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Generalized virial theorem in Palatini $f(\mathcal{R})$ gravity

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We use the collision-free Boltzmann equation in Palatini $f(\mathcal{R})$ gravity to derive the virial theorem within the context of the Palatini approach. It is shown that the virial mass is proportional to certain geometrical terms appearing in the Einstein field equations which contributes to gravitational energy and that such geometric mass can be attributed to the virial mass discrepancy in a cluster of galaxies. We then derive the velocity dispersion relation for clusters, followed by the metric tensor components inside the cluster as well as the $f(\mathcal{R})$ Lagrangian in terms of the observational parameters. Since these quantities may also be obtained experimentally, the $f(\mathcal{R})$ virial theorem is a convenient tool to test the viability of $f(\mathcal{R})$ theories in different models. Finally, we discuss the limitations of our approach in light of the cosmological averaging used and questions that have been raised in the literature against such averaging procedures in the context of the present work.

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I. INTRODUCTION

Dark matter is presently one of the most exciting open problems in cosmology. There is some compelling observational evidence for the existence of dark matter, for which the galaxy rotation curves and mass discrepancy in a cluster of galaxies are two prominent examples. According to Newtonian gravity, galaxy rotation curves give the velocity of matter rotating in a spiral disk as a function of the distance from the center of galaxy according to $v(r) = \sqrt{GM(r)/r}$. If we assume that the cluster mass obeys the relation $M(r) = \rho \frac{4\pi r^3}{3}$, where ρ is considered as a constant density in the cluster, then the velocity increases linearly within the cluster and drops off as the square root of r outside of the cluster. However, observation shows that the velocity remains approximately constant; that is $M \sim r$. This points to the possible existence of a new invisible matter which is referred to as dark matter and is distributed spherically around galaxies [1].

What is now known as the mass discrepancy of clusters can be understood when estimating the total mass of a cluster in two different ways; cluster masses can be deduced by summing the individual member masses which we shall call M in total. Alternatively, the virial theorem can be used to estimate the mass of a cluster M_V by studying the motions of each member of the cluster. As it turns out, M_V is nearly 20–30 times greater than M, and this difference is known as the virial mass discrepancy [1]. The prime tool in dealing with the above discrepancy is to postulate dark matter. There are several candidates for dark matter. One possible categorization tells us that dark matter can be baryonic or nonbaryonic. The main baryonic candidates are the massive compact halo objects which include brown dwarf stars and black holes. The nonbaryonic candidates are basically elementary particles which have nonstandard properties. Among the nonbaryonic candidates, we can point to axions as a solution to the strong *CP* problem. However, the largest class is the weakly interacting massive particle (WIMP) class which consists of hundreds of, as yet, unknown particles [2]. The most popular of these WIMPs is the neutralino from supersymmetry. The WIMP's interaction cross-section with normal baryonic matter is extremely small but nonzero, so their direct detection is a possibility. Neutrinos may also be considered as possible candidates for dark matter. Another important categorization tells us that dark matter can be hot or cold. A dark matter candidate is called hot if it was moving at relativistic speeds at the time when galaxies were just starting to form. It is called cold if it was moving nonrelativistically at that time. Of the above candidates, only the light neutrinos would be hot, while all others would be cold. There is, as of now, no nongravitational evidence for dark matter. Moreover, accelerator and reactor experiments do not support the scenarios in which dark matter emerges.

To deal with the question of dark matter, a great number of efforts has been concentrated on various modifications to the Einstein field equations [3]. One such modification is that of $f(\mathcal{R})$, where $\mathcal{R}$ is the Ricci scalar. Theories of $f(\mathcal{R})$ modified gravity have had some success in explaining the accelerated expansion of the Universe [4,5] and account for the existence of dark matter [6–8]. In this paper, we study $f(\mathcal{R})$ gravity in the context of the Palatini formalism. As it is well known, starting with the usual Einstein-Hilbert action, both the Palatini and metric approaches result in the same field equations. However, if the action is taken as a generic function of $\mathcal{R}$, then the two approaches result in different field equations [9]. Here, we study the virial theorem within the framework described above. In general, virial theorem plays an important role in astrophysical objects like galaxies, clusters, and super

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clusters. By using the virial theorem and studying the observational data from the velocity of each member, one can estimate the mean density of such objects, rendering the prediction of the total mass possible. The virial theorem also offers interesting predictions on the stability of the astrophysical objects. Several authors have studied the virial theorem in models with a cosmological constant [10,11], with brane world scenarios [12], and with metric $f(\mathcal{R})$ theories [13]. Our main purpose in this paper is to obtain the generalized form of the virial theorem in $f(\mathcal{R})$ Palatini formalism by using the collisionless Boltzmann equation. Of course, some extra terms emerge in virial theorem which are originated from the modified action and are geometric in nature. We show that one may account for the virial mass discrepancy by taking into account such extra terms. The components of the metric tensor inside the galaxies may also be derived in terms of physical observable quantities like the temperature of intracluster gas and the radius and density of the cluster core. Finding the components of the metric tensor leads to the form of the Lagrangian in $f(\mathcal{R})$ gravity in terms of observable quantities. Thus, the virial theorem is a convenient tool to test the validity of $f(\mathcal{R})$ models.

In what follows, we first give a brief review of the Palatini formalism, and the gravitational field equations are derived in the scalar-tensor representation of $f(\mathcal{R})$ gravity. Next, we introduce the relativistic Boltzmann equation from which we deduce the virial theorem with the aid of the field equations and discuss the limitations of the assumptions made. The geometric mass and density of a cluster is then identified in terms of the observable quantities, and the metric components are calculated inside the cluster. We then move on to study the velocity dispersion relation in galaxies. Finally, we present the Lagrangian which represents the $f(\mathcal{R})$ theory inside the cluster. Conclusions are drawn in the last section.

II. PALATINI $f(\mathcal{R})$ GRAVITY IN SCALAR-TENSOR REPRESENTATION

Let us start with the action

$$S_{\text{Palatini}} = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R}) + \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu}), \tag{1}$$

where $\mathcal{L}_m$ is the matter Lagrangian, and $\kappa = 8\pi G$. The Ricci scalar is written as $\mathcal{R}$ in the context of the Palatini formalism to point out that it is different from the Ricci scalar R in the context of metric $f(R)$ gravity. It is necessary to stress that $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma)$, where $\mathcal{R}_{\mu\nu}(\Gamma)$ is constructed from the connection which is independent of the metric. In addition, in the Palatini approach, the Lagrangian corresponding to matter does not depend on the connection. Varying the action with respect to the metric and connection, respectively, yields

$$F(\mathcal{R})\mathcal{R}_{\mu\nu} - \tfrac{1}{2}f(\mathcal{R})g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{2}$$

$$-\bar{\nabla}_\lambda(\sqrt{-g}F(\mathcal{R})g^{\mu\nu}) + \bar{\nabla}_\sigma(\sqrt{-g}F(\mathcal{R})g^{\sigma(\mu})\delta_\lambda^{\nu)} = 0, \tag{3}$$

where we have used $\delta\mathcal{R}_{\mu\nu} = \bar{\nabla}_\lambda\delta\Gamma^\lambda_{\mu\nu} - \bar{\nabla}_\nu\delta\Gamma^\lambda_{\mu\lambda}$ with $\bar{\nabla}_\mu$ being the covariant derivative which is defined with the independent connection; $(\mu\nu)$ and $[\mu\nu]$ define the symmetric and antisymmetric parts of the relevant parameter, and $T_{\mu\nu}$ is the energy-momentum tensor. We also denote $F(\mathcal{R}) = \frac{d}{d\mathcal{R}}f(\mathcal{R})$. By contracting Eqs. (2) and (3), the field equations can be written as

$$F(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = -\kappa T, \tag{4}$$

$$-\bar{\nabla}_\lambda(\sqrt{-g}F(\mathcal{R})g^{\mu\nu}) = 0. \tag{5}$$

It is useful to define a metric conformal to $g_{\mu\nu}$ as follows:

$$h_{\mu\nu} \equiv F(\mathcal{R})g_{\mu\nu}, \tag{6}$$

which yields

$$\sqrt{-h}h^{\mu\nu} = \sqrt{-g}F(\mathcal{R})g^{\mu\nu}. \tag{7}$$

Now, Eq. (3) becomes the definition of the Levi-Civita connection of $h_{\mu\nu}$ and can be solved algebraically to give

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}h^{\lambda\sigma}[\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}]. \tag{8}$$

By using Eq. (6), we can write the connection in terms of $g_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\sigma}}{2F(\mathcal{R})}[\partial_\mu(F(\mathcal{R})g_{\nu\sigma}) + \partial_\nu(F(\mathcal{R})g_{\mu\sigma}) - \partial_\sigma(F(\mathcal{R})g_{\mu\nu})]. \tag{9}$$

We now have an expression for $\Gamma^\lambda_{\mu\nu}$ in terms of $\mathcal{R}$ and $g_{\mu\nu}$, so the independent connection can be eliminated from the field equations. In fact, using conformal transformation in Eq. (6), the Riemann tensor, Ricci scalar, and Einstein tensor in Palatini formalism can be derived in terms of the metric ones [14]:

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} - \frac{3}{2}\frac{1}{[F(\mathcal{R})]^2}\nabla_\mu F(\mathcal{R})\nabla_\nu F(\mathcal{R}) + \frac{1}{F(\mathcal{R})}\left(\nabla_\mu\nabla_\nu + \frac{1}{2}g_{\mu\nu}\Box\right)F(\mathcal{R}). \tag{10}$$

Contracting Eq. (10) by $g^{\mu\nu}$ yields the Ricci scalar

$$\mathcal{R} = R - \frac{3}{2}\frac{1}{[F(\mathcal{R})]^2}\nabla_\mu F(\mathcal{R})\nabla^\mu F(\mathcal{R}) + \frac{3}{F(\mathcal{R})}\Box F(\mathcal{R}). \tag{11}$$

Substituting Eqs. (10) and (11) in (2), we may calculate the Einstein tensor

$$G_{\mu\nu} = -\frac{\kappa}{F}T_{\mu\nu} + \frac{3}{2}\frac{1}{F^2}\left[\nabla_\mu F\nabla_\nu F - \frac{1}{2}g_{\mu\nu}\nabla_\lambda F\nabla^\lambda F\right] - \frac{1}{F}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F - \frac{1}{2}g_{\mu\nu}\left(\mathcal{R} - \frac{f}{F}\right). \quad (12)$$

As a result, the independent connections disappear, and the theory is brought to the form of general relativity with a modified source which only depends on the metric and matter fields. We are now ready to introduce a Legendre transformation $\{\mathcal{R}, f\} \to \{\phi, V\}$ defined as

$$\phi \equiv F(\mathcal{R}), \qquad V(\phi) \equiv \mathcal{R}(\phi)F - f(\mathcal{R}(\phi)). \quad (13)$$

The theory in the new representation is given by the action

$$S_{\text{Palatini}} = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}[\phi\mathcal{R} - V(\phi)] + \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu}). \quad (14)$$

Let us write the field equations in terms of the new parameters [9]

$$G_{\mu\nu} = -\frac{\kappa}{\phi}T_{\mu\nu} + \theta_{\mu\nu}, \quad (15)$$

where

$$\theta_{\mu\nu} = \frac{3}{2\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\right) - \frac{1}{\phi}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi - \frac{V}{2\phi}g_{\mu\nu}. \quad (16)$$

It is now easy to realize that this result is the same as that of the Brans-Dicke theory with $\omega = \frac{3}{2}$. Introducing a modified action $f(\mathcal{R})$ results in the appearance of an effective gravitational constant. Note that $\kappa_{\text{eff}} = \frac{\kappa}{\phi} = 8\pi G_{\text{eff}}$. Therefore, we may find from Eq. (15) that $G_{\text{eff}} = \frac{G}{\phi}$. The $\theta_{\mu\nu}$ tensor on the right-hand side of Eq. (15) emerges as a new additional source for the gravitational field.

III. FIELD EQUATIONS FOR A SYSTEM OF IDENTICAL AND COLLISIONLESS POINT PARTICLES

Let us now consider an isolated and spherically symmetric cluster being described by a static and spherically symmetric metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2. \quad (17)$$

Suppose that the clusters are constructed from identical and collisionless point particles (galaxies), being described by the distribution function f_B. The energy-momentum tensor may be written in terms of f_B as [15]

$$T_{\mu\nu} = \int f_B m u_\mu u_\nu du, \quad (18)$$

where m is the cluster's member mass, u is the four velocity of the galaxy, and $du = \frac{du_r du_\theta du_\varphi}{u_t}$ is the invariant volume element of the velocity space. The energy-momentum tensor of the matter in a cluster can be represented in terms of an effective density ρ_{eff} and an effective anisotropic pressure, with radial p^r_{eff} and tangential $p^\perp_{\text{eff}}$ components [10]. In other words, we have

$$\rho_{\text{eff}} = \rho\langle u_t^2\rangle, \qquad p^{(r)}_{\text{eff}} = \rho\langle u_r^2\rangle, \\ p^{(\perp)}_{\text{eff}} = \rho\langle u_\theta^2\rangle = \rho\langle u_\varphi^2\rangle, \quad (19)$$

where $\langle\rangle$ represents the usual macroscopic averaging.

Before going any further, a word of caution is in order at this point. There is an interesting subtlety concerning the energy-momentum tensor appearing on the right-hand side of the Einstein field equations defined in terms of the averaged velocities; the fact that one can average over velocities to obtain an averaged energy-momentum tensor does not necessarily mean that there exists a corresponding averaged spacetime metric associated with that averaged energy-momentum tensor. This point was first raised by Flanagan [16] and Olmo [17]. Later, Motta and Shaw, using general averaging arguments [18], claimed that the averaged metric is almost indistinguishable from that of general relativity. Subsequently however, it was shown in [19] that such conclusions are not always correct because of the existence of counter examples; infrared corrected models do not allow such averaging of the metric. Models with high energy corrections do admit such averaging, but it is not guaranteed in general.

The above arguments boil down to the fact that in modified gravity theories such as $f(\mathcal{R})$ and within the context of the Palatini approach, it may not be reasonable to replace the microscopic energy-momentum tensor with its cosmological average. Modified gravity theories imply nonlinear correction terms to the energy-momentum tensor. When applying the field equations to cosmological scales, one may consider the microscopic structure of all particles in the system. The existence of nonlinear terms means that one should average over all of the microscopic structures of matter, and it seems as if in cosmological scales, the usual macroscopic averaging procedure is no longer valid. In Einstein general relativity, the field equations are approximately linear on microscopic scales. Therefore, the microscopic structure of matter is not particularly important on macroscopic scales. The condition in metric $f(R)$ gravity is the same as in Einstein general relativity. But what about the Palatini formalism? The Palatini formalism has been applied to $f(\mathcal{R})$ gravity in the Einstein frame [16,17,19], and the field equations thus obtained are nonlinear in terms of the energy-momentum tensor even on the smallest scales. In fact, this nonlinearity leads to the problem of averaging in Palatini theories. In other words, the standard averaging procedure may no longer be valid and results in incorrect predictions. In addition, if one takes into account all of the microscopic structures of the matter, the Palatini formalism of $f(\mathcal{R})$ gravity will be indistinguishable from the standard general relativity with a cosmological constant [18]. Interestingly, although Palatini theories were designed to modify gravity on large scales, they actually modify physics on the smallest scales and leave the large scales practically unaltered. In this work, we study clusters containing collisionless point particles (galaxies) within the Palatini formalism in the Jordan frame. Here we take the cosmological averaging of the energy-momentum tensor without studying the microscopic structure of the matter. Of course, the validity of our results should be taken in the light of the discussion presented above.

Using $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ and $u^\mu u_\mu = -1$ with the above definitions, the field equations become

$$e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = -\frac{\kappa}{\phi}\rho\langle u_t^2\rangle - \frac{3}{2\phi^2}\left(\nabla_t\phi\nabla^t\phi - \frac{1}{2}\nabla^\lambda\phi\nabla_\lambda\phi\right) + \frac{1}{\phi}(\nabla^t\nabla_t - \Box)\phi + \frac{V}{2\phi}, \tag{20}$$

$$e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} = -\frac{\kappa}{\phi}\rho\langle u_r^2\rangle + \frac{3}{2\phi^2}\left(\nabla_r\phi\nabla^r\phi - \frac{1}{2}\nabla^\lambda\phi\nabla_\lambda\phi\right) - \frac{1}{\phi}(\nabla^r\nabla_r - \Box)\phi - \frac{V}{2\phi}, \tag{21}$$

$$e^{-\lambda}\left[\frac{\nu' - \lambda'}{2r} - \frac{\nu'\lambda'}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4}\right] = -\frac{\kappa}{\phi}\rho\langle u_\theta^2\rangle + \frac{3}{2\phi^2}\left(\nabla_\theta\phi\nabla^\theta\phi - \frac{1}{2}\nabla^\lambda\phi\nabla_\lambda\phi\right) - \frac{1}{\phi}(\nabla^\theta\nabla_\theta - \Box)\phi - \frac{V}{2\phi}, \tag{22}$$

$$e^{-\lambda}\left[\frac{\nu' - \lambda'}{2r} - \frac{\nu'\lambda'}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4}\right] = -\frac{\kappa}{\phi}\rho\langle u_\varphi^2\rangle + \frac{3}{2\phi^2}\left(\nabla_\varphi\phi\nabla^\varphi\phi - \frac{1}{2}\nabla^\lambda\phi\nabla_\lambda\phi\right) - \frac{1}{\phi}(\nabla^\varphi\nabla_\varphi - \Box)\phi - \frac{V}{2\phi}. \tag{23}$$

Another useful equation is obtained by summing the above four equations:

$$e^{-\lambda}\left[\frac{2\nu'}{r} - \frac{\nu'\lambda'}{2} + \nu'' + \frac{\nu'^2}{2}\right] = -\frac{\kappa}{\phi}\rho\langle u^2\rangle - \frac{3}{2\phi^2}(2\nabla_t\phi\nabla^t\phi) + \frac{1}{\phi}(2\nabla^t\nabla_t + \Box)\phi - \frac{V}{\phi}, \tag{24}$$

where $\langle u^2\rangle = \langle u_t^2\rangle + \langle u_r^2\rangle + \langle u_\theta^2\rangle + \langle u_\varphi^2\rangle$. Since we are interested in the extra-galactic region, we assume a small deviation from standard general relativity in which we have $\phi = 1$. Let us take $\phi = 1 + \epsilon g'(\mathcal{R})$ in our case, where ϵ is a small quantity and $g'(\mathcal{R})$ describes the modification of the geometry due to the presence of tensor $\theta_{\mu\nu}$ [20]. Using $1/\phi \simeq 1 - \epsilon g'(\mathcal{R})$, Eq. (24) can be written as follows:

$$e^{-\lambda}\left[\frac{2\nu'}{r} - \frac{\nu'\lambda'}{2} + \nu'' + \frac{\nu'^2}{2}\right] = -\kappa\rho\langle u^2\rangle - \kappa\rho_\phi, \tag{25}$$

where

$$-\kappa\rho_\phi \simeq \kappa\rho\langle u^2\rangle\epsilon g'(\mathcal{R}) + \left\{-\frac{3}{2\phi^2}[2\nabla^t\phi\nabla_t\phi] + \frac{1}{\phi}[2\nabla^t\nabla_t + \Box]\phi - \frac{V}{\phi}\right\}\Bigg|_{\phi=1+\epsilon g'(\mathcal{R})}. \tag{26}$$

It is not difficult to see that Eq. (26) can be written as

$$e^{-\lambda}\left[\frac{\nu'}{r} - \frac{\nu'\lambda'}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4}\right] = -4\pi G\rho\langle u^2\rangle - 4\pi G\rho_\phi. \tag{27}$$

IV. THE VIRIAL THEOREM IN PALATINI $f(\mathcal{R})$ GRAVITY

To derive the virial theorem, we need the Boltzmann equation which governs the evolution of the distribution function. Integrating this equation on the velocity space when accompanied by the gravitational field equations can yield the virial theorem. We consider an isolated spherically symmetric cluster which is described by Eq. (17). The galaxies in the cluster behave like identical, collisionless point particles. The distribution function is denoted by f_B which obeys the general relativistic Boltzmann equation. Spacetime is a time oriented Lorentzian four-dimensional manifold. The tangent bundle $T(M)$ is a real vector bundle whose fibers at a point $x \in M$ are given by the tangent space $T_x(M)$. The state of a particle is given by the four momentum $p \in T_x(M)$ at an event $x \in M$. The one particle phase space $P_{\rm phase}$ is a subset of the tangent bundle given by [13,15]

$$P_{\rm phase} := \{(x, p) | x \in M, p \in T_x(M), p^2 = -m_0^2\}, \tag{28}$$

where m_0 is the particle mass. A state of a multiparticle system can be described by a continuous non-negative function $f(x, p)$. It is defined on $P_{\rm phase}$ and gives the number dN of the particles crossing the volume dV with momenta p laying within a corresponding three-surface element $d\vec{p}$ in the momentum space. The mean value of f_B is equal to the average number of occupied particle states (x, p) [13,15]. Let $\{x^\alpha\}$ with $\alpha = 0 \cdots 3$ be a local coordinate system in M, defined in some open set $U \subset M$. Note that ∂_t is timelike future directed and ∂_a, $a = 1, 2, 3$ are spacelike. Then $\{\frac{\partial}{\partial x^\alpha}\}$ is the natural basis for tangent vectors. Each tangent vector in U can be written as

$p = p^\alpha \frac{\partial}{\partial x^\alpha}$. We can define a system of local coordinates $\{z^A\}$, $A = 0, \ldots, 7$ in $T_U(M)$ as $z^\alpha = x^\alpha$, and $z^{\alpha+4} = p^\alpha$. A vertical vector field over $T(M)$ is given by $\pi = p^\alpha \frac{\partial}{\partial p^\alpha}$. The geodesic field σ, which can be constructed over the tangent bundle, is defined as

$$\sigma = p^\alpha \frac{\partial}{\partial x^\alpha} - p^\alpha p^\gamma \Gamma^\beta_{\alpha\gamma} \frac{\partial}{\partial p^\beta} = p^\alpha D_\alpha,$$

where $\Gamma^\beta_{\alpha\gamma}$ are the connection coefficients. Physically, σ describes the phase flow for a stream of particles whose motion through spacetime is geodesic. Therefore, the transport equation for the propagation of a particle in a curved arbitrary Riemannian spacetime is given by the Boltzmann equation [13,15]

$$\left(p^\alpha \frac{\partial}{\partial x^\alpha} - p^\alpha p^\beta \Gamma^i_{\alpha\beta} \frac{\partial}{\partial p^i}\right) f_B = 0, \tag{29}$$

where $i = 1, 2, 3$. In many applications, it is convenient to introduce an appropriate orthonormal frame or tetrad $e^a_\mu(x)$, $a = 0 \cdots 3$ which varies smoothly over some coordinates in the neighborhood of U and satisfies the condition $g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}$ for all $x \in U$. Any tangent vector p^μ at x can be expressed as $p^\mu = p^a e^\mu_a$, which defines the tetrad components p^a. In the case of the spherically symmetric line element given by Eq. (17), we introduce the following frame of orthonormal vectors [13,15]:

$$\begin{aligned} e^0_\mu &= e^{\nu/2}\delta^0_\mu, \qquad & e^1_\mu &= e^{\lambda/2}\delta^1_\mu, \\ e^2_\mu &= r\delta^2_\mu, \qquad & e^3_\mu &= r\sin\theta\,\delta^3_\mu, \end{aligned} \tag{30}$$

where u^μ is the four velocity of a typical galaxy, satisfying the condition $u^\mu u_\mu = -1$ with tetrad components $u^a = u^\mu e^a_\mu$. The relativistic Boltzmann equation in tetrad components can be written as

$$u^a e^\mu_a \frac{\partial f_B}{\partial x^\mu} + \gamma^i_{bc} u^b u^c \frac{\partial f_B}{\partial u^i} = 0, \tag{31}$$

where the distribution function $f_B = f_B(x^\mu, u^a)$ and $\gamma^a_{bc} = e^a_{\mu;\nu} e^\mu_b e^\nu_c$ are the Ricci rotation coefficients [10,13,15]. We may assume that f_B depends only on the radial coordinate r. Using Eq. (9), the relativistic Boltzmann equation in Palatini formalism is obtained as

$$u_1 \frac{\partial f_B}{\partial u_1} - \left(\frac{1}{2} u_0^2 \frac{\partial \nu}{\partial r} - \frac{u_2^2 + u_3^2}{r}\right)\frac{\partial f_B}{\partial u_1} - \frac{1}{r} u_1 \left(u_2 \frac{\partial f_B}{\partial u_2} + u_3 \frac{\partial f_B}{\partial u_3}\right) - \frac{1}{r} u_3 \cot\theta e^{\lambda/2}\left(u_2 \frac{\partial f_B}{\partial u_3} - u_3 \frac{\partial f_B}{\partial u_2}\right) - \frac{F'}{2F}\left[(u_0^2 + u_1^2 - u_2^2 - u_3^2)\frac{\partial f_B}{\partial u_1} + 2u_1 u_2 \frac{\partial f_B}{\partial u_2} + 2u_1 u_3 \frac{\partial f_B}{\partial u_3}\right] = 0. \tag{32}$$

Since we have assumed the system to be spherically symmetric, the term proportional to $\cot\theta$ must be zero. Let us take [15]

$$u_0 = u_t, \qquad u_1 = u_r, \qquad u_2 = u_\theta, \qquad u_3 = u_\varphi. \tag{33}$$

Multiplying Eq. (32) by $m u_r du$, integrating over the velocity space, and assuming that f_B vanishes rapidly enough as the velocities tend to $\pm\infty$, we obtain

$$\frac{\partial}{\partial r}[\rho\langle u_r^2\rangle] + \frac{1}{2}\frac{\partial \nu}{\partial r}\rho[\langle u_t^2\rangle + 2\langle u_r^2\rangle] - \frac{1}{r}\rho[\langle u_\theta^2\rangle + \langle u_\varphi^2\rangle] + \frac{2}{r}\rho\langle u_r^2\rangle + \frac{F'\rho}{2F}[\langle u_t^2\rangle + 9\langle u_r^2\rangle - \langle u_\theta^2\rangle - \langle u_\varphi^2\rangle] = 0. \tag{34}$$

It is worth mentioning that by integrating Eq. (32), we note that the functions F and F' depend on the average of the square of velocities according to Eq. (19) via Eq. (4), which are assumed to be constant quantities. Now, it is useful to multiply Eq. (34) by $4\pi r^2$ and integrate over the cluster volume to obtain

$$\int_0^R \rho[\langle u_r^2\rangle + \langle u_\theta^2\rangle + \langle u_\varphi^2\rangle] 4\pi r^2 dr - \frac{1}{2}\int_0^R \rho[\langle u_t^2\rangle + 2\langle u_r^2\rangle] \times \frac{\partial \nu}{\partial r} 4\pi r^3 dr - \int_0^R \frac{F'}{2F}\rho[\langle u_t^2\rangle + 9\langle u_r^2\rangle - \langle u_\theta^2\rangle - \langle u_\varphi^2\rangle] 4\pi r^3 dr = 0, \tag{35}$$

where R is the radius of the cluster.

At this point, it is appropriate to introduce some approximations. First, consider that λ' and ν' are small quantities. Then, the terms proportional to $\nu'\lambda'$ and ν'^2 in Eq. (27) may be ignored. Then, assuming that $e^{-\lambda} \approx 1$ inside the cluster [13], we can write Eq. (27) as

$$\frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \nu}{\partial r}\right) = -4\pi G\rho - 4\pi G\rho_\phi. \tag{36}$$

Second, consider that the galaxies in the cluster have velocities much smaller than the velocity of light. In other words, $\langle u_r^2\rangle \approx \langle u_\theta^2\rangle \approx \langle u_\varphi^2\rangle \ll \langle u_t^2\rangle \approx 1$. Now Eq. (35) can be written as

$$2K + \frac{1}{2}\int_0^R \rho \frac{\partial \nu}{\partial r} 4\pi r^3 dr + \int_0^R \frac{F'}{2F}\rho 4\pi r^3 dr = 0, \tag{37}$$

where

$$K = -\int_0^R \rho[\langle u_r^2\rangle + \langle u_\theta^2\rangle + \langle u_\varphi^2\rangle] 2\pi r^2 dr \tag{38}$$

is the kinetic energy of the galaxies. Multiplying Eq. (36) by r^2 and integrating yields

$$GM(r) = -\frac{1}{2}r^2\frac{\partial \nu}{\partial r} - GM_\phi(r), \tag{39}$$

where we have used $M = \int_0^R dM(r) = \int_0^R 4\pi\rho r^2 dr$ as the total mass, and we have also defined $M_\phi(r) = 4\pi\int_0^r \rho_\phi(r')r'^2 dr'$ as the geometric mass of the system. Now, consider the definitions

$$\Omega = -\int_0^R \frac{GM(r)}{r} dM(r), \tag{40}$$

and

$$\Omega_\phi = \int_0^R \frac{GM_\phi(r)}{r} dM(r). \tag{41}$$

Multiplying Eq. (39) by $\frac{dM(r)}{r}$, which is equal to $\frac{4\pi\rho r^2 dr}{r}$, and integrating gives

$$\Omega = \Omega_\phi + \frac{1}{2}\int_0^R \rho\frac{\partial \nu}{\partial r} 4\pi r^3 dr, \tag{42}$$

where Ω refers to the usual gravitational potential energy of the system. In the end, using Eq. (37) leads to the generalized virial theorem in Palatini formalism of $f(\mathcal{R})$ gravity:

$$2K + \Omega - \Omega_\phi + \int_0^R \frac{F'}{2F}\rho 4\pi r^3 dr = 0. \tag{43}$$

Recalling that $F = \phi$ in scalar-tensor representation of the theory, the virial theorem may be written in the form

$$2K + \Omega - \Omega_\phi + \int_0^R \frac{\phi'}{2\phi}\rho 4\pi r^3 dr = 0. \tag{44}$$

The fourth term on the left-hand side of the virial equation originates from the relativistic Boltzmann equation in Palatini formalism. This correction term does not exist in the metric variational approach. To represent the virial theorem in an alternative form, we write Eq. (43) as

$$2K - G\int_0^R \frac{M(r)dM}{r} - G\int_0^R \frac{M_\phi(r)dM}{r} + G\int_0^R \frac{F'}{2FG}r^2\frac{dM}{r} = 0, \tag{45}$$

or

$$2K - G\int_0^R \frac{M(r)dM}{r} - G\int_0^R \left[M_\phi(r) - \frac{F'}{2FG}r^2\right]\frac{dM}{r} = 0. \tag{46}$$

This equation becomes simpler by the definition $M_\phi^{\rm new} = M_\phi - \frac{F'}{2FG}r^2$:

$$2K - G\int_0^R \frac{M(r)dM}{r} - G\int_0^R M_\phi^{\rm new}\frac{dM}{r} = 0. \tag{47}$$

It is convenient to introduce the radii R_V and R_ϕ:

$$R_V = \frac{M^2}{\int_0^R \frac{M(r)}{r} dM(r)}, \tag{48}$$

and

$$R_\phi = \frac{(M_\phi^{\rm new})^2}{\int_0^R \frac{M_\phi^{\rm new}(r)}{r} dM(r)}. \tag{49}$$

In addition the virial mass, M_V is defined as [13]

$$2K = \frac{GMM_V}{R_V}. \tag{50}$$

Substituting these definitions in Eq. (47) yields

$$\frac{M_V}{M} = 1 + \frac{(M_\phi^{\rm new})^2 R_V}{M^2 R_\phi}. \tag{51}$$

For most of the observed galactic clusters, the relation $M_V/M > 3$ is true. Therefore, one can easily approximate the last equation

$$\frac{M_V}{M} \approx \frac{(M_\phi^{\rm new})^2 R_V}{M^2 R_\phi}. \tag{52}$$

One of the motivations of Palatini $f(\mathcal{R})$ gravity models is the possibility of explaining dark matter. We noted above that some geometric terms appear in the Einstein field equations which could effectively play a role in gravitational energy. These geometric terms may be attributed to a geometric mass at the galactic or extra-galactic level. They can be interpreted as dark matter, which originates from modified gravity theory. On the other hand, dark matter provides the main mass contribution to clusters. It means that one can ignore the mass contribution of the baryonic mass in the clusters and estimate the total mass of the cluster by $M_{\rm tot} \approx M_\phi^{\rm new}$. We also know that the virial mass is mainly determined by the geometric mass, so that the geometric mass could be a potential candidate for the virial mass discrepancy in clusters. As a result, we conclude that

$$M_\phi^{\rm new} \approx M_V \approx M_{\rm tot}. \tag{53}$$

Therefore, Eq. (52) can be written as

$$M_V \approx M\frac{R_\phi}{R_V}. \tag{54}$$

This shows that the virial mass is proportional to the normal baryonic mass of the cluster, whose proportionality constant has geometrical origins.

V. ASTROPHYSICAL APPLICATIONS

A. Typical values for the virial mass and radius

Virial mass density can be written as $\rho_V = \frac{3M_V}{4\pi R_V^3}$, where M_V and R_V are the virial mass and radius, respectively. Astrophysical observations, together with cosmological simulations, show that the virial mass is a measure of a fixed density, such as a critical density $\rho_c(z)$ at a special redshift. It means that the virial density can be represented as $\rho_V = \delta\rho_c(z)$, where $\delta \sim 200$. As it is well known, $\rho_c(z) = h^2(z)3H_0^2/8\pi G$. The Hubble parameter is therefore normalized to its local value $h^2(z) = \Omega_m(1+z)^3 + \Omega_\lambda$, where Ω_m and Ω_λ are the mass density and dark energy density parameters, respectively [21]. By knowing the integrated mass of the galaxy cluster as a function of the radius, one can estimate the appropriate physical radius for the mass measurement. The commonly used radii are r_{200} or r_{500}. These radii lie within the radii corresponding to the mean gravitational mass density of the matter $\langle\rho_{\rm tot}\rangle = 200\rho_c$ or $500\rho_c$. A useful radius is r_{200} to find the virial mass. By studying the values of r_{200} for various clusters, one can deduce that a typical value for r_{200} is approximately 2 Mpc. The corresponding masses for these radii are defined as M_{200} and M_{500}. In general, $M_V = M_{200}$ and $R_V = r_{200}$ are assumed as a virial mass and radius [22].

B. Geometric mass and geometric radius from galactic cluster observations

Intracluster gas has an important contribution to the baryonic mass of the clusters of galaxies. The following equation provides a reasonably good description of the observational date [13,22]:

$$\rho_g = \rho_0\left(1 + \frac{r^2}{r_c^2}\right)^{-(3\beta/2)}, \tag{55}$$

where r_c is the core radius, and ρ_0 and β are constants. If we assume that the observed x-ray emission from the hot, ionized intracluster gas is in isothermal equilibrium, then the pressure p_g of the gas satisfies the equation of state $p_g = (\frac{k_B T_g}{\mu m_g})\rho_g$, where k_B is the Boltzmann constant, T_g is the gas temperature, $\mu \approx 0.61$ is the mean atomic weight of the particles in the cluster gas, and m_p is the proton mass [13,22]. Using Jean's equation [1], the total mass distribution can be obtained as [12,13,22]

$$M_{\rm tot}(r) = -\frac{k_B T_g}{\mu m_p G} r^2 \frac{d}{dr}\ln\rho_g. \tag{56}$$

Substitution of the mass density of the cluster gas in Eq. (55) gives the total mass inside the cluster [13,22]

$$M_{\rm tot}(r) = \frac{3k_B\beta T_g}{\mu m_p G}\frac{r^3}{r_c^2 + r^2}. \tag{57}$$

Now, $f(\mathcal{R})$ theory tells us that the total mass of the cluster is $M_{\rm tot}(r) = 4\pi\int_0^r(\rho_g + \rho_\phi)r^2 dr$, where it is assumed that the main part of the baryonic mass of the cluster is in the form of intracluster gas. Therefore, $M_{\rm tot}(r)$ satisfies the relation

$$\frac{dM_{\rm tot}(r)}{dr} = 4\pi r^2\rho_g(r) + 4\pi r^2\rho_\phi(r). \tag{58}$$

Since we have estimated the quantities $M_{\rm tot}(r)$ and ρ_g, the expression for the geometric mass density can be readily obtained:

$$4\pi\rho_\phi(r) = \frac{3K_B\beta T_g(r^2 + 3r_c^2)}{\mu m_p(r_c^2 + r^2)^2} - \frac{4\pi G\rho_0}{(1 + \frac{r^2}{r_c^2})^{3\beta/2}}. \tag{59}$$

In the limit $r \gg r_c$, the geometric density takes the simple form

$$4\pi\rho_\phi(r) \approx \left[\frac{3K_B\beta T_g}{\mu m_p} - 4\pi G\rho_0 r_c^{3\beta} r^{2-3\beta}\right]\frac{1}{r^2}. \tag{60}$$

Also, using Eq. (59), we can easily write the geometric mass as

$$GM_\phi(r) = 4\pi\int_0^r r^2\rho_\phi(r)dr = \frac{3K_B\beta T_g}{\mu m_p}\frac{r}{1 + \frac{r_c^2}{r^2}} - 4\pi G\rho_0\int_0^r\frac{r^2 dr}{(1 + \frac{r^2}{r_c^2})^{3\beta/2}}. \tag{61}$$

In the limit $r \gg r_c$, the geometric mass takes the simple form

$$GM_\phi(r) \approx \left[\frac{3K_B\beta T_g}{\mu m_p} - \frac{4\pi G\rho_0 r_c^{3\beta} r^{2-3\beta}}{3(1-\beta)}\right]r. \tag{62}$$

Observations show that the intracluster gas has a small contribution to the total mass [13,21–24]. This means that the gas density and mass contributions can be neglected compared to the geometric density and mass, hence

$$4\pi G\rho_\phi(r) \approx \left(\frac{3K_B\beta T_g}{\mu m_p}\right)r^{-2}, \tag{63}$$

$$GM_\phi(r) \approx \left(\frac{3K_B\beta T_g}{\mu m_p}\right)r. \tag{64}$$

Now, it is easy to derive the metric tensor component e^ν inside the cluster. From Eq. (36), we deduce that $r^2\nu' = -2GM_{\rm tot}(r) \approx -2GM_\phi(r)$, which leads to

$$e^\nu \approx C_\nu r^{2s}, \tag{65}$$

where C_ν is the integration constant, and s is defined as

$$s = -\frac{3k_B\beta T_g}{\mu m_p}. \tag{66}$$

The other coefficient in the metric tensor, $e^{-\lambda}$, can be estimated approximately as

$$e^{-\lambda} \approx 1 - \frac{2GM_{\rm tot}}{r}, \tag{67}$$

which is the standard expression which, assuming $M_{\rm tot} \approx M_\phi$, may be written as $e^{-\lambda} \approx 1 - \frac{6k_B\beta T_g}{\mu m_p} = 1 + 2s$. Of course, there is no guarantee as to the accuracy of this approximation inside the cluster. As a result, astrophysical observations can be helpful in deriving the metric components inside the cluster in the context of $f(\mathcal{R})$ theory. However, how can we estimate the upper bound for the cutoff of the geometric mass? This happens in a special point where the decaying density profile of the geometric density associated with the cluster becomes smaller than the average energy density of the Universe. We assume that the two densities become equal at the radius $R_\phi^{\rm cr}$. In other words, we can write $\rho_\phi(R_\phi^{\rm cr}) = \rho_{\rm universe}$. Let us set $\rho_{\rm universe} = \rho_c = \frac{3H^2}{8\pi G} = 4.6975 \times 10^{-30} h_{50}^2\ {\rm g/cm^{-3}}$, where $H = 50h_{50}$ km/Mpc/s [13,22]. By using Eq. (63), we obtain

$$R_\phi^{(\rm cr)} = \left(\frac{3k_B\beta T_g}{\mu m_p G\rho_c}\right)^{1/2} = 91.33\sqrt{\beta}\left(\frac{k_B T_g}{5\ {\rm keV}}\right)^{1/2} h_{50}^{-1}\ {\rm Mpc}. \tag{68}$$

We may also derive the total geometric mass in Eq. (64) corresponding to this radius as

$$M_\phi^{(\rm cr)} = M_\phi(R_\phi^{(\rm cr)}) = 4.83 \times 10^{16}\beta^{3/2}\left(\frac{k_B T_g}{5\ {\rm keV}}\right)^{3/2} h_{50}^{-1} M_\odot. \tag{69}$$

This value is consistent with the mass distribution observations in clusters. Within $f(\mathcal{R})$ gravity, geometric mass effects are beyond the virial radius, which is about a few Mpc.

C. Radial velocity dispersion in galactic clusters

We can write the virial mass in terms of the characteristic velocity dispersion σ_1 as [24]

$$M_V = \frac{3}{G}\sigma_1^2 R_V, \tag{70}$$

where $3\sigma_1^2 = \sigma_r^2$. Let us consider an isotropic velocity dispersion. Then, we have $\langle u^2\rangle = \langle u_r^2\rangle + \langle u_\theta^2\rangle + \langle u_\varphi^2\rangle = 3\langle u_r^2\rangle = 3\sigma_r^2$, where σ_r^2 is the radial velocity dispersion. The radial velocity dispersion relation for clusters in $f(\mathcal{R})$ gravity can be obtained from Eq. (34) as

$$\frac{d}{dr}(\rho\sigma_r^2) + \frac{1}{2}\rho\frac{d\nu}{dr} + \frac{F'}{2F}\rho = 0, \tag{71}$$

from which one can deduce a relation for ν'. Also, Eq. (36) yields a relation for ν':

$$\nu' = -\frac{1}{r^2}[2GM_\phi(r) + 2GM(r) + 2c], \tag{72}$$

where c is an integration constant. By eliminating ν' from the last two equations, we obtain

$$\frac{d}{dr}(\rho\sigma_r^2) = -\frac{F'}{2F}\rho + \frac{1}{r^2}[GM_\phi(r) + GM(r) + c]\rho. \tag{73}$$

Integration now gives the solution as follows:

$$\sigma_r^2(r) = -\frac{1}{\rho}\int\frac{F'}{2F}\rho dr + \frac{1}{\rho}\int[GM_\phi(r) + GM(r) + c] \times \frac{\rho dr}{r^2} + \frac{c'}{\rho}. \tag{74}$$

We can apply Eq. (74) to a special case where the density ρ is chosen as [13]:

$$\rho(r) = \rho_0 r^{-\gamma}, \tag{75}$$

with ρ_0 and $\gamma \neq 1, 3$ being positive constants. As ρ is the normal matter density inside the cluster, it yields the normal matter mass profile $M(r) = 4\pi\rho_0 r^{3-\gamma}/(3-\gamma)$. According to Eq. (64), the geometric mass $GM_\phi(r) \approx q_0 r$, where $q_0 = 3k_B\beta T_g/\mu m_p$. The radial velocity dispersion for $\gamma \neq 1, 3$ will be

$$\sigma_r^2(r) = -r^\gamma\int\frac{F'}{2F}r^{-\gamma}dr - \frac{q_0}{\gamma} - \frac{2\pi G\rho_0}{(\gamma-1)(3-\gamma)}r^{2-\gamma} - \frac{c}{\gamma+1}\frac{1}{r} + \frac{c'}{\rho_0}r^\gamma. \tag{76}$$

For $\gamma = 1$ we find

$$\sigma_r^2(r) = -r\int\frac{F'}{2F}r^{-1}dr - q_0 + 2\pi G\rho_0 r\ln r - \frac{c}{2r} + \frac{c'}{\rho_0}r, \tag{77}$$

and for $\gamma = 3$ we have

$$\sigma_r^2(r) = -r^3\int\frac{F'}{2F}r^{-3}dr - \frac{q_0}{3} + \pi G\rho_0\left(\ln r + \frac{1}{4}\right) - \frac{c}{4}\frac{1}{r} + \frac{c'}{\rho_0}r^3. \tag{78}$$

The observed data can usually be translated into a specific function for the velocity dispersion relation. Then, one can compare the observed velocity dispersion with prediction in modified $f(\mathcal{R})$ gravity to compare the different theoretical scenarios.

VI. THE LAGRANGIAN

We recognized the possibility of finding the metric tensor components in $f(\mathcal{R})$ gravity using the virial theorem in the previous sections. One of the other consequences is to find the Lagrangian $f(\mathcal{R})$ of the theory, in other words, the form that $f(\mathcal{R})$ can take. We start from the field equations in the standard representation. Equations (20) and (21) then yield

$$F'' - \frac{\nu' + \lambda'}{2}F' + \frac{\nu' + \lambda'}{r}F - \frac{3}{2}\frac{F'^2}{F} = 0. \tag{79}$$

Also, from Eqs. (22) and (23) we have

$$-\frac{\nu'\lambda'}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{1}{r^2}(e^\lambda - 1) = -\frac{3}{4}\left(\frac{F'}{F}\right)^2 + \frac{1}{2}\frac{F''}{F} + \left(\frac{\nu' - \lambda'}{4} - \frac{1}{r}\right)\frac{F'}{F}. \tag{80}$$

Equation (2) with $\mu = \nu = 0$, using Eq. (10) leads to

$$f = Fe^{-\lambda}\left[\frac{\nu'\lambda'}{2} - \nu'' - \frac{\nu'^2}{2} - \frac{2\nu'}{r} + \left(\frac{3\nu' - \lambda'}{2} + \frac{2}{r}\right)\frac{F'}{F} + \frac{F''}{F}\right]. \tag{81}$$

Now, we can write the Ricci scalar from Eqs. (4) and (11) as follows:

$$R = \frac{2f}{F} - 3e^{-\lambda}\left[-\frac{1}{2}\left(\frac{F'}{F}\right)^2 + \left(\frac{F''}{F}\right) + \left(\frac{\nu' - \lambda'}{2} + \frac{2}{r}\right)\left(\frac{F'}{F}\right)\right]. \tag{82}$$

Here, R is derived in the context of metric formalism. In fact, we have replaced all of the parameters in terms of those appearing in the metric formalism which would then lead to the metric Ricci scalar. In the derivation of the above equations, we have neglected the contribution from the baryonic matter. Equation (65), together with the assumption $e^\lambda = \text{const}$ inside the clusters, then becomes reasonable to use. Now, Eq. (79) can be written as

$$FF'' - \frac{s}{r}F'F - \frac{3}{2}F'^2 + \frac{2s}{r^2}F^2 = 0, \tag{83}$$

for which a solution is

$$F = \frac{4(s^2 + 6s + 1)r^{(-s-1-\sqrt{s^2+6s+1})}}{c}, \tag{84}$$

where c is an integration constant, and we set $c = 1$. The component e^λ can be derived from Eq. (80) by using the expression for F:

$$e^\lambda = 1 + s - s^2 + \frac{(s + 1 + \sqrt{s^2 + 6s + 1})}{4} \times (5 - 3s - \sqrt{s^2 + 6s + 1}). \tag{85}$$

Now, Eq. (81) gives

$$\frac{f}{F} = \frac{e^{-\lambda}}{r^2}[-2s - 2s^2 + (s + 1 + \sqrt{s^2 + 6s + 1}) \times (-2s + \sqrt{s^2 + 6s + 1})], \tag{86}$$

and Eq. (82) results in

$$R = \frac{e^{-\lambda}}{r^2}\left[\frac{(s + 1 + \sqrt{s^2 + 6s + 1})(-5s + 3 + \sqrt{s^2 + 6s + 1})}{2} - 4s - 4s^2\right]. \tag{87}$$

Recall that $\mathcal{R}$ is the Ricci scalar in Palatini formalism, whereas R is the Ricci scalar in metric formalism. Finally, we can easily obtain f as a function of R as follows:

$$f = f_0 R^{s+3+\sqrt{s^2+6s+1}/2}, \tag{88}$$

where

$$f_0 = \frac{4(s^2 + 6s + 1)[-2s - 2s^2 + (s + 1 + \sqrt{s^2 + 6s + 1})(-2s + \sqrt{s^2 + 6s + 1})]}{[\frac{(s+1+\sqrt{s^2+6s+1})(-5s+3+\sqrt{s^2+6s+1})}{2} - 4s - 4s^2]^{s+3+\sqrt{s^2+6s+1}/2}} e^{\lambda(s+1+\sqrt{s^2+6s+1})/2}. \tag{89}$$

One may also obtain the action in terms of the Ricci scalar in Palatini formalism. In either case, by knowing the physical parameters such as gas density and temperature, one can deduce the action for modified gravity.

VII. CONCLUSIONS

Virial theorem is a convenient tool used to derive the mean density of galaxy clusters, and it can therefore predict the total mass of clusters. We have used the relativistic Boltzmann equation within the context of Palatini $f(\mathcal{R})$ theory to derive the virial theorem. To write the field equations, we have applied the standard cosmological averaging for the energy-momentum tensor. The virial mass is mainly determined by the geometric mass which is associated with the geometrical terms in the gravitational field equations. The generalized virial mass implicitly includes the effects of dark matter and may therefore be used to describe the dynamics of clusters. If Eqs. (54) and (68) accompany the assumption $R_\phi \approx R_\phi^{(cr)}$, one can estimate the virial mass as $M_V \approx 91.33\sqrt{\beta}(\frac{k_B T_g}{5\ \text{keV}})^{1/2} h_{50}^{-1} \frac{M}{R_V\ (\text{Mpc})}$ which, by having the physical parameters of a cluster, leads to a specific value. In fact, the virial mass can be approximated to give the total mass of clusters. In this paper, we have found that the virial mass or the total mass of clusters can also be obtained from the velocity dispersion relations. However, uncertainties in observational data suppress the exact value of the virial mass. It seems that the gravitational lensing of light in $f(\mathcal{R})$ theories gives more exact values for the total mass of clusters. In any case, since the virial theorem results can be compared with observational data, one can apply it to different theories to test their viability. We also derived, in effect, the Lagrangian of the theory in terms of the metric Ricci scalar. Finally, we discussed the limitations of our approach which is rooted in the assumption that there exists an averaged metric, which is the outcome of putting an averaged energy-momentum tensor on the right-hand side of the field equations.

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