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Regularizing role of teleparallelism

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The properties of the gravitational energy-momentum 3-form and of the superpotential 2-form are discussed in the covariant teleparallel framework, where the Weitzenböck connection represents inertial effects related to the choice of the frame. Because of its odd asymptotic behavior, the contribution of the inertial effects often yields unphysical (divergent or trivial) results for the total energy of the system. However, in the covariant teleparallel approach, the energy is always finite and nontrivial. The teleparallel connection plays a role of a regularizing tool which subtracts the inertial effects without distorting the true gravitational contribution. As a crucial test of the covariant formalism, we reanalyze the computation of the total energy of the Schwarzschild and the Kerr solutions.

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I. INTRODUCTION

The problem of defining an energy-momentum density for the gravitational field belongs to the oldest in modern theoretical physics. The concepts of energy and momentum are fundamental ones in classical field theory. Within the general Lagrange-Noether approach, conserved currents arise from the invariance of the classical action under transformations of fields and spacetime variables. In particular, energy and momentum are related to time and space translations. However, due to the geometric nature of the gravitational theory and because of the equivalence principle, which identifies locally gravitation and inertia, the definition of gravitational energy remained unsolved for many years. In general, there are no symmetries in Riemannian manifolds that can be used to generate the corresponding conserved energy-momentum currents. It is possible, though, to associate energy and momentum to asymptotically flat gravitational field configurations. The history of the problem and some of the corresponding achievements is described in reviews [1–5], for example.

On the other hand, although equivalent to general relativity, the gauge structure of teleparallel gravity gives rise to several conceptual and practical differences in relation to the geometric structure of general relativity. An important difference is that it is possible to distinguish gravitation and inertia [6]. Since inertia is in the realm of the pseudotensor behavior of the usual expressions for the gravitational energy-momentum density, it turns out possible in teleparallel gravity to write down a tensorial expression for such density [7]. With the purpose of getting a deeper insight into the covariant teleparallel formalism, as well as to test how it works, we reanalyze the computation of the total energy of the two important examples, namely, the Schwarzschild and Kerr solutions.

The paper is organized as follows. Using the language of exterior forms, we give in Sec. II an outline of the teleparallel approach to gravity. In Sec. III we present the covariant formalism for the gravitational energy-momentum. In simple terms, it means that a general relativistic system is described not by a single variable ϑ^α, as was done in the pure tetrad approach [8–14], but by the pair $(\vartheta^\alpha, \Gamma_\alpha{}^\beta)$. The tetrad ϑ^α is responsible for the gravitational effects, but its form also reflects the choice of the reference system. This inevitably brings in the inertial phenomena which are mixed up with the truly gravitational effects. The introduction of the teleparallel connection $\Gamma_\alpha{}^\beta$ makes it possible to deal with the inertial effects in a constructive way. Specifically, we demonstrate in Sec. IV that, due to an inconvenient choice of a reference system, the traditional computation of the total energy of the Schwarzschild solution can yield either a divergent or a vanishing result. With an account of the teleparallel connection, we can circumvent such results. In our covariant formalism, the Weitzenböck connection acts as a regularizing tool that separates the inertial contribution and provides the physically meaningful result for all reference frames. The notion of a proper tetrad, introduced in Sec. IV B, plays a central role in this approach. The results obtained are then further generalized to the case of the Kerr

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solution in Sec. V. Finally, in Sec. VI we summarize our results, namely, that the covariant teleparallel formalism automatically regularizes the computations, always yielding the physically relevant solution.

Our general notations are as in [15]. In particular, we use the Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and the Greek indices $\alpha, \beta, \ldots$ label (co)frame components. Particular frame components are denoted by hats, $\hat{0}$, $\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi \rfloor \Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta$. Using local coordinates x^i, we have $\vartheta^\alpha = h_i^\alpha dx^i$ and $e_\alpha = h^i_\alpha \partial_i$. We define the volume 4-form by $\eta := \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$. Furthermore, with the help of the interior product we define $\eta_\alpha := e_\alpha \rfloor \eta$, $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha$, $\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta}$ which are bases for 3-, 2- and 1-forms, respectively. Finally, $\eta_{\alpha\beta\mu\nu} = e_\nu \rfloor \eta_{\alpha\beta\mu}$ is the Levi-Civita tensor density. The η-forms satisfy the useful identities:

$$\vartheta^\beta \wedge \eta_\alpha = \delta_\alpha^\beta \eta, \tag{1.1}$$

$$\vartheta^\beta \wedge \eta_{\mu\nu} = \delta_\nu^\beta \eta_\mu - \delta_\mu^\beta \eta_\nu, \tag{1.2}$$

$$\vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta_\alpha^\beta \eta_{\mu\nu} + \delta_\mu^\beta \eta_{\nu\alpha} + \delta_\nu^\beta \eta_{\alpha\mu}, \tag{1.3}$$

$$\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta_\nu^\beta \eta_{\alpha\gamma\mu} - \delta_\mu^\beta \eta_{\alpha\gamma\nu} + \delta_\gamma^\beta \eta_{\alpha\mu\nu} - \delta_\alpha^\beta \eta_{\gamma\mu\nu}. \tag{1.4}$$

The line element $ds^2 = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$.

II. TELEPARALLEL GRAVITY

The teleparallel approach is based on the gauge theory of translations. Without going into the subtleties of the corresponding gauge-theoretic scheme (for an advanced reading, see [12,16–19], for example), one can view the coframe $\vartheta^\alpha = h_i^\alpha dx^i$ (tetrad) as a one-form that plays the role of the gauge translational potential of the gravitational field. Einstein's general relativity theory can be reformulated as the teleparallel theory. Geometrically, one can view the teleparallel gravity as a special (degenerate) case [20–22] of the metric-affine gravity in which the coframe ϑ^α and the local Lorentz connection $\Gamma_\alpha{}^\beta$ are subject to the distant parallelism constraint $R_\alpha{}^\beta = 0$. The torsion 2-form

$$T^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta, \tag{2.1}$$

arises as the gravitational gauge field strength, with $\Gamma_\beta{}^\alpha$ the Weitzenböck connection. As is well known, torsion T^α can be decomposed into three irreducible pieces: the tensor part, the trace, and the axial trace, given, respectively, by

$${}^{(1)}T^\alpha := T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha, \tag{2.2}$$

$${}^{(2)}T^\alpha := \frac{1}{3} \vartheta^\alpha \wedge (e_\beta \rfloor T^\beta), \tag{2.3}$$

$${}^{(3)}T^\alpha := \frac{1}{3} e^\alpha \rfloor (\vartheta^\beta \wedge T_\beta). \tag{2.4}$$

A Yang-Mills type Lagrangian is then constructed as a quadratic polynomial in torsion. In the so-called teleparallel equivalent gravity model, the Lagrangian reads

$$V = -\frac{1}{2\kappa} T^\alpha \wedge {}^\star\Big({}^{(1)}T_\alpha - 2{}^{(2)}T_\alpha - \frac{1}{2}{}^{(3)}T_\alpha\Big), \tag{2.5}$$

where $\kappa = 8\pi G/c^3$, and * denotes the Hodge dual in the metric $g_{\alpha\beta}$. The latter is assumed to be the flat Minkowski metric $g_{\alpha\beta} = o_{\alpha\beta} := \mathrm{diag}(+1, -1, -1, -1)$, and it is used to raise and lower the Greek (local frame) indices.

The teleparallel field equations are obtained from the variation of the total action with respect to the coframe:

$$DH_\alpha - E_\alpha = \Sigma_\alpha. \tag{2.6}$$

Here D denotes the covariant exterior derivative, i.e., $DH_\alpha = dH_\alpha - \Gamma_\alpha{}^\beta \wedge H_\beta$. The translational momentum and the canonical energy-momentum are, respectively:

$$H_\alpha = -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} {}^\star\Big({}^{(1)}T_\alpha - 2{}^{(2)}T_\alpha - \frac{1}{2}{}^{(3)}T_\alpha\Big), \tag{2.7}$$

$$E_\alpha = \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta. \tag{2.8}$$

In terms of H_α, the Lagrangian (2.5) is recast in the form

$$V = -\frac{1}{2} T^\alpha \wedge H_\alpha. \tag{2.9}$$

We remark that the resulting model is degenerate from the metric-affine viewpoint, because the variational derivatives of the action with the respect to the metric and connection are trivial. This means that the field equations are satisfied for any $\Gamma_\alpha{}^\beta$. However, as we are going to see, the presence of the connection field plays an important *regularizing* role. Furthermore, in its presence the teleparallel gravity becomes explicitly covariant under local Lorentz transformations of the coframe. In particular, the Lagrangian (2.5) is invariant under the changes

$$\begin{aligned} \vartheta'^\alpha &= L^\alpha{}_\beta \vartheta^\beta, \\ \Gamma'_\alpha{}^\beta &= (L^{-1})^\mu{}_\alpha \Gamma_\mu{}^\nu L^\beta{}_\nu + L^\beta{}_\gamma d(L^{-1})^\gamma{}_\alpha, \end{aligned} \tag{2.10}$$

with $L^\alpha{}_\beta \equiv L^\alpha{}_\beta(x) \in SO(1,3)$. In contrast to this, the Lagrangian of the pure tetrad gravity, which is obtained when we put $\Gamma_\alpha{}^\beta = 0$ for all frames, is only quasi-invariant—it changes by a total divergence.

The connection $\Gamma_\alpha{}^\beta$ can be decomposed into Riemannian and post-Riemannian parts as

$$\Gamma_\alpha{}^\beta = \tilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta. \tag{2.11}$$

Here, $\tilde{\Gamma}_\alpha{}^\beta$ is the purely Riemannian connection and $K_\alpha{}^\beta$ is the contortion which is related to the torsion by the identity

$$T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta. \tag{2.12}$$

One can then show that, due to geometric identities [23], the translational momentum (2.7) can be written as

$$H_\alpha = \frac{1}{2\kappa} K^{\mu\nu} \wedge \eta_{\alpha\mu\nu}. \tag{2.13}$$

A crucial property of the teleparallel framework is that the Weitzenböck connection $\Gamma_\alpha{}^\beta$ actually represents inertial effects that arise from the choice of the reference system. Because of its odd asymptotically behavior, the inertial contributions in many cases yield unphysical results for the total energy of the system, producing either trivial or divergent answers. We will show in this paper that the computation of the energy in the covariant teleparallel approach always yields finite and physically correct results. In this sense, we can say that the teleparallel connection acts as a regularizing tool which helps to eliminate the inertial effects without distorting the true gravitational contribution.

It is worthwhile to mention that the Lagrangian (2.9) differs from the Hilbert-Einstein Lagrangian by a total derivative (surface term). Correspondingly, the field Eq. (2.6) coincide with Einstein's gravitational field equation. In this sense, the physical contents of the two theories is the same.

III. ENERGY-MOMENTUM CONSERVATION

We begin by rewriting the field Eq. (2.6) in the Maxwell-type form:

$$DH_\alpha = E_\alpha + \Sigma_\alpha. \tag{3.1}$$

The analogy with the electromagnetism is obvious. The Maxwell 2-form $F = dA$ represents the gauge field strength of the electromagnetic potential 1-form A. From the Lagrangian $V(F)$, the 2-form of the electromagnetic excitations is defined by $H = -\partial V/\partial F$, and the field equation reads $dH = J$, where J is the 3-form of the electric current density of matter. In view of the nilpotency of the exterior differential, $dd \equiv 0$, the Maxwell equation yields the conservation law of the electric current, $dJ = 0$.

In contrast to electrodynamics, gravity is a self-interacting field, and the gauge field potential 1-form ϑ^α carries an "internal" index α. The gauge field strength 2-form $T^\alpha = D\vartheta^\alpha$ is now defined by the covariant derivative of the potential (compare with $F = dA$). The gravitational field excitation 2-form H_α is introduced by (2.7), in a direct analogy to the Maxwell theory (recall $H = -\partial V/\partial F$). Finally, we observe that as compared to the Maxwell field equation $dH = J$, the gravitational field Eq. (3.1) contains now the covariant derivative D, and in addition, the right-hand side is represented by a modified current 3-form, $E_\alpha + \Sigma_\alpha$. The last term is the energy-momentum of matter, and we naturally conclude that the 3-form E_α describes the *energy-momentum current of the gravitational field*. Its presence in the right-hand side of the field Eq. (3.1) reflects the self-interacting nature of the gravitational field, and such contribution is absent in the linear electromagnetic theory.

We can complete the analogy with electrodynamics by deriving the corresponding conservation law. Indeed, since $DD \equiv 0$ for the teleparallel connection, (3.1) tells us that the sum of the energy-momentum currents of gravity and matter, $E_\alpha + \Sigma_\alpha$, is covariantly conserved [7],

$$D(E_\alpha + \Sigma_\alpha) = 0. \tag{3.2}$$

This law is consistent with the covariant transformation properties of the currents E_α and Σ_α.

One can rewrite the conservation of energy-momentum in terms of the ordinary derivatives. Using the explicit expression $DH_\alpha = dH_\alpha - \Gamma_\alpha{}^\beta \wedge H_\beta$, the field Eq. (2.6) and (3.1) can be recast in an alternative form

$$dH_\alpha = \mathcal{E}_\alpha + \Sigma_\alpha, \tag{3.3}$$

where $\mathcal{E}_\alpha = E_\alpha + \Gamma_\alpha{}^\beta \wedge H_\beta$. Accordingly, (3.3) yields a usual conservation law with the ordinary derivative

$$d(\mathcal{E}_\alpha + \Sigma_\alpha) = 0. \tag{3.4}$$

The 3-form E_α describes the gravitational energy-momentum in a covariant way, whereas the 3-form $\mathcal{E}_\alpha$ is a noncovariant object. In terms of components, it gives rise to the energy-momentum pseudotensor. It is worthwhile to note that H_α plays the role of energy-momentum superpotential both for the covariant energy-momentum current $(E_\alpha + \Sigma_\alpha)$ and for the total (including inertia) noncovariant current $(\mathcal{E}_\alpha + \Sigma_\alpha)$.

The η-forms (defined above) serve as the basis of the spaces of forms of different rank, and when we expand the above objects with respect to the η-forms, the usual tensor formulation is recovered. Explicitly,

$$H_\alpha = \frac{1}{\kappa} S_\alpha{}^{\mu\nu} \eta_{\mu\nu}. \tag{3.5}$$

Here $S_\alpha{}^{\mu\nu} = -S_\alpha{}^{\nu\mu}$ is constructed from the contortion tensor in a usual way [20].

Analogously, we have explicitly for the gravitational energy-momentum

$$E_\alpha = \frac{1}{2}[(e_\alpha \rfloor T^\beta) \wedge H_\beta - T^\beta \wedge (e_\alpha \rfloor H_\beta)]. \tag{3.6}$$

Substituting here (3.5) and $T^\alpha = \frac{1}{2} T_{\mu\nu}{}^\alpha \vartheta^\mu \wedge \vartheta^\nu$, and using (1.2), (1.3), and (1.4), we find

$$E_\alpha = t_\alpha{}^\beta \eta_\beta, \qquad t_\alpha{}^\beta = \frac{1}{2\kappa}(4T_{\alpha\nu}{}^\lambda S_\lambda{}^{\beta\nu} - T_{\mu\nu}{}^\lambda S_\lambda{}^{\mu\nu} \delta_\alpha^\beta). \tag{3.7}$$

Similarly, we have

$$\mathcal{E}_\alpha = j_\alpha{}^\beta \eta_\beta, \qquad j_\alpha{}^\beta = \frac{1}{2\kappa}(4T_{\alpha\nu}{}^\lambda S_\lambda{}^{\beta\nu} - T_{\mu\nu}{}^\lambda S_\lambda{}^{\mu\nu}\delta_\alpha^\beta + 4\Gamma_{\nu\alpha}{}^\lambda S_\lambda{}^{\beta\nu}). \tag{3.8}$$

Now, whereas $t_\alpha{}^\beta$ is a true tensor, because it depends explicitly on the Weitzenböck connection $\Gamma_{\nu\alpha}{}^\lambda$, the current $j_\alpha{}^\beta$ is a pseudotensor. Since the Weitzenböck connection $\Gamma_{\nu\alpha}{}^\lambda$ represents the inertial effects related to the choice of the frame, we see clearly that the origin of the pseudotensor behavior of the usual energy-momentum densities is that they include those inertial effects [7].

Taking into account the analogous expansion of the matter energy-momentum, $\Sigma_\alpha = \Sigma_\alpha{}^\beta \eta_\beta$, which introduces the energy-momentum tensor $\Sigma_\alpha{}^\beta$, and using (3.5) and (3.7), we easily recover the field equation in tensor language (used, for example, in [24]). Note that the conservation laws (3.2) and (3.4) coincide when we put $\Gamma_\alpha{}^\beta = 0$. The last term in (3.8) then disappears, whereas torsion reduces to the anholonomity 2-form, $T^\alpha = F^\alpha = d\vartheta^\alpha$. We denote the corresponding energy-momentum and superpotential with a tilde:

$$\tilde{E}_\alpha = E_\alpha|_{\Gamma_\alpha{}^\beta=0}, \qquad \tilde{H}_\alpha = H_\alpha|_{\Gamma_\alpha{}^\beta=0}. \tag{3.9}$$

The properties of these quantities and their use for the computation of the total energy of the exact solutions was discussed in [22,25]. Explicitly, we have

$$\tilde{H}_\alpha = \frac{1}{2\kappa}\tilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \tag{3.10}$$

$$\tilde{E}_\alpha = \frac{1}{2}[(e_\alpha \rfloor d\vartheta^\beta) \wedge \tilde{H}_\beta - d\vartheta^\beta \wedge (e_\alpha \rfloor \tilde{H}_\beta)]. \tag{3.11}$$

IV. ENERGY OF THE SCHWARZSCHILD SOLUTION

In this section we will demonstrate the regularizing role of the teleparallel connection for the computation of the energy-momentum of the Schwarzschild solution. The generalization to the rotating Kerr configurations will be discussed separately in the next section.

We will consider several choices of the coframe. In order to show how the covariant formulation works, we will compare our results to the computations done in the pure tetrad formalism. The latter gives, depending on the choice of the reference system, either infinite or trivial answers. In contrast, the use of the covariant teleparallel framework always yields the physically meaningful result.

A. Schwarzschild metric: Naive choice of a tetrad

In accordance with the spherical symmetry of the Schwarzschild solution, we choose the spherical local coordinates, (t, r, θ, φ). We start our analysis by using the diagonal coframe:

$$\vartheta^{\hat 0} = \frac{1}{\alpha}c dt, \qquad \vartheta^{\hat 1} = \alpha dr, \qquad \vartheta^{\hat 2} = r d\theta, \qquad \vartheta^{\hat 3} = r\sin\theta d\varphi, \tag{4.1}$$

with $\alpha = \alpha(r)$. Actually, this class of metrics includes not only Schwarzschild, but also Reissner-Nordstrom (with electric charge) and Kottler (with a cosmological term) metrics. The pure Schwarzschild arises when

$$\alpha = \left(1 - \frac{2m}{r}\right)^{-(1/2)}, \tag{4.2}$$

with $m = GM/c^2$ (G is Newtonian gravitational constant).

If we take tetrad (4.1), as well as the trivial Weitzenböck connection $\Gamma_\alpha{}^\beta = 0$, and substitute them into (3.9), we find

$$\tilde{H}_{\hat 0} = \frac{\alpha}{\kappa}\cos\theta dr \wedge d\varphi - \frac{2r}{\kappa\alpha}\sin\theta d\theta \wedge d\varphi. \tag{4.3}$$

In particular, if we compute the total energy at a fixed time in the 3-space with a spatial boundary two-dimensional surface $\partial S = \{r = R, \theta, \varphi\}$, with $R \to \infty$, we obtain

$$\tilde{P}_{\hat 0} = \int_{\partial S}\tilde{H}_{\hat 0} = -\frac{2R}{\kappa\alpha}\int_{\partial S}\sin\theta d\theta \wedge d\varphi = -\frac{8\pi R}{\kappa\alpha}, \tag{4.4}$$

which diverges in the limit of $R \to \infty$ (note that $\alpha \to 1$ when the radius goes to infinity).

The physical reason that underlies such a result is obvious—the energy-momentum current and the superpotential contain an infinite contribution of the inertial effects that are present due to the inconvenient choice of the reference system. We have demonstrated in [25] how to regularize this result by subtracting the unphysical contribution with the help of the suitable choice of the flat background connection. Here we use a different regularization framework which is based on the covariance property. Namely, let us associate with the tetrad (4.1) a nontrivial teleparallel connection

$$\Gamma_{\hat 1}{}^{\hat 2} = d\theta, \qquad \Gamma_{\hat 1}{}^{\hat 3} = \sin\theta d\varphi, \qquad \Gamma_{\hat 2}{}^{\hat 3} = \cos\theta d\varphi. \tag{4.5}$$

The curvature obviously vanishes for this connection, but torsion $T^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$ is nontrivial. Substituting into (2.13), we then find

$$H_{\hat 0} = \frac{2r(1 - 1/\alpha)}{\kappa}\sin\theta d\theta \wedge d\varphi. \tag{4.6}$$

The integral over the spatial boundary yields

$$P_{\hat 0} = \int_{\partial S} H_{\hat 0} = M. \tag{4.7}$$

It is worthwhile to note that the account for the teleparallel connection removed the divergence and automatically produced the physical result. Another remark is in order about the choice of the connection (4.5). As one can immediately see, this is the same flat connection that was earlier used in [25] to subtract the inertial effects. This is not a mere coincidence. In Ref. [25] we have demonstrated the possibility of reinterpreting the background flat connection as a Weitzenböck connection in the metric-affine approach to the translational gauge gravity theory.

B. Schwarzschild metric: Proper tetrad

Instead of the inconvenient reference system of the previous section, we now choose a coframe that represents what we will call a *proper tetrad*. The definition will be provided later. Since a coframe is a basis of the cotangent space, it can be expanded with respect to a different basis. In other words, a new coframe can be always obtained from the old one with the help of the local Lorentz rotation.

We begin with an observation that the role of the teleparallel connection, that we used above, was to remove (or to separate) the inertial contribution from the truly gravitational one. By definition, since the teleparallel curvature is zero, the connection is a "pure gauge", that is

$$\Gamma_\alpha{}^\beta = (\Lambda^{-1})^\beta{}_\gamma d\Lambda^\gamma{}_\alpha, \tag{4.8}$$

The Weitzenböck connection always has the form (4.8). Since the Lorentz matrix $\Lambda^\alpha{}_\beta$ has to do with transformations among different frames, $\Gamma_\alpha{}^\beta$ turns out to describe inertial properties of a tetrad. In particular, it is easy to see that (4.5) is of the form (4.8), with the Lorentz matrix given explicitly by

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ 0 & -\cos\theta & \sin\theta & 0 \\ 0 & \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \end{pmatrix}. \tag{4.9}$$

Now, we are in a position to construct the proper tetrad. Qualitatively, it is clear what we need to do. The system described in the previous section is "spoiled" by the presence of the inertial effects, so that the teleparallel connection was required for the regularization of the energy-momentum. These inertial effects are encoded in the Lorentz matrix (4.9). Accordingly, in order to improve the situation, we need to go to a new reference system by performing the local Lorentz rotation that removes the drawbacks mentioned. In technical terms, we define a new tetrad $\overset{0}{\vartheta}{}^\alpha$ with the help of the Lorentz transformation

$$\overset{0}{\vartheta}{}^\alpha = \Lambda^\alpha{}_\beta \vartheta^\beta \tag{4.10}$$

with the rotation matrix (4.9). From (2.10) we can easily verify that the corresponding teleparallel connection, that is associated to the new tetrad, is trivial:

$$\overset{0}{\Gamma}{}_\alpha{}^\beta = (\Lambda^{-1})^\mu{}_\alpha \Gamma_\mu{}^\nu \Lambda^\beta{}_\nu + \Lambda^\beta{}_\gamma d(\Lambda^{-1})^\gamma{}_\alpha = 0. \tag{4.11}$$

In addition, the new coframe $\overset{0}{\vartheta}{}^\alpha$ has another important property: Let us "switch off" the gravitational effects. Technically, one can do it by putting equal zero the essential gravitational parameters that describe a given configuration; in this case $m = 0$. After doing this, we discover that such a "gravity switched-off" tetrad becomes *holonomic*:

$$\overset{0}{F}{}^\alpha = d\overset{0}{\vartheta}{}^\alpha|_{m=0} = 0. \tag{4.12}$$

Actually, it is easy to see that the tetrad (4.10) describes the Cartesian coordinate system when m vanishes. This means, in physical terms, that *this frame does not include inertial effects*. This is the definition of *proper tetrad*: it is a coframe that describes a reference system whose anholonomy has to do with gravitation only, not with inertial effects. It corresponds, in this sense, to the inertial frames of special relativity, and it reduces to a Cartesian frame in the absence of gravitation.

Let us calculate the energy-momentum. Using (4.10) and (4.11) in (3.5), we find

$$H_{\hat{0}} = \tilde{H}_{\hat{0}} = \frac{2r(1-1/\alpha)}{\kappa} \sin\theta d\theta \wedge d\varphi. \tag{4.13}$$

The total energy is found to be finite: $P_{\hat{0}} = \int H_{\hat{0}} = M$. The regularization is not needed. The energy-momentum is regular for the proper tetrad, which is consistent with the fact that the inertial effects, that are responsible for the bad behavior of the energy and momentum, are absent in the proper reference system.

C. Schwarzschild metric: Freely falling tetrad

Since Einstein with his famous thought experiments with an elevator, we know that gravity can be locally imitated by inertial effects, or alternatively, gravitational effects can be locally eliminated by using an appropriate noninertial reference system. A freely falling elevator is an example. This fact constitutes the contents of the strong equivalence principle which underlies Einstein's gravity theory.

A natural question then arises: What about the energy of the gravitational field? What happens to the energy when we "eliminate" gravity by going to a noninertial system? One possible answer was recently proposed in [26] in the framework of the pure tetrad (noncovariant) formulation. Here we will discuss the result of [26] and in the next section we will reanalyze the same question in the covariant formulation.

We start again from the diagonal tetrad (4.1), and construct a new coframe with the help of the Lorentz transformation $\overset{f}{\vartheta}{}^\alpha = \Lambda^\alpha{}_\gamma \Lambda'^\gamma{}_\delta \vartheta^\delta$, where $\Lambda^\alpha{}_\gamma$ is given by (4.9), and

$$\Lambda'^{\gamma}{}_{\delta} = \begin{pmatrix} \alpha & \alpha\beta & 0 & 0 \\ \alpha\beta & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.14}$$

with $\beta = \sqrt{1-\alpha^{-2}}$ (and hence $\alpha = 1/\sqrt{1-\beta^2}$). Specifically for the Schwarzschild metric, we have

$$\beta = \sqrt{\frac{2m}{r}}, \qquad \alpha = \left(1 - \frac{2m}{r}\right)^{-(1/2)}. \tag{4.15}$$

A direct computation shows that the superpotential is identically zero for this tetrad:

$$\tilde{H}_{\hat{0}} = 0. \tag{4.16}$$

This was demonstrated by Maluf *et al* [26]. In physical terms, such a tetrad describes a reference frame that is freely falling along the radial coordinate onto the attracting source. Gravity is "eliminated" in such a noninertial system, and at the first sight the trivial result (4.16) seems to be a natural outcome. However, it is interesting to ask whether we still can calculate the total energy of the source, after all, the latter did not disappear physically in the new reference system. We will come back to this question in the next section, whereas here our aim is to analyze the origin of the trivial result (4.16).

In order to do this, we will make use of the object that is often called the "generalized acceleration" [26,27]. Let us take the frame $e_\alpha = h^i_\alpha \partial_i$, dual to the coframe $\vartheta^\alpha = h^\alpha_i dx^i$. The zeroth leg of the frame

$$e_{\hat{0}} = u, \tag{4.17}$$

is usually interpreted as the 4-velocity of an observer, and the total frame then represents a comoving reference system of an observer. The "generalized acceleration" object is defined by

$$\Phi_\alpha{}^\beta := h^\beta_i \frac{\tilde{D} h^i_\alpha}{ds}, \tag{4.18}$$

where $\tilde{D} h^i_\alpha = dh^i_\alpha + \tilde{\Gamma}_j{}^i h^j_\alpha$ is a covariant derivative with respect to the Riemannian (Christoffel) connection. It acts on the vector index ^{i}, whereas the local tetrad index is just a label of the four legs of the frame.

By definition, this object is not a tensor, which is a well known fact. Indeed, by using the standard relation for the components of the connection in different frames, we easily find $h^\beta_i \tilde{D} h^i_\alpha = h^\beta_i dh^i_\alpha + h^\beta_i \tilde{\Gamma}_j{}^i h^j_\alpha = \tilde{\Gamma}_\alpha{}^\beta$. Consequently, we have explicitly

$$\Phi_\alpha{}^\beta = u \rfloor \tilde{\Gamma}_\alpha{}^\beta = \tilde{\Gamma}_{\hat{0}\alpha}{}^\beta. \tag{4.19}$$

In other words, the components of the "generalized acceleration" object coincide with some components of the Riemannian connection.

One can say that the condition

$$\Phi_\alpha{}^\beta = 0 \tag{4.20}$$

defines a kind of inertial reference system. It is easy to see that $\Phi_{\hat{0}}{}^\beta = a^\beta = h^\beta_i a^i$ with $a^i = u^k \tilde{\nabla}_k u^i$ the acceleration. Accordingly, when $\Phi_\alpha{}^\beta = 0$, the observer is "freely falling" without acceleration, and vanishing of the spatial components $\Phi_a{}^b$ ($a, b, \ldots = 1, 2, 3$) means that the comoving triad of an observer is not rotating.

Suppose that we have a reference system (a tetrad) with the property (4.20). Is this a sufficient condition for the energy to vanish? To find this out, we recall that in the pure tetrad formulation the energy is calculated with the help of the superpotential (3.10). We can straightforwardly see how the latter is related to the "generalized acceleration" object. We expand the connection 1-form with respect to the coframe basis, $\tilde{\Gamma}^{\beta\gamma} = \vartheta^\lambda \tilde{\Gamma}_\lambda{}^{\beta\gamma}$, and then find $\tilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = \tilde{\Gamma}_\alpha{}^{\beta\gamma} \eta_{\beta\gamma} + 2\tilde{\Gamma}_\lambda{}^{\beta\lambda} \eta_{\alpha\beta}$. Accordingly, the zeroth-component of the superpotential (3.10) reads

$$\begin{aligned} \tilde{H}_{\hat{0}} &= \frac{1}{2\kappa} (\tilde{\Gamma}_{\hat{0}}{}^{\beta\gamma} \eta_{\beta\gamma} + 2\tilde{\Gamma}_\lambda{}^{\beta\lambda} \eta_{\hat{0}\beta}) \\ &= \frac{1}{2\kappa} (\Phi^{ab} \eta_{ab} + 2\tilde{\Gamma}_b{}^{ab} \eta_{\hat{0}a}). \end{aligned} \tag{4.21}$$

Thus, we see that, contrary to the assumption of [26], the condition (4.20) of the vanishing "generalized acceleration" is not responsible for the "zero-energy" result (4.16). Instead, we find from (4.21) that the vanishing rotation is indeed needed: $\Phi^{ab} = 0$, a, $b = 1, 2, 3$. However, the absence of acceleration a^β is not necessary. In addition, however, one needs a rather curious condition for the 3D trace of the connection $\tilde{\Gamma}_b{}^{ab} = 0$. As a matter of fact, the 6 conditions

$$\Phi^{ab} = 0, \qquad \tilde{\Gamma}_b{}^{ab} = 0, \tag{4.22}$$

can be used to fix the choice of the tetrad, thus eliminating the freedom of the 6-parameter local Lorentz transformations. It is unclear though if this gauge is useful in practice. One can prove by a direct inspection that indeed the condition (4.22) is fulfilled for the freely falling tetrad $\overset{f}{\vartheta}{}^\alpha$. As we will demonstrate later, a similar freely falling tetrad can be constructed also for the Kerr solution.

D. Schwarzschild metric: Free fall in the covariant formulation

Let us now reanalyze the same question in the covariant formulation. We expect that taking appropriately into account the teleparallel connection (that is responsible for the inertial effects, as we already know), it will become possible to clear the gravitational energy of the contributions coming from the noninertial dynamics of the reference system. Indeed, this can be perfectly confirmed by explicit computations as follows.

Using (2.10), we straightforwardly find the teleparallel connection associated with the "freely falling" tetrad $\overset{f}{\vartheta}{}^{\alpha}$. It reads

$$\overset{f}{\Gamma}{}_{\alpha}{}^{\beta} = (\overset{f}{\Lambda}{}^{-1})^{\beta}{}_{\gamma} d\overset{f}{\Lambda}{}^{\gamma}{}_{\alpha}, \tag{4.23}$$

where $\overset{f}{\Lambda}{}^{\alpha}{}_{\beta} = \Lambda^{\alpha}{}_{\gamma}(\Lambda'^{-1})^{\gamma}{}_{\delta}(\Lambda^{-1})^{\delta}{}_{\beta}$. The Weitzenböck torsion for $(\overset{f}{\vartheta}{}^{\alpha}, \overset{f}{\Gamma}{}_{\alpha}{}^{\beta})$ has a rather complicated form, but using it in (3.5) we find for the superpotential

$$H_{\hat{0}} = \frac{2r(\alpha - 1)}{\kappa} \sin\theta d\theta \wedge d\varphi. \tag{4.24}$$

The total energy is thus $P_{\hat{0}} = \int H_{\hat{0}} = M$, as before. It is satisfactory to see that the final result is neither infinity nor zero. The teleparallel connection (4.23) has automatically regularized the situation. The contribution due to the noninertial motion of the "freely falling" reference system (that compensated the gravitational one in the purely tetrad formulation) is now subtracted and the correct total energy of the source is recovered.

V. ENERGY FOR THE ROTATING KERR SOLUTION

Although the Schwarzschild solution is a special case of the Kerr solution, we analyze these cases separately. The reason is that the Kerr metric is essentially more complicated and its study requires some specific techniques, which are not needed in the Schwarzschild case. Moreover, the final formulas are usually very nontrivial for the Kerr configuration and one needs to make various approximations (taking the limit of infinite radius, for example), whereas in the previous section it was possible to give the exact expressions.

In our discussion we use a spherical type local coordinate system (t, r, θ, φ) that is known also as the Boyer-Lindquist coordinate system.

A. Kerr metric: A naive tetrad

We will follow closely the scheme outlined earlier for the Schwarzschild metric, and choose the tetrad in the form

$$\vartheta^{\hat{0}} = \frac{\sqrt{\Sigma\Delta}}{\mathcal{A}} cdt, \qquad \vartheta^{\hat{1}} = \sqrt{\frac{\Sigma}{\Delta}} dr, \qquad \vartheta^{\hat{2}} = \sqrt{\Sigma} d\theta,$$
$$\vartheta^{\hat{3}} = \frac{\sin\theta}{\sqrt{\Sigma}}\left(\mathcal{A} d\varphi - \frac{2amr}{\mathcal{A}} dt\right). \tag{5.1}$$

Here the functions and constants are defined by

$$\Delta := r^2 + a^2 - 2mr, \tag{5.2}$$

$$\Sigma := r^2 + a^2\cos^2\theta, \tag{5.3}$$

$$m := \frac{GM}{c^2}, \tag{5.4}$$

$$\mathcal{A}^2 = \Delta\Sigma + 2mr(r^2 + a^2) \equiv (r^2 + a^2)^2 - a^2\sin^2\theta\Delta. \tag{5.5}$$

As we can immediately check, this tetrad reduces to the diagonal coframe (4.1) when we put the rotation parameter equal to zero: $a = 0$. After noticing this direct relation to the diagonal tetrad of the Schwarzschild solution, we can expect similar problems for the computation of the energy-momentum. As a matter of fact, this is indeed the case.

For the tetrad (5.1), accompanied by the trivial Weitzenböck connection $\Gamma_{\alpha}{}^{\beta} = 0$, we find the superpotential

$$\begin{aligned}\tilde{H}_{\hat{0}} = & \frac{am}{\kappa\mathcal{A}\sqrt{\sigma\Delta}} cdt \wedge (2r\cos\theta dr - \Delta\sin\theta d\theta) \\ & + \frac{\mathcal{A}\cos\theta}{\kappa\sqrt{\sigma\Delta}} dr \wedge d\varphi \\ & - \frac{\sqrt{\Delta}[2r(r^2 + a^2) + (m - r)a^2\sin^2\theta]}{\kappa\mathcal{A}\sqrt{\Sigma}} \\ & \times \sin\theta d\theta \wedge d\varphi.\end{aligned} \tag{5.6}$$

The last term has the leading behavior $\sim 2r$, just like the last term in (4.3), and thus the total energy (calculated as the integral over the sphere of infinite radius) is divergent.

As a check, we can straightforwardly verify that the tetrad (5.1) is not holonomic as such, and the tetrad that is obtained from it by "switching off" gravity (putting $m = 0$ and $a = 0$) is anholonomic too. This means that the inertial effects are again "spoiling" the picture.

The regularization is needed and as before this is achieved with the help of the same teleparallel connection (4.5). Substituting now the pair $(\vartheta^{\alpha}, \Gamma_{\alpha}{}^{\beta})$, where the tetrad is given by (5.1) and the connection by (4.5), into (2.13), we find

$$H_{\hat{0}} = \left(\frac{2m}{\kappa} + \ldots\right) \sin\theta d\theta \wedge d\varphi + \ldots. \tag{5.7}$$

The resulting expressions are rather complicated for the Kerr metric, so from now on we will display only the leading terms, whereas the terms that are proportional to $1/r^n$, $n \geq 1$ will be denoted by the dots. The integral over the spatial boundary then yields

$$P_{\hat{0}} = \int_{\partial S} H_{\hat{0}} = M. \tag{5.8}$$

B. Kerr metric: Proper tetrad

The proper tetrad is constructed along the same lines as we did in the previous section. Since the regularizing teleparallel connection has the form (4.8), we define the proper tetrad as the coframe $\overset{0}{\vartheta}{}^{\alpha}$ that is obtained from (5.1) with the help of the Lorentz transformation (4.9). It is easy to see that the resulting coframe indeed has the required properties: (i) the teleparallel connection is zero (4.11), (ii) the tetrad $\overset{0}{\vartheta}{}^{\alpha}$ becomes holonomic (4.12) when the gravity is "switched off" (for $m = 0$, $a = 0$).

The computation of the energy and momentum for the proper Kerr tetrad is straightforward. The result reads (again giving the leading terms only) as follows:

$$H_{\hat{0}} = \tilde{H}_{\hat{0}} = \frac{2m + (m^2 + a^2 - \frac{1}{2}a^2\sin^2\theta)/r + \mathcal{O}(1/r^2)}{\kappa} \times \sin\theta d\theta \wedge d\varphi + \dots. \tag{5.9}$$

The total energy is finite, $P_{\hat{0}} = \int H_{\hat{0}} = M$. Thus again we prove that for the proper tetrad one does not need a regularization, the result for the total energy-momentum is automatically finite and has the correct value.

C. Kerr metric: Freely falling tetrad

Here we study the possibility to find a noninertial reference system in which gravitation is eliminated by the inertia. Such a generalization of a freely falling tetrad from the Schwarzschild to the Kerr case can be indeed constructed.

As a first step, let us make a local Lorentz transformation

$$\overset{d}{\vartheta}{}^{\alpha} = \overset{d}{\Lambda}{}^{\alpha}{}_{\beta}\vartheta^{\beta}, \tag{5.10}$$

where $\overset{d}{\Lambda}{}^{\alpha}{}_{\beta} = (\Lambda_2)^{\alpha}{}_{\gamma}(\Lambda_1)^{\gamma}{}_{\beta}$, with

$$(\Lambda_1)^{\alpha}{}_{\beta} = \begin{pmatrix} \mathcal{A}/\sqrt{\Delta\Sigma} & \sqrt{2mr(r^2+a^2)}/\sqrt{\Delta\Sigma} & 0 & 0 \\ \sqrt{2mr(r^2+a^2)}/\sqrt{\Delta\Sigma} & \mathcal{A}/\sqrt{\Delta\Sigma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.11}$$

$$(\Lambda_2)^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\Sigma(r^2+a^2)}/\mathcal{A} & 0 & -a\sin\theta\sqrt{2mr}/\mathcal{A} \\ 0 & 0 & 1 & 0 \\ 0 & a\sin\theta\sqrt{2mr}/\mathcal{A} & 0 & \sqrt{\Sigma(r^2+a^2)}/\mathcal{A} \end{pmatrix}. \tag{5.12}$$

This brings us from the original tetrad (5.1) to the coframe

$$\overset{d}{\vartheta}{}^{\hat{0}} = cdt + \frac{\sqrt{2mr(r^2+a^2)}}{\Delta}dr, \tag{5.13}$$

$$\overset{d}{\vartheta}{}^{\hat{1}} = \sqrt{\frac{2mr}{\Sigma}}(cdt - a\sin^2\theta d\varphi) + \frac{\sqrt{\Sigma(r^2+a^2)}}{\Delta}dr, \tag{5.14}$$

$$\overset{d}{\vartheta}{}^{\hat{2}} = \sqrt{\Sigma}d\theta, \tag{5.15}$$

$$\overset{d}{\vartheta}{}^{\hat{3}} = \sin\theta\left(\sqrt{r^2+a^2}d\varphi + \frac{a\sqrt{2mr}}{\Delta}dr\right). \tag{5.16}$$

We will call this new coframe a Doran tetrad because (5.13), (5.14), (5.15), and (5.16) is closely related to an alternative representation of the Kerr metric given by Doran in [28]. We can simplify the above formulas by making the coordinate transformations

$$cdt_d = cdt + \frac{\sqrt{2mr(r^2+a^2)}}{\Delta}dr, \tag{5.17}$$

$$d\varphi_d = d\varphi + \frac{a}{\Delta}\sqrt{\frac{2mr}{r^2+a^2}}dr. \tag{5.18}$$

In the new "Doran coordinates" $(t_d, r, \theta, \varphi_d)$, the Kerr metric will reduce to the form described in [28]. Indeed, the coframe (5.13), (5.14), (5.15), and (5.16) in the new coordinates reads

$$\overset{d}{\vartheta}{}^{\hat{0}} = cdt_d, \tag{5.19}$$

$$\overset{d}{\vartheta}{}^{\hat{1}} = \sqrt{\frac{2mr}{\Sigma}}(cdt_d - a\sin^2\theta d\varphi_d) + \sqrt{\frac{\Sigma}{r^2+a^2}}dr, \tag{5.20}$$

$$\overset{d}{\vartheta}{}^{\hat{2}} = \sqrt{\Sigma}d\theta, \tag{5.21}$$

$$\overset{d}{\vartheta}{}^{\hat{3}} = \sin\theta\sqrt{r^2+a^2}d\varphi_d. \tag{5.22}$$

This exactly reproduces the line element of Doran, see Eq. (18) in [28].

One can verify that the zeroth leg of the dual frame

$$u = \overset{d}{e}_{\hat{0}} = \frac{\mathcal{A}}{c\Sigma\Delta}\partial_t - \frac{\sqrt{2mr(r^2+a^2)}}{\Sigma}\partial_r + \frac{2amr}{\Sigma\Delta}\partial_\varphi = \frac{1}{c}\partial_{t_d} - \frac{\sqrt{2mr(r^2+a^2)}}{\Sigma}\partial_r \tag{5.23}$$

is a geodesic vector field, the acceleration is zero $u^i\nabla_i u^j = 0$, or $\Phi_{\hat{0}}{}^a = 0$. However, this frame has a nontrivial rotation, namely

$$\Phi^{\hat{1}\hat{2}} = a^2 \sin\theta\cos\theta \frac{\sqrt{2mr}}{\Sigma^2}. \tag{5.24}$$

This reference system thus does not satisfy the "compensation conditions" (4.22).

However, we can improve the situation if we make an additional Lorentz transformation

$$\overset{f}{\vartheta}{}^\alpha = (\Lambda_4)^\alpha{}_\gamma (\Lambda_3)^\gamma{}_\beta \overset{d}{\vartheta}{}^\beta, \tag{5.25}$$

where the matrices

$$(\Lambda_3)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\sqrt{(r^2+a^2)/\Sigma} & -\sin\theta r/\sqrt{\Sigma} & 0 \\ 0 & \sin\theta r/\sqrt{\Sigma} & \cos\theta\sqrt{(r^2+a^2)/\Sigma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.26}$$

and

$$(\Lambda_4)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\varphi_d & -\sin\varphi_d \\ 0 & 0 & \sin\varphi_d & \cos\varphi_d \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.27}$$

Note that the transformation (5.26) is the same in both coordinate systems, in the original (t, r, θ, φ) and in the Doran coordinates $(t_d, r, \theta, \varphi_d)$. However the transformation (5.27) refers to the Doran coordinate system. Of course one can apply it also in the original coordinates, but keep in mind that φ_d is a function defined by the integral from (5.18).

One can verify that the "generalized acceleration" object vanishes for the final coframe, $\Phi_\alpha{}^\beta = 0$. This means that indeed the resulting tetrad does not have acceleration and rotation, i.e., it describes a "freely falling" reference system. However, still $\tilde{\Gamma}_b{}^{ab} \neq 0$ for this frame. Namely,

$$\tilde{\Gamma}_b{}^{\hat{1}b} = \frac{a\sin\varphi_d \sin\theta}{\Sigma}\sqrt{\frac{2m}{r}}, \tag{5.28}$$

$$\tilde{\Gamma}_b{}^{\hat{2}b} = \frac{-a\cos\varphi_d \sin\theta}{\Sigma}\sqrt{\frac{2m}{r}} \tag{5.29}$$

As a result, the conditions (4.22) are not satisfied for this frame, and the tetrad energy-momentum does not vanish, in contrast to (4.16). Instead, the direct computation of the $\tilde{H}_\alpha$ for the final coframe $\overset{f}{\vartheta}{}^\alpha$ yields:

$$\tilde{H}_{\hat{0}} = a\sin\theta\left[\frac{2m}{\Sigma}(c\,dt_d - a\sin 2\theta\, d\varphi_d) + \sqrt{\frac{2m}{r(r^2+a^2)}}dr\right] \wedge d\theta, \tag{5.30}$$

$$\tilde{H}_{\hat{1}} = \frac{\sin^2\theta}{\Sigma}[\cos\varphi_d\sqrt{2mr}(a^2 - 2r^2(r^2+a^2)/\Sigma) + \sin\varphi_d am\sqrt{r^2+a^2}]d\varphi_d \wedge d\theta + \ldots, \tag{5.31}$$

$$\tilde{H}_{\hat{2}} = \frac{\sin^2\theta}{\Sigma}[\sin\varphi_d\sqrt{2mr}(a^2 - 2r^2(r^2+a^2)/\Sigma) - \cos\varphi_d am\sqrt{r^2+a^2}]d\varphi_d \wedge d\theta + \ldots, \tag{5.32}$$

$$\tilde{H}_{\hat{3}} = -\frac{\sin\theta\cos\theta}{\Sigma^2}\sqrt{m}[2r(r^2+a^2)]^{3/2} d\varphi_d \wedge d\theta + \ldots. \tag{5.33}$$

The expression (5.30) is exact, whereas in (5.31), (5.32), and (5.33) the dots denote other terms which are irrelevant for the computation of the total conserved quantities in a sphere of an arbitrary radius. It is easy to see that integration over the spherical angles ($\int_0^{2\pi} d\varphi_d \int_0^\pi d\theta$, note that $d\varphi_d = d\varphi$ on a sphere) yields zero for the components (5.31), (5.32), and (5.33).

Thus, the local energy density is nontrivial despite the fact that the tetrad is nonaccelerating and nonrotating. Nevertheless, since in the limit of the large radius we have the leading behavior $\tilde{H}_{\hat{0}} \cong r^{-2}$, the *total energy* is obviously zero for this coframe:

$$P_{\hat{0}} = \int \tilde{H}_{\hat{0}} = 0. \tag{5.34}$$

This result again originates from the contribution of the essentially noninertial behavior of the reference system.

D. Kerr metric: Covariant treatment of the freely falling frame

As in the Schwarzschild case, the situation can be improved if the reanalyze the same problem in the covariant teleparallel framework. In this case we start with the original coframe (5.1) and the corresponding regularizing teleparallel connection (4.5), and construct for the final coframe $\overset{f}{\vartheta}{}^{\alpha}$ the corresponding Weitzenböck connection from the transformation law (2.10):

$$\overset{f}{\Gamma}{}_{\alpha}{}^{\beta} = (\overset{f}{\Lambda}{}^{-1})^{\mu}{}_{\alpha}\Gamma_{\mu}{}^{\nu}\overset{f}{\Lambda}{}^{\beta}{}_{\nu} + \overset{f}{\Lambda}{}^{\beta}{}_{\gamma}d(\overset{f}{\Lambda}{}^{-1})^{\gamma}{}_{\alpha}, \qquad (5.35)$$

where $\overset{f}{\Lambda}{}^{\alpha}{}_{\beta} = (\Lambda_4)^{\alpha}{}_{\mu}(\Lambda_3)^{\mu}{}_{\nu}(\Lambda_2)^{\nu}{}_{\lambda}(\Lambda_1)^{\lambda}{}_{\beta}$.

The Weitzenböck torsion for the final pair of fields $(\overset{f}{\vartheta}{}^{\alpha}, \overset{f}{\Gamma}{}_{\alpha}{}^{\beta})$ is much more complicated than that of the Schwarzschild metric. Nevertheless, it is straightforward to substitute it in (2.13) and (3.5) and to get the superpotential

$$H_{\hat{0}} = \frac{2m + (3m^2 + a^2 - \frac{3}{2}a^2\sin^2\theta)/r + \mathcal{O}(1/r^2)}{\kappa} \times \sin\theta d\theta \wedge d\varphi + \ldots . \qquad (5.36)$$

The total energy is thus $P_{\hat{0}} = \int H_{\hat{0}} = M$, as before. We see again that the end result is neither infinity nor zero. The teleparallel connection (5.35) has automatically regularized the energy for the Kerr solution in the same way it worked for the Schwarzschild case. Namely, the contribution due to the noninertial motion of the generalized "freely falling" reference system (in which gravity is locally eliminated) is again correctly subtracted and the physically meaningful value for the total energy of the source is again recovered.

VI. DISCUSSION AND CONCLUSION

Although equivalent to general relativity, teleparallel gravity has several conceptual differences with respect to general relativity. One of these differences is that the Weitzenböck connection represents only inertial effects related to the frame. As a consequence of this property, one can separate gravitation from inertial effects. It becomes then possible to write down a tensorial expression for the energy-momentum density of gravity. Because of the fact that the frame-related inertial contribution to the conserved quantities is always properly subtracted by the Weitzenböck connection, the covariant teleparallel approach naturally yields regularized solutions for the energy and momentum.

As a test of the regularizing property of teleparallelism, we have considered in this paper two concrete examples: the Schwarzschild and Kerr solutions. For these two important cases, we have computed the total energy for different frames, and have shown that the covariant teleparallel approach always yields the physically correct result. We can thus say that the Weitzenböck connection acts as a regularizing tool which separates the inertial energy-momentum density, leaving the tensorial, physical energy-momentum density of the system untouched.

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