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Anti–de Sitter 5D black hole solutions with a self-interacting bulk scalar field: A potential reconstruction approach

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We construct asymptotically AdS black hole solutions, with a self-interacting bulk scalar field, in the context of 5D general relativity. As the observable universe is characterized by spatial flatness, we focus on solutions where the horizon of the black hole, and subsequently all 3D hypersurfaces for fixed radial coordinate, have zero spatial curvature. We examine two cases for the black hole scalar hair: (a) an exponential decaying scalar field profile and (b) an inverse power scalar field profile. The scalar black hole solutions we present in this paper are characterized by four functions $f(r)$, $a(r)$, $\phi(r)$, and $V(\phi(r))$. Only the functions $\phi(r)$ and $a(r)$ are determined analytically, while the functions $f(r)$ and $V(\phi(r))$ are expressed semianalytically by integral formulas in terms of $a(r)$. We present our numerical results and study in detail the characteristic properties of our solutions. We also note that the potential we obtain has a nonconvex form in agreement with the corresponding ''no hair theorem'' for AdS spacetimes.

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I. INTRODUCTION

According to the ''no hair conjecture'' by R. Ruffini and J. A. Wheeler [1], conventional black holes, namely, vacuum solutions of the Einstein-Maxwell system, are fully described by the parameters of mass, electric and magnetic charge, and angular momentum. Parameters such as multipole moments, which are introduced when spherical symmetry is broken, are absent after gravitational collapse of matter sources inside the black hole event horizon.

Although black hole solutions are severely restricted in the case of the standard Einstein-Maxwell action, new solutions can be obtained for generalized actions, which include, for example, scalar or non-Abelian Gauge fields. The so-called ''no hair theorems'' are formulated for specific models, under certain symmetry and asymptotic behavior considerations for the metric, and set concrete restrictions to the corresponding vacuum solutions.

A neutral scalar field with a self-interaction potential term $V(\phi)$ is served as a first example of a study beyond the Einstein-Maxwell action. The corresponding ''no hair theorem'' has been formulated long ago by Bekenstein in Ref. [2]. Accordingly, there is no asymptotically flat black hole solution with a nontrivial continuous scalar ''hair'' for a convex potential $V(\phi)$ ($V'(\phi) \geq 0$). However, nontrivial scalar black hole solutions can be obtained if we relax some of the above conditions of the ''no hair theorem.'' Recent (analytical or numerical) asymptotically flat solutions in four dimensions with ''scalar hair'' are presented in Refs. [3–8], while asymptotically AdS solutions can be found, for example, in Refs. [9–15]. The issue of the stability against linear perturbation around the scalar field also has been examined, see Refs. [4,11–13,16,17].

We would like to emphasize that the scalar black hole solutions, which are presented in the above mentioned references, do not introduce new quantities which could characterize the black hole, beyond the standard ones of mass, electric (or magnetic) charge, and angular momentum. Hence, the meaning of the ''no hair conjecture'' remains, see also the relative discussions in Refs. [18,19]. Note that in Ref. [20] an analytical solution with a scalar field conformally coupled to gravity which introduces a new quantity (in particular a scalar charge) is presented, but the scalar field blows up on the event horizon of the black hole, see also the comments in Ref. [8].

In this paper we study 5D asymptotically AdS black hole solutions with scalar hair in a semianalytical way. We focus to solutions where all 3D hypersurfaces for fixed radial coordinate r have zero spatial curvature, in contrast with the usual case where the horizon of the black hole is characterized by spherical symmetry (or with positive spatial curvature). In the present approach, the self-interaction potential $V(\phi)$ of the scalar field is assumed to be an undetermined function, and there is a freedom in the choice of the scalar field profile $\phi(r)$. Note that the functions $a(r)$ and $f(r)$ which appear in the black hole metric, as well as the potential $V(\phi)$, depend on the specific choice of the scalar field. We have examined two cases for the scalar field: (a) an exponential decaying profile and (b) an inverse power profile, of Eqs. (11) and (12) below correspondingly. Note that these profiles [(a) and (b) above] have been studied previously by Lechtenfeld and coauthors in Refs. [3,4], for 4D black hole solutions in asymptotically flat space-time, with an analogous methodology that we follow in this paper. In the solutions, we present the functions $\phi(r)$ and $a(r)$, which are determined analytically, while the functions $f(r)$ and

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$V(\phi(r))$ are expressed semianalytically by integral formulas in terms of $a(r)$. In the main part of this work we present our results in figures and the characteristic properties of our solutions are discussed. We see that the reconstructed potential $V(\phi)$, for the specific choices of the scalar field (a) and (b), has a nonconvex form in agreement with the corresponding "no hair theorem" for AdS_5 space-times which is presented in Sec. IV of this work.

The interest for five-dimensional AdS black holes is motivated by extra-dimensional theories, and mainly by the so-called brane world models [21–25] which promise a resolution for the hierarchy problem. In the case that we examine, our world is assumed to be trapped in a hypersurface of fixed radius r in the background of a five-dimensional AdS black hole vacuum [26]. This also explains why we have focused on solutions for which spatial 3D sections are characterized by zero curvature, in agreement with the current astrophysical phenomenology. Note that in contrast with the standard vacuum of Randall-Sundrum [24,25], the five-dimensional AdS black hole vacuum does not preserve 4D Lorentz invariance on the brane, which may have interesting phenomenological implications, see, for example, Refs. [26,27]. Note also that in the framework of AdS/CFT correspondence, an AdS_5 black hole background is of particular interest as it can trigger a thermal conformal field theory on the AdS_5 boundary (brane), see Ref. [28].

II. 5D ADS BLACK HOLES WITH A SELF-INTERACTING BULK SCALAR FIELD

We consider the following 5D action

$$S = \int d^5x\sqrt{|g|}\left(\frac{1}{2\kappa_5}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right),$$

$$\mu, \nu = 0, 1, 2, 3, 5 \qquad (1)$$

for general relativity, with a bulk self-interacting scalar field with a potential $V(\phi)$, where $\kappa_5 = 8\pi G_5$ (G_5 is the 5D Newton constant). Note that the extra dimension is parametrized by the coordinate $x^5 = r$ (radius of the black hole), while the other coordinates x^0, x^1, x^2, x^3 correspond to the usual 4D space-time. In addition, we have assumed that a negative cosmological constant Λ is incorporated in the potential of the scalar field, according to the equation $\Lambda = V(0)$ ($V(0) < 0$).

The Einstein equations for the above action read

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa_5 T^{(\phi)}_{\mu\nu}, \qquad (2)$$

and the energy-momentum tensor $T^{(\phi)}_{\mu\nu}$ for the bulk scalar field is

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}[\tfrac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + V(\phi)]. \qquad (3)$$

If we use Eqs. (2) and (3) we obtain the equivalent equation:

$$R_{\mu\nu} = \kappa_5(\partial_\mu\phi\partial_\nu\phi + \tfrac{2}{3}g_{\mu\nu}V(\phi)). \qquad (4)$$

Now for the metric of the black hole solution we make the following ansatz

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + a^2(r)d\mathbf{x}^2, \qquad (5)$$

where $d\mathbf{x}^2$ is the metric of the spatial 3-section, which in our case is assumed to have zero curvature, in agreement with the current astrophysical phenomenology, pointing toward spatial flatness of the observable universe.

In the case of the metric of Eq. (5), if we use Eq. (4) we find the following three independent differential equations

$$f''(r) + 3\frac{a'(r)}{a(r)}f'(r) + \frac{4}{3}V(\phi) = 0, \qquad (6)$$

$$\frac{a'(r)}{a(r)}f'(r) + \left(2\frac{(a'(r))^2}{a^2(r)} + \frac{a''(r)}{a(r)}\right)f(r) + \frac{2}{3}V(\phi) = 0, \qquad (7)$$

$$f''(r) + 3\frac{a'(r)}{a(r)}f'(r) + 6\left(\frac{a''(r)}{a(r)} + \frac{1}{3}(\phi'(r))^2\right)f(r) + \frac{4}{3}V(\phi) = 0. \qquad (8)$$

It is worth noting that when the Einstein equations are satisfied, the equation for the scalar field [Eq. (37) below] is satisfied automatically. In particular, this equation is equivalent to the conservation equation $\nabla^\mu T_{\mu\nu} = 0$, where $T_{\mu\nu}$ is given by Eq. (3) above. However, we could derive Eq. (37) (for the scalar field) straightforwardly from Eqs. (6)–(8). This is left as an exercise for the interested reader.

All the quantities, in the above equations, have been rendered dimensionless via the redefinitions $\sqrt{\kappa_5}\phi \to \phi$, $\kappa_5\ell^{-2}V \to V$, and $r/l \to r$, where the AdS_5 radius l in the above rescaling is defined as $l = \sqrt{-6/(\kappa_5\Lambda)}$.

If we eliminate the potential $V(\phi)$ from the above equations we obtain

$$a''(r) + \tfrac{1}{3}(\phi'(r))^2a(r) = 0, \qquad (9)$$

$$f''(r) + \frac{a'(r)}{a(r)}f'(r) - \left(4\frac{(a'(r))^2}{a^2(r)} + 2\frac{a''(r)}{a(r)}\right)f(r) = 0, \qquad (10)$$

where the potential can be determined from Eq. (6) if the functions $a(r)$ and $f(r)$ are known.

We observe that in the above two differential equations (9) and (10) we have three unknown functions, hence we have the freedom to choose one of them; for example, the scalar field $\phi(r)$, and the other two functions $a(r)$ and $f(r)$ can be determined subsequently. We will consider continuous scalar field deformations that are localized in a small region of r, while for large values of r they tend rapidly to zero. Mainly, we aim to study the following two cases for the scalar field: (a) an exponential profile and (b) an inverse power profile, which are given by the following equations

$$\phi_1(r) = \phi_0 e^{-(r/d)}, \tag{11}$$

$$\phi_2(r) = \frac{q}{r^n}. \tag{12}$$

As we will see in the next section the above choices of Eqs. (11) and (12) can lead to analytic solutions for the function $a(r)$ by solving Eq. (9). However, the other functions $f(r)$ and $V(\phi)$ can be determined semianalytically. Note that these profiles have been studied previously by the authors of Refs. [3,4] in a similar problem for 4D black hole solutions with a self-interacting phantom scalar field.

Now, if the function $a(r)$ has been obtained for a specific choice of the scalar field $\phi(r)$, the function $f(r)$ can be determined by Eq. (10), for which the general solution can be expressed as a linear combination of two independent solutions:

$$f(r) = C_1 a^2(r) + C_2 a^2(r) \int_r^{+\infty} \frac{dr'}{a^5(r')}. \tag{13}$$

In order to fix the constants of integration C_1 and C_2, we assume that for large r the solution approaches the well-known AdS_5 Schwarzschild black hole solution. For the asymptotic behavior, see for example [26,27], and references therein. Thus we will look for solutions with the following asymptotic behavior:

$$a(r) \to r, \tag{14}$$

$$f(r) \to r^2 - \frac{\mu}{r^2}, \tag{15}$$

where μ is a dimensionless constant of integration.[1] Note that the previous consideration for the asymptotic behavior of the black hole solution is reasonable because the scalar field vanishes in the infinity and the scalar field potential $V(\phi)$ approaches a negative nonzero value $\Lambda = V(0)$, which corresponds to the 5D cosmological constant. If we compare Eq. (13) and (15) in the limit of large r we find

$$C_1 = 1, \qquad C_2 = -4\mu, \tag{16}$$

hence we obtain the formula

$$f(r) = a^2(r)\left[1 - \int_r^{+\infty} \frac{4\mu}{a^5(r')} dr'\right]. \tag{17}$$

Note that the integration over r' is restricted in the range $r' > r_s$, where r_s is the largest zero of the function $a(r)$ that represents the physical singularity of the black hole. The horizon of the black hole r_h can be determined via the equation $f(r_h) = 0$, or equivalently by the equation

$$\int_{r_h}^{+\infty} \frac{1}{a^5(r')} dr' = \frac{1}{4\mu}, \tag{18}$$

which always has a unique positive solution r_h larger that r_s. This means that, independently of the choice of the scalar field profile and from the mass of the black hole μ, the singularity r_s is protected by a horizon r_h. In order to establish the existence of a root of Eq. (18) we consider the function

$$G(r) = \int_r^{+\infty} \frac{1}{a^5(r')} dr'. \tag{19}$$

This function is continuous in the open interval $(r_s, +\infty)$. In addition, we will assume that the function $a(r)$, for $r > r_s$, has a profile which is given by Fig. 1, or equivalently that it is a strictly monotonically increasing function in the interval $(r_s, +\infty)$. From this we can conclude that $G(r)$ is also a strictly monotonically increasing function in the interval $(r_s, +\infty)$. In addition, we can see that for $r \to +\infty \Rightarrow G \to 0$ and for $r \to r_s \Rightarrow G \to +\infty$. From the intermediate value theorem of real analysis there is a coordinate value r_h, in the interval $(r_s, +\infty)$, for which $G(r_h) = 1/4\mu$ as $0 < 1/4\mu < +\infty$. The uniqueness of r_h is demonstrated by the fact that $G(r)$ is a strictly monotonically increasing function. We would like to emphasize that in this proof we have considered scalar fields of the form of Eqs. (11) and (12), or more specifically scalar fields that are nonzero for $r = 0$ and tend monotonically to zero for $r \to +\infty$. This guarantees that $a(r)$ is a strictly monotonically increasing function, as it is required for this theorem.

By replacing Eq. (17) into Eq. (6), the scalar field potential $V(\phi(r))$ can be expressed in terms of the function $a(r)$, according to the equation

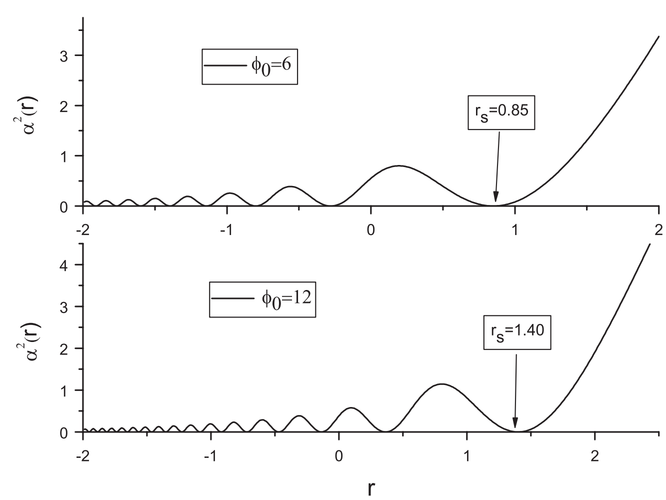

FIG. 1. The factor $a^2(r)$ as a function of r for $d = 1$ and $\phi_0 = 6, 12$. We observe that the position of the physical singularity r_s becomes larger when ϕ_0 increases.

[1]The mass of the black hole, without the scalar hair, is $m = (8\pi G_5)^{-1}\ell^2\mu$.

$$V(\phi(r)) = -6\mu \frac{a'(r)}{a^4(r)} + \left(4\mu \int_r^{+\infty} \frac{1}{a^5(r)} dr - 1\right) \times \left(6(a'(r))^2 + \frac{3}{2} a''(r) a(r)\right). \quad (20)$$

III. NUMERICAL ANALYSIS

We will study two classes of solutions for the profiles of Eqs. (11) and (12) above. The warp factor $a(r)$ can be determined analytically by the differential equation (9), while the black hole factor $f(r)$ and the potential $\hat{V}(r) = V(\phi(r))$ will be computed by the integral formulas of Eqs. (17) and (20). The potential $V(\phi)$ as a function of the scalar field ϕ can be determined by the replacement $r = r^{-1}(\phi)$ in the function $\hat{V}(r)$.

A. The exponential profile of Eq. (11)

In the case of the exponential profile $\phi_1(r) = \phi_0 e^{-(r/d)}$, from Eq. (9) we get the analytic solution

$$a(r) = \tilde{C}_1 J_0\left(\frac{\phi_0 e^{-(r/d)}}{\sqrt{3}}\right) + \tilde{C}_2 Y_0\left(\frac{\phi_0 e^{-(r/d)}}{\sqrt{3}}\right), \quad (21)$$

where the constants of integration $\tilde{C}_1$ and $\tilde{C}_2$ are fixed if we take into account the asymptotic behavior $a(r)$ in the large r limit, see Eq. (14) above. If we expand the Bessel functions $J_0(x)$, $Y_0(x)$ for small argument x (large r) we find

$$\tilde{C}_1 = d\left[\ln\left(\frac{\phi_0}{2\sqrt{3}}\right) + \gamma\right], \qquad \tilde{C}_2 = -\frac{\pi d}{2}. \quad (22)$$

As we see in Fig. 1 the function $a^2(r)$ is an oscillating function with an infinite number of zeroes. Note that r can be negative, as it is just a coordinate and not the "real" radius of the black hole, which is represented by the function $a(r)$. The next step is to determine the largest zero r_s of $a^2(r)$ ($a(r_s) = 0$), which is the physical singularity of the black hole. It is reasonable to ignore completely the part of $a(r)$ for $r \leq r_s$, which has no physical meaning, and to keep only the region for $r > r_s$, which corresponds to the 5D AdS black hole solution. Also in Fig. 1 we see that, for fixed d, the position r_s of the singularity becomes larger when ϕ_0 increases.

As the physical space of the parameter r is restricted for $r > r_s$, the maximum value of the scalar field $\phi(r)$ is not ϕ_0, the maximum value for the scalar field $\phi_{\max}$ is given by the equation

$$\phi_{\max} = \phi_0 e^{-(r_s/d)}, \quad (23)$$

which is exponentially suppressed by the factor $e^{-(r_s/d)}$. We have checked numerically that $\phi_{\max}$ is an increasing function but depends very weakly on ϕ_0. For example, for $\phi_0 = 50$ we obtain $\phi_{\max} = 2.68$, and for $\phi_0 = 1000$ we obtain $\phi_{\max} = 3.74$. This can be explained if we take into account that the coefficient $\tilde{C}_1$ in Eq. (21) is logarithmically dependent on ϕ_0.

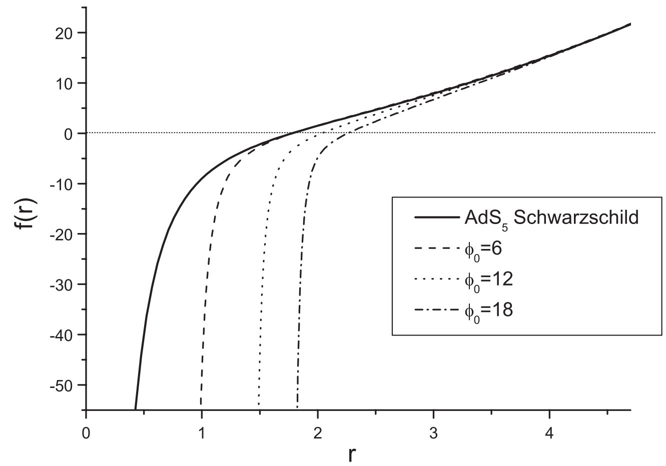

FIG. 2. The factor $f(r)$ as a function of r for $\mu = 10$, $d = 1$, and $\phi_0 = 6, 12, 18$. We observe that for $r \to +\infty$ we recover the well-known AdS_5 Schwarzschild black hole solution.

In Fig. 2 we have plotted the function $f(r)$ for several values of ϕ_0 assuming that the value of d is fixed. Note that $f(r)$ has a singularity and an event horizon as a conventional black hole solution. In this figure we see that outside the horizon, $f(r)$ is weakly dependent on the parameter ϕ_0. Such a behavior is expected because the parameter that has an impact on $f(r)$ is the maximum value of the scalar $\phi_{\max}$ and not the parameter ϕ_0. Note that $\phi_{\max}$ weakly depends on ϕ_0 and remains comparatively small even for very large values of ϕ_0, as is mentioned in the previous paragraph.

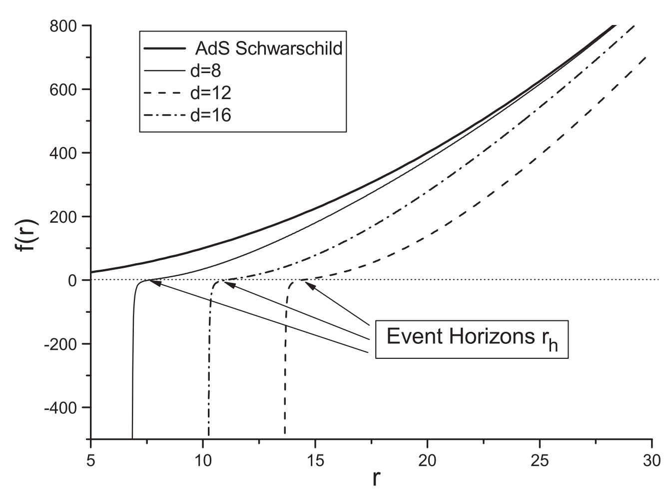

FIG. 3. The factor $f(r)$ as a function of r for $\mu = 10$, $\phi = 6$, and $d = 8, 12, 16$. We observe that when $d \gg 1$, outside the event horizon, the scalar black hole differs significantly from the AdS_5 Schwarzschild black hole (bold line in the figure). Although it is not displayed in the figure, we have checked that for enough large r we recover the well-known AdS_5 Schwarzschild black hole solution, even for $d = 12$ and $d = 16$.

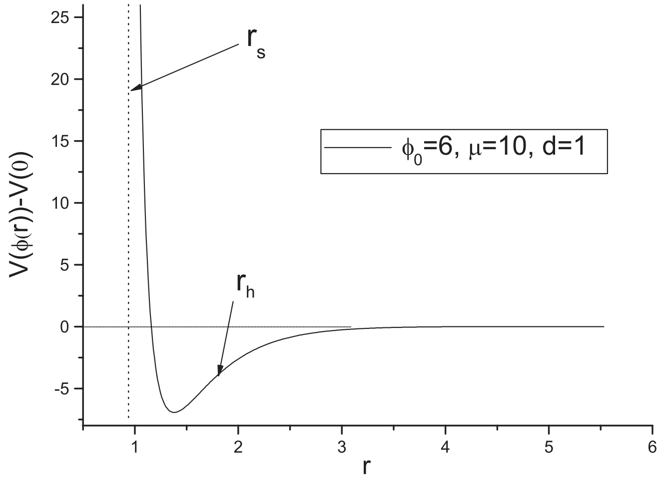

FIG. 4. The potential $V(\phi(r))$ as a function of r for $\mu = 10$, $d = 1$, and $\phi_0 = 6$. It blows up for $r \to r_s$, while for large r it tends to a constant value equal to 5D cosmological constant. We also see that minimum value of the potential is inside the black hole horizon r_h.

On the other hand, in Fig. 3 we see that the function $f(r)$ is strongly dependent on d, for fixed ϕ_0. For large d (d is an estimate for the size of the scalar field), in particular, for $d \gg \sqrt[4]{\mu}$, we observe that outside the horizon, the black hole solution becomes significantly different from the corresponding AdS_5 black hole solution without the scalar field (bold line in the figure). Note that for $d \sim \sqrt{[4]}\mu$, as we see in Fig. 2, $f(r)$ tends rapidly to its asymptotic behavior $r^2 - \mu/r^2$ for $r > r_h$ (r_h is the position of the event horizon of the black hole with the scalar field).

The potential $V(\phi(r))$, if we subtract the contribution from the cosmological constant $\Lambda = V(0)$, has been plotted as a function of the radius coordinate r in Fig. 4. It has two characteristic features: (a) it blows up near the singularity of the black hole r_s, and (b) it tends to a constant value equal to the 5D cosmological constant Λ when r tends to infinity. Now we can obtain the potential as a function of ϕ by constructing the parametric plot $(\phi(r), V(\phi(r)) - V(0))$, as it is presented in Fig. 5. In the left panel we see that the potential becomes infinitely large near $\phi(r_s)$, hence the scalar field is restricted in the region $0 \leq \phi \leq \phi(r_s)$. In the right panel we see that the potential has minimum and the difference $V(\phi(r)) - V(0)$ becomes negative in a large region near the axes origin, namely, the potential has a nonconvex form in agreement with Bekenstein 's "no hair theorem."

Also we have performed numerical computation by covering other ranges of the free parameters of the model, and we have checked that the above general features, which are exhibited in Figs. 1, 2, 4, and 5, are preserved.

In order to establish the singularity nature of r_s [r_s is the largest zero of the function $a(r)$] it is not enough to know that the metric tensor, or the Riemann tensor, becomes infinite for $r = r_s$, as it may be an artifact of the specific coordinate system we use. Hence, scalar quantities like the Ricci scalar R, or higher order scalar quantities such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$, have to be considered. In the case of the standard 5D AdS Schwarzschild black hole the Ricci scalar R is constant ($R = -20/l^2$). Note that $R_{\mu\nu}R^{\mu\nu} = 80/l^4$ is also constant. Thus, in order to establish the singularity nature of the black hole at the point

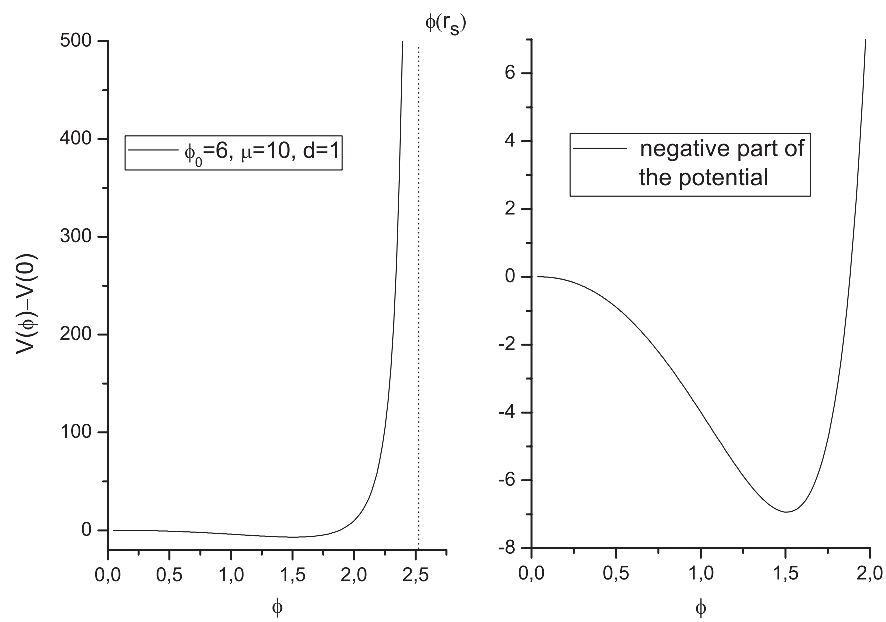

FIG. 5. The potential $V(\phi) - V(0)$ as a function of ϕ for $\mu = 10$, $d = 1$, and $\phi_0 = 6$. In the left panel we observe that the potential blows up at the point where the scalar field becomes equal to $\phi(r_s)$. In the right panel we have plotted in detail the negative part of the potential $[V(\phi) - V(0)]$, and we see that it possesses a local minimum in the range $\phi(r_h) < \phi < \phi(r_s)$.

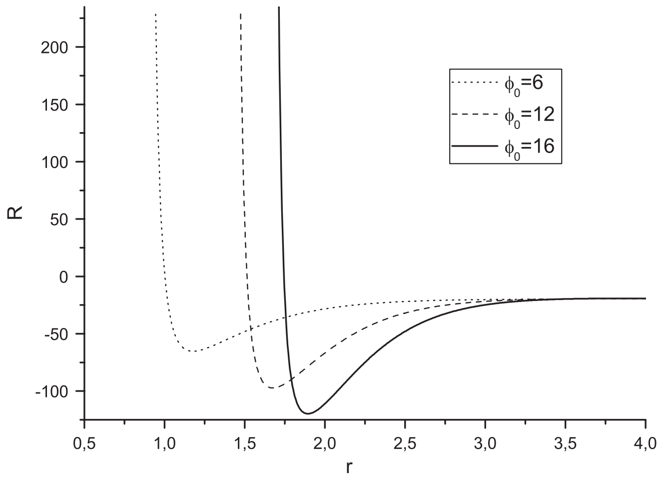

FIG. 6. The Ricci scalar as a function of the radius r for $\mu = 10$, $d = 1$, and $\phi_0 = 6$, 12, 18. We see that the Ricci scalar blows up near the singularity point r_s.

$r = r_s$ we have to compute the Kretschmann scalar $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$. We obtain that

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{40}{l^2} + 72\frac{\mu}{r^8}, \tag{24}$$

which blows up for $r = 0$. However, in the case of the black hole solutions with the scalar field, we see in Fig. 6 that the Ricci scalar scalar blows up for $r = r_s$, contrary to the usual case. Thus it is not necessary to compute the Kretschmann scalar in order to show that the point $r = r_s$ is a singularity. Note that we have performed similar numerical calculations for several of our solutions, beyond those which are presented in Fig. 6, and we have seen that the infinite value of the Ricci scalar at the point $r = r_s$ is a general feature.

B. The inverse power profile of Eq. (12)

For the inverse power profile $\phi_2(r) = q/r^n$ the differential equation (9) can be solved analytically with solution

$$a(r) = \tilde{C}_3\sqrt{r}J_{-(1/2n)}\left(\frac{q}{\sqrt{3}r^n}\right) + \tilde{C}_4\sqrt{r}Y_{-(1/2n)}\left(\frac{q}{\sqrt{3}r^n}\right). \tag{25}$$

A comparison with the asymptotic behavior of Eq. (14) implies

$$\tilde{C}_4 = 0, \qquad \tilde{C}_3 = \Gamma\left(1 - \frac{1}{2n}\right)\left(\frac{q}{2\sqrt{3}}\right)^{1/2n}, \tag{26}$$

where we have assumed that $n \neq 1/2(k+1)$, ($k = 0, 1, 2..$). Note that for $n = 1$ the function $a(r)$ takes the simple form

$$a(r) = r\cos\left(\frac{q}{\sqrt{3}r}\right). \tag{27}$$

However, as we will see subsequently, the value of n is restricted according to the relation $n \geq 2$.

The functions $a(r)$, $f(r)$, and $V(\phi)$ for the scalar field with the inverse power profile have similar characteristic features with those of the exponential profile scalar field that was examined in the previous subsection. For this reason we have restricted the number of figures in this section. However, in the case of the inverse power profile an additional analysis is required in order to satisfy the asymptotic behavior of $a(r)$ and $f(r)$, as it is given by Eqs. (14) and (15) above, or we can write

$$a(r) \to r, \qquad f(r) \to r^2 - \frac{\mu}{r^2} + O\left(\frac{1}{r^c}\right), \qquad c \geq 2. \tag{28}$$

Note that the next to leading term r^2, in Eq. (28) for the function $f(r)$, is assumed to tend to zero like $1/r^2$. Now if we take for granted an asymptotic behavior q/r^n for the scalar field we can satisfy Eq. (28) only if we set concrete restrictions to the power n. For large r, the function $a(r)$ can be written as

$$a(r) \simeq r + \delta h(r), \tag{29}$$

where $\delta h(r)$ is assumed to be a small correction if it is compared with r ($\delta h(r) \ll r$). If we replace $a(r)$ in Eq. (9), in the case of the inverse power profile scalar field $\phi(r) = q/r^n$, we obtain that the correction $\delta h(r)$ is given by the equation

$$\delta h(r) = -\frac{c_n}{r^{2n-1}}, \qquad c_n = \frac{nq^2}{6(2n-1)}. \tag{30}$$

In the special case where $n = 1/2$ we obtain that

$$\delta h(r) = \frac{q^2}{12}\ln(r) \tag{31}$$

For $n > 1/2$ we have $2n - 1 > 0$, and then $\delta h \to 0$ for $r \to +\infty$. This is a first restriction for the asymptotic behavior of the scalar field. However, from $f(r)$ we will set a stronger restriction as we will see below.

By replacing $a(r) \simeq r + \delta h(r)$ in Eq. (17), if we keep only linear terms in $\delta h(r)$, we obtain the corresponding asymptotic formula for $f(r)$:

$$f(r) \simeq r^2 - \frac{\mu}{r^2} - \frac{2c_n}{r^{2n-2}} + \ldots, \tag{32}$$

where we have neglected the terms that vanish faster than $1/r^{2n-2}$. In order to guarantee the behavior of the AdS_5 Schwarzschild black hole $r^2 - \mu/r^2$ in the large r limit, we have to impose the stronger restriction $n > 2$, hence the additional term $1/r^{2n-2}$ in Eq. (32) vanishes faster than $1/r^2$. Note that in the case where $n = 2$ the asymptotic behavior is of the form $r^2 - \mu_{\mathrm{eff}}/r^2$, where we have defined an effective mass $\mu_{\mathrm{eff}} = \mu + 2c_n$.

In Fig. 7 we have plotted the function $f(r)$ in the case of the inverse power scalar field profile, for $n = 0.2$, 1, 2. We present this figure in order to check the asymptotic formula of Eq. (32) above. We see that for $n = 0.2$ the asymptotic

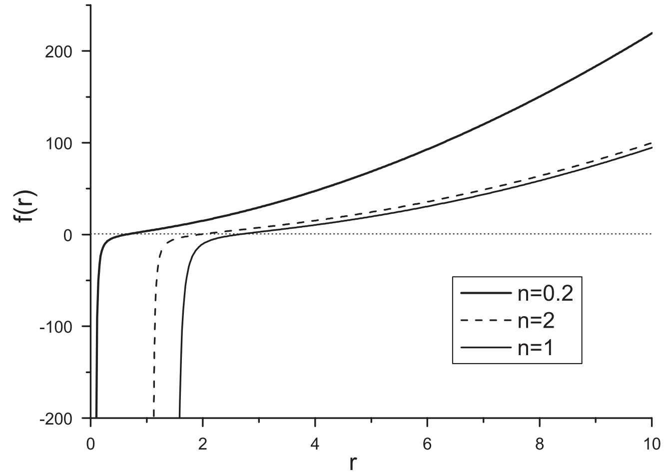

FIG. 7. The function $f(r)$ for the inverse power type scalar field, for $n = 0.2$, 1, 2, $q = 4$, and $\mu = 10$. We observe that for $n = 0.2$ and $n = 1$ the asymptotic behavior of the black hole is not of the standard form $r^2 - \mu/r^2$.

behavior is not of the form $r^2 - \mu/r^2$, but we have an asymptotic behavior $r^2 + 2|c_{0.2}|r^{1.6}$ which is in agreement with the formula of Eq. (32). In particular, the deviation between the curves with $n = 0.2$ and $n = 2$ in Fig. 4 is due exactly to the term $r^{1.6}$ in the previously mentioned asymptotic behavior. For $n = 1$ we observe an asymptotic behavior of the form $r^2 - 2c_1$. This behavior can be confirmed in Fig. 7 if we compare the $n = 1$ and $n = 2$ curves for $f(r)$ which have a constant difference. Note that the curve with $n = 2$ exhibits an asymptotic behavior of the form $r^2 - \mu/r^2$ (AdS_5 Schwarzschild black hole) as we see in Fig. 8. In the same figure we see also that for $n = 3$ we have a similar asymptotic behavior.

Now by replacing $a(r) = r + \delta h(r)$ in Eq. (20) we can obtain the asymptotic formula for the potential $V(\phi)$ when $\phi \to 0$,

$$V(\phi) \simeq V(0) + \tfrac{1}{2}m_n^2\phi^2 + \ldots, \qquad m_n^2 = n(n-4). \quad (33)$$

We have neglected higher order terms of the form ϕ^c with $c > 2$. Note that for $n > 4$ the mass term in Eq. (33) becomes positive, however the nonconvex nature of the potential arises for larger values of ϕ, as we have checked numerically. For $n = 4$ the mass term becomes zero and the higher order terms become significant, but even in this case we obtained numerically that the potential is nonconvex.

Now it is worthwhile to compare with the exponential profile scalar field of Eq. (11), for which we find

$$a(r) \simeq r + \delta h(r), \qquad \delta h(r) = \frac{\phi_0^2}{12}(d + r)e^{-(2r/d)} \quad (34)$$

while the corresponding asymptotic formula for $f(r)$ is

$$f(r) \simeq r^2 - \frac{\mu}{r^2} + \frac{\phi_0^2}{6}(dr + r^2)e^{-(2r/d)} + \ldots. \quad (35)$$

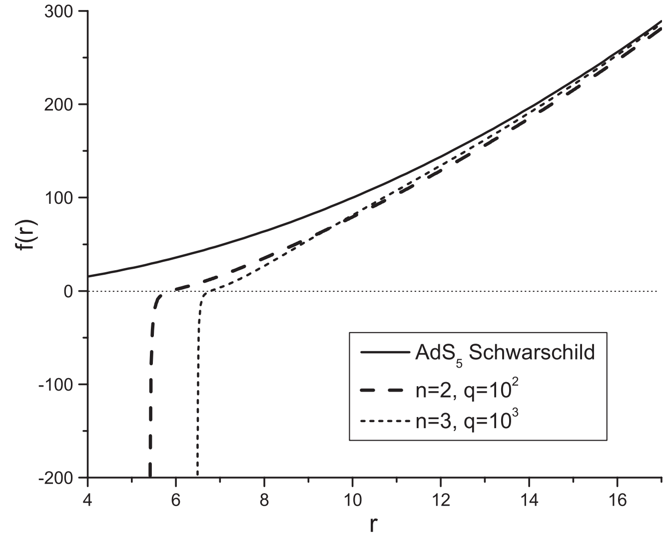

FIG. 8. The function $f(r)$ for the inverse power type scalar field for $n = 2$, $q = 10^2$ and $n = 3$, $q = 10^3$, and $\mu = 10$. We see that for $n = 2$, 3 the asymptotic behavior of the black hole is of the standard form $r^2 - \mu/r^2$, as it is expected by the asymptotic formula of Eq. (32).

For $\phi(r) = \phi_0 e^{-r/d}$ we see that the functions $a(r)$ and $f(r)$ approach their asymptotic behavior in an exponentially fast way independently from the values of the parameters ϕ_0 and d, in contrast with the case of the inverse power profile $1/r^n$ for which we have to set restrictions to the parameter n ($n \geq 2$). The potential $V(\phi)$, in the case of the exponential profile scalar field, for $\phi \to 0$ is

$$V(\phi) \simeq V(0) + \frac{1}{2}\left[\ln\left(\frac{\phi}{\phi_0}\right)\right]^2\phi^2 + \ldots, \quad (36)$$

where we have neglected terms of the form $\ln(\phi)\phi^2$ and ϕ^2 as they tend to zero faster than $[\ln(\phi)]^2\phi^2$. Note that the above equation for small ϕ implies a positive derivative $V'(\phi)$, however, it becomes negative for slightly larger values of ϕ, with $r > r_h$, as we have confirmed by numerical calculations. In the left panel of Fig. 5 we have plotted the negative part of $V(\phi) - V(0)$ for small ϕ. Note that the region where $V'(\phi)$ is positive, due to the term $[\ln(\phi)]^2\phi^2$ in Eq. (36), is very close to the axes origin and it is not visible in this figure.

IV. THE NONCONVEX NATURE OF THE SCALAR FIELD POTENTIAL

The equation for the scalar field

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - V'(\phi) = 0 \quad (37)$$

can be used in order to demonstrate a "no hair theorem" like that of Bekenstein in the special case of 5D AdS spacetime we examine (see also [3] in the case of Minkowski space-time). If we multiply by ϕ and integrate by parts, from r_h (horizon of the black hole) to infinity, we obtain

$$(g^{rr}\sqrt{g}\phi\partial_r\phi)^{+\infty}_{r_h} - \int_{r_h}^{+\infty} drg^{rr}\sqrt{g}(\partial_r\phi)^2 = \int_{r_h}^{+\infty} dr\sqrt{g}\phi V'(\phi). \qquad (38)$$

The boundary term vanishes.[2] Because in the previous paragraph, in order to satisfy the asymptotic behavior of Eq. (28) for the function $f(r)$, we impose the restriction $\phi(r) \simeq 1/r^n$ with $n > 2$, for $r \to +\infty$ (namely $\phi(r)$ tends to zero faster than $1/r^2$). In addition, the scalar field $\phi(r)$ and the first derivative of the scalar field $\partial_r\phi(r)$ are regular on the horizon $r = r_h$, while the metric component g^{rr} vanishes on the horizon and becomes positive ($g^{rr} > 0$) for $r > r_h$. From Eq. (38) We obtain that

$$\int_{r_h}^{+\infty} drg^{rr}\sqrt{g}\phi(\partial_r\phi)^2 = -\int_{r_h}^{+\infty} dr\sqrt{g}\phi V'(\phi). \qquad (39)$$

As we have assumed that $\phi > 0$, from the above equation we conclude that $V'(\phi(r)) < 0$, at least in a region of the radius r with $r > r_h$. This confirms the nonconvex nature of the potential $V(\phi)$ for 5D scalar black hole solutions which behave asymptotically like an AdS_5 Schwarzschild black hole ($n \geq 2$). For $0 < n < 2$, as we have mentioned previously, the Einstein equation possesses black hole solutions (with a horizon and a singularity, see Fig. 7) but their asymptotic behavior is not of the form $r^2 - \mu/r^2$. Even in this case ($0 < n < 2$) we have checked numerically that the potential does not possess a convex form.

V. CONCLUSIONS

We studied 5D scalar black hole solutions, with flat 3D slices, by using a semianalytical way. In our approach, first we chose the scalar field $\phi(r)$ and subsequently the corresponding self-interacting potential $V(\phi)$ was determined.

We focused on continuous scalar field configurations, which are nonzero in a small region of r near the origin and vanish rapidly in the large r limit. In particular, two cases for the black hole scalar hair were examined: (a) an exponential decaying scalar field profile $\phi = \phi_0 e^{-r/d}$ and (b) an inverse power scalar field profile $\phi = q/r^n$. The constants of integration in the metric of the scalar black hole solution are fixed by assuming an asymptotic behavior identical with that of an AdS_5 Schwarzschild black hole. We show that this asymptotic behavior can be achieved only if the scalar field vanishes asymptotically like $1/r^2$ or faster. However, we constructed black hole solutions [with a horizon and a singularity for the profile (b)] even in the interval $0 < n < 2$, but we show that they do not have an asymptotic behavior of the form $r^2 - \mu/r^2$.

We found that the reconstructed potential $V(\phi)$, for the cases (a) and (b), has a double well form (see Fig. 4 above). However, as we see, it blows up near the singularity of the black hole $\phi(r_s)$, hence the scalar field is restricted in the region $0 \leq \phi \leq \phi(r_s)$. We also demonstrated (and checked numerically) a "no hair theorem" for our case, which requires a nonconvex form for the potential $V(\phi)$, or equivalently that $V'(\phi(r))$ is negative at least in a region of r with $r > r_h$, in order to have a nontrivial solution for the scalar field. Although this theorem has been proven only when the scalar field vanishes asymptotically like $1/r^2$ or faster, our numerical results show that the nonconvex nature of the potential remains even in the region $0 < n < 2$ for the profile (b).

Finally, we would like to state that the stability of our solutions, against linear scalar field perturbations, has not been examined in this work, and it is left for further investigation.

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[2]For $n = 2$ the scalar field behaves like $1/r^2$ for large r. In this case the boundary term is nonzero but it possesses a finite negative value. In particular, we find that $\int_{r_h}^{+\infty} dr\sqrt{g}\phi V'(\phi) = -\int_{r_h}^{+\infty} drg^{rr}\sqrt{g}(\partial_r\phi)^2 - 2$, or equivalently that $V'(\phi) < 0$. This means that the potential is nonconvex even in the limiting case where $n = 2$

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