, \frac{1}{2}) + (1, -\frac{1}{2}) = (2, 0)$ and is the prime candidate for the magnetically charged object which leads to confinement in QCD(adj) in the $L\Lambda \ll 1$ regime.

There is an immediate puzzle with this proposal. The BPS and $\overline{\text{KK}}$ monopoles interact electromagnetically via Coulomb repulsion, hence in order to have a bound state, there must exist an attraction which may overcome the Coulomb repulsion.[2] In the QCD(adj) vacuum, a pairing mechanism arguably as strange as the BCS theory [11] takes place. An attraction which overwhelms the Coulomb repulsion between BPS and $\overline{\text{KK}}$ is generated via (an even number of) fermion exchange. In $n_f = 1$ QCD(adj) (i.e., SYM), this is a fermion pair exchange. In $n_f > 1$ QCD(adj), it is the exchange of $2n_f$ fermions. The attractive potential is a logarithmical one

$$V_{\text{eff}}(r) = 4n_f \log r + \frac{1}{4\pi r}, \qquad r \gg 1 \tag{1.6}$$

and it easily overcomes the repulsive Coulomb force. This forces the BPS and $\overline{\text{KK}}$ monopoles to form a charged bound state. We refer to this molecule as a *magnetic bion*, and to the $\overline{\text{BPS}}$-KK molecule as *antibion*. The important point that is worth repeating is that the net topological charge of the BPS-$\overline{\text{KK}}$ pair is identically zero: $\int_{\mathbb{R}^3\times S^1} F\tilde{F} = 0$, even though for individual (isolated) BPS it is $\int_{\mathbb{R}^3\times S^1} F\tilde{F} = \frac{1}{2}$, and for $\overline{\text{KK}}$ it is $\int_{\mathbb{R}^3\times S^1} F\tilde{F} = -\frac{1}{2}$. Consequently, bions do not have fermion zero-modes attached to them, and they are the leading contribution to the effective bosonic potential for the dual photon.

Considerations along these lines also provide dynamical explanations for the absence of confinement in a Yang-Mills *noncompact* Higgs system with adjoint Dirac fermions formulated on $\mathbb{R}^3$. Affleck, Harvey, and Witten in Ref. [12] showed that such systems do not confine despite the presence of magnetic monopoles. Their argument is based on symmetries and index theorems. Without much recourse to the microscopic theory, they showed that the photon arises as a Goldstone boson of spontaneously broken fermion number symmetry, hence remains massless nonperturbatively. Here, we give a microscopic derivation of this beautiful symmetry argument based on the dynamics of monopoles (and bions). In one sentence, the absence of magnetically charged, but topologically null configurations (which may be the only source of a mass gap for a dual photon in the presence of fermions) implies the absence of confinement in the $SU(2)$ application. We also provide a dynamical explanation for the absence of confinement in the $\mathcal{N} = 2$ SYM theory on $\mathbb{R}^3$ based on a similar rationale.[3]

The discussion of nonsupersymmetric QCD(adj) can also be applied to $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times S^1$ with only cosmetic changes. All one needs to be careful about is the extra massless scalar, and keep it in the effective theory. In fact, the long-distance effective theory for SYM (which is a supersymmetric affine Toda theory) was derived far before our work on the subject [13–16].[4] In spite of that, the fact that confinement was induced not due to (self-dual) monopoles, but rather via (non-self-dual) magnetic bions was not understood earlier. Remarkably, the mechanism of confinement for $\mathcal{N} = 1$ SYM and nonsupersymmetric QCD(adj) is one and the same in the small S^1 regime.

The second part of the paper discusses the $SU(N)$ generalization of the nonsupersymmetric QCD(adj), and derives the long-distance Lagrangian. The biggest surprise is that the bosonic sector of QCD(adj) maps into an integrable system, intimately related to possible integrable generalization of the affine Toda theories. We identify magnetic bions as bound states of magnetic monopoles with charge α_j and antimonopoles with charge $-\alpha_{j+1}$. The net effect of bions can be encoded into a *prepotential*,

[2]This situation is analogous to the BCS theory of superconductivity. There must exist a net attraction between electron pairs which overcomes the shielded, yet repulsive Coulomb potential. Such an attractive force is provided through the exchange of phonons of the crystal lattice. A novel pairing mechanism is at work in QCD(adj) formulated on small $S^1 \times \mathbb{R}^3$. Aswill be seen explicitly, the pairing in QCD(adj) is a real-space phenomena, unlike the BCS theory.

[3]These theories (formulated on $\mathbb{R}^3$) are as important as QCD(adj) on $\mathbb{R}^3 \times S^1$. They exhibit that if massless fermions are present, having monopoles is not sufficient to have confinement.

[4]Our derivation of the bosonic potentials in SYM differs from earlier work, which was based on using supersymmetry as a completion device to obtain superpotential (hence bosonic potential) from the monopole inducedfermionic terms. We instead chose to delineate on the microscopic origin of the bosonic potential, and obtained it directly without any recourse to supersymmetry. The final result is the same of earlier work [13–16]. The real payoff of our approach is in its applicability to nonsupersymmetric theories.

out of which we may derive the potential. Interestingly enough, the relation between the prepotential and potential is the same as the relation between the superpotential and potential in $\mathcal{N} = 1$ SYM, modulo the absence of the Higgs scalar in the former (where it is massive). We give the analytic derivations of characteristic sizes of chromo-electric flux tubes in QCD(adj) in the small S^1 regime.

Let us complete the introduction by saying that closer and deeper inspection of nonsupersymmetric QCD-like theories may also be used to build the relation between the inner goings-on of the supersymmetric and nonsupersymmetric gauge theories. Suffice it to say that the integrable systems which emerge in the QCD(adj) are variants of the affine Toda systems [17–19], which also appeared in the discussions of $\mathcal{N} = 1$, 2 SYM, and elliptic curves [20]. This direction will not be explored in this paper, but is potentially interesting.[5]

II. DYNAMICS OF $SU(2)$ QCD(ADJ) ON SMALL $S^1 \times \mathbb{R}^3$

A. Perturbation theory

First, we wish to give the microscopic derivation of the dual theory (1.2). The action of $SU(N)$ QCD(adj) defined on $\mathbb{R}^3 \times S^1$ is

$$S = \int_{\mathbb{R}^3 \times S^1} \frac{1}{g^2} \mathrm{tr}\left[\frac{1}{4} F_{MN}^2 + i\bar{\lambda}^I \bar{\sigma}^M D_M \lambda_I\right] \tag{2.1}$$

where $\lambda_I = \lambda_{I,a} t_a$, $a = 1, \ldots, N^2 - 1$ is the Weyl fermion in adjoint representation, F_{MN} is the non-Abelian gauge field strength, and I is the flavor index, and the generators are normalized as $\mathrm{tr} t^a t^b = \delta^{ab}$. The classical theory possesses a $U(n_f)$ flavor symmetry whose $U(1)_A$ part is anomalous. The symmetry of the quantum theory is

$$(SU(n_f) \times \mathbb{Z}_{2Nn_f})/\mathbb{Z}_{n_f}. \tag{2.2}$$

The quantum theory has the dynamical strong scale Λ, which arises via dimensional transmutation, and is given by $\Lambda^{b_0} = \mu^{b_0} e^{-8\pi^2/g^2(\mu)N}$ where μ is the renormalization group scale and $b_0 = (11 - 2n_f)/3$. We consider small n_f so that asymptotic freedom is preserved. The $n_f = 1$ case (SYM) will be discussed separately. We first discuss $N = 2$ QCD(adj), and $N \geq 3$ will be discussed in Sec. III.

At small S^1 ($L\Lambda \ll 1$), due to asymptotic freedom, the gauge coupling is small, and a perturbative Coleman-Weinberg analysis is reliable [28]. Let $U(x) = Pe^{i\int dx_4 A_4(x,x_4)}$ be the path-ordered holonomy of the spatial Wilson line wrapping the S^1, and sitting at the point $x \in \mathbb{R}^3$. Integrating out the heavy KK-modes along the S^1 circle, $|\omega_n| \geq \omega_1$ where $\omega_n = \frac{2\pi}{L} n$, $n \in \mathbb{Z}$, induces a nontrivial effective potential for $U(x)$ [29].

$$\begin{aligned} V^+[U] &= (-1 + n_f)\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} |\mathrm{tr} U^n|^2, \\ U(x) &= Pe^{i\int dx_4 A_4(x,x_4)} \equiv e^{iL\Phi}. \end{aligned} \tag{2.3}$$

Note that the stability of the center symmetry is induced by massless adjoint fermions with periodic boundary conditions. In this sense, this theory *does not* require the double-trace deformations to achieve phases of unbroken center symmetry [30–32]. The action for the classical zero-modes reduces to

$$\begin{aligned} S = \int_{\mathbb{R}^3} \frac{L}{g^2} \mathrm{tr}\Big[&\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(D_\mu \Phi)^2 + g^2 V(|\Phi|) \\ &+ i\bar{\lambda}^I(\bar{\sigma}^\mu D_\mu + \bar{\sigma}_4[\Phi, \,])\lambda_I\Big]. \end{aligned} \tag{2.4}$$

The minimum of the potential V_{eff} is located at $L|\Phi| = \frac{\pi}{2}$, hence

$$U = \begin{pmatrix} e^{i\pi/2} & \\ & e^{-i\pi/2} \end{pmatrix} \quad \text{or} \quad L\Phi = \begin{pmatrix} \pi/2 & \\ & -\pi/2 \end{pmatrix}. \tag{2.5}$$

Since $\mathrm{tr} U = 0$, the $\mathbb{Z}_2$ center symmetry is preserved. By the Higgs mechanism, the gauge symmetry is broken down as

$$SU(2) \longrightarrow U(1). \tag{2.6}$$

Because of adjoint Higgs mechanism, the neutral fields aligned with U along the Cartan subalgebra $(A_{3,\mu}, \lambda_3^I)$ remain massless, and off-diagonal components acquire mass, given by the separation between the eigenvalues of the Wilson line

$$m_{W^\pm} = m_{\lambda^{I,\pm}} = \pi/L \tag{2.7}$$

where $\pm$ refers to the charges under unbroken $U(1)$. Therefore, in perturbation theory, the low-energy theory is a $d = 3$ dimensional Abelian $U(1)$ gauge theory with n_f massless fermions with a free action

$$S = \int_{\mathbb{R}^3} \frac{L}{g^2}\left[\frac{1}{4} F_{3,\mu\nu}^2 + i\bar{\lambda}_3^I \bar{\sigma}^\mu \partial_\mu \lambda_{3,I}\right]. \tag{2.8}$$

At distances shorter than L, the coupling constant flows according to the four-dimensional renormalization group.

[5] There are also recent, interesting works on the dynamics of four-dimensional gauge theories, in particular, for pure Yang-Mills, see [21–23], and for lattice works, see [1,24] and references therein. Also, good reviews covering different aspects of monopoles and instantons can be found in [25–27].

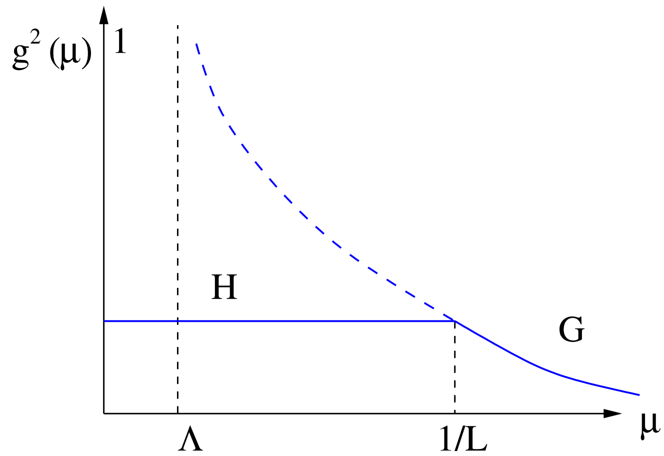

FIG. 1 (color online). Summary of perturbative analysis: Solid line indicates the running of the gauge coupling in QCD(adj) compactified on a small circle S^1 with circumference L, and dashed line is the usual running on $\mathbb{R}^4$. In the regime $1/L \gg \Lambda$ perturbative Coleman-Weinberg analysis is reliable, and leads to a radiatively induced gauge symmetry breaking $G \to H$ where $G = SU(2)$ and $H = U(1)$. To all orders in perturbation theory, the long-distance theory described by H is free due to absence of charged massless excitations. This is reminiscent of the $\mathcal{N} = 2$ SYM theory on $\mathbb{R}^4$, for which gauge symmetry breaking takes place on the semiclassical domain of the moduli space.

Since the heavy $W^\pm$, $\lambda^{I,\pm}$ which are charged under $U(1)$ decouple from the long-distance physics at scale L and above, the coupling constant ceases to run at $1/L \gg \Lambda$ much before the strong coupling sets in, see Fig. 1. In perturbation theory, this is the whole story.

B. Nonperturbative effects and Abelian duality

Nonperturbatively, the perturbatively free infrared fixed point is unstable. This follows from the existence of monopoles (strictly speaking, these are monopole-instantons or fractional instantons), at the cores of which the $U(1)$ symmetry of the free theory enhances to the whole non-Abelian $SU(2)$.

Because of gauge symmetry breaking via a *compact* adjoint Higgs field, there are two types of monopoles, BPS and KK, as well as their antimonopoles $\overline{\text{BPS}}$, $\overline{\text{KK}}$ [9,10,33,34].[6] These four types of monopoles are distinguished by their quantized magnetic and topological charges ($\int F$, $\int F\tilde{F}$) normalized as

$$\text{BPS}: \left(+1, \frac{1}{2}\right), \qquad \overline{\text{BPS}}: \left(-1, -\frac{1}{2}\right),$$
$$\text{KK}: \left(-1, \frac{1}{2}\right), \qquad \overline{\text{KK}}: \left(+1, -\frac{1}{2}\right). \tag{2.9}$$

[6]Were the gauge symmetry broken by a noncompact Higgs field, the KK monopole would not be there. As we will discuss, this is the case in the extension of the Polyakov model in the presence of adjoint fermions, a theory which does not confine.

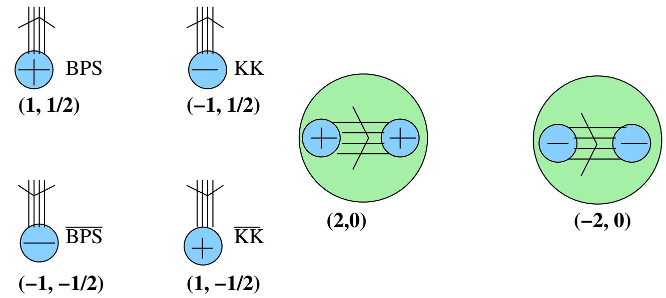

FIG. 2 (color online). (Left) Magnetically and topologically charged monopoles carry compulsory fermion zero-modes. Consequently, they cannot induce a bosonic potential for the dual photon. (Right) Topologically null, magnetically charged bions have no external fermionic legs. Hence, they induce the leading bosonic potential, which implies mass for the dual photon and confinement. The figure is for $SU(2)$ with $n_f = 2$. The combination of the BPS-KK monopoles (which is not depicted) is an instanton (or caloron). It is present in confined phase, but is not the source of the dual photon mass term.

Because of the chiral anomaly relation [35],

$$\partial_M J^{M5} = \frac{g^2(2Nn_f)}{32\pi^2} \operatorname{tr} F_{MN}\tilde{F}^{MN}, \tag{2.10}$$

each object with a nonvanishing topological charge is associated with a certain number of fermionic zero-modes. Integrating both sides over the space, we find

$$\Delta Q_5 = n_\lambda - n_{\bar{\lambda}} = 4n_f \int \frac{g^2}{32\pi^2} \operatorname{tr} F_{MN}\tilde{F}^{MN}$$
$$= \begin{cases} 4n_f(\frac{1}{2}) = 2n_f & \text{for BPS} \quad \text{or} \quad \text{KK} \\ 4n_f(-\frac{1}{2}) = -2n_f & \text{for } \overline{\text{BPS}} \quad \text{or} \quad \overline{\text{KK}} \end{cases} \tag{2.11}$$

where the term inside the parenthesis is the topological charge. As it should be clear, $4n_f$ is the number of fermionic zero-modes associated with a four-dimensional instanton, whose topological charge is $+1$. Since the topological charges of monopoles are a fraction of the one of the instanton, they are sometimes referred as fractional instantons. Clearly, a BPS-KK pair has the correct quantum numbers to be the constituents of the instanton [33,34].

By Abelian duality [6,36], we know that the functional integral in a gauge theory in the presence of a single monopole with charge ± 1 located at the position x is equivalent to the insertion of an operator $e^{\pm i\sigma(x)}$ in the path integral of the dual theory. However, the index version of the chiral anomaly relations (2.11) tells us that a monopole acts as it contains a source for every fermion flavor, and an antimonopole acts as if it contains a sink for all fermion flavors. Adapting a combination of techniques developed by 't Hooft [37] and by Polyakov [6] to our problem, we can sum up all the monopole effects. The functional integral (with a source) in the presence of a monopole

$$\int DA_\mu D\psi^I D\bar{\psi}^I e^{-S_{\text{one mon.}}(A,\psi,\bar{\psi})+J\det\psi^I\psi^J+\bar{J}\det\bar{\psi}^I\bar{\psi}^J} \tag{2.12}$$

is the same as having

$$e^{-S_0}\int D\sigma D\psi^I D\bar{\psi}^I e^{-S_{d,0}(\sigma,\psi,\bar{\psi})+J\det\psi^I\psi^J+\bar{J}\det\bar{\psi}^I\bar{\psi}^J} e^{i\sigma(x)} \times \det_{I,J}\psi^I\psi^J \tag{2.13}$$

where $S_{d,0}(\sigma, \psi, \bar{\psi}) = \int_{\mathbb{R}^3}[\frac{1}{2}(\partial\sigma)^2 + i\bar{\psi}^I\gamma_\mu\partial_\mu\psi_I]$ is the free kinetic term. Hence, a functional integral in the presence of a monopole can be translated into having a monopole vertex $e^{i\sigma(x)}$ with accompanying fermionic zero-modes. We can insert the monopole at any $x \in \mathbb{R}^3$, and we can consider an arbitrary number of them. The sum over all possible monopole configurations is

$$\begin{aligned}\sum_{n_{\text{BPS}}=0}^{\infty}\sum_{n_{\overline{\text{BPS}}}=0}^{\infty}\sum_{n_{\text{KK}}=0}^{\infty}\sum_{n_{\overline{\text{KK}}}=0}^{\infty}&\frac{e^{-(n_{\text{BPS}}+n_{\overline{\text{BPS}}}+n_{\text{KK}}+n_{\overline{\text{KK}}})S_0}}{n_{\text{BPS}}!n_{\overline{\text{BPS}}}!n_{\text{KK}}!n_{\overline{\text{KK}}}!}\\ &\times\left[\int d^3x e^{i\sigma(x)}\det_{I,J}\psi^I\psi^J\right]^{n_{\text{BPS}}}\\ &\times\left[\int d^3x e^{-i\sigma(x)}\det_{I,J}\bar{\psi}^I\bar{\psi}^J\right]^{n_{\overline{\text{BPS}}}}\\ &\times\left[\int d^3x e^{-i\sigma(x)}\det_{I,J}\psi^I\psi^J\right]^{n_{\text{KK}}}\\ &\times\left[\int d^3x e^{i\sigma(x)}\det_{I,J}\bar{\psi}^I\bar{\psi}^J\right]^{n_{\overline{\text{KK}}}}.\end{aligned} \tag{2.14}$$

Performing the summation yields monopole induced terms of order e^{-S_0} in our effective Lagrangian

$$\exp\left[\int d^3x e^{-S_0}(e^{i\sigma} + e^{-i\sigma})(\det_{I,J}\psi^I\psi^J + \det_{I,J}\bar{\psi}^I\bar{\psi}^J)\right]. \tag{2.15}$$

Therefore, the combined effect of BPS and KK monopoles is $\cos\sigma\det\psi^I\psi^J$. This vertex is manifestly invariant under continuous $SU(n_f)$ flavor symmetry, acting as $\psi \to U\psi$ where $U \in SU(n_f)$. The microscopic theory also possesses a $\mathbb{Z}_{4n_f}$ discrete chiral symmetry.[7] The effective theory, in order to respect the $\mathbb{Z}_{4n_f}$ discrete chiral symmetry, intertwines it with a discrete shift symmetry of the dual photon:

$$\psi^I \to e^{i2\pi/(4n_f)}\psi^I, \qquad \sigma \to \sigma + \pi \tag{2.16}$$

both of which acts as negation on the determinantal fermion vertex and cosine combinations

$$\det_{I,J}\psi^I\psi^J \to -\det_{I,J}\psi^I\psi^J, \qquad \cos\sigma \to -\cos\sigma \tag{2.17}$$

respectively, so that the effective theory respects the real symmetries of the underlying theory.

In the effective Lagrangian, this is the set of all nonperturbative effects at order e^{-S_0} in the e^{-S_0} expansion. However, the discrete $\mathbb{Z}_2$ shift symmetry $\sigma \to \sigma + \pi$, unlike a continuous shift symmetry, cannot prohibit a mass term for the scalar σ. Clearly, a term $e^{-S_0}\cos\sigma$ is forbidden by $\mathbb{Z}_2$. But its square is an allowed operator. If fermions were not present,

$$e^{-S_0}\cos\sigma \sim e^{-S_0}(e^{i\sigma} + e^{-i\sigma}) \tag{2.18}$$

would be an allowed term as in the Polyakov's discussion of the Georgi-Glashow model, and would induce a mass term of order $e^{-S_0/2}$ for dual photon. However, because of the index theorem (2.11), a monopole must come with fermion zero-modes, and a term such as $e^{i\sigma}$ cannot appear on its own, but must appear in combination $e^{i\sigma}\det_{I,J}\psi^I\psi^J$.

Symmetry principles also tell us that, at the e^{-2S_0} order, we can write

$$[e^{-S_0}\cos\sigma]^2 \sim e^{-2S_0}(1 + 1 + e^{2i\sigma} + e^{-2i\sigma}) \tag{2.19}$$

and this would generate a mass term for the dual photon, hence leading to confinement. We wish to understand the dynamical origin of this potential.

Let us first forget about the issues about fermion zero-modes, and decide on the basis of quantum numbers, which topological excitations may contribute to the nonperturbative potential. Since we know that, due to index theorem, such an object cannot be a monopole, let us enlist all possible pairs of monopoles, the magnetic and topological charges of constituents and pairs, and the types of the long-range Coulomb interactions, repulsive or attractive. In nonsupersymmetric QCD(adj) with $2 \leq n_f \leq 4$, the list of all Coulomb interaction channels for monopoles is given by

[7]More generally, consider $SU(N)$ QCD(adj) with n_f flavors. The chiral symmetry is $[SU(n_f) \times \mathbb{Z}_{2Nn_f}]/\mathbb{Z}_{n_f}$, where the common $\mathbb{Z}_{n_f}$ is factored out to prevent double counting. The $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_{2N}$ is $(-1)^F$ fermion number modulo 2, which cannot be spontaneously broken so long as Lorentz symmetry is unbroken. Thus, the only genuine discrete chiral symmetry of $SU(N)$ QCD(adj) which may potentially be broken is the remaining $\mathbb{Z}_N$, irrespective of the number of flavors. In small S^1, we explicitly demonstrate the existence of N vacua, and spontaneous breaking of chiral $\mathbb{Z}_N$ symmetry (which is intertwined with the discrete shift symmetry of the photon). This $\mathbb{Z}_N$ symmetry should not be confused with the spatial center symmetry, $\mathcal{G}_s = \mathbb{Z}_N$, which is unbroken in spatial compactification of QCD(adj).

Type	Type	σ−int.	$\left(\int F, \int F\widetilde{F}\right)$
BPS − $e^{+i\sigma}$	BPS − $e^{+i\sigma}$	rep.	$(+1,+\frac{1}{2})+(+1,+\frac{1}{2})=(2,1)$
BPS	$\overline{\text{BPS}}$ − $e^{-i\sigma}$	att.	$(+1,+\frac{1}{2})+(-1,-\frac{1}{2})=(0,0)$
BPS	KK − $e^{-i\sigma}$	att.	$(+1,+\frac{1}{2})+(-1,+\frac{1}{2})=(0,1)$
BPS	$\overline{\text{KK}}$ − $e^{+i\sigma}$	rep.	$(+1,+\frac{1}{2})+(+1,-\frac{1}{2})=(2,0)$
$\overline{\text{BPS}}$	$\overline{\text{BPS}}$	rep.	$(-1,-\frac{1}{2})+(-1,-\frac{1}{2})=(-2,-1)$
$\overline{\text{BPS}}$	KK	rep.	$(-1,-\frac{1}{2})+(-1,+\frac{1}{2})=(-2,0)$
$\overline{\text{BPS}}$	$\overline{\text{KK}}$	att.	$(-1,-\frac{1}{2})+(+1,-\frac{1}{2})=(0,-1)$
KK	KK	rep.	$(-1,+\frac{1}{2})+(-1,+\frac{1}{2})=(-2,1)$
KK	$\overline{\text{KK}}$	att.	$(-1,+\frac{1}{2})+(+1,-\frac{1}{2})=(0,0)$
$\overline{\text{KK}}$	$\overline{\text{KK}}$	rep.	$(+1,-\frac{1}{2})+(+1,-\frac{1}{2})=(2,-1)$

(2.20)

In the presence of the fermion zero-modes, the (bosonic) potential must arise due to the sector of the theory with zero topological charge so that there will not be any fermion zero-mode insertions in it. In other words, the objects which may contribute to the potential must be topologically indistinguishable from the perturbative vacuum.

This immediately rules out the four possible monopoles, and six of the ten pairs in our list from contributing to the bosonic potential. In particular, the two identical monopole configurations such as BPS-BPS with $(1,\frac{1}{2})+(1,\frac{1}{2})=(2,1)$ have the correct magnetic charge, but their topological charge does not permit them to contribute to the bosonic potential. Another interesting combination which does not lead to the confining potential is a BPS-KK pair. The BPS-KK pair in fact constitutes an instanton (sometimes called a caloron, [33,34]) with charge $(1,\frac{1}{2})+(-1,\frac{1}{2})=(0,1)$ and does not induce a mass term for the dual photon.

The monopole and antimonopole pairs such as BPS-$\overline{\text{BPS}}$ are topologically null, but also magnetically neutral. Their contribution to the effective potential can only be an uninteresting constant. There remains a single option: a bound state of BPS monopole, and KK antimonopole, BPS-$\overline{\text{KK}}$, and its conjugate. Such an object has the correct quantum numbers $(\pm1,\frac{1}{2})+(\pm1,-\frac{1}{2})=(\pm2,0)$. We referred to this object as a magnetic bion (see Fig. 2). Consequently, the bion is the prime candidate which may lead to confinement in QCD(adj) in the $L\Lambda \ll 1$ regime.

However, there is an immediate puzzle with this proposal. There is a long-range Coulomb repulsion between BPS-$\overline{\text{KK}}$ constituents of the bion. If we wish to have a bound state, there must exist an attractive interaction which overcomes the repulsive Coulomb force. Happily, there is!

C. Pairings and attractive multifermion exchanges

The presence of fermion zero-modes changes things drastically. We will demonstrate that for the pairs with net topological charge zero, there exists an attractive $V_{\text{eff}} \sim \log r$ interaction between the constituents due to fermion pair exchanges. For the pairs with a nonvanishing topological charge, the constituents do not interact at all due to chirality at leading order.

Let us first show the first assertion: Consider BPS and $\overline{\text{KK}}$ monopoles located at $x, y \in \mathbb{R}^3$, where $|x-y| \gg 1$. (x, y are dimensionless coordinates in units of L.) We can extract their interactions from the connected correlator of the BPS vertex $V_{\text{BPS}}(x)$, and $\overline{\text{KK}}$ vertex $V_{\overline{\text{KK}}}(y)$ in the free dual theory with action $S_{d,0}(\sigma, \psi, \bar{\psi})$

$$\begin{aligned}\langle V_{\text{BPS}}(x)V_{\overline{\text{KK}}}(y)\rangle_0 &= \langle e^{i\sigma(x)}\det_{IJ}\psi^I\psi^J(x)e^{+i\sigma(y)} \\ &\quad\times \det_{I'J'}\bar{\psi}^{I'}\bar{\psi}^{J'}(y)\rangle_0 \\ &= \langle e^{i\sigma(x)}e^{i\sigma(y)}\rangle_0\langle\det_{IJ}\psi^I\psi^J(x) \\ &\quad\times \det_{I'J'}\bar{\psi}^{I'}\bar{\psi}^{J'}(y)\rangle_0 \\ &\sim e^{-G(x-y)}[S_F(x-y)]^{2n_f} \qquad (2.21)\end{aligned}$$

where $G(x-y)=\frac{1}{4\pi|x-y|}$ is the Coulomb potential, which is the position space propagator of the σ field, $G(x)=\int\frac{d^3p}{(2\pi)^3}e^{ipx}\frac{1}{p^2}$ and $S(x)=\frac{\sigma^\mu x^\mu}{4\pi|x|^3}$ is the $d=3$ dimensional free fermion propagator $S(x)=\sigma^\mu\frac{\partial}{\partial x_\mu}G(x)$. The static interaction potential between the BPS and $\overline{\text{KK}}$ pair is

$$\begin{aligned}V_{\text{eff}}(x-y) &= -\log\langle V_{\text{BPS}}(x)V_{\overline{\text{KK}}}(y)\rangle_0 \\ &= \frac{1}{4\pi|x-y|}+4n_f\log|x-y|. \qquad (2.22)\end{aligned}$$

Asymptotically, $4n_f\log|x-y|$ is the dominant attractive interaction term, and it easily overcomes the Coulomb repulsion. Therefore, there exists a stable bion bound state with the total magnetic and topological charge $(+2, 0)$, and antibion with charge $(-2, 0)$. It should be noted that the stability of the magnetic bion relies on the masslessness (or lightness) of the adjoint fermions. In this case, the fermion induced attraction overcomes the Coulomb repulsion for a small range of (light) fermion mass. (See Fig. 2.) For more details, see Sec. III B.

It should also be noted that similar fermion zero-mode induced pairings of topological excitations were discussed earlier in the literature by Callan, Dashen, and Gross and others [27,38] in the context of instantons on $\mathbb{R}^4$. The pairing mechanism is similar to what we have found above, in that case instanton and anti-instanton form molecules due to attraction induced by fermions. Interestingly, the form of the attractive interaction is the same both in $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$, and is a logarithmically attractive interaction proportional to the number of flavors, $n_f \log r$. As these instanton–anti-instanton molecules are magnetically and topologically neutral, they play no role in confinement and chiral symmetry realization in the small S^1 regime of QCD(adj). In our topological semiclassical expansion, they appear at order e^{-2NS_0} and are a negligible effect.

Analogously, the net interaction between a BPS-$\overline{\text{BPS}}$ pair is attractive in both interaction channels, either Coulomb, or fermion exchange interactions. The long-distance attraction has the form $-\log\langle V_{\text{BPS}}(x)V_{\overline{\text{BPS}}}(y)\rangle_0 = -\frac{1}{4\pi|x-y|} + 4n_f \log|x-y|$.

Because of chirality of the underlying theory, the interaction between pairs with the same topological charge vanishes identically: $\langle V_{\text{BPS}}(x)V_{\text{BPS}}(y)\rangle_0 = \langle V_{\text{BPS}}(x)V_{\text{KK}}(y)\rangle_0 = 0$.

Since the topological charge of the magnetic bion is zero, it does not have any fermion zero-mode attached to it. Since magnetic bions and antibions have ± 2 magnetic charges, they will lead to Debye phenomena. The appropriate effective potential induced by bions is indeed what we wrote based on symmetry arguments:

$$V(\sigma) = [e^{-S_0}\cos\sigma]^2 \sim e^{-2S_0}(1 + 1 + e^{2i\sigma} + e^{-2i\sigma}). \tag{2.23}$$

The terms in the potential have an interpretation as the contribution of respectively $\text{BPS}\overline{\text{BPS}} + \text{KK}\overline{\text{KK}} + \text{BPS}\overline{\text{KK}} + \text{KK}\overline{\text{BPS}}$.

More precisely, the interaction terms in the Lagrangian are due to monopole and bion contributions. The monopole contributions necessarily involve the fermion interactions. Schematically, the nonperturbatively induced interaction terms will always be

$$L_{\text{int}} = \underbrace{\sum_{\text{bions}} V_{\text{bion}}}_{\int F\tilde{F}=0} + \underbrace{\sum_{\text{monopoles}} V_{\text{monopole}}}_{\int F\tilde{F}=\pm(1/2)}. \tag{2.24}$$

Therefore, the dual QCD Lagrangian for $SU(2)$ QCD(adj) on small $S^1 \times \mathbb{R}^3$ is given by

$$L^{\text{dQCD}} = \frac{1}{2}(\partial\sigma)^2 - be^{-2S_0}\cos 2\sigma + i\bar{\psi}^I\gamma_\mu\partial_\mu\psi_I + ce^{-S_0}\cos\sigma(\det_{I,J}\psi^I\psi^J + \text{c.c.}) \tag{2.25}$$

up to higher-order (insignificant) terms in e^{-S_0}.

The potential term for the dual photon, when expanded around one of its two minima (located at 0 and π), provides a mass term for the dual photon. From the point of view of Euclidean field theory, the photon mass is the inverse Debye screening length in the plasma of magnetic bions. On a fixed time slice of a timelike Wilson loop, the inverse photon mass is the thickness of the chromoelectric flux tube formed between two external electric test charges. Just like the Polyakov model [6] on $\mathbb{R}^3$, the QCD(adj) on small $S^1 \times \mathbb{R}^3$ exhibits linear confinement,

$$V_{\text{linear}}(R) \sim e^{-S_0}R, \tag{2.26}$$

and the potential energy of a pair of the electric source separated by a distance R grows linearly with separation.

Remark: The results and approach of this work should not be confused with 't Hooft's Abelian projection scheme [39], which only leaves an $U(1)^{N-1}$ gauge symmetry. Hence, monopoles in that case are gauge artifacts, which is fine in the prescribed gauge. In our case, the gauge symmetry breaking $SU(N) \to U(1)^{N-1}$ is dynamical, and is a well-controlled effect due to the radiatively induced Coleman-Weinberg potential. The QCD(adj) in the $L\Lambda \ll 1$ regime tells us that, in the presence of fermions, the idea of monopole condensation no longer holds due to fermion zero-modes. Despite this fact, the qualitative and beautiful idea of dual superconductivity of 't Hooft and Mandelstam [39,40] is still realized at a quantitative level, albeit via condensation of the pairs with combined magnetic and topological charges $(\pm 2, 0)$.

As emphasized, the presence of monopoles is not sufficient to induce confinement, or monopole condensation. Better appreciation of the above picture can come with the study of a Yang-Mills Higgs system with adjoint fermions on $\mathbb{R}^3$, a system with monopoles and yet no confinement.

D. Noncompact Higgs with adjoint fermions on $\mathbb{R}^3$, and the lack of confinement

Affleck, Harvey, and Witten studied extensions of Polyakov's model in the presence of an adjoint Dirac fermion on $\mathbb{R}^3$ [12]. The generalization of their argument to multiple flavors is obvious. They analyzed (among other things) a Yang-Mills Higgs system which possesses the same action as Eq. (2.4), except the fact that the compact adjoint Higgs field in Eq. (2.4) is substituted by a noncompact one.

$$V_{\text{eff}}^{\text{compact}}(|\Phi|) \to V_{\text{eff}}^{\text{noncompact}}(|\Phi|). \tag{2.27}$$

Since the chiral anomaly is absent in odd dimensions, the noncompact model has a genuine $U(n_f)$ symmetry whose $U(1)$ part is a fermion number. Reference [12] showed quite explicitly that such a model does *not* confine. Photons remain at infinite range nonperturbatively, and it is indeed the Goldstone boson of the spontaneously broken $U(1)$ fermion number symmetry. Their arguments are essentially based on symmetries, and index theorems by Callias [41], and explicit zero-mode construction by Rebbi and Jackiw [42]. Here, we wish to provide a simple dynamical explanation for this phenomena.

Since gauge symmetry breaking occurs via a *noncompact* adjoint Higgs field, there is no longer a KK monopole. Thus, in order to obtain the long-distance effective action from our discussion in previous section, we must delete all KK monopole related terms from our effective action. Hence, the interaction Lagrangian is $L_{\rm int} \sim V_{\rm BPS} + V_{\overline{\rm BPS}} + V_{\rm BPS\overline{BPS}}$. Consequently,

$$L_{\rm eff}^{\rm noncompact} = \frac{1}{2}(\partial\sigma)^2 + i\bar{\psi}^I\gamma^\mu\partial_\mu\psi_I + ae^{-S_0}(e^{i\sigma}\det\psi^I\psi^J + {\rm c.c.}) \qquad (2.28)$$

where we ignored a trivial cosmological constant which may be induced by a BPS $\overline{\rm BPS}$ pair. This is indeed the generalization of Ref. [12] to multiflavor ($n_f > 1$). The effective action is respectful to all the symmetries of the underlying theory, in particular $SU(n_f) \times U(1)$ symmetry, where the former is manifest. The $U(1)$ fermion number symmetry acts as

$$\psi^I \to e^{i\alpha}\psi^I, \qquad \bar{\psi}^I \to e^{-i\alpha}\bar{\psi}^I, \qquad \sigma \to \sigma - 2n_f\alpha \qquad (2.29)$$

and prohibits any kind of mass term (or potential) for the dual photon. This is the symmetry which breaks down spontaneously, and the dual photon is the Goldstone boson.

Clearly, the only topologically neutral object (which may contribute to the bosonic potential) is the BPS$\overline{\rm BPS}$ pair. But such an object has vanishing magnetic charge. Since there are no topologically null but magnetically charged carriers in the vacuum of the model studied in [12], the Debye mechanism is not possible. Hence, the photon remains at infinite range nonperturbatively. The inability to form magnetically charged bions is the dynamical reason for the absence of confinement in the extension of Polyakov's model in the presence of adjoint fermions.

This discussion also shows that the presence of monopoles in the Yang-Mills Higgs systems with adjoint fermions is a necessary but insufficient condition to have confinement. In particular, it also exhibits that, in such systems, condensations of objects with nonvanishing topological charge (monopole condensation) do not occur.

E. Magnetic bions in $\mathcal{N} = 1$ SYM on small $S^1 \times \mathbb{R}^3$

The generalization of the discussion in Sec. II C to $SU(2)$ $\mathcal{N} = 1$ supersymmetric gauge theory is easy, yet important. All one needs to take care of is an extra massless scalar which remains massless in perturbation theory. Hence it should be incorporated into long-distance physics. With the inclusion of the ϕ-scalar, the monopoles may interact via ϕ-exchange, σ-exchange, and fermion pair exchange channels:

Type	Type	σ−int	ϕ−int.	combined	$\left(\int F, \int F\tilde{F}\right)$
BPS − $e^{-\phi+i\sigma}$	BPS − $e^{-\phi+i\sigma}$	rep.	att.	0	$(1,\frac{1}{2}) + (+1,+\frac{1}{2}) = (2,1)$
BPS	$\overline{\rm BPS}$ − $e^{-\phi-i\sigma}$	att.	att.	2(att.)	$(1,\frac{1}{2}) + (-1,-\frac{1}{2}) = (0,0)$
BPS	KK − $e^{+\phi-i\sigma}$	att.	rep.	0	$(1,\frac{1}{2}) + (-1,+\frac{1}{2}) = (0,1)$
BPS	$\overline{\rm KK}$ − $e^{+\phi+i\sigma}$	rep.	rep.	2(rep.)	$(1,\frac{1}{2}) + (+1,-\frac{1}{2}) = (2,0)$
...	...	...	...	...	...

(2.30)

Incorporating the scalar field ϕ into monopole operators, we find

$$\begin{aligned} &{\rm BPS}: e^{-\phi+i\sigma}\psi\psi, \qquad {\rm KK}: e^{+\phi-i\sigma}\psi\psi, \\ &\overline{\rm BPS}: e^{-\phi-i\sigma}\bar{\psi}\bar{\psi}, \qquad \overline{\rm KK}: e^{+\phi+i\sigma}\bar{\psi}\bar{\psi}. \end{aligned} \qquad (2.31)$$

The bosonic potential is due to the sector of the theory with net zero topological charge, so that there will not be any fermion zero-mode insertion in it. Thus

$$\begin{aligned} {\rm BPS}\,\overline{\rm BPS} + {\rm KK}\overline{\rm KK} + {\rm BPS}\overline{\rm KK} + {\rm KK}\overline{\rm BPS} &= e^{-2S_0}(e^{-2\phi} + e^{+2\phi} - e^{i2\sigma} - e^{-2i\sigma}) \\ &= e^{-2S_0}|e^z - e^{-z}|^2 \end{aligned} \qquad (2.32)$$

where we defined $z = -\phi + i\sigma$. Remarkably, the magnetic bions already know that there is an underlying superpotential, given by[8]

$$\mathcal{W}(z) = e^{-S_0}(e^{z} + e^{-z}). \qquad (2.33)$$

The long-distance effective action for SYM on small $S^1 \times \mathbb{R}^3$ is

$$L_{\rm eff}^{\rm SYM} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\phi)^2 - c^2 e^{-2S_0}(\cos 2\sigma - \cosh 2\phi) + i\bar{\psi}\gamma_\mu\partial_\mu\psi + ce^{-S_0}[(e^{-\phi+i\sigma} + e^{+\phi-i\sigma})\psi\psi + (e^{-\phi-i\sigma} + e^{+\phi+i\sigma})\bar{\psi}\,\bar{\psi}]. \qquad (2.34)$$

The $\mathbb{Z}_{2N} = \mathbb{Z}_4$ discrete chiral symmetry of the original theory is also manifest in the effective theory

$$\psi^I \to e^{i2\pi/4}\psi^I, \qquad \sigma \to \sigma + \pi. \qquad (2.35)$$

This symmetry breaks down spontaneously to $\mathbb{Z}_2 = (-1)^F$ where F is the fermion number leading to the appearance of two isolated vacua.

The dynamics of the $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times S^1$ is previously analyzed by imbedding it into F theory in Ref. [13], and by using the elliptic curves of $\mathcal{N} = 2$ SYM combined with the mass deformation in [14]. The works of Davies *et al.* [15,16] provided a clear field theory exposition of the nonperturbatively induced effects in such theories. The general strategy of these papers was to calculate the monopole operator first, then use supersymmetry as a completion device to find the superpotential, hence bosonic potential. For fermionic terms, our strategy is the same as in these earlier works. For the bosonic potential, our strategy is different. Rather then using supersymmetry as a completion tool to derive bosonic potential, we preferred to delineate on its microscopic (physical) origin. In essence, we identified topologically null configurations which are topologically indistinguishable from the perturbative vacuum, and hence can contribute to the potential. Summing up their contributions gives us the bosonic potential, which can also be derived from the superpotential.

These two approaches in the case of $\mathcal{N} = 1$ SYM are identical. The latter approach has a higher value in our opinion due to the fact that it does not make any reference to supersymmetry, and works for nonsupersymmetric QCD-like theories. Our analysis makes it manifest that the mechanism of confinement in $\mathcal{N} = 1$ SYM is not monopole condensation, i.e., condensation of excitations with topological charge $\pm\frac{1}{2}$, rather of objects with topological charge 0. This physical fact was not understood in earlier important works on the subject [13–16]. Up to our knowledge, our work is the first analytic demonstration of confinement induced by non-self-dual topological excitations. Needless to say, even the issue of presence or absence of such topological excitations was not discussed. We conclude this section by pointing out that the mechanism of the confinement in supersymmetric $\mathcal{N} = 1$ SYM is same as the one in nonsupersymmetric QCD(adj) theories in the $L\Lambda \ll 1$ regime, both of which are magnetic bion condensation, a new class of (non-self-dual) topological excitations.

In the dimensional reduction of $\mathcal{N} = 1$ SYM down to $\mathbb{R}^3$, confinement does not occur as shown in [12]. The distinctions are so important that it is worthwhile rederiving their results following the consideration of this paper, and explaining the absence of confinement on dynamical grounds.

F. The $\mathcal{N} = 2$ SYM on $\mathbb{R}^3$ and lack of confinement, again

Delete all the terms in the effective action (2.34) which are related to KK monopole. (This is the same statement as the ϕ field becomes noncompact on the $\mathbb{R}^3$ limit.) This leaves us with BPS and $\overline{\rm BPS}$ induced operators (involving fermion bilinears) and a BPS$\overline{\rm BPS}$ induced term in the bosonic potential in the Lagrangian (2.34):

$$L_{\rm eff}^{\rm n.c.} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\phi)^2 - c^2 e^{-2S_0}e^{-2\phi} + i\bar{\psi}\gamma_\mu\partial_\mu\psi + ce^{-S_0}[e^{-\phi+i\sigma}\psi\psi + e^{-\phi-i\sigma}\bar{\psi}\,\bar{\psi}] \qquad (2.36)$$

which is the same as the Lagrangian in [12]. The $\mathbb{Z}_{2N}$ discrete chiral symmetry of SYM on locally four-dimensional settings elevates to the full $U(1)$ fermion number on $\mathbb{R}^3$ due to absence of chiral anomaly in odd dimensions. The continuous $U(1)$ symmetry acts as

$$\psi^I \to e^{i\alpha}\psi^I, \qquad \bar{\psi}^I \to e^{-i\alpha}\bar{\psi}^I, \qquad \sigma \to \sigma - 2\alpha \qquad (2.37)$$

and prohibits any kind of explicit mass term (or potential) for the dual photon. This is the symmetry which breaks down spontaneously, and the dual photon is the Goldstone boson. The runaway potential $e^{-2\phi}$ does not have a vacuum at finite ϕ.

On dynamical grounds, the absence of confinement is due to the inability to form long-range magnetic bions in SYM vacuum on $\mathbb{R}^3$. The BPS$\overline{\rm BPS}$ pairs are neutral, and the photon remains at infinite range in a medium of neutral molecules. In other words, it remains massless nonperturbatively as demanded from a Goldstone particle, and this implies the absence of confinement.

III. $SU(N)$ QCD(ADJ), BIONS, AND SECRET INTEGRABILITY?

The $SU(N)$ QCD(adj) theory undergoes gauge symmetry breaking on sufficiently small spatial S^1 due to a perturbative Coleman-Weinberg potential. The gauge symmetry breaking is $SU(N) \to U(1)^{N-1}$. For simplicity, we will add a decoupled "center of mass" degree of freedom to the original theory and consider gauge symmetry breaking of the form $U(N) \to U(1)^N$. This is a technical trick,

[8]Strictly speaking, this superpotential is the form acquired after the super-Higgs mechanism.

and in the spontaneously broken gauge theory, the center of mass mode decouples from the dynamics. Hence, our goal is to determine the dynamics of the $N-1$ modes $\frac{U(1)^N}{U(1)_{\rm c.m.}}$.

The monopoles may be described by their magnetic charges, topological charge, and their action. The magnetic charges of the N types of (BPS and KK) monopoles under unbroken gauge symmetry $U(1)^N$ are proportional to the simple roots and affine root of the Lie algebra, respectively. The simple roots are given by

$$\begin{aligned} \alpha_1 &= (1,-1,0,\ldots,0) = e_1 - e_2 \\ \alpha_2 &= (0,1,-1,,\ldots,0) = e_2 - e_3 \\ \alpha_i &= (0,\ldots,1,-1,\ldots 0) = e_i - e_{i+1} \\ &\ldots \\ \alpha_{N-1} &= (0,\ldots,,0,1,-1) = e_{N-1} - e_N \end{aligned} \tag{3.1}$$

and the affine root is

$$\alpha_N \equiv -\sum_{j=1}^{N-1} \alpha_j = (-1,0,0,\ldots,1) = e_N - e_1. \tag{3.2}$$

It is convenient to define the simple Δ^0 and affine (extended) $\Delta^0_{\rm aff}$ root systems of the associated Lie algebra:

$$\begin{aligned} \Delta^0 &\equiv \{\alpha_1,\alpha_2,\ldots,\alpha_{N-1}\}, \\ \Delta^0_{\rm aff} &\equiv \{\alpha_1,\alpha_2,\ldots,\alpha_{N-1},\alpha_N\}. \end{aligned} \tag{3.3}$$

The latter is the one relevant for QCD(adj) on $\mathbb{R}^3\times S^1$. More generally, in the Yang-Mills Higgs systems with adjoint fermions, if the Higgs field is noncompact, the monopole and antimonopole charges are valued in Δ^0 and $-\Delta^0$, respectively. If the Higgs field is compact, then there is an extra monopole, and the charges take values in $\pm\Delta^0_{\rm aff}$.

The topological charges $\int F\tilde{F}$ are correlated with the sign of the two sets $\pm\Delta^0_{\rm aff}$. Thus, the quantized magnetic and topological charges are

$$\begin{aligned} \int_{S^2} F^i &= \pm\frac{2\pi}{g}\alpha^i, \\ \int F\tilde{F} &\equiv \frac{g^2}{32\pi^2}\int {\rm tr} F_{MN}\tilde{F}^{MN} = \pm\frac{1}{N}. \end{aligned} \tag{3.4}$$

The action of a monopole with charge α_i and topological charge $\int F\tilde{F} = \pm\frac{1}{N}$ is given by $S_{0,i} = \frac{8\pi^2}{g^2}\int F\tilde{F} = \frac{8\pi^2}{g^2 N}$. Because of the presence of the effective potential for the Wilson line, the monopoles of QCD(adj) theory (except for $n_f = 1$ which is supersymmetric) do not saturate the BPS bound. But the corrections are perturbative in g^2 and we will neglect them.

The long-range Coulomb interaction of monopoles (in the absence of fermions) is given by[9]

$$\begin{aligned} V(\alpha_i,\pm\alpha_j,r) &= \frac{\alpha_i.(\pm\alpha_j)}{4\pi r} = \pm\frac{2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}}{4\pi r}, \\ &\quad i,j = 1,\ldots N \end{aligned} \tag{3.5}$$

which translates to self and nearest neighbor interaction between monopoles in the Dynkin space. The inner product of the roots of the associated Lie algebra is a basis independent statement, though the above choice of the basis (3.1) is due to its visual simplicity.

We are now ready to generalize the derivation of effective potential for $SU(2)$ QCD(adj) to $SU(N)$ with $1 < n_f \leq 4$. Our discussion will be brief.

Were the adjoint fermions absent, a monopole with charge α_j would be associated with operator $e^{i\alpha_j\sigma}$. Because of index theorem (2.11), any object with a nonvanishing topological charge $(1/N)$ must have $\Delta Q_5 = 2n_f$ fermions attached to it. As discussed in footnote 7, the underlying QCD(adj) theory has $[SU(n_f)\times\mathbb{Z}_{2Nn_f}]/\mathbb{Z}_{n_f}$ continuous and discrete chiral symmetries. The manifestly $SU(n_f)$ invariant fermion vertex with $2n_f$ fermion insertion is given by $\det_{IJ}\alpha_i\psi^I\alpha_i\psi^J$ where the determinant is over the flavor index. Here, we use a vector notation

$$\begin{aligned} \sigma = (\sigma_1,\ldots,\sigma_N), \qquad \psi^I &= (\psi^I_1,\ldots,\psi^I_N), \\ \alpha_i\sigma &= \sigma_i - \sigma_{i+1}. \end{aligned} \tag{3.6}$$

As stated earlier, the center of mass mode is extraneous and decouples from the dynamics completely. Hence, the appropriate monopole and antimonopole operators are

$$\begin{aligned} V_{\alpha_i} &= e^{i\alpha_i\sigma}\det_{IJ}\alpha_i\psi^I\alpha_i\psi^J, \\ V_{-\alpha_i} &= e^{-i\alpha_i\sigma}\det_{IJ}\alpha_i\bar{\psi}^I\alpha_i\bar{\psi}^J. \end{aligned} \tag{3.7}$$

This means, the interaction Lagrangian at $O(e^{-S_0})$ is given by

$$\begin{aligned} e^{-S_0}\sum_{\alpha_i\in\Delta^0_{\rm aff}} &(e^{i\alpha_i\sigma}\det_{IJ}\alpha_i\psi^I\alpha_i\psi^J \\ &+ e^{-i\alpha_i\sigma}\det_{IJ}\alpha_i\bar{\psi}^I\alpha_i\bar{\psi}^J). \end{aligned} \tag{3.8}$$

This vertex is invariant under $(SU(n_f)\times\mathbb{Z}_{2Nn_f})/\mathbb{Z}_{n_f}$ as desired. The discrete chiral symmetry acts as

$$\begin{aligned} \psi^I \to e^{i2\pi/(2Nn_f)}\psi^I, \qquad \bar{\psi}^I &\to e^{-i2\pi/(2Nn_f)}\bar{\psi}^I, \\ \sigma \to \sigma - \frac{2\pi}{N}\sum_{j=1}^{N-1}\mu_k \end{aligned} \tag{3.9}$$

where μ_k are the $N-1$ fundamental weights (not the

[9]We set $\frac{2\pi}{g}$ to unity as in our discussion of $SU(2)$ to lessen the clutter in expressions. All physical quantities are measured in units of L, which is also set to unity. We will restore both quantities if necessary.

weight of fundamental representation) of the associated Lie algebra. They are defined by the reciprocity relation,

$$\frac{2\alpha_i\mu_j}{\alpha_i^2} = \alpha_i\mu_j = \delta_{ij}. \tag{3.10}$$

The shift in the photon field is called the Weyl vector, and we will often abbreviate it as

$$\rho \equiv \sum_{j=1}^{N-1}\mu_j, \quad \text{such that } e^{i((2\pi)/N)\rho\alpha_j} = e^{i((2\pi)/N)}, \\ j = 1,\ldots,N. \tag{3.11}$$

The action of the discrete chiral symmetry on $SU(n_f)$ singlets is a $\mathbb{Z}_N$ symmetry transformation,

$$\det_{I,J}\alpha_i\psi^I\alpha_i\psi^J \to e^{i2\pi/N}\det_{I,J}\alpha_i\psi^I\alpha_i\psi^J, \\ e^{i\alpha_i\sigma} \to e^{-i2\pi/N}e^{i\alpha_i\sigma}. \tag{3.12}$$

Consequently, the monopole induced interaction terms (which are of order e^{-S_0}) are respectful the discrete (and continuous) symmetries of the underlying theory.

Exactly as in the $SU(2)$ discussion, this is the net effect of the topologically nontrivial sector of the theory which saturates the Lagrangian at order e^{-S_0}. In particular, a would-be (confining) potential term for the σ field

$$e^{-S_0}\sum_{\alpha_i\in\Delta_{\text{aff}}^0}(e^{i\alpha_i\sigma} + e^{-i\alpha_i\sigma}) \tag{3.13}$$

is forbidden by the $\mathbb{Z}_N$ shift symmetry $\sigma - \frac{2\pi}{N}\sum_{j=1}^{N-1}\mu_k$ of the dual photon. This is a consequence of having adjoint fermions in the system. In the absence of fermions, such as a pure Yang-Mills compact Higgs system, this *is* the leading term which renders all the photons massive, with masses of order $e^{-S_0/2}$. We will see that in QCD(adj), the masses of photons are of order e^{-S_0}, and there is a $\mathbb{Z}_N$ shift symmetry respecting potential at order e^{-2S_0}.

A. Attractive channels, bions, and a prepotential

We must examine the combinations of the monople-antimonopole pairs with magnetic charges from the two sets Δ_{aff}^0 and $-\Delta_{\text{aff}}^0$ with respective topological charges $\frac{1}{N}$ and $-\frac{1}{N}$. Because of the presence of many possible pairs that one can construct, this may *a priori* seem arbitrary. However, the theory does something remarkable. At order e^{-2S_0}, the fermion zero-mode exchanges only pair the monopoles with charge α_j with their nearest neighbor antimonopoles, with charges $-\alpha_{j\pm1}$ in the Dynkin space. These combinations are the magnetic bion states. (There are also neutral monopoles and antimonopole pairing of the same kind, but the magnetic charge of such an object is zero and not so interesting in nonsupersymmetric QCD(adj). It has an effect in SYM as discussed in Sec. II E.)

Let us first find the attractive channels. We can extract the interaction of a monopole with charge α_i and antimonopole with charge $-\alpha_j$ by inspecting its connected correlator in the functional integral of the free theory with the action $S_{d,0}(\sigma,\psi,\bar\psi)$.

$$\begin{aligned}\langle V_{\alpha_i}(x)V_{-\alpha_j}(y)\rangle_0 &= \langle e^{i\alpha_i\sigma(x)}\det_{IJ}\alpha_i\psi^I\alpha_i\psi^J(x)e^{-i\alpha_j\sigma(y)}\det_{I'J'}\alpha_j\bar\psi^{I'}\alpha_j\bar\psi^{J'}(y)\rangle_0 \\ &= \langle e^{i\alpha_i\sigma(x)}e^{-i\alpha_j\sigma(y)}\rangle_0\langle\det_{IJ}\alpha_i\psi^I\alpha_i\psi^J(x)\det_{I'J'}\alpha_j\bar\psi^{I'}\alpha_j\bar\psi^{J'}(y)\rangle_0 \sim e^{+\alpha_i.\alpha_j G(x-y)}(\alpha_i\alpha_j)^{2n_f}[S_F(x-y)]^{2n_f}.\end{aligned} \tag{3.14}$$

The connected correlator is only nonzero if $\alpha_i\alpha_j$ is nonzero, and induces a logarithmic binding potential of the form

$$V_{\text{eff}}(x-y) = \begin{cases} +\frac{1}{4\pi|x-y|} + 4n_f\log|x-y| & \text{for } i = j\pm1 \\ -\frac{2}{4\pi|x-y|} + 4n_f\log|x-y| & \text{for } i = j \\ 0 & \text{otherwise.}\end{cases} \tag{3.15}$$

If $i = j$, then both Coulomb and fermion zero-mode exchange induced forces are attractive. If $i = j\pm1$, then the Coulomb interaction is repulsive, but the attractive fermion exchange term easily dominates.

Now, we are ready to define the magnetic bions in the spontaneously broken $SU(N)$ gauge theory. A bion is a bound state of the monopole associated with magnetic charge α_i and antimonopole associated with charge $-\alpha_{i+1}$ with null topological charge:

$$Q_i = \alpha_i - \alpha_{i-1} = 2e_i - e_{i+1} - e_{i-1}, \\ \int F\tilde F = 0 \qquad i = 1,\ldots N. \tag{3.16}$$

Restoring the prefactors and writing more explicitly, the magnetic bion (antibion) charges are given under the $U(1)^N$ gauge group as

$$Q_i = \pm\frac{2\pi}{g}(0,\ldots,\underbrace{-1}_{i-1},\underbrace{2}_{i},\underbrace{-1}_{i+1},\ldots,0). \tag{3.17}$$

This means, bions interact via a next-to-nearest neighbor interaction in the Dynkin space: For high-rank gauge groups ($N\geq5$),

$$Q_iQ_j = 6\delta_{ij} - 4\delta_{i,j+1} - 4\delta_{i,j-1} + \delta_{i,j+2} + \delta_{i,j-2}, \\ N\geq5. \tag{3.18}$$

In order to find the bion-bion interactions in low-rank gauge groups $N \leq 4$, we need to identify nodes $j \equiv j + N$ in the (affine) Dynkin diagram as there are less than five nodes. Consequently,

$$Q_i Q_j = 6\delta_{ij} - 4\delta_{i,j+1} - 4\delta_{i,j-1} + 2\delta_{i,j+2}, \qquad N = 4$$
$$Q_i Q_j = 6\delta_{ij} - 3\delta_{i,j+1} - 3\delta_{i,j-1}, \qquad N = 3$$
$$Q_i Q_j = 8\delta_{ij} - 8\delta_{i,j+1}, \qquad N = 2. \tag{3.19}$$

The long-range interactions of magnetic bions are given by Coulomb's potential and are equal to

$$V(Q_i, \pm Q_j, r) = \frac{Q_i.(\pm Q_j)}{4\pi r} = \pm \frac{6\delta_{ij} - 4\delta_{i,j+1} - 4\delta_{i,j-1} + \delta_{i,j+2} + \delta_{i,j-2}}{4\pi r}. \tag{3.20}$$

The meaning of this formula is clear. Two magnetic bions with charges (Q_i, Q_i) repel, $(Q_i, Q_{i\pm1})$ attract, $(Q_i, Q_{i\pm2})$ repel, and no interactions for pairs (Q_i, Q_{i+k}) with $k > 2$. The overall sign of the interactions is reversed for the bion-antibion pairs.

Now, we can convert the Coulomb gas of magnetic bions into a field theory following Polyakov's treatment [6]. We only quote the result, since the manipulations are standard. The operator appropriate for a bion molecule located at $x \in \mathbb{R}^3$ is

$$e^{iQ_i\sigma(x)} = e^{i\alpha_i\sigma(x)} e^{-i\alpha_{i-1}\sigma(x)}. \tag{3.21}$$

Clearly, this is manifestly invariant under the $\mathbb{Z}_N$ shift symmetry of the photon which acts as $e^{i\alpha_i\sigma(x)} \rightarrow e^{-i2\pi/N} e^{i\alpha_i\sigma(x)}$. The bosonic effective potential is a sum over all bion and antibion contributions given by

$$V(\sigma) = -e^{-2S_0} \sum_{i=1}^{N} (e^{iQ_i\sigma} + e^{-iQ_i\sigma}) = -2e^{-2S_0} \sum_{i=1}^{N} \cos Q_i\sigma. \tag{3.22}$$

There is something remarkable about this potential, in fact surprising. It can be derived from a *prepotential*, just like a bosonic potential in the supersymmetric system may be derived from a superpotential. In order to see this, rewrite the potential $V(\sigma)$ as

$$V(\sigma) = -e^{-2S_0} \sum_{i=1}^{N} (e^{i\alpha_i\sigma} e^{-i\alpha_{i-1}\sigma} + e^{-i\alpha_i\sigma} e^{i\alpha_{i-1}\sigma}) = e^{-2S_0} \sum_{i=1}^{N} |e^{i\alpha_i\sigma} - e^{i\alpha_{i-1}\sigma}|^2 + \text{constant} \tag{3.23}$$

where constant is unimportant. Define the prepotential as

$$\mathcal{W}(\sigma) = e^{-S_0} \sum_{\alpha_i \in \Delta^0_{\text{aff}}} e^{i\alpha_i\sigma}. \tag{3.24}$$

Hence, the potential may be written as

$$V(\sigma) = \sum_{i=1}^{N} \left| \frac{\partial \mathcal{W}}{\partial \sigma_i} \right|^2 = e^{-2S_0} \sum_{i=1}^{N} |e^{i\alpha_i\sigma} - e^{i\alpha_{i-1}\sigma}|^2, \quad \text{QCD(adj) } n_f > 1. \tag{3.25}$$

The reader familiar with the supersymmetric affine Toda theories will recognize the form of our (nonsupersymmetric) prepotential as the superpotential. In order to describe the infrared of $\mathcal{N} = 1$ SYM on small S^1, one must incorporate the extra massless scalars into the potential: All one needs to do is a holomorphic completion of our formula. Not surprisingly,

$$V(z, \bar{z}) = \sum_{i=1}^{N} \left| \frac{\partial \mathcal{W}}{\partial z_i} \right|^2 = \sum_{i=1}^{N} |e^{i\alpha_i z} - e^{i\alpha_{i-1} z}|^2, \qquad \text{SYM.} \tag{3.26}$$

The fact that the potential can be derived from a prepotential as above implies that the classical equations of motions for the σ field can be reduced to afirst-order one.

Let us finalize this section by writing the final form of the dual of the QCD(adj) Lagrangian on small $S^1 \times \mathbb{R}^3$ with $1 < n_f \leq 4$ flavors:

$$\begin{aligned} L^{\text{dQCD}} = {} & \frac{1}{2}(\partial\sigma)^2 - b e^{-2S_0} \sum_{\alpha_i \in \Delta^0_{\text{aff}}} |e^{i\alpha_i\sigma} - e^{+i\alpha_{i-1}\sigma}|^2 \\ & + i\bar{\psi}^I \gamma_\mu \partial_\mu \psi_I \\ & + c e^{-S_0} \sum_{\alpha_i \in \Delta^0_{\text{aff}}} (e^{i\alpha_i\sigma} \det_{IJ} \alpha_i \psi^I \alpha_i \psi^J \\ & + e^{-i\alpha_i\sigma} \det_{IJ} \alpha_i \bar{\psi}^I \alpha_i \bar{\psi}^J). \end{aligned} \tag{3.27}$$

The dual QCD Lagrangian and the physics it encapsulates, which will be discussed next, are the essential result of this paper.

B. Brief comparison to deformed YM theory

In this section, we will briefly outline the main difference between the deformed YM theory (to be abbreviated as YM*) studied in [30] and QCD(adj). In YM*, due to the absence of fermionic matter, the monopole operators do not carry any fermionic zero-modes. Thus, the dual theory can be obtained by summing over all monopole operators. The dual description of YM* theory is

$$L^{\mathrm{dYM}^*} = \frac{1}{2}(\partial\sigma)^2 - e^{-S_0}\sum_{\alpha_i\in\Delta^0_{\mathrm{aff}}}(e^{i\alpha_i\sigma} + e^{-i\alpha_i\sigma}). \quad (3.28)$$

On the other hand, for QCD(adj) with massless fermions, the dual description is given in (3.27). In particular, in QCD(adj), monopole operators do not contribute a mass gap for the dual photons. The bosonic potential which renders the dual photon massive is effectively the square of the potential given for YM*. If we just write the dual of the gauge sector for QCD(adj), the difference is more transparent.

$$L^{\mathrm{dQCD}} = \frac{1}{2}(\partial\sigma)^2 - e^{-2S_0}\sum_{\alpha_i\in\Delta^0_{\mathrm{aff}}}(e^{i(\alpha_i-\alpha_{i-1})\sigma} + e^{-i(\alpha_i-\alpha_{i-1})\sigma}). \quad (3.29)$$

Consequently, the functional form of the mass gap for gauge fluctuations is different in these two classes of theories as it will be compared in Sec. III C 1.

If one keeps center symmetry stable and turns on a mass term for the adjoint fermion, the magnetic bion induced confinement mechanism should be replaced by magnetic monopole induced confinement. In particular, for heavy adjoint fermions, the index theorem on $S^1\times\mathbb{R}^3$ does not apply. Thus, the theory must reduce to YM*.

The more interesting case is the light adjoint fermions. In principle, the fermion zero-modes may be (softly) lifted by the mass term. As a result, the modified monopoles operators will also contribute to mass gap and confinement. For sufficiently light fermions, the bion mechanism dominates. It would be interesting to examine the transition from magnetic bion induced confinement to magnetic monopole induced confinement in more detail in future work.

C. The vacuum structure of QCD(adj)

The bosonic potential of nonsupersymmetric QCD(adj) has N gauge inequivalent isolated vacua, aligned along the Weyl vector ρ

$$\sigma = \left\{0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{(N-1)2\pi}{N}\right\}\rho \quad (3.30)$$

in the field space. This is the same as $\mathcal{N}=1$ SYM studied in [16]. Since each component of σ is a periodic variable with periodicity 2π, there exists a physical congruence between σ and σ' which is separated by an element of the root lattice Λ_r.

$$\sigma \equiv \sigma + 2\pi\alpha \quad \text{for some } \alpha\in\Lambda_r. \quad (3.31)$$

Since the sum of all fundamental weights is a root, $\rho = \sum_{j=1}^{N-1}\mu_j \in \Lambda_r$, this implies there only exist N gauge inequivalent vacua when the (global) gauge symmetry redundancies are removed. Let us the abbreviate and label the vacuum states in Hilbert space as

$$|\Omega_{((2\pi k)/N)\rho+\Lambda_r}\rangle \equiv |\Omega_k\rangle \equiv |\Omega_{k+N}\rangle, \qquad k=0,\ldots N-1$$
$$\text{Ground states} = \{|\Omega_0\rangle, |\Omega_1\rangle, \ldots, |\Omega_{N-1}\rangle\} \quad (3.32)$$

which form a one-dimensional representation of $\mathbb{Z}_N$ shift symmetry, (which is intertwined with $\mathbb{Z}_N$ discrete chiral symmetry, see footnote 7.) This means, the (large) physical Hilbert space spits into N superselection sectors, each of which may be built upon the associated vacuum. The choice of the vacuum breaks the $\mathbb{Z}_N$ discrete chiral symmetry (which is same as $\mathbb{Z}_N$ shift symmetry of the dual photon) spontaneously. Note that QCD(adj) also possess a $\mathcal{G}_s = \mathbb{Z}_N$ spatial center symmetry which remains unbroken regardless of the size of the S^1, and which should not be confused with the $\mathbb{Z}_N$ axial or equivalently, $\mathbb{Z}_N$ shift symmetry of dual photons.

1. Mass gap in the gauge sector

The small fluctuations around one of the N minima of the $-\cos Q_i\sigma$ potential shows that the $N-1$ dual photon acquires masses proportional to e^{-S_0}. In order to see this, let us expand the nonperturbative bion induced potential to quadratic order in dual photon σ

$$\begin{aligned} V(\sigma_i) &= -e^{-2S_0}\sum_{i=1}^{N}\cos Q_i\sigma \\ &= -e^{-2S_0}\sum_{i=1}^{N}\cos(2\sigma_i - \sigma_{i+1} - \sigma_{i-1}) \\ &= \frac{1}{2}e^{-2S_0}\sum_i(6\sigma_i^2 - 4\sigma_i\sigma_{i+1} - 4\sigma_i\sigma_{i-1} + \sigma_i\sigma_{i+2} \\ &\quad + \sigma_i\sigma_{i-2}) \qquad \text{bion induced.} \end{aligned} \quad (3.33)$$

If the fermions were absent, and the gauge symmetry was still broken by a compact adjoint Higgs field as in YM* [30], the quadratic fluctuations would be described by the nearest neighbor coupled harmonic oscillator

$$V(\sigma_i) = \frac{1}{2}e^{-S_0}\sum_i(2\sigma_i^2 - \sigma_i\sigma_{i+1} - \sigma_i\sigma_{i-1}) \qquad \text{monopole induced, YM}^* \quad (3.34)$$

which is not the case in QCD(adj). The bion induced "hopping" terms are next-to-nearest neighbor and of order e^{-2S_0} as opposed to the monopole induced hopping terms which are just nearest neighbor, and of order e^{-S_0}.

The quadratic fluctuations can be diagonalized by using the discrete Fourier transform $\sigma_p = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\omega^{jp}\sigma_j$ in Dynkin space:

$$V(\sigma_p) = \frac{1}{2} e^{-2S_0} \sum_p (6 - 4\omega^{-p} - 4\omega^p + \omega^{-2p} + \omega^{2p}) \times \sigma_p \sigma_{-p} = \frac{1}{2} e^{-2S_0} \sum_p (\omega^{p/2} - \omega^{-p/2})^4 \sigma_p \sigma_{-p} = \frac{1}{2} e^{-2S_0} \sum_p \left(2 \sin\frac{p\pi}{N}\right)^4 \sigma_p \sigma_{-p}. \tag{3.35}$$

Restoring the dimensions, we obtain the mass spectrum of the $N-1$ dual photons as

$$m_p^{\mathrm{QDC(adj)}} \sim (\Lambda(\Lambda L)^{b_0 - 1} = \Lambda(\Lambda L)^{(8-2n_f)/3}) \times \left(2 \sin\frac{p\pi}{N}\right)^2, \quad p = 1, \ldots, N-1. \tag{3.36}$$

This result implies that the gauge sector of the QCD(adj) theory is quantum mechanically gapped due to nonperturbative effects, and permanently confines external electric charges at small $S^1 \times \mathbb{R}^3$ limit. Note that the analogous formula for the mass gap in the gauge sector of YM* is given by

$$m_p^{\mathrm{YM}^*} \sim \Lambda(\Lambda L)^{5/6} \sin\frac{p\pi}{N}, \qquad p = 1, \ldots, N-1 \tag{3.37}$$

in the $\Lambda L \ll 1$ regime.

The masses are graded according to the $\mathbb{Z}_N$ center group of $SU(N)$ in one to one correspondence with the representations $\mathcal{R}_p$ of $SU(N)$ under the center group. There are two equivalent physical interpretations for the mass gap: one as the inverse Debye screening length in a magnetic conductor (in a Euclidean setting), and the other is the inverse thickness of the chromoelectric flux tubes in a magnetic superconductor (at a fixed time in a Minkowski setting). (See [38] for a parallel discussion in the context of the Polyakov model.)

Imagine a large, planar Wilson loop in a representation with charge p under the center $\mathbb{Z}_N$, In the small S^1 regime (where gauge symmetry is broken to the Abelian subgroup), we may regard the Wilson loop as carrying an electric current along the contour of the loop. Hence, by Maxwell's equation, the current generates a magnetic field along the axis perpendicular to the plane of the loop, within the boundary C of the loop surface Σ. The external magnetic field cannot penetrate into the magnetic conductor above a penetration depth, due to Debye screening. The mobile magnetic charge carriers (bions) form a dipole layer in the vicinity of the surface Σ to prevent the penetration of the external magnetic field into the magnetic conductor, which is the vacuum of QCD(adj) from Euclidean viewpoint. The thickness of the dipole layer for the Wilson loop with $\mathbb{Z}_N$ charge p is the inverse of the photon mass m_p^{-1}.

We may visualize a Wilson loop at a fixed time slice. This is a system with $\pm p$ $\mathbb{Z}_N$ chromoelectric sources located at two boundaries of the fixed time slice of the Wilson loop. There exists a stable chromoelectric flux tube in between the two. Since the dual superconductor expels the electric field, the flux lines are trapped within tubes with quantized flux. The $N-1$ classes of the photon masses are indeed the inverse characteristic sizes of the $N-1$ types of the chromoelectric flux tubes, both of which are a class function of the $\mathbb{Z}_N$ center group. In a weakly coupled regime, making L larger reduces the thicknesses of the stable flux tubes

$$l_p \sim \Lambda^{-1}(\Lambda L)^{-(8-2n_f)/3} \times \left(2 \sin\frac{p\pi}{N}\right)^{-2}, \quad p = 1, \ldots, N-1. \tag{3.38}$$

We expect it to saturate to an L independent value above the scale of gauge symmetry restoration. Also, intermediate N-ality tubes seem to be much more slimmer than the small and large N-ality ones.

Because of compactification, in the weakly coupled regime, the characteristic size of the flux tubes and their tensions are no longer parametrically related. In the next section, we explicitly calculate the string tensions.

2. Domain wall tensions and area law of confinement

Domain walls: Any theory which exhibits spontaneous breaking of a discrete symmetry will have discrete isolated vacua and stable domain walls which interpolate in between. QCD(adj) possesses both continuous and discrete axial chiral symmetry. As discussed in Sec. III C, the discrete chiral symmetry is broken at any radius, thus the theory possesses stable domain walls. Note that in the small S^1 regime, the discrete chiral symmetry $\mathbb{Z}_N \in \mathbb{Z}_{2Nn_f}$ is intertwined with the $\mathbb{Z}_N$ shift symmetry of the dual photon (3.9).

The domain wall on $\mathbb{R}^4$ is a tthree-dimensional infinite hypersurface $\mathbb{R}^3$. If $\mathbb{R}^4$ is compactified down to $\mathbb{R}^3 \times S^1$ and the pattern of the discrete chiral symmetry breaking remains invariant as a function of radius, which is the case in QCD(adj), the domain wall curls over itself with an $\mathbb{R}^2 \times S^1$ geometry. Therefore, in the long-distance description, the domain wall is an $\mathbb{R}^2$ filling surface embedded into $\mathbb{R}^3$. Let us assume that the wall lies on $x, y \in \mathbb{R}^2$ plane and is centered at $z = 0$ with a profile which extrapolates from $z = -\infty$ to $z = +\infty$. The topological charge of such a k-wall (kink) is

$$t = \int_{-\infty}^{\infty} dz \frac{d\sigma}{dz} = \sigma(\infty) - \sigma(-\infty) = \frac{2\pi k}{N} \rho. \tag{3.39}$$

As stated earlier, the fact that the potential may be derived from a prepotential leads to the reduction of the equations of motions of the solitons to the first order (Prasad-Sommerfield type). This, combined with Bogomolny's trick, allows us to find the global minimum of the action in each topologically distinct sector of the effective theory.

We have

$$\langle \Omega_k | e^{-zH} | \Omega_0 \rangle \equiv \int_{\sigma_{(z=-\infty)}=0}^{\sigma_{(z=+\infty)}=((2\pi k)/N)\rho} D\sigma e^{-S(\sigma)} = e^{-\mathrm{Area}(\mathbb{R}^2) S_k^*}, \qquad \Sigma \sim \mathbb{R}^2. \tag{3.40}$$

Thus, the k-wall tension is proportional to the global minimum of the action (divided by the area of the $\mathrm{Area}(\mathbb{R}^2)$), i.e., $T_k^{\mathrm{DW}} \equiv S_k^*$, given by

$$T_k^{\mathrm{DW}} = |\mathcal{W}(\sigma(\infty)) - \mathcal{W}(\sigma(-\infty))| = \left| \mathcal{W}\left(\frac{2\pi k}{N}\rho\right) - \mathcal{W}(0) \right| \tag{3.41}$$

in terms of prepotential. Hence,

$$T_k^{\mathrm{DW}} L = \frac{1}{L^2} e^{-S_0} N |e^{i((2\pi k)/N)} - 1| = \frac{1}{L^2} e^{-S_0} 2N \sin\frac{\pi k}{N} \qquad k = 1, \dots N-1. \tag{3.42}$$

Restoring the dimensions and using the one-loop renormalization group result for the strong scale, we obtain

$$T_k^{\mathrm{DW}} \sim (\Lambda^3 (\Lambda L)^{b_0 - 3} = \Lambda^2 (\Lambda L)^{2(1-n_f)/3}) \times 2N \sin\frac{\pi k}{N}. \tag{3.43}$$

Note that, for $n_f = 1$, this gives a new derivation of the domain wall tension in $\mathcal{N} = 1$ SYM, a result obtained earlier by Dvali and Shifman [43]. This tension is independent of the radius. For $n_f > 1$ confining gauge theories, we expect the L dependence to disappear around the strong scale, $L\Lambda \sim 1$, and expect the domain wall tension to saturate to $T_k^{\mathrm{DW}} \sim 2N\Lambda^3 \sin\frac{\pi k}{N}$ in the decompactification limit.

Area law of confinement: We wish to exhibit the area law of confinement for all but adjoint representations $\mathcal{R}_p$ of the $SU(N)$ gauge group. The representations of the Wilson loops C under the center group $\mathbb{Z}_N$ are in one to one correspondence with the monodromies, $\int_{C'} d\sigma$ in the dual theory [36], where C' is any closed curve whose linking number with C is one. In QCD(adj), both form a representation of $\mathbb{Z}_N$.

The evaluation of a Wilson loop in a representation with charge k under the $\mathbb{Z}_N$ center group in the original theory translates into finding the field configurations for the dual scalar theory with monodromies equal to $2\pi\mu_k$ in the dual theory where μ_k is the fundamental weight corresponding to external charge. Note that $\mu_k = k\mu_1 + \alpha$, for some α valued in root lattice Λ_r, and weights differing by elements of Λ_r are identified. Thus, we need to find the action of the soliton configurations for which $\Delta\sigma = 2\pi\mu_k$ across the Wilson loop interface, or equivalently,

$$\int_{C'} d\sigma = \int_{z=0^-}^{z=0^+} dz \frac{d\sigma}{dz} = \sigma(0^+) - \sigma(0^-) = 2\pi\mu_k, \qquad \mathrm{linking}(C, C') = 1. \tag{3.44}$$

The reader should note that the monodromy given in (3.44) is not related to the topological charge of the domain wall kink given in (3.39). In particular,

$$\int_{C'} d\sigma \neq \int_{-\infty}^{\infty} dz \frac{d\sigma}{dz} \tag{3.45}$$

although both objects, in the case of QCD(adj), are $\mathbb{Z}_N$ valued due to the fact that both the center group and discrete axial symmetry group are $\mathbb{Z}_N$. These two $\mathbb{Z}_N$ are unrelated to each other. For generic representations, this coincidence disappears. Even in the case of QCD(adj), $\int_{C'} d\sigma$ is not parallel to $\int_{-\infty}^{\infty} dz \frac{d\sigma}{dz}$ in the root space. The former corresponds to interpolations between fundamental weights on one and the same vacuum and the latter integral is tied with interpolations between discrete isolated vacua.

The expectation values of the Wilson loop fall into N categories, and translate, in the path integral formulation into

$$\lim_{A(\Sigma)\to\infty} \langle W_{\mathcal{R}_k}(C) \rangle|_{C=\partial\Sigma} = \int_{\sigma_{(z=-\infty)}=\sigma_{(z=+\infty)}} D\sigma e^{-S(\sigma)}|_{\Delta\sigma(0)=2\pi\mu_k}. \tag{3.46}$$

Thus, the string tension is

$$T_k = \lim_{A(\Sigma)\to\infty} \frac{\log\langle W_{\mathcal{R}_k}(C)\rangle}{\mathrm{Area}(\Sigma)} = \min_{\sigma(z)} \frac{S(\sigma)}{\mathrm{Area}(\mathbb{R}^2)} \Bigg|_{\Delta\sigma(0)=2\pi\mu_k}. \tag{3.47}$$

For general $SU(N)$, we believe that the string tension in QCD(adj) should be calculable by using the techniques similar to [18]. Because of its technical nature, we will perform this calculation in a separate publication. The expected result is

$$T_k \sim (\Lambda^2 (\Lambda L)^{b_0 - 2} = \Lambda^2 (\Lambda L)^{(5-2n_f)/3}) \times 2N \sin\frac{\pi k}{N}. \tag{3.48}$$

On the other hand, it is evident that T_k is nonzero. This is sufficient to exhibit the area law of permanent confinement in QCD(adj) in the $L\Lambda \ll 1$ regime, and the existence of the linearly confining potential between two external electric sources with charges $\pm k \in \mathbb{Z}_N$

$$V_k(R) = T_k R, \qquad \text{linear confinement.} \tag{3.49}$$

We expect the tension to saturate to a size independent value, a c-number times Λ^2 for $L\Lambda > 1$.

To summarize, in QCD(adj), the domain wall tensions, the string tensions, and thicknesses of flux tubes (which are the inverse masses of the dual photons) are class functions of the center group $\mathbb{Z}_N$. The class functions depend on the N-ality of the source, but are blind to the particular representative of a class. Also, exchanging (color) source and sink is just the mirror image, and tells us that class functions must obey $X_k = X_{N-k}$, where X is any class function.

Interesting physical quantities (which are all measurable in lattice) are the ratios of the string tensions, (inverse) string thicknesses, and their energy densities given by

$$\frac{T_p}{T_1} = \frac{\sin\frac{p\pi}{N}}{\sin\frac{\pi}{N}}, \qquad \frac{m_p}{m_1} = \left(\frac{\sin\frac{p\pi}{N}}{\sin\frac{\pi}{N}}\right)^2, \qquad \frac{\mathcal{E}_p}{\mathcal{E}_1} = \left(\frac{\sin\frac{p\pi}{N}}{\sin\frac{\pi}{N}}\right)^5. \tag{3.50}$$

These observables obey

$$X_p \equiv X_{N+p}, \qquad X_p = X_{N-p}, \qquad p = 1, \ldots N-1. \tag{3.51}$$

Therefore, there are $[\frac{N}{2}]$ types of flux tubes, where a bracket labels the integer part of the $N/2$. The ratio of the string tensions yields the ''sine-law'' for the tensions.

In the $n_f = 1$ case, the sine law for tension has previously been derived by Douglas and Shenker [5] on $\mathbb{R}^4$ by deforming the $\mathcal{N} = 2$ theory by a perturbative mass term for the chiral multiplet, and by Hanany *et al.* [44] by realizing the same deformation in the M-theory five-brane version, referred as mQCD.[10] Both [5,44] achieve a weakly coupled $\mathcal{N} = 1$ SYM theory on $\mathbb{R}^4$ by adding extra matter into the theory.[11] In our derivation, no extra matter is needed. But in order to achieve a weakly coupled formulation, we compactify the theory on $\mathbb{R}^3 \times S^1$ and benefit from asymptotic freedom. In both cases, the physics is rather similar, it is spontaneously broken $U(1)^{N-1}$ gauge theory, and Abelian duality in $d = 3$ and $d = 4$ plays a fundamental role. The formula receives $O(e^{-S_0})$ corrections, which is insignificant in the $L\Lambda \ll 1$ regime, but will be essential at large radius. Consequently, our result does not imply that the tension will obey a sine law in large S^1 or in $\mathbb{R}^4$, even in the $n_f = 1$ case which is $\mathcal{N} = 1$ pure SYM.

Remark on other QCD-like theories: Either the mass gap in the gauge sector or the area law for large Wilson loops are equally valid indicators of confinement for theories in which the only dynamical degrees of freedom are adjoint fermions. For theories such as QCD with two adjoint and one fundamental fermions (which also breaks its gauge symmetry at small S^1), the mass gap should still emerge, but area law must become a perimeter law. The theory should still be confining, but the ability to form stable flux tubes must be lost due to the fact that charged fermions can be pair created out of the vacuum, and break the flux tube to reduce its energy. It would be interesting to examine this class of theories in the future.

3. Chiral symmetry realizations

The choice of the vacuum state $|\Omega_k\rangle$ spontaneously breaks the $\mathbb{Z}_N$ shift symmetry, which is intertwined with the $\mathbb{Z}_N$ discrete chiral symmetry. The chiral order parameter which is a singlet under continuous flavor symmetry, and which only probes the discrete chiral symmetry, is the determinantal condensate $\det\mathrm{tr}\lambda^I\lambda^J$ in the original theory. In the infrared of the theory on small S^1, the off-diagonal modes of the λ^I are heavy due to gauge symmetry breaking and cannot contribute to the determinantal chiral condensate. We may decompose $\lambda^I = \lambda^{I,a}t^a$ into massless components along the Cartan subalgebra and heavy off-diagonal modes, $\mathrm{tr}\lambda^I\lambda^J \sim L^{-3}\sum_j(\alpha_j\psi^J)(\alpha_j\psi^J) +$ heavy, where L^{-3} is due to dimensional reasons. The vacuum expectation value of the flavor singlet chiral condensate in $SU(N)$ QCD(adj) with $1 \le n_f \le 4$ flavor can be found by integrating over the zero-mode wave functions (which are essentially proportional to monopole profiles) in the background of a monopole in the small S^1 regime, where the gauge symmetry is broken. On large S^1, we do not know a reliable analytical technique in the $1 < n_f \le 4$ case to evaluate the condensate. However, we expect the modulus of the chiral condensate to saturate to a c-number times Λ^{3n_f}. Consequently,

$$\langle\Omega_k|\det\mathrm{tr}\lambda^I\lambda^J|\Omega_k\rangle \sim \begin{cases} \Lambda^{3n_f}(\Lambda L)^{(11/3)(1-n_f)}e^{((i2\pi k)/N)} & L \ll L_c \\ \Lambda^{3n_f}e^{((i2\pi k)/N)}, & L > L_c \end{cases} \tag{3.52}$$

where the phase is $\mathbb{Z}_N$ valued. In the $n_f = 1$ case, this produces the correct L independence of chiral condensate (which is due to supersymmetry) [16], and N isolated vacua. We believe that the scale at which the determinantal condensate becomes L independent is the scale of the gauge symmetry restoration.

In the far infrared of the QCD(adj), since σ is massive, the long-distance theory further reduces to a purely fermionic theory, which schematically looks like an NJL-type Lagrangian:

$$L_{\mathrm{NJL}} = \sum_{j=1}^{N}[i\bar{\psi}^I_j\gamma_\mu\partial_\mu\psi^I_j + ce^{-S_0}(\det_{I,J}\alpha_j\psi^I\alpha_j\psi^J + \mathrm{c.c.})]. \tag{3.53}$$

The Lagrangian is invariant under $SU(n_f) \times \mathbb{Z}_{2n_f}$ chiral symmetry. The $\mathbb{Z}_{2n_f}$ is the unbroken subgroup of the $\mathbb{Z}_{2Nn_f}$ discrete symmetry. We wish to know whether the continuous chiral symmetry is broken spontaneously.

[10] Our result for nonsupersymmetric theories is new, and directly testable on the lattice in the appropriate regime. Our derivation for the SYM is also different from earlier work [5,44] and does not make any reference to supersymmetry, or the underlying theory being realizable in string theory. Because of the generality of our approach, it is applicable to nonsupersymmetric QCD-like theories which are more interesting.

[11] An important issue here is to realize that this theory is not pure $\mathcal{N} = 1$ SYM in $\mathbb{R}^4$. As the authors of [5] discuss, this mechanism holds so long as $m/\Lambda \ll 1$, a perturbation. In order to obtain pure $\mathcal{N} = 1$ SYM in the IR, we must take $m \gg \Lambda$, which is not a perturbation, and calculational control of the softly broken $\mathcal{N} = 2$ does get lost. Currently, there is no analytical derivation of mass gap or confinement in pure $\mathcal{N} = 1$ SYM on $\mathbb{R}^4$.

At small S^1, we believe the continuous chiral symmetry is unbroken, based on studies on related $d = 3$ dimensional NJL-type models. Such models have generically a weakly coupled chirally symmetric phase and a chirally asymmetric strong coupling phase. (See the review in Ref. [45]). Our dimensionless coupling constant is $g \sim e^{-S_0}$, far too small to induce a chiral transition. Hence, the chiral symmetry must be unbroken, and there must be massless fermions (protected by chiral symmetry) in the spectrum within the region of validity of our long-distance effective theory, ($L\Lambda \ll 1$). We believe the naive extrapolation of the NJL Lagrangian Eq. (3.53) will exhibit the continuous chiral transition in an expected regime of the underlying QCD theory. (See Fig. 3.) However, this will happen outside the region of validity of our effective theory. Consequently, this does not tell us that the monopole operator is the sole origin of the continuous chiral symmetry breaking, even though it is the origin of the discrete chiral symmetry breaking in the small S^1 regime. In the large S^1 regime, nondilute monopoles with fermionic zero-modes play the major role in continuous chiral symmetry breaking.

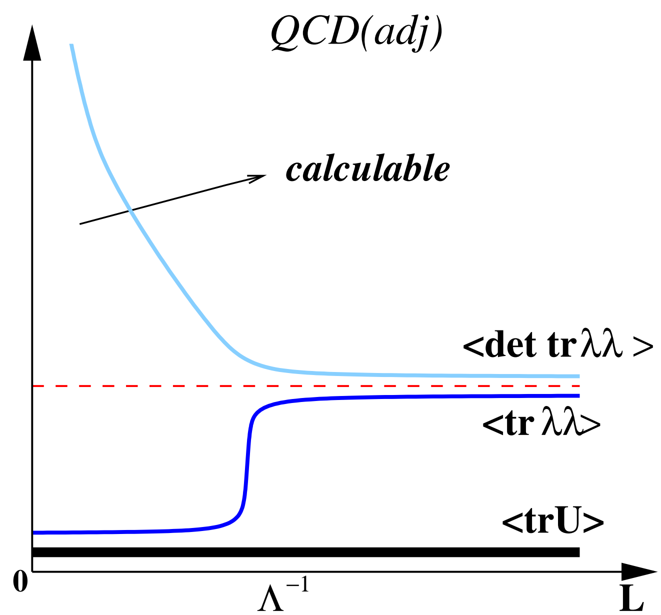

FIG. 3 (color online). The cartoon of the behavior of the center, discrete, and continuous chiral symmetry realization in QCD(adj), for $SU(N)$ where $N =$ few, $n_f = 2$, and $n_f = 1$ ($\mathcal{N} = 1$ SYM). The spatial center symmetry is unbroken at any L in both cases $\langle \mathrm{tr} U \rangle = 0$. In $n_f = 2$, the continuous chiral symmetry is unbroken at small S^1 and broken at large S^1, and discrete chiral symmetry is always broken. The red (dotted) line is the chiral condensate in $\mathcal{N} = 1$ SYM, and the discrete chiral symmetry is always broken. In the small S^1 regime, the string tensions and thicknesses (the inverse mass gap in gauge sector) are calculable, and $n_f = 2$ theory exhibits confinement without continuous chiral symmetry breaking. The lines slightly on top of the horizontal axis are all zero and are split to guide the eye.

The absence of the continuous chiral symmetry breaking in a weak coupling regime can also be seen by an independent argument. In the small S^1 regime where theory is weakly coupled, we have control over all nonperturbative objects. A BPS or KK monopole, which may in principle contribute to the condensate, has a minimum of $2n_f$ fermionic zero-modes. However, our order parameter $\mathrm{tr}\lambda^I\lambda^J$ can only soak up two zero-modes. This implies it cannot acquire a nontrivial vacuum expectation value. The minimal operator which may acquire a condensate must have $2n_f$ fermion insertion, and this is indeed the determinantal condensate $\langle \mathrm{dettr}\lambda^I\lambda^J\rangle$. The reliability of this argument is tied with weak coupling, and in fact, it does not hold at strong coupling.

At large S^1 (and $\mathbb{R}^4$), the common lore is that the chiral symmetry is spontaneously broken down to $SO(n_f) \times \mathbb{Z}_2$ by the formation of the chiral condensate

$$\langle \Omega_k | \mathrm{tr}\lambda^I\lambda^J | \Omega_k \rangle \sim \begin{cases} 0 & L < L_c \\ \Lambda^3 e^{((i2\pi\kappa)/Nn_f)}, & L > L_c \end{cases}. \quad (3.54)$$

Consequently, there must exist N isolated coset spaces each of which is $SU(n_f)/SO(n_f)$. In this expression, κ ranges in $[0, Nn_f)$. Denote $\kappa = \eta N + k$ where $k = 0, \ldots N-1$ and $\eta = 0, \ldots n_f - 1$. For a given k, there are n_f many η for which the determinant of the condensate is invariant. Thus, they reside in the same coset space, and there are consequently N isolated coset spaces.

The continuous chiral transition in QCD(adj) is very different from its thermal counterparts. In particular, it occurs in the absence of any change in its spatial center symmetry realization. This is a quantum phase transition at absolute zero temperature, induced solely due to quantum fluctuations. We do not know the order of the phase transition.

Finally, we wish to conjecture that the scale of the chiral phase transition L_c in QCD(adj) is associated with the restoration of the spontaneously broken gauge symmetry. Consequently, we believe that the chiral symmetry breaking is a strong coupling phenomena. Confinement is not necessarily so.[12]

D. Noncompact versus compact adjoint Higgs, final pass

Let us reconsider the $SU(N)$ gauge theory with a *noncompact* adjoint Higgs field and with one Dirac fermion in adjoint representation on $\mathbb{R}^3$. (Multiflavor generalization is obvious.) The theory possess a $U(1)$ fermion number symmetry. The generalization of the argument of Ref. [12] shows that the $U(1)$ symmetry is spontaneously broken, and consequently, there only exists one gapless excitation

[12] In subsequent work, I showed the natural scale of chiral symmetry breaking at arbitrary N is Λ^{-1}/N. Figure 3 is for $N =$ few for which there is no parametric separation between Λ^{-1} and Λ^{-1}/N.

by Goldstone's theorem. The other $N-2$ photons of the spontaneously broken gauge symmetry must acquire masses. We wish to know how this is realized in the microscopic description.

When the $SU(N)$ gauge symmetry breaks down to $U(1)^{N-1}$ via an noncompact adjoint Higgs field rather than a compact one (which was the case in QCD(adj)), monopoles only come in $N-1$ varieties. The KK monopole is now absent. We may still define the magnetic bions in the spontaneously broken $SU(N)$ gauge theory for $N \geq 3$, but there are only $N-2$ of them. As before, a bion is a bound state of the monopole associated with magnetic charge α_i and antimonopole associated with charge $-\alpha_{i+1}$ with null topological charge. The magnetic charge of a bion is

$$Q_i = \alpha_i - \alpha_{i-1}, \qquad i = 2, \ldots N-1. \tag{3.55}$$

Hence, there are only $N-2$ types of magnetic bions. In other words, the absence of the $\alpha_N \equiv \alpha_0$ KK monopole removes two would-be bions of the compact theory. Thus, the potential for the σ field is a sum over $N-2$ bions and their conjugates given by

$$V(\sigma) = -e^{-2S_0} \sum_{i=2}^{N-1} (e^{iQ_i\sigma} + \text{c.c}). \tag{3.56}$$

The potential generates mass terms only for $N-2$ dual photons. The massless photon is the Goldstone boson. Equivalently, we may say the sum in the prepotential is restricted to the root system Δ_0, $\mathcal{W}(\sigma) = e^{-S_0}\sum_{\alpha_i \in \Delta^0} e^{i\alpha_i\sigma}$, and from the study of the analogous supersymmetric theory, we know that the reduction from affine Toda to nonaffine Toda renders the gapped theory gapless [13,16].

IV. OUTLOOK: CONFINEMENT AND NON-SELF-DUAL TOPOLOGICAL EXCITATIONS

A microscopic derivation of the mechanism which provides confinement in QCD(adj) quantized on small $S^1 \times \mathbb{R}^3$ is given. This is a QCD-like theory with no elementary scalars in its Lagrangian, and no special properties such as supersymmetry (except the $n_f = 1$ case). We believe the solution provides a significant contribution to our current understanding of QCD-like gauge theories, and teaches us many valuable lessons. We also found the underlying dynamical reasons behind the lack of confinement in Yang-Mills noncompact Higgs systems with adjoint fermions formulated on $\mathbb{R}^3$. Let us quote our main result for the $SU(2)$ gauge group:

(i) New non-self-dual topological excitations that we referred to as magnetic bions exist in the QCD(adj) vacuum and are the source of confinement. A mechanism by non-self-dual excitations was not suspected in QCD-like theories by the wisdom gained from other analytically solvable theories, such as Polyakov model or Seiberg-Witten theory. Even the existence of such stable topological excitations is surprising as they are topologically neutral, just like perturbative vacuum! But they carry a magnetic charge.

(ii) QCD(adj) exhibits permanent confinement even at arbitrarily weak coupling (small S^1). In other words, in asymptotically free confining gauge theories, confinement is not *necessarily* a strong coupling phenomena.

(iii) In the presence of massless adjoint dynamical fermions, the monopole operators must have a compulsory fermion zero-mode attached to them. Hence, they induce fermion-fermion and fermion-dual photon interactions, neither of which can appear in the bosonic potential of the dual photon. Our arguments rule out monopoles and monopole condensation as the microscopic mechanism of the confinement in QCD-like theories with *dynamical fermions* in general.

(iv) The beautiful and qualitative idea of dual superconductivity is quantitatively realized in the vacuum of QCD(adj), but not in terms of self-dual monopoles, or instantons. Non-self-dual magnetic bions with magnetic and topological charge $(\pm 2, 0)$ generate a mass gap in the gauge sector and confinement.

(v) Magnetic bions are composites of BPS and $\overline{\text{KK}}$ monopoles, and their stability is due to a dynamical fermionic pairing mechanism. The repulsive Coulomb repulsion between the bion constituents (with charges $(1, +\frac{1}{2})$ and $(1, -\frac{1}{2})$) is overwhelmed by a attractive logarithmic force. The pairing mechanism responsible for the bound state is induced by $2n_f$-fermion exchange in n_f flavor theory.

(vi) This rationale also explains why the Yang-Mills with *noncompact* adjoint Higgs field and adjoint fermions does not confine on $\mathbb{R}^3$ despite the presence of monopoles. The same rationale is also true for $\mathcal{N} = 2$ SYM on $\mathbb{R}^3$. These are examples as important as QCD(adj) itself, because we believe it is equally important to understand the lack of confinement in order to understand confinement.

(vii) In the general $SU(N)$ case, we demonstrated the area law of confinement for Wilson loops in arbitrary representations. The dual theory hints at an integrable (generalized Toda) system behind QCD(adj), in the e^{-S_0} expansion of the action at order e^{-2S_0}. We do not know whether this extends to higher order if we were to find higher-order terms in e^{-S_0} expansion. We also do not know whether there may be integrability behind QCD(adj) on $\mathbb{R}^4$.

We wish to express that we are optimistic of future progress which will reveal more on the inner goings-on of general QCD-like theories:

Incorporating fundamental representation fermions: For example, in a theory with two adjoint and one fundamental fermions (*mixed action*), the backreaction of the fermion is insufficient to induce center symmetry breaking in the small S^1 regime. This theory has both magnetic monopoles and massless electric charges within the weak coupling regime examined in this paper. This system should teach us something which may be relevant to the real QCD. Unfortunately, our techniques are not directly applicable to pure Yang-Mills or QCD with fundamental fermions due to breaking of (temporal or spatial) center symmetry at small S^1.

Confinement on QCD-like theories on $\mathbb{R}^4$: The techniques of this paper are strictly valid in the gauge symmetry broken phase of the QCD(adj). However, we believe that certain assertions are generalizable to $\mathbb{R}^4$, and direct progress will occur in QCD(adj) on $\mathbb{R}^4$, where strong coupling necessarily occurs.

Lattice gauge theory: Many assertions made in this paper are directly testable in lattice simulations with available technologies. In particular, the string tensions and characteristic sizes of flux tubes (3.36) and (3.50) can be extracted from the lattice simulations of QCD(adj) as in [46]. QCD(adj) also undergoes a zero temperature quantum chiral transition in the absence of any change in center symmetry realization. This should be directly testable on the lattice by modifying the existing simulations (such as [47]) appropriately. It would also be useful to construct the duality between QCD(adj) on $\mathbb{R}^3 \times S^1$ with Lagrangian (2.1) and dual QCD defined in (3.27) directly in lattice formulations.

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Note added.—There was a large time delay between the arXiv version of this paper and its submission to a journal. In the meantime, new useful techniques, such as center stabilizing double-trace deformations, which allows a smooth connection of small and large S^1 physics, and the relevant index theorem for generic topological excitations on $S^1 \times \mathbb{R}^3$ have been found. These techniques enabled us to study nonperturbative dynamics of all vectorlike and even chiral theories on $S^1 \times \mathbb{R}^3$. In all chiral theories and QCD-like theories with two index matter representations, we now understand that magnetic bions or similar composite non-self-dual excitations are the root cause of confinement. For a review of these developments and related works, see the recent preprint [48].

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