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Black-hole quasinormal resonances: Wave analysis versus a geometric-optics approximation

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel,
and The Hadassah Institute, Jerusalem 91010, Israel
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It has long been known that null unstable geodesics are related to the characteristic modes of black holes—the so-called quasinormal resonances. The basic idea is to interpret the free oscillations of a black hole in the eikonal limit in terms of null particles trapped at the unstable circular orbit and slowly leaking out. The real part of the complex quasinormal resonances is related to the angular velocity at the unstable null geodesic. The imaginary part of the resonances is related to the instability time scale (or the inverse Lyapunov exponent) of the orbit. While this geometric-optics description of the black-hole quasinormal resonances in terms of perturbed null *rays* is very appealing and intuitive, it is still highly important to verify the validity of this approach by directly analyzing the Teukolsky wave equation which governs the dynamics of perturbation *waves* in the black-hole spacetime. This is the main goal of the present paper. We first use the geometric-optics technique of perturbing a bundle of unstable null rays to calculate the resonances of near-extremal Kerr black holes in the eikonal approximation. We then directly solve the Teukolsky wave equation (supplemented by the appropriate physical boundary conditions) and show that the resultant quasinormal spectrum obtained directly from the wave analysis is in accord with the spectrum obtained from the geometric-optics approximation of perturbed null rays.

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The no-hair conjecture [1] asserts that the external field of a perturbed black hole relaxes to a Kerr-Newman spacetime, characterized solely by three parameters: the black-hole mass, charge, and angular momentum. This implies that perturbation fields left outside the black hole would either be radiated away to infinity, or be swallowed by the black hole.

The relaxation phase in the dynamics of perturbed black holes is characterized by ‘‘quasinormal ringing,’’ damped oscillations with a discrete spectrum (see e.g. [2] for reviews). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [3–6]. Quasinormal resonances are expected to play a prominent role in gravitational radiation emitted by a variety of astrophysical scenarios involving black holes. Being the characteristic ‘‘sound’’ of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters, especially the mass and angular momentum of the black hole.

The dynamics of black-hole perturbations is governed by the Regge-Wheeler equation [7] in the case of a spherically symmetric Schwarzschild black hole, and by the Teukolsky equation [8] for rotating Kerr-Newman spacetimes. The black hole quasinormal modes (QNMs) correspond to solutions of the wave equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [9]. Such boundary conditions single out a discrete set of black-hole resonances $\{\omega_n\}$ (assuming a time dependence of the form $e^{-i\omega t}$). In analogy with standard scattering theory, the QNMs can be regarded as the scattering resonances of the black-hole spacetime. Thus they correspond to poles of the transmission and reflection amplitudes of a standard scattering problem in a black-hole spacetime.

In accord with the spirit of the no-hair conjecture [1], the external perturbation fields would either fall into the black hole or radiate to infinity. This implies that the perturbation decays with time and the corresponding QNM frequencies are therefore *complex*. It turns out that there exists an infinite number of quasinormal modes, characterizing oscillations with decreasing relaxation times (increasing imaginary part); see [10–13] and references therein. The mode with the smallest imaginary part (known as the fundamental mode) determines the characteristic dynamical time scale for generic perturbations to decay [14–19].

In most cases of physical interest [20], the black-hole QNMs must be computed *numerically* by solving the Teukolsky equation supplemented by the appropriate physical boundary conditions. However, Mashhoon [21] has suggested an *analytical* technique of calculating the QNMs in the geometric-optics (eikonal) limit. The basic idea is to interpret the black-hole free oscillations in terms of null particles trapped at the unstable circular orbit and slowly leaking out [21–23]. The real part of the complex quasinormal resonances is related to the angular velocity at the unstable null geodesic, while the imaginary part of the resonances is related to the instability time scale of the orbit (or the inverse Lyapunov exponent of the geodesic [23]).

It should be emphasized that, Mashhoon's approach of computing black-hole quasinormal frequencies in the eikonal limit is somewhat indirect: instead of explicitly solving the Teukolsky wave equation which describes the dynamics of perturbation waves in the black-hole spacetime, Mashhoon analyzed perturbations of test null rays in the unstable circular orbit of the black hole. While this geometric-optics description of black-hole quasinormal resonances in terms of perturbed null rays is very appealing and intuitive, it is still important to verify the validity of this approach by directly solving the Teukolsky master equation which governs the dynamics of perturbation waves in the Kerr black-hole spacetime. Filling this gap is the main goal of the present paper. We shall first use the geometric-optics technique of perturbing a bundle of unstable null geodesics to calculate the resonances of near-extremal Kerr black holes in the eikonal limit. We shall then directly solve the Teukolsky wave equation (supplemented by the appropriate physical boundary conditions) and show that the resultant resonances of the wave equation agree with those obtained from Mashhoon's indirect approach of perturbing null rays in the appropriate eikonal approximation.

We start with Mashhoon's approach of calculating the black-hole QNM resonances in the eikonal limit $l = m \gg 1$, where l is the angular momentum of the perturbed null rays, and m is the azimuthal harmonic index of the rays. According to Mashhoon's analysis of perturbed null geodesics, the Kerr QNM frequencies in the $l = m \gg 1$ limit are given by [21] (we use natural units in which $G = c = \hbar = 1$)

$$\omega_n = m\omega_+ - i(n + \tfrac{1}{2})\beta\omega_+; \qquad n = 0, 1, 2, \ldots, \quad (1)$$

where

$$\omega_+ \equiv \frac{M^{1/2}}{r_{\rm ph}^{3/2} + aM^{1/2}} \quad (2)$$

is the Kepler frequency for null rays in the unstable equatorial circular orbit of the black hole, and

$$r_{\rm ph} \equiv 2M\{1 + \cos[\tfrac{2}{3}\cos^{-1}(-a/M)]\} \quad (3)$$

is the limiting circular photon orbit. The function β is given by

$$\beta = \frac{(12M)^{1/2}(r_{\rm ph} - r_+)(r_{\rm ph} - r_-)}{r_{\rm ph}^{3/2}(r_{\rm ph} - M)}; \quad (4)$$

see Ref. [21] for details. Here M and a are the mass and angular momentum per unit mass of the black hole, respectively.

We now focus on the near-extremal limit $T_{\rm BH} \to 0$, where

$$T_{\rm BH} = \frac{(M^2 - a^2)^{1/2}}{4\pi M[M + (M^2 - a^2)^{1/2}]} \quad (5)$$

is the Bekenstein-Hawking temperature of the black hole. Let $r_\pm = M \pm \epsilon$ with $\epsilon/M \ll 1$, where $r_\pm = M + (M^2 - a^2)^{1/2}$ are the black-hole event and inner horizons. The Bekenstein-Hawking temperature of the black hole now reads $T_{\rm BH} = \epsilon/4\pi M^2 + O(\epsilon^2/M^3)$. After some algebra, we find from Eq. (3)

$$r_{\rm ph} = M + \frac{2\epsilon}{\sqrt{3}} + O(\epsilon^2/M). \quad (6)$$

This, in turn, implies

$$\omega_+ = \frac{1}{2M} - \frac{\sqrt{3}\epsilon}{4M^2} + O(\epsilon^2/M^3), \quad (7)$$

and

$$\beta = \frac{\epsilon}{M} + O(\epsilon^2/M^2). \quad (8)$$

Substituting Eqs. (7) and (8) into Eq. (1), one finds that the black-hole quasinormal resonances in the eikonal approximation $l = m \gg 1$ are given by

$$\omega_n = m\Omega + 2\pi T_{\rm BH}\left(1 - \frac{\sqrt{3}}{2}\right)m - i2\pi T_{\rm BH}\left(n + \frac{1}{2}\right) + O(MT_{\rm BH}^2), \quad (9)$$

where

$$\Omega \equiv \frac{a}{r_+^2 + a^2} = \frac{1}{2M} - 2\pi T_{\rm BH} + O(MT_{\rm BH}^2) \quad (10)$$

is the angular velocity of the black-hole event horizon.

We shall now study directly the Teukolsky wave equation in order to determine the fundamental (least-damped) resonant frequencies of the Kerr black hole. The Teukolsky equation is amenable to an analytical treatment in the near-extremal limit $(M^2 - a^2)^{1/2} \ll a \lesssim M$.

In order to determine the black-hole resonances, we shall analyze the scattering of massless waves in the Kerr spacetime. The dynamics of a perturbation field Ψ in the rotating Kerr spacetime is governed by the Teukolsky equation [8]. One may decompose the field as

$$\Psi_{slm}(t, r, \theta, \phi) = e^{im\phi}S_{slm}(\theta; a\omega)\psi_{slm}(r)e^{-i\omega t}, \quad (11)$$

where (t, r, θ, ϕ) are the Boyer-Lindquist coordinates, ω is the (conserved) frequency of the mode, l is the spheroidal harmonic index, and m is the azimuthal harmonic index with $-l \le m \le l$. The parameter s is called the spin weight of the field and is given by $s = \pm 2$ for gravitational perturbations, $s = \pm 1$ for electromagnetic perturbations, $s = \pm\frac{1}{2}$ for massless neutrino perturbations, and $s = 0$ for scalar perturbations. (We shall henceforth omit the indices s, l, m for brevity.) With the decomposition (11), ψ and S obey radial and angular equations, both of the confluent Heun type [24,25], coupled by a separation constant $A(a\omega)$.

The angular functions $S(\theta; a\omega)$ are the spin-weighted spheroidal harmonics which are solutions of the angular equation [8,25]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)+\left[a^2\omega^2\cos^2\theta-2a\omega s\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+s+A\right]S=0. \quad (12)$$

The angular functions are required to be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions pick out a discrete set of eigenvalues A_l labeled by an integer l. [In the $a\omega \ll 1$ limit, these angular functions become the familiar spin-weighted spherical harmonics with the corresponding angular eigenvalues $A = l(l+1) - s(s+1) + O(a\omega)$.] The angular Eq. (12) can be solved analytically in the $l = m \gg 1$ limit to yield

$$A = m^2 + O(m); \quad (13)$$

see Ref. [26].

The radial Teukolsky equation is given by

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d\psi}{dr}\right)+\left[\frac{K^2-2is(r-M)K}{\Delta}-a^2\omega^2+2ma\omega-A+4is\omega r\right]\psi=0, \quad (14)$$

where $\Delta \equiv r^2 - 2Mr + a^2$ and $K \equiv (r^2 + a^2)\omega - am$. For the scattering problem, one should impose physical boundary conditions of purely ingoing waves at the black-hole horizon and a mixture of both ingoing and outgoing waves at infinity (these correspond to incident and scattered waves, respectively). That is,

$$\psi \sim \begin{cases} e^{-i\omega y} + \mathcal{R}(\omega)e^{i\omega y} & \text{as } r \to \infty (y \to \infty); \\ \mathcal{T}(\omega)e^{-i(\omega - m\Omega)y} & \text{as } r \to r_+ (y \to -\infty), \end{cases} \quad (15)$$

where the "tortoise" radial coordinate y is defined by $dy = [(r^2 + a^2)/\Delta]dr$. The coefficients $\mathcal{T}(\omega)$ and $\mathcal{R}(\omega)$ are the transmission and reflection amplitudes for a wave incident from infinity. They satisfy the usual probability conservation equation $|\mathcal{T}(\omega)|^2 + |\mathcal{R}(\omega)|^2 = 1$.

The discrete quasinormal frequencies are the scattering resonances of the black-hole spacetime. Thus they correspond to poles of the transmission and reflection amplitudes. (The pole structure reflects the fact that the QNMs correspond to purely outgoing waves at spatial infinity.) These resonances determine the ringdown response of a black hole to external perturbations. Teukolsky and Press [27] and also Starobinsky and Churilov [28] have analyzed the black-hole scattering problem in the double limit $a \to M$ and $\omega \to m\Omega$. Detweiler [29] then used that solution to obtain a resonance condition for near-extremal Kerr black holes. Define

$$\sigma \equiv \frac{r_+ - r_-}{r_+}; \qquad \tau \equiv M(\omega - m\Omega); \qquad \hat{\omega} \equiv \omega r_+. \quad (16)$$

Then the resonance condition obtained in [29] for $\sigma \ll 1$ and $\tau \ll 1$ is

$$-\frac{\Gamma(2i\delta)\Gamma(1+2i\delta)\Gamma(1/2+s-2i\hat{\omega}-i\delta)\Gamma(1/2-s-2i\hat{\omega}-i\delta)}{\Gamma(-2i\delta)\Gamma(1-2i\delta)\Gamma(1/2+s-2i\hat{\omega}+i\delta)\Gamma(1/2-s-2i\hat{\omega}+i\delta)} = (-2i\hat{\omega}\sigma)^{2i\delta}\frac{\Gamma(1/2+2i\hat{\omega}+i\delta-4i\tau/\sigma)}{\Gamma(1/2+2i\hat{\omega}-i\delta-4i\tau/\sigma)}, \quad (17)$$

where $\delta^2 \equiv 4\hat{\omega}^2 - 1/4 - A - a^2\omega^2 + 2ma\omega$. Taking cognizance of Eq. (13), one finds

$$\delta = \frac{\sqrt{3}}{2}m + O(1), \quad (18)$$

for near-extremal Kerr black holes in the $l = m \gg 1$ limit.

The left-hand side of Eq. (17) has a well defined limit as $a \to M$ and $\omega \to m\Omega$. We denote that limit by $\mathcal{L}$. Since $\delta \simeq \sqrt{3}m/2 \gg 1$, one has $(-i)^{-2i\delta} = e^{-i\sqrt{3}m\ln(-i)} = e^{-i\sqrt{3}m\ln e^{-i\pi/2}} = e^{-i\sqrt{3}m(-i\pi/2)} = e^{-\sqrt{3}\pi m/2} \ll 1$, which implies $\lambda \equiv (-2i\hat{\omega}\sigma)^{-2i\delta} \ll 1$. Thus, a consistent solution of the resonance condition, Eq. (17), may be obtained if $1/\Gamma(1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma) = O(\lambda)$. Suppose

$$1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma = -n + \eta\lambda + O(\lambda^2), \quad (19)$$

where $n \geq 0$ is a non-negative integer, and η is an unknown constant to be determined below. Then one has

$$\Gamma(1/2+2i\hat{\omega}-i\delta-4i\tau/\sigma) \simeq \Gamma(-n+\eta\lambda) \simeq (-n)^{-1}\Gamma(-n+1+\eta\epsilon) \simeq \cdots \simeq [(-1)^n n!]^{-1}\Gamma(\eta\lambda), \quad (20)$$

where we have used the relation $\Gamma(z+1) = z\Gamma(z)$ [30]. Next, using the series expansion $1/\Gamma(z) = \sum_{k=1}^{\infty} c_k z^k$ with $c_1 = 1$ (see Eq. (6.1.34) of [30]), one obtains

$$1/\Gamma(1/2+2i\hat{\omega}-i\delta-4i\tau/\sigma) = (-1)^n n!\eta\lambda + O(\lambda^2). \quad (21)$$

Substituting this into Eq. (17), one finds $\eta = \mathcal{L}/[(-1)^n n!\Gamma(-n+2i\delta)]$.

Finally, substituting $4\tau/\sigma = (\omega - m\Omega)/2\pi T_{\text{BH}}$, $2i\hat{\omega} = im + O(mMT_{\text{BH}})$ for $\omega = m\Omega + O(mT_{\text{BH}})$, and $\delta = \sqrt{3}m/2 + O(1)$ into Eq. (19), one obtains the resonance condition

$$(\omega - m\Omega)/2\pi T_{\text{BH}} = i\left[-n+\eta\lambda-1/2+im\left(\frac{\sqrt{3}}{2}-1\right)\right]. \quad (22)$$

The black-hole quasinormal resonances in the $l = m \gg 1$ limit are therefore given by the formula

$$\omega_n = m\Omega + 2\pi T_{\mathrm{BH}}\left(1 - \frac{\sqrt{3}}{2}\right)m - i2\pi T_{\mathrm{BH}}\left(n + \frac{1}{2}\right) + O(MT_{\mathrm{BH}}^2), \tag{23}$$

where $n = 0, 1, 2, \cdots$. We emphasize that this result, obtained from the direct wave analysis, coincides with the previously derived spectrum Eq. (9) of the ray analysis (the geometric-optics approximation).

In summary, we have studied analytically the quasinormal spectrum of rapidly-rotating Kerr black holes. We first used the technique of perturbing a bundle of unstable equatorial null geodesics to calculate the quasinormal resonances of Kerr black holes in the eikonal limit $l = m \gg 1$. We then used the alternative (and the more direct) approach of solving the Teukolsky wave equation which governs the dynamics of perturbation waves in the Kerr black-hole spacetime. We have shown that the resonance spectrum (23) obtained directly from the wave analysis is in accord with the spectrum (9) which was obtained from the geometric-optics approximation of perturbed null rays.

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