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Initial conditions for small-field inflation

Michał Spaliński

Sołtan Institute for Nuclear Studies, Warsaw, Poland
and Physics Department, University of Białystok, Białystok, Poland[*]

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Small-field inflation typically requires rather special initial conditions to commence. It is proposed that in models where the inflaton is an axionlike field, with a periodic contribution to the potential, there is a possibility of significantly enhancing the chances of inflation without any fine-tuning of initial conditions and with no additional fine-tuning of the dynamics beyond what is needed for the potential to support inflation in the first place.

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I. INTRODUCTION

It is often said that the enormous expansion of the Universe during inflation erases any memory of its pre-inflationary state. While this is likely to be true, it is also the case that in many models of inflation one needs rather special initial conditions for inflation to start. This fine-tuning of initial conditions is quite separate from any fine-tuning of the dynamics that may be necessary.

Not all inflationary scenarios are afflicted with this malady; notably, chaotic inflation [1,2] requires essentially no fine-tuning of initial conditions. Models of this sort (frequently referred to as large-field models) are attractive partly due to their simplicity. They are also of great phenomenological interest as they can easily accommodate increasingly accurate observational data. Furthermore, since the inflaton traverses a trans-Planckian distance during inflation, large-field models typically predict potentially measurable levels of tensor perturbations (as suggested by the Lyth bound [3]). In some sense, the simplest models of chaotic inflation are the most attractive incarnation of the idea of inflation. On the theoretical side there are, however, some puzzles due to the necessarily trans-Planckian expectation values of the inflaton. Some of the arguments appearing in this context invoke effective field theory ideas while others reflect expectations based on string theory.

There are also many models of inflation which do not involve trans-Planckian expectation values of the inflaton (such models are sometimes referred to as small-field models). An interesting class of inflationary models of the small-field type has recently been discussed in the context of *D*-brane inflation [4–7] in the supersymmetric standard model [8–10] and in supergravity [11,12]. These scenarios envision inflation taking place close to an inflection point of the inflaton potential. In each of these cases some fine-tuning of the effective inflaton potential is involved; a recent treatment of these issues [5,13] considers inflection points arising when the inflaton potential has a pair of extrema which can be tuned to coincide. This requires fine-tuning at the level of one part in 10^{-3}.

If one accepts as inevitable the fine-tuning of the potential, inflection point inflationary scenarios have a number of attractive features, such as the fact that they admit a low scale of inflation which may be desirable from some points of view. As a small-field model, inflection point inflation predicts a very low level of tensor perturbations, which can be regarded either as a positive or negative consequence. In the end, this question will be settled by observation [14].

A serious conceptual drawback of inflection point inflation is that it requires fine-tuning of initial conditions as well as of the dynamics. An aspect of this is often referred to as the overshoot problem [15]: for generic initial conditions, the evolving inflaton field will simply miss the inflection point as if it were not there at all [5,6,16]. The only way for inflation to start is if some agent causes the inflaton expectation value to be very close to the inflection point with negligible velocity. This is a potentially disturbing aspect of this class of models [17]. It is the purpose of this short paper to suggest a small-field scenario which does not involve any fine-tuning of the initial conditions, and no additional fine-tuning of the dynamics beyond what is required to have an approximate saddle point.

II. INFLECTION POINT INFLATION

Inflation near an inflection point has been discussed in a number of recent articles [5,6,13], so only a very brief summary is included here. Consider the scalar potential $V(\phi)$ in the vicinity of a point ϕ_0, where it is assumed that $V'(\phi_0) = V''(\phi_0) = 0$. In this region the potential can be approximated by

$$V(\phi) = V_0 + V_1(\phi - \phi_0) + V_3(\phi - \phi_0)^3 + \dots, \quad (1)$$

where $V_1 = 0$ for an exact saddle point, but the linear term could be present without spoiling anything provided it is small enough. Allowing for $V_1 \neq 0$, to get at least 60 e-folds, one needs [5,13]

$$3M_P^4 V_1 V_3 < 10^{-3} V_0^2. \quad (2)$$

[*]michal.spalinski@fuw.edu.pl

To simplify the formulas in this section, V_1 will be neglected: more general expressions can be found in [5,13].

The slow roll parameters are given by

$$\epsilon = \frac{1}{2} M_P^2 \left(\frac{V'}{V}\right)^2 \approx \frac{9}{2} M_P^2 \left(\frac{V_3}{V_0}\right)^2 (\phi - \phi_0)^4, \tag{3}$$

$$\eta = \frac{1}{2} M_P^2 \frac{V''}{V} \approx M_P^2 \frac{6V_3}{V_0} (\phi - \phi_0). \tag{4}$$

Clearly, one has $\epsilon \ll \eta$ in models of this type.

In the slow roll approximation, the number of e-folds is given by

$$N = \frac{1}{M_P^2} \int_{\phi_i}^{\phi} d\phi \frac{V}{V'} \approx \frac{1}{M_P^2} \frac{V_0}{3V_3} \frac{1}{\phi_0 - \phi}, \tag{5}$$

and the scalar perturbation spectrum is characterized by the spectral index

$$n_s - 1 = 1 - 6\epsilon + 2\eta \approx 1 - \frac{4}{N}. \tag{6}$$

Taking $N = 60$ here and below (with standard assumptions about reheating), this gives $n_s = 0.93$, which is consistent with current limits. In the slow roll approximation, the amplitude of the scalar perturbation spectrum is given by

$$P_S^2 = \frac{1}{12\pi^2 M_P^6} \frac{V^3}{V'^2}, \tag{7}$$

evaluated at horizon crossing. For the present case, using $P_S^2 = 2.5 \times 10^{-9}$ [19], this gives the condition

$$\frac{V_3^2}{V_0} = \frac{4\pi^2}{3M_P^2 N^4} P_S^2 = \frac{2.5 \times 10^{-15}}{M_P^2}. \tag{8}$$

Another piece of information is provided by limits on the ratio of tensor to scalar perturbations $r = 16\epsilon$. Imposing $r < 0.25$ [19], translates into

$$\frac{V_0^2}{V_3^2} = \frac{9}{8} M_P^6 N^4 r < 3.6 \times 10^6 M_P^6. \tag{9}$$

The analysis summarized above (using the slow roll approximation) is somewhat simplified—to get a realistic picture, one should also account for quantum effects near the inflection point [6,13].

The major difficulty, as discussed in the introduction, is to motivate the rather stringent initial conditions required for inflation to start. Numerical investigation shows that to avoid overshoot, ϕ needs to be sufficiently close to the inflection point, and furthermore $\dot{\phi}$ has to be negligible. In short, the initial condition needs to be in a suitable, small region of phase space. It is clearly a problem to explain why the inflaton field expectation value would reside in such a special region in the pre-inflationary epoch. The scenario proposed in the following section is basically trying to make this region of phase space appear less special.

III. THE “STAIRCASE” SCENARIO

The basic idea is that the dynamics of the inflaton are governed by a potential which over some range of field space can be visualized as an ascending staircase: an array of inflection points. A suitable potential could arise as follows. Suppose that in some region of field space the potential energy is dominated by two types of contributions: one of them periodic and the other linear in the inflaton field. The periodic contribution is naturally interpreted as an instanton-generated energy density of an axionlike field identified with the inflaton. This contribution, periodic with period $2\pi f$, defines a scale. The other key element of the scenario is the presence of a second, nonperiodic [20] contribution $u(\phi)$, so that the energy density has the form

$$V(\phi) = \mu^4 P\left(\frac{\phi}{f}\right) + u(\phi), \tag{10}$$

where P is a periodic function (with period 2π) and μ is a parameter with dimension of mass. The present proposal assumes that the scale of variation of $u(\phi)$ is much smaller than that set by f. If this is the case, it will be a good approximation to linearize this contribution over some potentially large number of cycles of the inflaton. If there is enough freedom in a specific model for these two contributions to be fine-tuned, the resulting potential acquires, due to the periodicity of the first contribution, a sequence of inflection points. The total potential in this region resembles a smoothed staircase.

An idealized realization, which can be envisaged to appear as an approximation in many instances (including the specific examples described below), is to assume the inflaton potential in the form

$$V(\phi) = \mu^4 \left(-\sin\left(\frac{\phi}{f}\right) + (1 + \alpha)\frac{\phi}{f} + \lambda\right), \tag{11}$$

where α and λ are constant parameters. If it is possible to tune $\alpha \ll 1$, the potential has a sequence of approximate saddle points at

$$\phi_k = 2k\pi f, \tag{12}$$

with k assuming values in a range of integers such that the ϕ_k lie in the region of field space where (11) is valid. Close to any of these points the potential takes the form (1) with

$$V_0 = \mu^4(\lambda + 2k\pi), \qquad V_1 = \alpha \frac{\mu^4}{f}, \qquad V_3 = \frac{\mu^4}{6f^3}. \tag{13}$$

The question of precisely how much fine-tuning of α is required will be revisited at the end of this section.

In the context of chaotic inflation, it is usually assumed that in the pre-inflationary universe the distribution of the inflaton field is essentially random, apart from the assumption that the energy density remains sub-Planckian so that field theory notions may be applicable [1]. In the original proposal for chaotic inflation, a spacial domain of extent $L \sim H^{-1} \sim M_P^{-1}$ is considered where the inflaton field is homogeneous. The chaotic inflation scenario assumes a large initial inflaton expectation value and negligible velocity, but one can also consider the situation where the initial inflaton velocity is large. This will lead to an ascending trajectory where the inflaton stops at some point before it starts rolling back down. If the velocity of the inflaton is sufficiently large [21] to scale at least one step of the staircase and as long as there is a large number of steps, any initial condition of this sort will cause the inflaton to stop somewhere on the staircase. If the density of inflection points is sufficient, this turning point will be close enough to one of them *with vanishing velocity* so that (at least intuitively) there is a very good chance that inflation will commence. Making that last statement precise would require adopting a satisfactory notion of measure on the space of initial conditions.

If inflation takes place at the k-th step, one can impose current observational bounds to see if the constraints are reasonable. The condition on the tensor ratio (9) gives

$$(\lambda + 2k\pi)^2\left(\frac{f}{M_P}\right)^6 < 1.0 \times 10^5, \tag{14}$$

which is a very weak bound: already for k of the order 10 and λ of order 1 this implies the axion decay constant f has to be below the Planck scale, which is, in any case, to be expected. Next, applying the COBE normalization (8) yields

$$\mu^4 M_P^2 = f^6(\lambda + 2k\pi)9.0 \times 10^{-14}. \tag{15}$$

This fixes the scale of the potential μ well below the Planck scale for sensible values of the axion decay constant and assuming that the parameter λ is not large. This is a reasonable assumption, given that large enough λ would lead to a model of chaotic inflation which would be analyzed differently. It is clear from Eq. (11) that μ determines the spacing of the steps in energy. Since the constraints (14) and (15) limit μ from above, there is no obstruction from the observational side to making the steps in energy quite dense.

One also needs to revisit the fine-tuning condition (2), which can be expressed as

$$\alpha < \frac{1}{3}\left(\frac{f}{M_P}\right)^4(\lambda + 2k\pi)^2 \times 10^{-3}. \tag{16}$$

For a given k, this is a fine-tuning condition on alpha. However, if one regards α, f, and λ as determined by the underlying theory, the relation (16) can be interpreted as a condition on k, the number of "stairs" the inflaton has to scale for sustained inflation to commence. This depends on the initial conditions, so (16) can be reinterpreted as a requirement that the initial inflaton "velocity" be large enough.

Despite Hubble friction, the scale of inflation depends not only on the potential, but also on the initial conditions. There is, however, no need to tune them: all that is required for inflation to commence is that the turning point is in a region of field space where the approximate form of the potential (11) is valid. Note, also, that if (11) is an approximate form of a symmetric potential (valid in some range of ϕ), then the sign of the initial inflaton velocity is not relevant.

It is interesting to ask whether any traces of the "uphill" phase can be detectable. This is very unlikely, indeed, since if inflation takes place at an inflection point typically very large numbers of e-folds ensue [13], significantly exceeding the 55–70 e-folds of expansion since visible perturbations were generated. Thus, in general, there should be no memory of any initial transient.

IV. POST-INFLATIONARY EVOLUTION

After the Universe inflates at the uppermost inflection point reached, the inflaton expectation value continues to move down. The potential gradient accelerates the inflaton, but since there is another inflection point nearby it is important to know whether another stage of inflation is possible as the inflaton approaches that point. To answer this question, one needs to resort to numerical analysis. The evolution of the inflaton is governed by Einstein's equations, which reduce to

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0, \qquad 3M_P^2H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{17}$$

It is convenient to use the number of e-folds, N, as the evolution parameter [13]. Using $dN = Hdt$ and denoting derivatives with respect to N by a prime, the resulting equations can be written in first-order form as

$$\begin{aligned} \phi' &= u, \\ u' &= -\frac{1}{H^2}\left(\frac{uV(\phi)}{M_P^2} + V_{,\phi}(\phi)\right), \\ H' &= -\frac{1}{2M_P^2}Hu^2, \end{aligned} \tag{18}$$

with the Friedman constraint

$$\left(3M_P^2 - \frac{1}{2}u^2\right)H^2 = V(\phi). \tag{19}$$

It is straightforward to integrate these equations numerically and consider inflaton velocities at the point where inflation ends. For example, taking one of the inflection points considered by Linde and Westphal [13] with $V_0 = 2.7 \times 10^{-23}$, $V_1 = 7.29 \times 10^{-32}$, and $V_3 = 1.0 \times 10^{-20}$

(in Planck units), one finds that if the inflaton starts at rest with an initial value not exceeding 7×10^{-5} one gets hundreds of e-folds. The inflaton velocity u at the time when inflation ends depends very weakly on the initial condition and is of the order 3×10^{-2}. This is much too large for inflation to commence at the next inflection point—at the parameter values cited, the initial inflaton velocity cannot exceed 10^{-4} for inflation to start. Exploring various choices of parameters leads to the conclusion that on the way down, once the inflaton makes it past the inflection point where inflation takes place, it attains a velocity which causes all the remaining inflection points to be overshot. Thus only one stage of inflation is possible in this scenario: at the uppermost inflection point reached.

While the background evolution is essentially unaffected by the presence of inflection points on the way down, they may leave observable traces in the cosmic microwave background since they induce small variations of the inflaton velocity. Eventually, the staircase region will end (assuming that the nonperiodic contribution $u(\phi)$ in (10) was bounded below). To formulate a realistic scenario, the total potential should have a minimum such that the Universe can reheat. This part of the story depends on the behavior of $u(\phi)$ in the region of field space where the linear approximation is no longer valid and thus will differ from model to model.

V. EXAMPLES

To construct specific models, one needs an axion (to be identified with the inflaton) and an instanton-generated potential with a reasonable scale and periodicity. This potential restricts the shift symmetry of the axion to a discrete subgroup. The other essential ingredient is a source of additional symmetry breaking which eliminates the residual discrete shift symmetry. It is natural to consider models of axion inflation in the framework of string theory (e.g. [22,23]). In fact, string theory vacua provide a setting where the key ingredients of the staircase scenario are easily found since they typically involve numerous fields with periodic potentials. These fields are usually called axions since it is expected that the QCD axion, which resolves the strong CP problem, can be found among them. String theory axions arise by dualizing second-rank antisymmetric fields, which arise in the process of compactification in heterotic as well as type II vacua.

Before considering string theory, one can of course formulate suitable models purely in field theory [24]. One simple way that a staircase potential could appear is if there were two periodic contributions to the axion potential with disparate periods:

$$V = \mu^4 \cos\left(\frac{\phi}{f}\right) + \mu'^4 \cos\left(\frac{\phi}{f'} + \gamma\right), \qquad (20)$$

where γ is a phase. If $f \ll f'$, then the contribution with the longer period could be approximated by a linear potential over many cycles of the first term. This way a potential of the form (11) appears. Models like this were discussed by Freese, Liu, and Spolyar [26] (and recently in [27]). In that work, the relative contribution of the periodic and quasilinear pieces were chosen such that a sequence of minima ensued, leading to a realization of "chain inflation" [28]. Here, the two terms in (20) would have to be fine-tuned to give a sequence of inflection points leading to a staircase potential in some region of field space. The staircase is finite in this example so that the initial conditions, while not fine-tuned, need to be such that the inflaton turns back in the staircase region, i.e. the initial inflaton velocity could not be too large.

String theory offers good prospects for finding staircase potentials. An example which has the right ingredients appeared quite recently [29] in connection with efforts to construct a string theory model of large-field inflation. The inflaton is identified with an axion arising from a 2-form field integrated over a 2-cycle Σ_2 in the usual way [30]. The potential receives an instanton contribution (due to Euclidean $D1$-branes) which gives rise to a standard periodic term. The authors of [29] have, however, also identified a nonperiodic contribution due to branes wrapping the 2-cycle Σ_2. This term (basically the Dirac-Born-Infeld action action of the wrapped brane) is nonperiodic—it contributes to the energy increasing without bound as a function of the axion field. The ensuing potential is of the form (10); specifically

$$V(\phi) = \mu^4 \cos\left(\frac{\phi}{f}\right) + \mu'^3\sqrt{v^2 + \phi^2}, \qquad (21)$$

where μ, μ', and v depend on the geometry. For inflaton values much larger than v^2, the potential again takes the form (11). In [29], the geometry was assumed such that the instanton contribution to the potential was suppressed relative to the nonperiodic term. The relative magnitude of the two terms is determined by the geometry, but there appears to be no obstruction to fine-tuning the coefficients so that the two terms lead to an infinite staircase for $\phi \gg v$. It would certainly be interesting to verify this and explore the observable consequences of this class of models.

VI. CONCLUSIONS

Inflation at an inflection point is an attractive implementation of the inflationary paradigm, which appears in a number of contexts. A major concern with such models is ensuring that inflation actually begins. The scenario described here addresses this issue. The basic observation is that if there is any chance at all of inflation starting in a model with a single point of inflection, that chance should increase if there is a whole sequence of such points. A reasonably natural way such a structure could come about is by "tilting" a periodic potential due to the breaking of discrete shift symmetry. Apart from the case of inflation at inflection points, this could also be relevant for other instances of small-field inflation (e.g. hilltop inflation [31]). This note has focused on presenting the universal aspects of the idea.

Apart from making inflection point inflation somewhat more plausible, the scenario proposed here suggests observational consequences which could eventually support or refute it. The main source of information is, of course, the cosmic microwave background. The uphill evolution of the inflaton could leave traces in the cosmic microwave background in the form of large deviations from scale invariance [6]. This would, however, only be observable if the visible perturbations were generated right at the start of inflation, that is, if inflation lasted only for about 60 e-folds, which is very unlikely in the scenario discussed above. Beyond this possibility, one could expect clear signatures if multiple inflection points were traversed by the inflaton after inflation. Even though the background evolution in the "downhill phase" is hardly affected by the inflection points, the inflaton velocity would vary periodically as it crosses the steps on the way down which would be reflected in the cosmic microwave background as k-dependent oscillations in the spectrum of primordial density perturbations. The impact of a "feature" in the potential on the perturbation spectra has been the subject of numerous studies (e.g. [32,33]). The observable effects of a sequence of steps induced by a duality cascade in the context of brane inflation were analyzed recently by Bean *et al.* [34]. The observational imprint of inflaton oscillations of the inflaton potential was also recently considered in [35,36]. These studies have considered various forms of the potential, but not quite the kind discussed here. Clearly, it would be very interesting to explore the signatures of a staircase of inflection points.

Finally, it would be important to construct explicit, consistent models of "staircase inflation" in string theory, perhaps by making the second of the two examples discussed above more precise. At this point, it is not completely clear that staircase inflation can be realized in string theory, but the essential elements required are certainly present given the ubiquity of axions and the rather generic mechanism for breaking the shift symmetry identified in [29]. Although it is not obvious *a priori*, it appears that these models have enough freedom to attain the requisite fine-tuning while retaining control over the approximations made. Since string theory has many axion-type fields, it is a very natural setting for staircase inflation.

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