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Shearfree cylindrical gravitational collapse

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We consider diagonal cylindrically symmetric metrics, with an interior representing a general nonrotating fluid with anisotropic pressures. An exterior vacuum Einstein-Rosen spacetime is matched to this using Darmois matching conditions. We show that the matching conditions can be explicitly solved for the boundary values of metric components and their derivatives, either for the interior or exterior. Specializing to shearfree interiors, a static exterior can only be matched to a static interior, and the evolution in the nonstatic case is found to be given in general by an elliptic function of time. For a collapsing shearfree isotropic fluid, only a Robertson-Walker dust interior is possible, and we show that all such cases were included in Cocke's discussion. For these metrics, Nolan and Nolan have shown that the matching breaks down before collapse is complete, and Tod and Mena have shown that the spacetime is not asymptotically flat in the sense of Berger, Chrusciel, and Moncrief. The issues about energy that then arise are revisited, and it is shown that the exterior is not in an intrinsic gravitational or superenergy radiative state at the boundary.

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I. INTRODUCTION AND SUMMARY

Many papers have considered cylindrical solutions, with or without matching: see e.g. [1–4] and Stephani *et al.* [5], Chap. 22. Here we take diagonal metrics in both the interior and exterior, and initially allow an anisotropic fluid interior. The exterior is a vacuum Einstein-Rosen (ER) solution [6]. Our aim was to develop a solution where one could explicitly see the relation between source motion and gravitational radiation, albeit in a physically unrealistic case. We were therefore interested in collapse, though for nonstatic cases one can easily reverse the sense of time so that collapse becomes expansion and vice versa.

We set out the metrics and the Darmois matching conditions (which preclude surface shells in the boundary) for a timelike boundary in Secs. II and III, and show that the junction conditions can be explictly solved for the boundary values for the exterior in terms of interior quantities and vice versa. This extends the work of [7].

Specializing to the shearfree case in Sec. IV, we are able to give a first-order ordinary differential equation for the time evolution of the interior whose solution is in general an elliptic function. It is shown that a static exterior implies a static interior.

Further specializing to a isotropic fluid in Sec. V, we prove that only a Robertson-Walker (RW) dust interior is possible and that all such interiors are included in the discussion of Cocke [8]. The matching then leads in Sec. VI to specific behavior of the ER functions at the boundary. However, previous work of Nolan and Nolan [9] shows that the matching breaks down before collapse is complete, and Tod and Mena [10] showed that the solutions cannot be asymptotically flat in the sense of Berger, Chrusciel, and Moncrief [11], which makes them unsatisfactory for our purposes.

Finally in Sec. VII we consider whether there are waves in the exterior by asking if there is energy transport. Because a cylindrically symmetric spacetime cannot be asymptotically flat, we cannot employ the usual global definition of radiation for isolated bodies due to [12], and various alternatives are discussed. We conclude that the exterior of a cylindrical region of a collapsing RW dust solution cannot be in an intrinsic gravitational or superenergy radiative state at the boundary, and infer that no radiation is transferred to or from the interior.

The specializations made here did not lead to solutions of the type we hoped for. This is a consequence of the additional restrictions imposed in the hope of avoiding the full complexity of the problem in the general anisotropic case. Nevertheless we believe that it is necessary to have the results obtained in these more restricted cases as a first step: we hope in the future to undertake further study of the shearing dust and shearfree anisotropic fluid interiors,

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which may establish whether such results as the breakdown of the matching in the Friedmann-Robertson-Walker (FRW) case hold more generally.

II. COLLAPSING ANISOTROPIC FLUID CYLINDERS

We consider a collapsing cylinder filled with anisotropic nondissipative fluid bounded by a timelike cylindrical surface Σ and with energy-momentum tensor given by

$$T^{-}_{\alpha\beta} = (\mu + P_r)V_\alpha V_\beta + P_r g_{\alpha\beta} + (P_z - P_r)S_\alpha S_\beta + (P_\phi - P_r)K_\alpha K_\beta, \tag{1}$$

where μ is the energy density, P_r, P_z, and P_ϕ are the principal stresses and V_α, S_α, and K_α are vectors satisfying

$$V^\alpha V_\alpha = -1, \qquad S^\alpha S_\alpha = K^\alpha K_\alpha = 1, \qquad V^\alpha S_\alpha = V^\alpha K_\alpha = S^\alpha K_\alpha = 0. \tag{2}$$

We assume the general time dependent diagonal nonrotating cylindrically symmetric metric

$$ds^2_- = -A^2(dt^2 - dr^2) + B^2 dz^2 + C^2 d\phi^2, \tag{3}$$

where A, B, and C are functions of t and r. To represent cylindrical symmetry, we impose the following ranges on the coordinates:

$$-\infty \leq t \leq \infty, \qquad 0 \leq r, \qquad -\infty < z < \infty, \qquad 0 \leq \phi \leq 2\pi, \tag{4}$$

where we assume $C = 0$ at $r = 0$ which is a nonsingular axis. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = z$, and $x^3 = \phi$, and we choose the fluid to be comoving in this coordinate system; hence from (2) and (3)

$$V_\alpha = -A\delta^0_\alpha, \qquad S_\alpha = B\delta^2_\alpha, \qquad K_\alpha = C\delta^3_\alpha. \tag{5}$$

Calculating the motion of the fluid according to its expansion Θ and shear $\sigma_{\alpha\beta}$,

$$\Theta = V^\alpha{}_{;\alpha}, \tag{6}$$

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + V_{(\alpha;\gamma}V^\gamma V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta), \tag{7}$$

by using (3) and (4) we obtain for the expansion,

$$\Theta = \frac{1}{A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{8}$$

and for the nonzero components of the shear,

$$\sigma_{11} = \frac{A}{3}\left(2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{9}$$

$$\sigma_{22} = \frac{B^2}{3A}\left(2\frac{\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right), \tag{10}$$

$$\sigma_{33} = \frac{C^2}{3A}\left(2\frac{\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right), \tag{11}$$

where the overdot stands for differentiation with respect to t. The Einstein field equations, $G_{\alpha\beta} = \kappa T_{\alpha\beta}$, for (1), (3), and (5), have the nonzero components,

$$G^-_{00} = \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{B''}{B} - \frac{C''}{C} + \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) - \frac{B'}{B}\frac{C'}{C} = \kappa\mu A^2, \tag{12}$$

$$G^-_{01} = -\frac{\dot{B}'}{B} - \frac{\dot{C}'}{C} + \frac{\dot{A}}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) \tag{13}$$

$$+ \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\frac{A'}{A} = 0, \tag{14}$$

$$G^-_{11} = -\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) + \frac{B'}{B}\frac{C'}{C} = \kappa P_r A^2, \tag{15}$$

$$G^-_{22} = \left(\frac{B}{A}\right)^2\left[-\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} + \left(\frac{\dot{A}}{A}\right)^2 + \frac{A''}{A} + \frac{C''}{C} - \left(\frac{A'}{A}\right)^2\right] = \kappa P_z B^2, \tag{16}$$

$$G^-_{33} = \left(\frac{C}{A}\right)^2\left[-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \left(\frac{\dot{A}}{A}\right)^2 + \frac{A''}{A} + \frac{B''}{B} - \left(\frac{A'}{A}\right)^2\right] = \kappa P_\phi C^2, \tag{17}$$

where the prime stands for differentiation with respect to r.

From (1), (2), and (5), we have the nontrivial components of the Bianchi identities $T^\alpha{}_{\beta;\alpha} = 0$,

$$\dot{\mu} + (\mu + P_r)\frac{\dot{A}}{A} + (\mu + P_z)\frac{\dot{B}}{B} + (\mu + P_\phi)\frac{\dot{C}}{C} = 0, \tag{18}$$

$$P_r' + (\mu + P_r)\frac{A'}{A} + (P_r - P_z)\frac{B'}{B} + (P_r - P_\phi)\frac{C'}{C} = 0. \tag{19}$$

For the exterior we take an ER spacetime [6]

$$ds^2_+ = -e^{2(\gamma-\psi)}(dT^2 - dR^2) + e^{2\psi}dz^2 + e^{-2\psi}R^2 d\phi^2, \tag{20}$$

where γ and ψ are functions of T and R, and for the vacuum field equations we have

$$\psi_{,TT} - \psi_{,RR} - \frac{\psi_{,R}}{R} = 0, \tag{21}$$

and

$$\gamma_{,T} = 2R\psi_{,T}\psi_{,R}, \qquad \gamma_{,R} = R(\psi_{,T}^2 + \psi_{,R}^2). \tag{22}$$

Equation (21) is the cylindrically symmetric wave equation in an Euclidean spacetime, suggesting the presence of a gravitational wave field.

III. JUNCTION CONDITIONS

As the boundary must be comoving with the fluid interior, it will be timelike, and given by $r =$ constant in the interior metric (3) and a curve $R(T)$ in the ER metric (20). Matching the collapsing cylinder at Σ to the ER spacetime, Darmois's junction conditions [7,13] give us, after a little algebra,

$$e^{\gamma-\psi}\left[1 - \left(\frac{dR}{dT}\right)^2\right]^{1/2} dT \overset{\Sigma}{=} Adt \overset{\Sigma}{=} d\tau, \tag{23}$$

$$e^{\psi} \overset{\Sigma}{=} B, \tag{24}$$

$$R \overset{\Sigma}{=} BC, \tag{25}$$

$$P_r \overset{\Sigma}{=} 0, \tag{26}$$

$$e^{\psi}(R_{,\tau}\psi_{,T} + T_{,\tau}\psi_{,R}) \overset{\Sigma}{=} \frac{B'}{A}, \tag{27}$$

$$T_{,\tau} \overset{\Sigma}{=} \frac{(BC)'}{A}. \tag{28}$$

In this form it is easy to see that if the exterior is known, the boundary values of B and C, and $1/A$ times their first derivatives with respect to t and r (equivalently, the derivatives with respect to proper time in the surface and proper distance orthogonal to it), can be solved for. The value of A is not fixed, as one can redefine the t and r coordinates, but its evolution can be found from the interior field equations.

We note also that in order for the surface to be timelike we require $(dR/dT)^2 < 1$.

One can reorganize these equations, together with (22), to conversely give the exterior functions' values on the boundary in terms of the interior. Rewriting (23), and differentiating (24) and (25) with respect to τ in the surface, we obtain

$$e^{2\gamma-2\psi}(T_{,\tau}^2 - R_{,\tau}^2) \overset{\Sigma}{=} 1, \tag{29}$$

$$e^{\psi}(T_{,\tau}\psi_{,T} + R_{,\tau}\psi_{,R}) \overset{\Sigma}{=} \frac{\dot{B}}{A}, \tag{30}$$

$$R_{,\tau} \overset{\Sigma}{=} \frac{(BC)^{\cdot}}{A}. \tag{31}$$

Solving (30) and (27) for $\psi_{,T}$ and $\psi_{,R}$, and substituting in (29), the results are (24) and

$$\psi_{,T} \overset{\Sigma}{=} \frac{B_{,t}(BC)_{,r} - B_{,r}(BC)_{,t}}{B[(BC)_{,r}^2 - (BC)_{,t}^2]}, \tag{32}$$

$$\psi_{,R} \overset{\Sigma}{=} \frac{B_{,r}(BC)_{,r} - B_{,t}(BC)_{,t}}{B[(BC)_{,r}^2 - (BC)_{,t}^2]}, \tag{33}$$

$$e^{\gamma} \overset{\Sigma}{=} \frac{AB}{[(BC)_{,r}^2 - (BC)_{,t}^2]^{1/2}}, \tag{34}$$

$$\gamma_{,T} \overset{\Sigma}{=} \frac{2C[B_{,t}(BC)_{,r} - B_{,r}(BC)_{,t}][(BC)_{,r}B_{,r} - (BC)_{,t}B_{,t}]}{B[(BC)_{,r}^2 - (BC)_{,t}^2]^2}, \tag{35}$$

$$\gamma_{,R} \overset{\Sigma}{=} \frac{C\{[B_{,t}(BC)_{,r} - B_{,r}(BC)_{,t}]^2 + [(BC)_{,r}B_{,r} - (BC)_{,t}B_{,t}]^2\}}{B[(BC)_{,r}^2 - (BC)_{,t}^2]^2}. \tag{36}$$

The radius $\mathcal{R}$ of the collapsing cylinder as measured by the circumference in the exterior ER spacetime is given by

$$\mathcal{R} \overset{\Sigma}{=} e^{-\psi}R \overset{\Sigma}{=} C. \tag{37}$$

IV. SHEARFREE COLLAPSING SOLUTION

Now let the motion of the collapsing cylindrical fluid be shearfree, $\sigma_{\alpha\beta} = 0$. Then we can integrate (9)–(11) and obtain

$$B = b(r)A, \qquad C = c(r)A, \tag{38}$$

where $b(r)$ and $c(r)$ are arbitrary functions of r and the metric (3) becomes

$$ds_-^2 = A^2(-dt^2 + dr^2 + b^2dz^2 + c^2d\phi^2). \tag{39}$$

We observe that by contrast the shearfree condition in a collapsing spherical distribution of matter in a comoving frame leaves two unknown functions of time and radial coordinate in the metric [14].

Substituting (39) into (14) we obtain

$$\frac{\dot{A}'}{A} - 2\frac{\dot{A}}{A}\frac{A'}{A} = 0, \tag{40}$$

which can be integrated producing

$$A = \frac{1}{w(t) + a(r)}, \tag{41}$$

where w is an arbitrary function of t and a is an arbitrary function of r. The expansion (8) for (39) and (41) becomes

$$\Theta = -3\dot{w}, \tag{42}$$

which, as for the shearfree isotropic fluid spherical collapse, depends also only on t [14].

The field equations (12)–(17) with (39) and (41) become

$$\kappa\mu = 3\dot{w}^2 + 2(w+a)a'' - 3a'^2 + 2(w+a)a'\left(\frac{b'}{b} + \frac{c'}{c}\right) - (w+a)^2\left(\frac{b''}{b} + \frac{c''}{c} + \frac{b'}{b}\frac{c'}{c}\right), \tag{43}$$

$$\kappa P_r = 2(w+a)\ddot{w} - 3\dot{w}^2 + 3a'^2 - 2(w+a)a'\left(\frac{b'}{b} + \frac{c'}{c}\right) + (w+a)^2\frac{b'}{b}\frac{c'}{c}, \tag{44}$$

$$\kappa P_z = 2(w+a)\ddot{w} - 3\dot{w}^2 - 2(w+a)a'' + 3a'^2 - 2(w+a)a'\frac{c'}{c} + (w+a)^2\frac{c''}{c}, \tag{45}$$

$$\kappa P_\phi = 2(w+a)\ddot{w} - 3\dot{w}^2 - 2(w+a)a'' + 3a'^2 - 2(w+a)a'\frac{b'}{b} + (w+a)^2\frac{b''}{b}. \tag{46}$$

From the junction condition (26) we have that on the boundary (44) is given by

$$2\Omega\ddot{\Omega} - 3\dot{\Omega}^2 + c_2\Omega^2 + c_1\Omega + c_0 \overset{\Sigma}{=} 0, \tag{47}$$

where

$$\Omega \overset{\Sigma}{=} w + a, \qquad c_0 \overset{\Sigma}{=} 3a'^2, \qquad c_1 \overset{\Sigma}{=} -2a'\left(\frac{b'}{b} + \frac{c'}{c}\right), \qquad c_2 \overset{\Sigma}{=} \frac{b'c'}{bc}. \tag{48}$$

We can integrate (47) producing

$$\dot{\Omega}^2 = w_0\Omega^3 + c_2\Omega^2 + \frac{c_1}{2}\Omega + \frac{c_0}{3}, \tag{49}$$

where w_0 is an integration constant. Thus in general the time dependence is given by an elliptic function whose parameters are fixed by values at the boundary.

If the exterior spacetime is static, $\psi_{,T} = 0$, then the field equations (21) and (22) reduce to the static Levi-Civita spacetime,

$$e^{\psi} = \frac{a_1}{R^{a_2}}, \tag{50}$$

and

$$e^{\gamma} = R^{a_2^2}, \tag{51}$$

where a_1 and a_2 are integration constants: for a positive mass source we need $a_2 > 0$ [15,16]. For the solution (39) and (41) we have the following relations on Σ. From (24) and (50) we have

$$\frac{a_1}{R^{a_2}} \overset{\Sigma}{=} \frac{b}{w+a}, \tag{52}$$

and from (24) and (25)

$$R \overset{\Sigma}{=} \frac{bc}{(w+a)^2}. \tag{53}$$

With (52) and (53) we obtain

$$a_1(w+a)^{2a_2+1} \overset{\Sigma}{=} b^{a_2+1}c^{a_2}; \tag{54}$$

since w is a function of t this relation can hold only if w is constant. Hence, for shearfree cylindrically symmetric anisotropic fluids if the exterior spacetime is static, i.e. the Levi-Civita spacetime, the cylindrical source must be static too.

Not all the junction conditions have been used, since no specific model of the interior has been given (and static shearfree fluid solutions may not all contain a suitable matching surface—for instance they may not contain a surface where $P_r = 0$). However, the remaining junction conditions can be satisfied for anisotropic and isotropic fluids, shells, and other suitable choices of interior (see, for example, [3,16–18]).

V. CYLINDRICALLY COLLAPSING ISOTROPIC FLUID

If the collapsing cylinder is filled with isotropic fluid, then $P_r = P_z = P_\phi$ and from (44)–(46) we have

$$(w+a)\left[2\left(a'' - a'\frac{b'}{b}\right) - (w+a)\left(\frac{c''}{c} - \frac{b'}{b}\frac{c'}{c}\right)\right] = 0, \tag{55}$$

$$(w+a)\left[2\left(a'' - a'\frac{c'}{c}\right) - (w+a)\left(\frac{b''}{b} - \frac{b'}{b}\frac{c'}{c}\right)\right] = 0. \tag{56}$$

Then from (55) and (56), assuming w has nontrivial time dependence,

$$a'' = a'\frac{b'}{b} = a'\frac{c'}{c}, \qquad \frac{b''}{b} = \frac{c''}{c} = \frac{b'}{b}\frac{c'}{c}, \tag{57}$$

which can be shown by direct calculation to reduce the Weyl tensor to $C_{\alpha\beta\gamma\delta} = 0$; i.e. the spacetime inside the cylinder is conformally flat. All conformally flat perfect fluid solutions are known, and all are shearfree [5]. If a conformally flat perfect fluid has a barotropic equation of state, then (Trümper, cited in [19]) it is RW. We now show directly that this is the case, i.e. that the interior must be RW and thus conformally flat, without assuming barotropy, using regularity at the axis instead.

If $a' \neq 0$, then we must have

$$\frac{a''}{a'} = \frac{b'}{b} = \frac{c'}{c},$$

from which it easily follows, using (57), that a, b, and c are all proportional to $e^{\beta r}$ for some nonzero constant β. This is not consistent with having a nonsingular axis where $C = 0$ at $r = 0$.

If $a' = 0$ then we can set $a = 0$ by redefining w, and from (57) we find that b'/c and c'/b are each constant, whence

$$\frac{b'}{b}\frac{c'}{c} = -\epsilon, \tag{58}$$

where ϵ is a constant. Writing $A = 1/w$ in the field equations (43)–(46), they become

$$3(A_{,\tau}^2 + \epsilon) = \kappa\mu A^2, \tag{59}$$

$$-2AA_{,\tau\tau} - A_{,\tau}^2 - \epsilon = \kappa P A^2, \tag{60}$$

which are easily recognizable as the usual equations for RW spacetimes, ϵ being the spatial curvature parameter usually denoted k. It is known that the most general solutions with zero shear, rotation, and acceleration are the Robertson-Walker solutions (see e.g. Ellis [19], p. 135). Moreover from the junction condition (26), i.e. $P \overset{\Sigma}{=} 0$, we have $P = 0$ and consequently the fluid is a homogeneous collapsing dust, which has no acceleration, i.e. a Friedman solution.

Hence we can state that a collapsing cylinder with a nonsingular axis filled with shearfree irrotational isotropic fluid must be an RW solution and if it is matched to an ER solution the fluid must be dust. We again compare our result to the corresponding isotropic spherical shearfree collapse [14]. There the general solution cannot be obtained since for the complete integration of the system further equations of state are required [e.g. an equation of state of the form $P = P(\mu)$ [20]].

Now we show that the form for this discussed by Cocke [8] is the most general one by direct coordinate transformations. One could reach the final metric form more immediately by integrating (57) and imposing regularity at the axis: the extra information below is that of the coordinate transformations.

The RW metric can be expressed in spherical coordinates as

$$ds^2 = -dt^2 + A^2\left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{61}$$

where A is a function only of t, $k = -1$, 0, or 1, and the ranges of the coordinates are

$$-\infty \leq t \leq \infty, \qquad 0 \leq \bar{r} < \infty, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \phi \leq 2\pi, \tag{62}$$

except when $k = 1$ where instead $0 \leq \bar{r} < 1$ and $\bar{r} = 1$ is the antipode of the origin. To write (61) in cylindrical coordinates we make the transformation

$$\hat{r} = \bar{r}\sin\theta, \qquad z = f(\bar{r}, \theta), \tag{63}$$

which yields

$$d\hat{r} = \sin\theta d\bar{r} + \bar{r}\cos\theta d\theta, \qquad dz = f_{,\bar{r}}d\bar{r} + f_{,\theta}d\theta. \tag{64}$$

The inverse transformation is

$$d\bar{r} = \frac{f_{,\theta}d\hat{r} - \bar{r}\cos\theta dz}{\sin\theta f_{,\theta} - \bar{r}\cos\theta f_{,\bar{r}}}, \qquad d\theta = \frac{\sin\theta dz - f_{,\bar{r}}d\hat{r}}{\sin\theta f_{,\theta} - \bar{r}\cos\theta f_{,\bar{r}}}. \tag{65}$$

In order to write (61) transformed by (63) we first observe that

$$\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\theta^2 = \frac{[(\frac{f_{,\theta}^2}{1-k\bar{r}^2} + \bar{r}^2 f_{,\bar{r}}^2)d\hat{r}^2 - 2\bar{r}(\frac{\cos\theta f_{,\theta}}{1-k\bar{r}^2} + \bar{r}\sin\theta f_{,\bar{r}})d\hat{r}dz + \frac{\bar{r}^2(1-k\bar{r}^2\sin^2\theta)}{1-k\bar{r}^2}dz^2]}{(\sin\theta f_{,\theta} - \bar{r}\cos\theta f_{,\bar{r}})^2}. \tag{66}$$

Imposing the requirement that the transformation does not produce cross terms we must choose

$$f_{,\theta} = -\bar{r}(1 - k\bar{r}^2)\frac{\sin\theta}{\cos\theta}f_{,\bar{r}}. \tag{67}$$

Substituting (67) back into (66) we obtain

$$\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\theta^2 = \frac{[d\hat{r}^2 + \frac{\cos^2\theta}{(1-k\bar{r}^2)f_{,\bar{r}}^2}dz^2]}{1 - k\bar{r}^2\sin^2\theta}. \tag{68}$$

With (63) and (68) we can write (61) as

$$ds^2 = -dt^2 + A^2\left(\frac{d\hat{r}^2}{1 - k\hat{r}^2} + h^2 dz^2 + \hat{r}^2 d\phi^2\right), \tag{69}$$

where

$$h = \frac{\cos\theta}{(1 - k\bar{r}^2\sin^2\theta)^{1/2}(1 - k\bar{r}^2)^{1/2}f_{,\bar{r}}}. \tag{70}$$

Then from (67) and (70) we have

$$f_{,\bar{r}} = \frac{\cos\theta}{h(1 - k\bar{r}^2\sin^2\theta)^{1/2}(1 - k\bar{r}^2)^{1/2}}, \tag{71}$$

$$f_{,\theta} = -\frac{\bar{r}\sin\theta(1 - k\bar{r}^2)^{1/2}}{h(1 - k\bar{r}^2\sin^2\theta)^{1/2}}. \tag{72}$$

Requiring that h is a function only of $\hat{r}$ we have

$$h_{,\bar{r}} = \sin\theta\frac{dh}{d\hat{r}}, \qquad h_{,\theta} = \bar{r}\cos\theta\frac{dh}{d\hat{r}}. \tag{73}$$

Differentiating (71) and (72) with respect to θ and $\bar{r}$, respectively, and using (73) we obtain

$$f_{,\bar{r}\theta} = -\frac{1}{h(1-k\bar{r}^2\sin^2\theta)^{1/2}(1-k\bar{r}^2)^{1/2}} \times \left[\frac{(1-k\bar{r}^2)\sin\theta}{1-k\bar{r}^2\sin^2\theta} + \frac{\bar{r}\cos^2\theta}{h}\frac{dh}{d\hat{r}}\right], \quad (74)$$

$$f_{,\theta\bar{r}} = -\frac{\sin\theta}{h(1-k\bar{r}^2\sin^2\theta)^{1/2}}\left[\frac{1-2k\bar{r}^2}{(1-k\bar{r}^2\sin^2\theta)^{1/2}} + \frac{k\bar{r}^2\sin^2\theta(1-k\bar{r}^2)^{1/2}}{1-k\bar{r}^2\sin^2\theta} - \frac{\bar{r}\sin\theta(1-k\bar{r}^2)^{1/2}}{h}\frac{dh}{d\hat{r}}\right]. \quad (75)$$

Equating the two expressions (74) and (75), i.e. imposing the integrability condition $f_{,\bar{r}\theta} = f_{,\theta\bar{r}}$, and using (63) we obtain

$$\frac{1}{h}\frac{dh}{d\hat{r}} = -\frac{k\hat{r}}{1-k\hat{r}^2}, \quad (76)$$

and integrating this we obtain

$$h = \gamma(1-k\hat{r}^2)^{1/2}, \quad (77)$$

where γ is an integration constant. Now substituting (77) into (69) and rescaling z we finally have

$$ds^2 = -dt^2 + A^2\left[\frac{d\hat{r}^2}{1-k\hat{r}^2} + (1-k\hat{r}^2)dz^2 + \hat{r}^2 d\phi^2\right]. \quad (78)$$

The metric form (78) was obtained by Cocke [8], but here we have proved that it is the general RW metric in cylindrical coordinates.

In order to write (78) in the coordinate system employed in (3) we rescale t and transform $\hat{r}$ so that

$$r = \int \frac{d\hat{r}}{(1-k\hat{r}^2)^{1/2}}, \quad (79)$$

and we obtain

$$ds^2 = A^2(t)(-dt^2 + dr^2 + g'^2 dz^2 + g^2 d\phi^2), \quad (80)$$

or

$$ds^2 = -d\tau^2 + A^2(\tau)(dr^2 + g'^2 dz^2 + g^2 d\phi^2), \quad (81)$$

with $g = \sinh r$, r, or $\sin r$ accordingly as $k = -1, 0$, or 1.

The metric (39), with (41) and (58), is the general RW metric (80) where $k = \epsilon$ and $w = 1/A$, and it satisfies (59) and (60).

VI. MATCHING FRW SPACETIME TO ER SPACETIME

Matchings between a cylindrical homogeneous perfect fluid interior and a vacuum exterior (or vice versa) have been studied by Mena, Tavakol, and Vera [21], who showed that matching to a static vacuum is impossible which is a special case of our result above, by Nolan and Nolan [9], and by Tod and Mena [10]. In the last of these papers, the matching of ER and $k = 0$ FRW metrics is given, using coordinates in which the exterior metric takes the form

$$ds_+^2 = -e^{2(\gamma-\psi)}(d\hat{T}^2 - d\rho^2) + e^{2\psi}dz^2 + e^{-2\psi}R^2 d\phi^2, \quad (82)$$

where $R = R(\hat{T}, \rho)$: the main difference from our treatment is that the coordinates $(\hat{T}, \rho)$ are chosen so that the boundary is at $\rho =$ constant. Here $\hat{T}$ has replaced the T of the original paper to avoid confusion. Tod and Mena [10] study the global structure and conclude that the spacetime is not asymptotically flat in the sense of Berger, Chrusciel, and Moncrief [11], but instead has a singular Cauchy horizon.

In [9], the matching of a general cylindrically symmetric vacuum to FRW is studied: the exterior is then shown, as one might expect, to be of ER form. Here we shall find the trajectory of the boundary as an equation relating T and R, and then display the conditions satisfied there by γ and ψ. Then we comment further on the results of Tod and Mena [10] and Nolan and Nolan [9].

The well-known solutions to (59) and (60) are

$$k = 1, \qquad A = m\sin^2\Psi/2, \qquad 2\tau = m(\Psi - \sin\Psi), \quad (83)$$

$$k = 0, \qquad A = (3\sqrt{m}\tau/2)^{2/3}, \quad (84)$$

$$k = -1, \qquad A = m\sinh^2\Psi/2, \qquad 2\tau = m(\sinh\Psi - \Psi), \quad (85)$$

where m is a constant giving the mass density [see e.g. [5], Eq. (14.6)]. These are given in a form expanding as τ increases from 0. Reversing the sense of τ in (83)–(85), so that we have collapse, and introducing a constant T_0 where the singularity occurs, we use these in $R = A^2 gg'$ and $T_{,\tau} = A(gg')'$ [which follow from (25), (28), and (81)]. We can then integrate for T, and where necessary eliminate Ψ, to get

$$2\alpha_-(T_0 - T) \stackrel{\Sigma}{=} R^{1/4}\left(R^{1/2} - \frac{3}{2}R_0\right)(R^{1/2} + R_0)^{1/2} + \frac{3}{2}R_0^2 \operatorname{arcsinh}\left(\frac{R^{1/4}}{R_0^{1/2}}\right), \quad (86)$$

$$T_0 - T \stackrel{\Sigma}{=} R_0 R^{5/4}, \quad (87)$$

$$2\alpha_+(T_0 - T) \stackrel{\Sigma}{=} -R^{1/4}\left(R^{1/2} + \frac{3}{2}R_0\right)(R_0 - R^{1/2})^{1/2} + \frac{3}{2}R_0^2 \arcsin\left(\frac{R^{1/4}}{R_0^{1/2}}\right), \quad (88)$$

where T_0 and R_0 are integration constants, and

$$\alpha \overset{\Sigma}{=} \frac{gg'}{(gg')'}, \tag{89}$$

evaluated on Σ, and thus from (80) for $\epsilon = -1$ and 1, respectively,

$$\alpha_- \overset{\Sigma}{=} \frac{\tanh r_0}{1 + \tanh^2 r_0}, \tag{90}$$

$$\alpha_+ \overset{\Sigma}{=} \frac{\tan r_0}{1 - \tan^2 r_0}, \tag{91}$$

where $r \overset{\Sigma}{=} r_0$, while for $k = 0$, $\alpha \overset{\Sigma}{=} r_0$.

On this moving boundary we know from (24) and (32)–(34) that we must have (noting that for all three possible g, $g'^2 - gg'' = 1$)

$$e^{\psi} \overset{\Sigma}{=} Ag', \tag{92}$$

$$\psi_{,T} \overset{\Sigma}{=} \frac{A_{,\tau}}{A^2((gg')')^2[1 - (2\alpha A_{,\tau})^2]}, \tag{93}$$

$$\psi_{,R} \overset{\Sigma}{=} \frac{-g(k(gg')' + 2A_{,\tau}^2 g'^2)}{A^2((gg')')^2[1 - (2\alpha A_{,\tau})^2]}, \tag{94}$$

$$e^{\gamma} \overset{\Sigma}{=} \frac{g'}{(gg')'\sqrt{1 - (2\alpha A_{,\tau})^2}}, \tag{95}$$

where the functions of r have to be evaluated at r_0.

One can give a general solution of (21) as a sum of separable solutions, but we have not been able to determine the specific solution which matches to RW even in the simplest ($k = 0$) case.

Our condition that R be a spatial coordinate in the exterior metric is violated when T is sufficiently close to T_0: in fact as T increases,

$$\left|\frac{dR}{dT}\right| \to 1 \Leftrightarrow 5R_0^{4/5}(T_0 - T)^{1/5} \to 4.$$

(Note that this can also be expressed as $1 = |2\alpha A_{,\tau}|$.) For larger T our coordinates, and hence our matching, do not apply at Σ. This agrees with the results of Nolan and Nolan [9], who showed that such a breakdown is inevitable, essentially because the collapsing source always leads to trapped cylinders, whereas the ER vacuum region cannot contain trapped cylinders [22]. They describe the matching as impossible, meaning that it cannot be carried right up to the FRW singularity, but note that it can be used up to some finite time.

One can continue the discussion of trapped cylinders by using the coordinates of Tod and Mena [10]. The bound on applicability of our matching corresponds to their conclusion that in the $k = 0$ FRW case the boundary becomes a marginally trapped surface when $1 = |2\alpha A_{,\tau}|$ [in their notation, with hats added for clarity, at $\hat{T} = \hat{\alpha} - (4\hat{r}_0/3)^3$] so at larger T the surface is trapped. Such a trapped surface is not consistent with "asymptotic flatness" in the sense of Berger, Chrusciel, and Moncrief [11] (see their Proposition 2.3).

Tod and Mena further show that $u = \sqrt{R}\psi$ (in their coordinates) has a divergent derivative at the Cauchy horizon (essentially the past null cone of the FRW singularity) and hence conclude that this horizon is singular. The argument for these conclusions applies also to the $k = \pm 1$ cases, with changes in formulas. Tod and Mena infer that there is incoming gravitational radiation: however, this does not seem to be supported by the calculations in the following section.

VII. ENERGY, SUPERENERGY, RADIATION, AND BOUNDARY CONDITIONS

Part of our motivation was a search for an exact cylindrical solution for interior and exterior enabling one to study exactly how gravitational radiation arises. Our ansätze for the interior turned out to allow only FRW, which we would expect to be nonradiative in any definition. Thus we expect the exterior to also be nonradiative (or possibly to show waves coming in from infinity and totally reflected at the boundary with the interior).

A cylindrically symmetric spacetime cannot be asymptotically flat, due to the behavior in directions parallel to the axis, so one cannot employ the usual global definition of radiation for isolated bodies due to Bondi, van der Burg, and Metzner [12]. Moreover, it appears from Proposition 2.3 of Berger, Chrusciel, and Moncrief [11] that simple cylindrical solutions with collapsing cores could not even be "asymptotically flat" in their modified sense, because one would expect trapped cylindrical surfaces to arise in any collapse which does not halt or reverse, and such surfaces are not compatible with "asymptotic flatness."

So for more detailed study we need some definition of radiation, or energy, or energy density, other than the one from asymptotic flatness. This must be (quasi)local, at least in z, to avoid the problem that integration for z from $-\infty$ to ∞ would obviously give an infinite answer for any nonzero energy density.

There is also the possibility of energy being transferred to or from any given region by transport along the axis. Bondi [1] showed that there is no conserved mass per unit length as a result of "intangible" gravitational induction arising from axial motion (and distinct from the work done by axial pressure).

It is well-known that in relativity there cannot be any covariant local definition of energy, as this would violate the equivalence principle. There are a number of quasilocal definitions, using either integrals over surfaces, or pseudotensors integrated over volumes (which in general also reduce to surface integrals, provided the interior volumes do not have discontinuities or singularities of the pseudotensor). Clavering [23] has shown that none of the latter agree with the various quasilocal surface integrals discussed by Szabados [24], and none of them are satisfactory, for a variety of reasons, the best being Möller's definition. We therefore do not calculate the pseudotensorial energies.

In his study of definitions of standing waves, Stephani [25] considered cylindrical systems. His results suggest that for the ER solutions Thorne's ''C energy'' [22] is the least unsatisfactory. (A recent indication of the unsatisfactoriness of C energy, even in vacuum, has been given in [26], where it is shown that it can be nonvanishing in Minkowski space.) We consider, following Chiba [27] and Hayward [28], the modified C energy defined in the ''Note added in proof'' on pp. B256–257 of Thorne's paper. Taking the generic cylindrically symmetric metric in the form

$$ds^2 = -e^{2(\gamma-\psi)}(dT^2 - dr^2) + e^{2\psi}dz^2 + e^{-2\psi}R^2d\phi^2, \tag{96}$$

where $R = R(r, t)$, this energy is

$$E = \frac{1}{8}(1 - (R'^2 - \dot{R}^2)e^{-2\gamma}), \tag{97}$$

where the prime and dot refer to differentiation with respect to r and t, respectively.

Hayward has shown, in an elegant formulation, that one can define an invariant tensor (in his notation, Θ_{ab}) such that the sum of this and the usual energy-momentum tensor T_{ab} is conserved: Θ may then be interpreted as the energy-momentum of gravitational waves. One can derive the conservation in a simple way from the fact that for the metric (96), when matter is present, the equations (22) generalize to

$$\tfrac{1}{2}[(R'^2 - \dot{R}^2)e^{-2\gamma}]' = -RR'(\kappa T_{00} + \psi'^2 + \dot{\psi}^2) + R\dot{R}(\kappa T_{01} + 2\psi'\dot{\psi}), \tag{98}$$

$$\tfrac{1}{2}[(R'^2 - \dot{R}^2)e^{-2\gamma}]^{\cdot} = -RR'(\kappa T_{01} + 2\psi'\dot{\psi}) + R\dot{R}(\kappa T_{11} + \psi'^2 + \dot{\psi}^2) \tag{99}$$

[cf. Eqs. (A41) and (A42), of [28]]: the result is then just the integrability condition for the left sides, the relevant terms of Θ being the terms in ψ on the right sides. That these terms are invariantly defined (provided there are no more translational Killing vectors) arises because the two Killing vectors, ∂_ϕ and ∂_z, and their lengths, are uniquely defined up to rescaling of z, so ψ is fixed up to a constant and R up to a constant factor.

Hayward's discussion also brings out the fact that for the form (96), if μ and ρ are the divergences of the incoming and outgoing null normals to timelike cylinders of symmetry, then

$$\rho\mu = \frac{1}{8R^2}e^{2(\psi-\gamma)}(R'^2 - \dot{R}^2) \tag{100}$$

with an obvious relation to (97).

This is also related to one of the best known quasilocal energy definitions by an integral on a surface, the Hawking mass. For a closed surface S with surface area element $d\mathbf{S}$ this mass is

$$m = \frac{1}{(4\pi)^{3/2}}\left(\oint_S d\mathbf{S}\right)^{1/2}\left(2\pi - \oint_S \mu\rho d\mathbf{S}\right).$$

To make a closed cylindrical surface we would have to add (e.g.) surfaces $z =$ constant at some finite z values, but $\mu\rho$ has the same values for all z on such surfaces and so we could ignore them for very large cylindrical surfaces and think of E as giving a Hawking mass per unit length in z.

Unfortunately this does not lead to a satisfactory account of energy lost or gained by the collapsing dust. The Darmois conditions imply that E is continuous at Σ (one can check this by direct calculation, but it is easy to understand because there is no surface layer and hence no immediate change in geodesic deviations at Σ, so $\mu\rho$ is continuous, and the continuity of $g_{\phi\phi}$ and g_{zz} then shows the same is true for E). Calculating on the dust side of Σ, using (80), we have

$$E = \frac{1}{8}\left(1 - \frac{((gg')')^2}{g'^2} + 4g^2\frac{\dot{A}^2}{A^2}\right).$$

[As one expects [28], this is $O(r^2)$ as $r \to 0$.] At a fixed r, only the $\dot{A}^2/A^2 = A^2_{,\tau}$ term changes, and it increases as the dust collapses. This is consistent with Cocke's calculation [8] of (unmodified) C-energy flux, normal to cylinders of constant R, in our notation, in the Einstein-Rosen region, where he found an inward flux.

It would, however, be physically very odd if this had to be interpreted as the exterior giving energy to the dust, since the behavior of the dust is exactly the same as in a uniform universe, where a cylinder cannot easily be thought of as taking energy from the rest of the universe. Probably the result is better considered as another indication of the unsatisfactoriness of C energy.

Other local characterizations of the presence of radiation are given by the decomposition of the Bel-Robinson tensor. The behavior of this tensor in an ER spacetime was considered in [29], where the obvious unit vector in the form (20), i.e. the one parallel to $\partial/\partial T$, was used to define the decomposition. In that reference it was shown that for a pulse of radiation, well behind the front of the pulse, there is an incoming flux of superenergy, which seems to agree with Cocke's result. However, to define states of intrinsic radiation one has to consider all possible timelike vectors, and we now show, using the timelike vector parallel to the boundary, that our solutions do not have intrinsic radiation there according to this definition.

Bel [30,31] defined the tensor

$$T^{abcd} = R^{aecf} R_e{}^b{}_f{}^d + {}^*R^{aecf*} R_e{}^b{}_f{}^d + R^{*aecf} R^{*bd}_{ef} + {}^*R^{*aecf*} R^*{}_e{}^b{}_f{}^d, \tag{101}$$

which in the vacuum case is referred to as the Bel-Robinson tensor. Here the star operation is the usual Hodge dual (see e.g. [5], Chap. 3). Bonilla and Senovilla [32] pointed out that with the usual decomposition in terms of the Weyl and Ricci tensors, C_{abcd} and R_{ab}, i.e.

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd}, \tag{102}$$

where

$$E_{abcd} \equiv \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}), \tag{103}$$

$$G_{abcd} \equiv \frac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}) \equiv \frac{1}{12}Rg_{abcd}, \tag{104}$$

$$S_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}, \qquad R \equiv R^a{}_a, \tag{105}$$

the Bel tensor can be written as

$$T^{abcd} = C^{aecf} C_e{}^b{}_f{}^d + {}^*C^{aecf*} C_e{}^b{}_f{}^d + \frac{R}{6}(C^{acbd} + C^{adbc}) + M^{abcd}, \tag{106}$$

where the matter contribution is

$$M^{abcd} = S^{ab}S^{cd} + \frac{1}{2}S^{ae}S^b{}_e g^{cd} + \frac{1}{2}S^{ce}S^d{}_e g^{ab} - 2S^{e(a}g^{b)(c}S^{d)}_e + \frac{1}{4}S^{ef}S_{ef}(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd}) + \frac{R^2}{144}(2g^{ac}g^{bd} + 2g^{ad}g^{bc} - g^{ab}g^{cd}). \tag{107}$$

One can decompose the Bel tensor in a 3 + 1 formalism [33] relative to a unit timelike vector n^a, giving, among the parts, the superenergy W, the super-Poynting vector P^a, and the tensor Q_{abc} defined by

$$W \equiv T^{abcd}n_a n_b n_c n_d, \qquad P^d \equiv T^{abce}n_a n_b n_c h^d{}_e, \qquad Q^{bcd} \equiv -T^{aefg}n_a h^b{}_e h^c{}_f h^d{}_g, \tag{108}$$

where $h_{ab} = g_{ab} + n_a n_b$. Bel [31] defined a state of intrinsic gravitational radiation (at a point p) to be one in which $P^a \neq 0$ for any choice of n^a (at p). Garcia-Parrado Gomez-Lobo [33] similarly defines an intrinsic superenergy radiative state to be one where $Q^{abc} \neq 0$ for any choice of n^a. In vacuum,

$$P^a = 2B_p{}^l E_{ql}\eta^{apq}, \qquad Q_{cdb} = h_{cd}P_b - 2(B_{da}E_{cf} - B_{ca}E_{df})\eta_b{}^{af}, \tag{109}$$

where, as usual, $E_{ab} = C_{acbd}n^b n^d$, $B_{ab} = {}^*C_{acbd}n^b n^d$, and $\eta_{abc} = \eta_{abcd}n^d$, η_{abcd} being the usual volume 4-form density.

We now consider how these quantities behave at the (timelike) boundary between regions of spacetime. It is simplest to describe this using an orthonormal tetrad $\{\boldsymbol{e}_a, a = 0\ldots3\}$ chosen such that the timelike unit vector $\boldsymbol{e}_0$ lies on the boundary surface [and will be used as n^a in (108)] and $\boldsymbol{e}_1$ is the normal to the surface. From the equations (80) of Mars and Senovilla [34] we can straightforwardly show that if the Darmois junction conditions are satisfied then the following combinations of Riemann tensor components are continuous:

$$E_{12}, \quad R_{12}, \quad E_{13}, \quad R_{13}, \quad R_{01}, \tag{110}$$

$$B_{11}, \quad B_{22} \quad (\text{and hence } B_{33}), \quad B_{23}, \tag{111}$$

$$B_{13} + \frac{1}{2}R_{02}, \quad B_{12} - \frac{1}{2}R_{03}, \quad E_{23} - \frac{1}{2}R_{23} \tag{112}$$

$$-E_{11} - \frac{1}{6}R + \frac{1}{2}(R_{22} + R_{33}), \qquad E_{22} + \frac{1}{6}R + \frac{1}{2}(R_{00} - R_{22}), \qquad E_{33} + \frac{1}{6}R + \frac{1}{2}(R_{00} - R_{33}). \tag{113}$$

Note that one consequence is (26), the continuity of T_{11}.

From (111) and (112) we see that (assuming we align $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ with S_a and K_b) with the energy-momentum form assumed in (1), B_{ab} is continuous at the boundary. Moreover, the reflection symmetries in ϕ and z imply that $P_2 = P_3 = 0$ (and $B_{12} = B_{13} = 0$) in both interior and exterior (note that since reversing one axis also reverses the orientation and hence the sign of the dual, it does not follow that $B_{23} = 0$).

From this, it is easy to see that for an FRW interior, which is conformally flat, $B_{ab} = 0$ on both sides of the boundary, so from (109) $P_a = 0$ and $Q_{abc} = 0$ there in the frame defined above (i.e. with $n^a = \boldsymbol{e}_0^a$ and under Darmois boundary conditions). Hence at the boundary the exterior does not have intrinsic gravitational radiation or intrinsic superenergy radiation. Thus it is reasonable to conclude that any possible radiation in the exterior spacetime is not produced by the source. This agrees with our expectation that such an interior should not radiate or absorb radiation. Note that we have not excluded the possibility of total reflection at the boundary if at a reflecting surface these indicators (P^a and Q_{cdb}) would show no intrinsic radiation, and it is therefore possible that there could be nonzero incoming and/or outgoing radiation in the exterior. The conclusion that no radiation crosses the boundary seems in conflict with the discussion given by Tod and Mena [10].

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