PHYSICAL REVIEW D **80,** 064011 (2009)

Quasinormal modes of a Schwarzschild white hole

Nigel T. Bishop and Amos S. Kubeka

Department of Mathematical Sciences, University of South Africa, P.O. Box 392, Unisa 0003, South Africa

(Received 10 July 2009; published 8 September 2009)

We investigate perturbations of the Schwarzschild geometry using a linearization of the Einstein vacuum equations within a Bondi-Sachs, or null cone, formalism. We develop a numerical method to calculate the quasinormal modes, and present results for the case $\ell = 2$. The values obtained are different than those of a Schwarzschild black hole, and we interpret them as quasinormal modes of a Schwarzschild white hole.

DOI: 10.1103/PhysRevD.80.064011 PACS numbers: 04.70.Bw, 04.30.Db

I. INTRODUCTION

The theory of a linear perturbation of a black hole was developed some time ago [1–4]; see also Ref. [5] and the review in[6]. The essential idea is that the vacuum Einstein equations are linearized about the Schwarzschild (or Kerr) geometry described by the usual (t, r, θ, ϕ) coordinates. Then a standard separation of variables ansatz is applied, with metric quantities behaving as an unknown function of $r \times Y_{\ell m}(\theta, \phi) \exp(i\nu t)$ (actually, the angular dependence is somewhat more complicated, and the technical details can be found in the literature). There results an ordinary differential equation in r, and the quasinormal modes are obtained by finding the special values of ν for which solutions exist that satisfy appropriate boundary conditions in the neighborhood of the event horizon, and of infinity.

Quasinormal mode theory has become a cornerstone of modern general relativity theory. They have been seen in numerical relativity simulations of binary black hole coalescence. And, while not yet actually observed, it is strongly expected that they will be measured by the LIGO collaboration, and certainly by Laser Interferometer Space Antenna (LISA), yielding precise information about the parameters describing a black hole from some coalescence event.

In the usual approach to linear perturbations of a black hole, the linearization is performed using standard Schwarzschild (or Kerr) coordinates (t, r, θ, ϕ). It is also possible to perform the linearization using Bondi-Sachs coordinates, which is a coordinate system based on outgoing null cones. This has been done in previous work in order to obtain analytic solutions of the linearized Einstein equations for the purpose of testing numerical relativity codes. As with the usual approach, one ends up with a second order ordinary differential equation involving ℓ and ν as parameters, Eq. (5). However, when the quasinormal modes were calculated for this equation, it was found that they are not the standard ones. Different physical problems are considered in the two cases, as illustrated in the Penrose diagram of Schwarzschild spacetime (Fig. 1). K is a typical hypersurface used in finding the quasinormal modes of a black hole, and the direction of wave propagation at the boundaries of K is shown by arrows. On the other hand, N is a typical hypersurface used in finding the quasinormal modes of Eq. (5). From the direction of wave propagation on N, the resulting quasinormal modes can be interpreted as being those of a white hole.

The plan of this paper is as follows. Section II summarizes previous work on the Bondi-Sachs metric and linearized solutions within that framework. Section III describes our approach to calculating the quasinormal modes, and Sec. IV presents the results. We end with a conclusion in Sec. V.

II. BACKGROUND MATERIAL

The Bondi-Sachs formalism uses coordinates $x^i = (u, r, x^A)$ based upon a family of outgoing null hypersurfaces. We label the hypersurfaces by $u =$ const, null rays by x^A $(A = 2, 3)$, and the surface area coordinate by r. In this coordinate system, the Bondi-Sachs metric [7–9] takes the form

$$ds^2 = -\left[e^{2\beta}\left(1 + \frac{W}{r}\right) - r^2 h_{AB} U^A U^B\right] du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B, \tag{1}$$

where $h^{AB} h_{BC} = \delta^A_B$ and $\det(h_{AB}) = \det(q_{AB})$, with q_{AB}

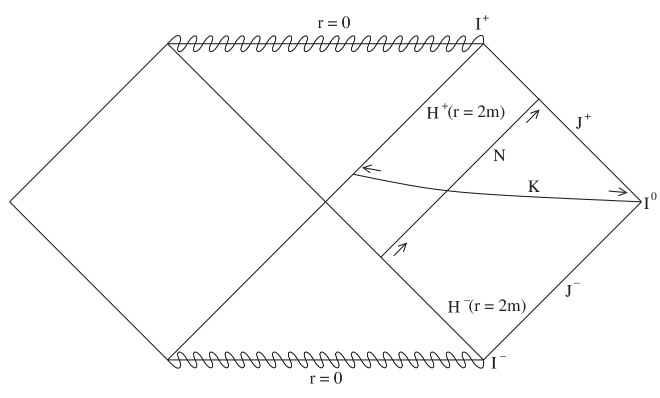

FIG. 1. Penrose diagram illustrating the differences, in terms of location and boundary conditions, between the hypersurfaces K and N.

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being a unit sphere metric. We represent q_{AB} by means of a complex dyad q_A. For example, in the case that the angular coordinates are spherical polar (θ, ϕ), the dyad takes the form

$$q_A = (1, i\sin\theta). \tag{2}$$

For an arbitrary Bondi-Sachs metric, h_{AB} can be represented by its dyad component

$$J = h_{AB}q^A q^B/2. \tag{3}$$

We also introduce the spin-weighted field $U = U^A q_A$ as well as the (complex differential) eth operators $\eth$ and $\bar{\eth}$ [10]. In Schwarzschild spacetime, $W = -2M$, $\beta = 0$, $U^A = 0$, and $J = 0$.

We use $Z_{\ell m}$, rather than $Y_{\ell m}$, as spherical harmonic basis functions, where [11] the $Z_{\ell m}$ have orthonormal properties similar to those of the $Y_{\ell m}$, and are real.

We assume the following ansatz, representing a small perturbation of the Schwarzschild geometry:

$$\begin{aligned} \beta &= \mathrm{Re}(\beta_0(r)e^{i\nu u})Z_{\ell m}, \\ U &= \mathrm{Re}(U_0(r)e^{i\nu u})\eth Z_{\ell m}, \\ J &= \mathrm{Re}(J_0(r)e^{i\nu u})\eth^2 Z_{\ell m}, \\ W &= -2M + \mathrm{Re}(w_0(r)e^{i\nu u})Z_{\ell m}. \end{aligned} \tag{4}$$

Using the above ansatz, Ref. [11] constructed the resulting linearized Einstein vacuum equations. As expected, the angular and time dependence factored out, and a system of ordinary differential equations (in r) was obtained. As discussed in Ref. [11], the system can be manipulated to give

$$\begin{aligned} &x^3(1 - 2xM)\frac{d^2 J_2}{dx^2} + 2\frac{dJ_2}{dx}(2x^2 + i\nu x - 7x^3 M) \\ &\quad - 2(x(\ell^2 + \ell - 2)/2 + 8Mx^2 + i\nu)J_2 = 0, \end{aligned} \tag{5}$$

where $J_2(x) = d^2 J_0/dx^2$ and $x = 1/r$. [Actually, Ref. [11] gave Eq. (5) only in the case $\ell = 2$, and here we give the formula for general ℓ.]

III. PROBLEM SPECIFICATION

We note that Eq. (5) has singularities at $x = 0$ and $x = 0.5M$. The problem is to find values of ν for which there exists a solution to Eq. (5) that is regular everywhere in the interval $[0, 0.5M]$; these values of ν are the quasinormal modes. This is the same situation that is faced when finding the quasinormal modes of a black hole. The first solution to this problem was obtained by using series solutions around the singular points, and a numerical solution of an ordinary differential equation within the interior of the interval [4]. Subsequently, it was shown [12] how the theory of 3-term recurrence relations [13] for the series solution about $x = 0.5M$ could be used to determine the quasinormal modes.

It is straightforward to write Eq. (5) with the origin transferred to $x = 0.5M$, and then to evaluate the recurrence relation satisfied by a regular solution [see Eqs. (24) and (25) below]. We find a 4-term, rather than a 3-term, recurrence relation. While it may be that the quasinormal modes could be found by an approach similar to that of [12], this is not a practical option since there does not seem to be available a well-developed mathematical theory of 4-term recurrence relations.

Instead, we proceed along the lines used in [4]. We construct the asymptotic series about the essential singularity at $x = 0$, and use it to find a solution to within a specified tolerance at a point $x_0 > 0$. We then use this solution as initial data for a numerical solution of Eq. (5) in the range (x_0, x_c) where $x_c < 0.5M$; actually, as in [4], we do not integrate Eq. (5) directly but first convert it to first-order Ricatti form. Finally, we construct the regular series solution about $x = 0.5M$ and use it to find a solution at $x = x_c$. Then a value of ν is a quasinormal mode if the difference at $x = x_c$ between the regular series solution and the numerical solution vanishes.

A. Asymptotic series solution about the essential singularity at $x = 0$

Since the singularity is essential, the resulting series solution has a radius of convergence zero, although it is asymptotic. We use [14] to determine rigorous bounds on the error of approximating the solution by its first n terms. Note that a series solution $J_2(x) = \sum_{n=1}^{\infty} a_n x^n$ to Eq. (5) can be generated by the recurrence relation

$$a_n = -a_{n-1}\frac{n^2 + n - 6}{2i\nu(n-1)} + a_{n-2}M\frac{2n(n+2)}{2i\nu(n-1)}, \tag{6}$$

with $a_1 = 1$, $a_2 = 0$.

In order to use the theory developed in [14], we must first transform Eq. (5) to an asymptotic form by

$$x \to z = \frac{1}{x} \tag{7}$$

and investigate the solution about the singularity at infinity. We find

$$\begin{aligned} &z^2(z-2)\frac{d^2 J_2(z)}{dz^2} - z(2z^2 i\nu + 2z - 10)\frac{dJ_2(z)}{dz} \\ &\quad - (2z^2 i\nu + 4z + 16)J_2(z) = 0, \end{aligned} \tag{8}$$

where we have normalized the scaling of z by setting $M = 1$. We evaluate quantities used in [14]:

$$\begin{aligned} &f = -\frac{2(z - 5 + i\nu z^2)}{z(z-2)}, \qquad g = -\frac{2(2z + 8 + i\nu z^2)}{z^2(z-2)}, \\ &f_0 = -2i\nu, \qquad f_1 = -2 - 4i\nu, \qquad g_0 = 0, \\ &g_1 = -2i\nu, \qquad \rho = i\nu, \qquad \sigma = 2 + 2i\nu. \end{aligned} \tag{9}$$

Then the solutions can be written as

$$J_{2j}(z) \asymp \exp(\lambda_j z) z^{\mu_j} \sum_{s=0}^{\infty} \frac{a_{s,j}}{z^s}, \tag{10}$$

where

$$\lambda_1 = 0, \qquad \mu_1 = -1, \qquad \lambda_2 = 2i\nu, \qquad \mu_2 = 3 + 4i\nu. \tag{11}$$

Following [14], we let the solution to Eq. (8) be

$$J_2(z) = L_n(z) + \epsilon_n(z), \tag{12}$$

where

$$L_n(z) = \exp(\lambda_1 z) z^{\mu_1} \sum_{s=0}^{n-1} \frac{a_{s,1}}{z^s} \tag{13}$$

and define the residual $R_n(z)$ by

$$\frac{d^2 L_n(z)}{dz^2} + f(z)\frac{dL_n(z)}{dz} + g(z)L_n(z) = \frac{R_n(z)}{z}, \tag{14}$$

with

$$|R_n(z)| \le \frac{B_n}{z^{n+1}} \tag{15}$$

in some region $|z| > b$ and where B_n is calculable. Reference [14] obtains a bound on $\epsilon_n(z)$, provided the quantity $C(n, b, \nu)$ defined immediately below satisfies $C < 1$, where

$$C(n, b, \nu) = \frac{\beta\sqrt{\pi}\Gamma(\frac{1}{2}(n+1)+1))}{|2i\nu|\Gamma(\frac{1}{2}(n+1)+\frac{1}{2}))(n+1)}, \tag{16}$$

where β is bounded by

$$\beta \le |4i\nu| + \left|8\frac{1+i\nu}{b-2}\right| + \left|32\frac{1}{b(b-2)}\right| + |2i\nu|\left(|2+4i\nu| + \left|2\frac{3-4i\nu}{b-2}\right|\right). \tag{17}$$

Given ν and b, we use numerics to determine conditions on n such that $C < 0.99$ and then we bound $\epsilon_n(z)$ by

$$|\epsilon_n(z)| \le \frac{2B_n}{\beta(1 - C(n, b, \nu)|z|^{n+1}}. \tag{18}$$

We also need to bound the error $\epsilon_n'(x)$ in using a finite series to estimate $\frac{dJ_2(x)}{dx}$. Noting that

$$\frac{dJ_2(x)}{dx} = -z^2\frac{dJ_2(z)}{dz}, \tag{19}$$

the bound on the error is

$$|\epsilon_n'(x)| \le \frac{2|i\nu|B_n}{\beta(1 - C(n, b, \nu)|z|^{n-1}}. \tag{20}$$

Numerical implementation

We have written MATLAB code that takes as input ν and b, and then finds β and the lowest value of n such that $C < 0.99$. Then the code finds the maximum of the absolute values of $\epsilon_n(b)$ and $\epsilon_n'(x = 1/b)$. A bisection method program takes ν as input and refines b until the absolute value of the maximum error is in the range $(0.5, 1) \times$ machine precision (about 2×10^{-16}). The code returns the values of $1/b$ and $L_n(b)/L_n'(x = 1/b)$.

B. Numerical integration of Eq. (5)

The first step is to transform Eq. (5) into first-order Ricatti form. Defining a new dependent variable $v(x)$ by

$$J_2(x) \to v(x) = \frac{1}{J_2(x)}\frac{dJ_2(x)}{dx}, \tag{21}$$

we obtain

$$x^3(1-2x)\left(\frac{dv}{dx} + v^2\right) + 2x(2x + i\nu - 7x^2)v - 2(2x + 8x^2 + i\nu) = 0. \tag{22}$$

The numerical integration of Eq. (22) near the singularity at $x = 0$ can be tricky because we need the results to be as accurate as possible. We found that a fourth order Runge-Kutta scheme (ODE45 in MATLAB) performed better than the stiff schemes, provided stringent tolerance conditions were used (specifically, $\mathrm{RelTol} = 10^{-12}$, $\mathrm{AbsTol} = 10^{-12}$, $\mathrm{MaxStep} = 2 \times 10^{-6}$). Under these conditions, each integration to x_c ($= 0.25$) takes the of order of 100 s.

C. Series solution about the regular singularity at $x = 0.5M$

We first make the transformation

$$x \to s = 1 - 2x \tag{23}$$

to Eq. (5) and obtain

$$s(1-s)^3\frac{d^2J_2(s)}{ds^2} - (1-s)(4i\nu - 3 + 10s - 7s^2)\frac{dJ_2(s)}{ds} - 4(i\nu + 3 - 5s + 2s^2)J_2(s) = 0. \tag{24}$$

This equation has a series solution $\sum_0^\infty a_n s^n$ that satisfies the recurrence relation

$$\begin{aligned} a_0 &= 1, \qquad a_1 = 4\frac{3+i\nu}{3-4i\nu}, \\ a_2 &= \frac{15(4+3i\nu)}{2(1-i\nu)(3-4i\nu)}, \\ a_n &= a_{n-1}\frac{4ni\nu - 8i\nu - 5 - 3n^2 - 4n}{n(4i\nu - n - 2)} \\ &\quad + a_{n-2}\frac{4+3n^2+2n}{n(4i\nu - n - 2)} + a_{n-3}\frac{(1-n)(1+n)}{n(4i\nu - n - 2)}. \end{aligned} \tag{25}$$

The radius of convergence of the above series is $s < 1$, and, given ν, the numerical evaluation of the coefficients, and then of the series, is straightforward. Using $x_c = 0.25$ means that we need to evaluate the series at $s = 0.5$. We terminate summation of the series at the first term smaller than 10^{-18} (typically, about 60 terms), and thus expect the result to be accurate to within machine precision (about 2×10^{-16}).

IV. RESULTS

We have written a MATLAB program that, given a value of ν, first uses the asymptotic series to find the value v_0 of $v(x)$ [as defined in Eq. (21)] at $x = x_0 = 1/b$, and then integrates numerically Eq. (22) between x_0 and $x_c = 0.25$, obtaining a complex number $v_+ = v(x_c)$; and second uses the regular series about $x = 0.5$ to find $v_- = v(x_c)$. Defining

$$g_\nu = v_+ - v_-, \tag{26}$$

the quasinormal modes are those values of ν such that g_ν is indistinguishable from zero.

We calculated g_ν for values of ν in the range $\nu = a + ib$, $0.1 \le a \le 1.07$, $0.05 \le b \le 0.89$, in increments of 0.03. The results are shown in the contour plot in Fig. 2. The black, dotted line is the zero contour of $\mathrm{Im}(g_\nu)$, the red, solid line is the zero contour of $\mathrm{Re}(g_\nu)$, and the blue, dashed line is the boundary of a region where the computation is probably unreliable (because the computed curve oscillates, indicating that a smaller step length is required). Clearly, the quasinormal modes lie at the intersection of a red and a green line, and from the plot we can read off an estimate for the lowest mode, $\nu = 0.9 + 0.63i$. We then applied a secant method, obtaining a final estimate for the lowest quasinormal mode at

$$\nu = 0.883 + 0.614i. \tag{27}$$

In this case, $x_0 = 0.036\,493\,228\,795\,438$, $v_0 = 0.036\,838\,521\,818\,950 + 0.000\,637\,428\,772\,012i$, $\beta = 0.988\,517\,790\,240\,599$, and 62 terms were used in the asymptotic series. The contour plot indicates another quasinormal mode at about $\nu = 1.06 + 0.63i$, but we did not investigate further.

We now use the value in Eq. (27), and vary the numerical methods so as to determine the accuracy with which g_ν has been determined. In Fig. 3 the integration between x_0 and x_c is carried out with different values of MAXSTEP, 2×10^{-6}, 10^{-6}, and 5×10^{-7}, and also an error of an amount $(1 + i) \times 10^{-15}$ is introduced into the value of v_0 at x_0 in the case MaxStep $= 2 \times 10^{-6}$. Also, numerical integration of Eq. (22) as well as a series solution is used in the range $(x_c, 0.5)$. The various curves lie on top of each other and are visually indistinguishable. Taking all these options into account, the maximum value noted for g_ν was $(6.02 + 5.87i) \times 10^{-4}$. Using intermediate results from the secant root-finding process to estimate

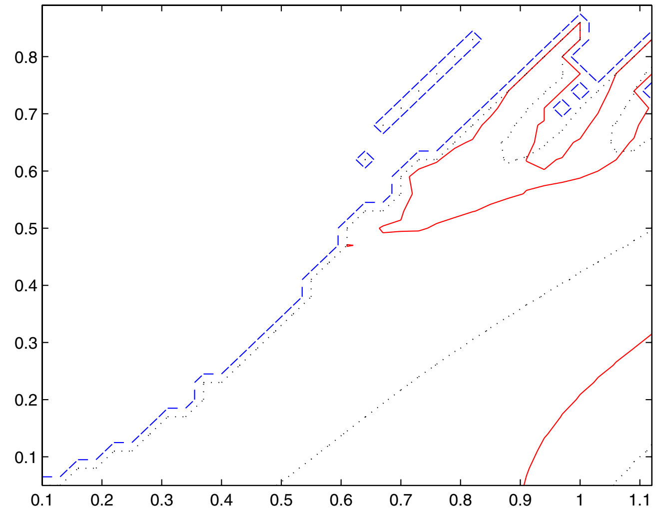

FIG. 2 (color online). Contour plot in the complex plane of ν showing the contours where $\mathrm{Re}(g_\nu) = 0$ (red, solid line) and $\mathrm{Im}(g_\nu) = 0$ (black, dotted line) as well as the boundary of the region of unreliable computation (blue, dashed line).

$$\frac{\partial \nu}{\partial g_\nu} = 3.95 + 0.69i, \tag{28}$$

it follows that the possible error in Eq. (27) is

$$|(3.95 + 0.69i) \times (6.02 + 5.87i) \times 10^{-4}| = 0.003, \tag{29}$$

so that Eq. (27) should be amended to read

$$\nu = 0.883 + 0.614i + 0.003k, \tag{30}$$

where k is a complex number satisfying $|k| \le 1$.

The lowest quasinormal mode of a Schwarzschild black hole is at $\nu = 0.373\,67 + 0.088\,96i$. We have used this value in our code, and obtained Fig. 4 from which it is clear that this value of ν is not a quasinormal mode of Eq. (5).

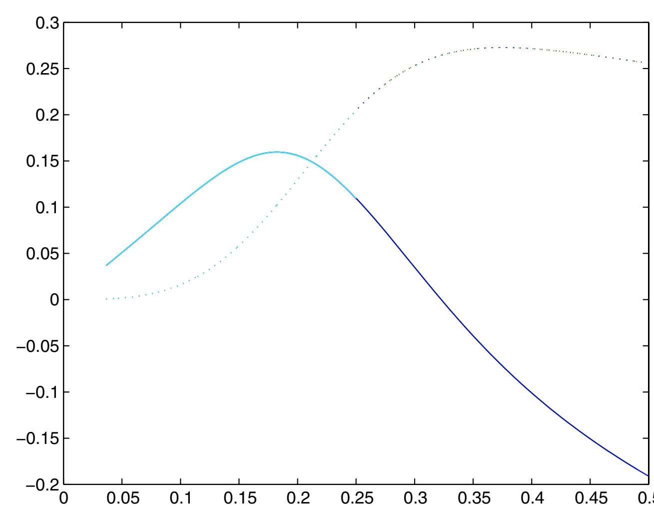

FIG. 3 (color online). The real (solid line) and imaginary (dotted line) parts of $v(x)$ in the quasinormal mode case $\nu = 0.883 + 0.614i$.

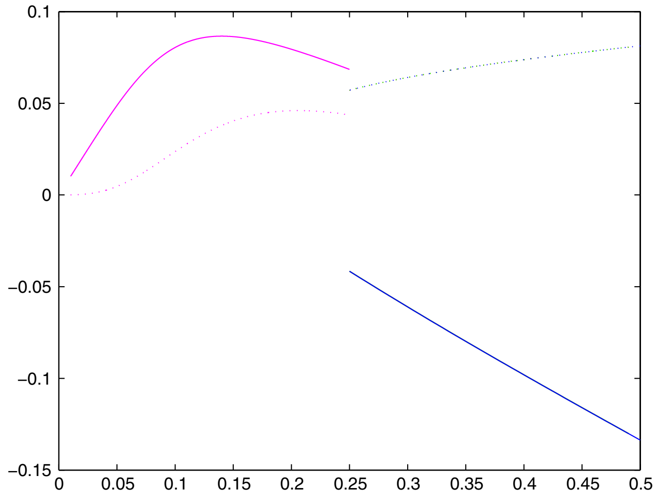

FIG. 4 (color online). The real (solid line) and imaginary (dotted line) parts of $v(x)$ in the case $\nu = 0.3\,7367 + 0.08\,896i$, indicating that the lowest quasinormal mode of a Schwarzschild black hole is not a quasinormal mode of Eq. (5).

V. CONCLUSION

Using a linearization of the vacuum Einstein equations about the Schwarzschild geometry, within a Bondi-Sachs framework, we have constructed a numerical procedure to calculate the quasinormal modes. The value of the lowest mode in the case $\ell = 2$ is not a quasinormal mode of a Schwarzschild black hole, and further the lowest quasinormal mode of a Schwarzschild black hole is not a quasinormal mode of Eq. (5). As discussed in the Introduction, this apparent discrepancy can be avoided by interpreting the quasinormal modes of Eq. (5) as being those of a white hole rather than those of a black hole.

The results obtained depend crucially on the validity of Eq. (5), and thus it is important to discuss the extent to which this has been verified. Equation (5) was derived in [11], and there has been no subsequent, independent derivation. Nevertheless, Eq. (5) has been subject to some consistency checks since Ref. [11] confirmed that solutions obtained also satisfy the remaining Einstein equations (the constraint equations). Further in the case $M = 0$, solutions based on Eq. (5) have been used as analytic solutions for the testing of numerical relativity codes based on the Bondi-Sachs metric, and the expected order of convergence was observed [15,16].

The evidence for the existence of black holes is now very strong. However, the question about the existence of white holes is much more problematic since such objects cannot form from regular initial data, but instead must have been created as part of the creation of the Universe. The present work provides a possible observational signature of a white hole, since it is, in principle, possible for a gravitational wave detector to extract the parameters of a quasinormal mode from a gravitational wave signal.

ACKNOWLEDGMENTS

N. T. B. and A. S. K. would like to thank the National Research Foundation of South Africa for financial support. N. T. B. thanks the Max-Planck-Institut für Gravitationsphysik for its hospitality. We thank E. Rosinger for discussion and for drawing our attention to Ref. [14].

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