

Anisotropic singularities in modified gravity models

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We show that the common singularities present in generic modified gravity models governed by actions of the type $S = \int d^4x \sqrt{-g} f(R, \phi, X)$, with $X = -\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi$, are essentially the same anisotropic instabilities associated to the hypersurface $F(\phi) = 0$ in the case of a nonminimal coupling of the type $F(\phi)R$, enlightening the physical origin of such singularities that typically arise in rather complex and cumbersome inhomogeneous perturbation analyses. We show, moreover, that such anisotropic instabilities typically give rise to dynamically unavoidable singularities, precluding completely the possibility of having physically viable models for which the hypersurface $\frac{\partial f}{\partial R} = 0$ is attained. Some examples are explicitly discussed.

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I. INTRODUCTION

In the absence of a more fundamental physical model based on first-principles for the description of the cosmic acceleration discovered more than a decade ago [1] (see, for reviews, [2]), many dark energy phenomenological models have been proposed and investigated in detail. In particular, the questions about the stability against small perturbations in the initial conditions and in the model parameters are always the first requirement demanded to assure the physical viability of any cosmological model. The most part of such dark energy models belong to the general class of cosmological models governed by an action of the type (see, for instance, [3])

$$S = \int d^4x \sqrt{-g} f(R, \phi, X), \quad (1)$$

where R stands for the spacetime scalar curvature, ϕ is a scalar field, $X = -\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi$, and f is a smooth function. Quintessence models [4], for instance, correspond to the choice $f(R, \phi, X) = \frac{1}{16\pi} R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + V(\phi)$. Nonminimally coupled models [5–7], on the other hand, are typically of the type

$$f(R, \phi, X) = F(\phi)R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + V(\phi). \quad (2)$$

Many other models discussed in the literature correspond yet to the case $f(R, \phi, X) = g(R, \phi) + h(\phi, X)$, including k -essence [8] and the string-inspired case of a Dirac-Born-Infeld tachyonic action [9]. (For more recent works, see [10].) The particular case of pure modified gravity $f(R, \phi, X) = f(R)$ (see, for a recent review, [11]) has been intensively investigated as an alternative to quintessence. Some primordial inflationary models [12] are also

described by actions of the type (1). Since one of the proposals of any cosmological model is to describe our universe without finely-tuned parameters, a given dark energy or inflationary model would be physically viable only if it is robust against small perturbations in the initial conditions and in the model parameters. This is the question to be addressed here.

Nonminimally coupled models of the type (2) are known to be plagued by anisotropic singularities in the phase space region corresponding to $F(\phi) = 0$. For instance, Starobinski [13] was the first to identify the singularity corresponding to the hypersurfaces $F(\phi) = 0$, for the case of conformally coupled anisotropic solutions. Futamase and coworkers [5] identified the same kind of singularity in the context of chaotic inflation in $F(\phi) = 1 - \xi \phi^2$ theories (See also [6]). In [14], it is shown that such kind of singularities are generically related to anisotropic instabilities.

Many authors have described different singularities corresponding to $\frac{\partial f}{\partial R} = 0$ in general models like (1) (see, for instance, [3]) or, more commonly, in pure $f(R)$ gravity models (see, for instance, [11,15]). Such singularities appear typically in rather complex and cumbersome inhomogeneous perturbation analyses, obscuring their physical origin and cause. In this work, we show that these singularities are essentially due to anisotropic instabilities, in a similar way to those ones described in [14] for models of the type (2). Moreover, we show that such instabilities typically give rise to dynamically unavoidable singularities, rendering the original model physically nonviable.

One can advance that there are some geometrically special regions on the phase space of the model in question by an elementary analysis of the equations derived from the action (1). They are the generalized Klein-Gordon equation

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$$D_a(f_{,X}\partial^a\phi) + f_{,\phi} = 0, \quad (3)$$

and the Einstein equations

$$FG_{ab} = \frac{1}{2}(f - RF)g_{ab} + D_a D_b F - g_{ab}\square F - \frac{1}{2}f_{,X}\partial_a\phi\partial_b\phi, \quad (4)$$

where $F = F(R, \phi, X) \equiv \frac{\partial f}{\partial R}$. We will consider here the simplest anisotropic homogeneous cosmological model, the Bianchi type I, whose spatially flat metric is given by

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2. \quad (5)$$

The dynamically relevant quantities in this case are

$$H_1 = \frac{\dot{a}}{a}, \quad H_2 = \frac{\dot{b}}{b}, \quad \text{and} \quad H_3 = \frac{\dot{c}}{c}. \quad (6)$$

For such a metric and with a homogeneous scalar field $\phi = \phi(t)$, Einstein Eq. (4) can be written as

$$FG_{00} = -\frac{1}{2}(f - FR) - (H_1 + H_2 + H_3)\dot{F} + \frac{1}{2}f_{,X}\dot{\phi}^2, \quad (7)$$

$$\frac{FG_{11}}{a^2} = \frac{1}{2}(f - FR) + (H_2 + H_3)\dot{F} + \ddot{F}, \quad (8)$$

$$\frac{FG_{22}}{b^2} = \frac{1}{2}(f - FR) + (H_1 + H_3)\dot{F} + \ddot{F}, \quad (9)$$

$$\frac{FG_{33}}{c^2} = \frac{1}{2}(f - FR) + (H_1 + H_2)\dot{F} + \ddot{F}, \quad (10)$$

and the generalized Klein-Gordon equation will read

$$\frac{d}{dt}(f_{,X}\dot{\phi}) + (H_1 + H_2 + H_3)f_{,X}\dot{\phi} - f_{,\phi} = 0. \quad (11)$$

Notice that (11) is a second order differential equation for ϕ , while Eqs. (8)–(10) form a higher order system of ordinary differential equations. Since $F = F(R, \phi, X)$, the term corresponding to \ddot{F} involves, in fact, second derivatives of R and, consequently, third derivatives of H_i , $i = 1, 2, 3$. Thus, the corresponding phase space \mathcal{M} is 11-dimensional and spanned by the variables $(\phi, \dot{\phi}, H_1, \dot{H}_1, \ddot{H}_1, H_2, \dot{H}_2, \ddot{H}_2, H_3, \dot{H}_3, \ddot{H}_3)$. Equation (7) corresponds to the energy constraint. It restricts the solutions of (8)–(11) on a certain (vanishing energy) hypersurface \mathcal{E} of \mathcal{M} . Thus, effectively, the solutions of (8)–(11) are constrained to the 10-dimensional manifold $\mathcal{E} \in \mathcal{M}$.

It is quite simple to show that Eqs. (8)–(10) are not compatible, in general, on the hypersurface \mathcal{F} of \mathcal{M} corresponding to the region where $F(R, \phi, X) = 0$. Subtracting (9) and (10) from (8) we have, respectively, on such hypersurface

$$(H_1 - H_2)\dot{F} = 0, \quad \text{and} \quad (H_1 - H_3)\dot{F} = 0. \quad (12)$$

Hence, Eqs. (8)–(10) cannot be fulfilled in general for anisotropic metrics. As it will be shown, the hypersurface

\mathcal{F} indeed corresponds a geometrical singularity for anisotropic spacetimes which cannot be dynamically prevented in general by requiring, for instance, that $\dot{F} = 0$ on the hypersurface \mathcal{F} as suggested naively from (12). Furthermore, the Cauchy problem for the Eqs. (8)–(11) is ill-posed on this hypersurface, since one cannot choose general initial conditions on it.

II. THE SINGULARITY

In order to study the geometrical nature of the singular hypersurface \mathcal{F} , let us consider the Einstein Eqs. (7)–(10) in detail. For the metric (5), we have the following identities

$$G_{00} = H_1H_2 + H_2H_3 + H_1H_3, \quad (13)$$

$$G_{11} = a^2(\dot{H}_1 + H_1(H_1 + H_2 + H_3) - \frac{1}{2}R), \quad (14)$$

$$G_{22} = b^2(\dot{H}_2 + H_2(H_1 + H_2 + H_3) - \frac{1}{2}R), \quad (15)$$

$$G_{33} = c^2(\dot{H}_3 + H_3(H_1 + H_2 + H_3) - \frac{1}{2}R), \quad (16)$$

$$R = 2(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_2H_3 + H_1H_3). \quad (17)$$

Now, we introduce the new dynamical variables $p = H_1 + H_2 + H_3$, $q = H_1 - H_2$, and $r = H_1 - H_3$. Notice that

$$R = 2\dot{p} + \frac{2}{3}(2p^2 + q^2 + r^2 - qr), \quad (18)$$

implying that \ddot{R} involves terms up to third order derivative in p and up to second order in q and r . In terms of the new dynamical variables, Einstein Eqs. (8)–(10) can be cast in the form

$$3\ddot{F} = (\dot{p} + p^2)F - \frac{3}{2}f - 2p\dot{F}, \quad (19)$$

$$q\dot{F} = -(\dot{q} + qp)F, \quad (20)$$

$$r\dot{F} = -(\dot{r} + rp)F. \quad (21)$$

As to the energy constraint (7), we have

$$\frac{1}{3}(p^2 + qr - q^2 - r^2)F + p\dot{F} + \frac{1}{2}(f - FR) = \frac{1}{2}f_{,X}\dot{\phi}^2, \quad (22)$$

and the generalized Klein-Gordon Eq. (3) reads simply

$$(f_{,X} + f_{,XX}\dot{\phi}^2)\ddot{\phi} + (f_{,XR}\dot{R} + f_{,X\phi}\dot{\phi} + pf_{,X})\dot{\phi} - f_{,\phi} = 0. \quad (23)$$

Notice that the Eqs. (19)–(23) do not involve the terms \ddot{q} and \ddot{r} . Moreover, Eqs. (20) and (21) are, respectively, first order differential equations for q and r , from which the terms involving first and second derivative of q and r present in the terms \dot{F} and \ddot{F} of (19) and (22) can be evaluated directly. The order reduction of the system of

differential equations attained with the introduction of the new dynamical variables implies that the phase space \mathcal{M} is not 11, but 7-dimensional and spanned by the variables $(\phi, \dot{\phi}, p, \dot{p}, \dot{q}, r)$. The solutions are still constrained to the hypersurface $\mathcal{E} \in \mathcal{M}$ corresponding to the energy constraint (22). It is clear, however, that the manifold \mathcal{E} is, in fact, 6-dimensional.

There is still a further dynamical restriction on the solutions of (19)–(23). From (20) and (21), one has

$$r\dot{q} - q\dot{r} = 0, \quad (24)$$

implying that $q(t)/r(t)$ is a constant of motion fixed only by the initial conditions. Suppose the initial ratio is $q(0)/r(0) = \gamma$: this would imply that $(H_1 - H_2) = \gamma(H_1 - H_3)$ for all t , leading to, for instance, $c^\gamma(t) \propto a^{\gamma-1}(t)b(t)$ in the metric (5). This simplification is a consequence of the scalar character of our homogeneous source field, and it is also present [14] in the nonminimally coupled case given by actions of the form (2). Let \mathcal{Q} be the hypersurface corresponding to q/r constant. Finally, the solutions of (19)–(23) are necessary restricted to the 5-dimensional submanifold $\mathcal{Q} \cap \mathcal{E}$ of \mathcal{M} .

A closer analysis of Eqs. (20) and (21) reveals the presence of the singularity. They can be written as

$$\dot{q} = -\left(p + \frac{\dot{F}}{F}\right)q, \quad (25)$$

$$\dot{r} = -\left(p + \frac{\dot{F}}{F}\right)r. \quad (26)$$

In general, the right-hand side of these equations diverge on the hypersurface \mathcal{F} corresponding to $F(R, \phi, X) = 0$, unless $q = r = 0$. The first observation is that (25) and (26) imply, in general, that, if q (or r) vanishes for some t , it will vanish for any t . This is why such kind of singularity can be evaded in homogeneous and isotropic situations. We will return to this point in the next section, with an explicit example. For any physically viable cosmological model, small amounts of anisotropy, corresponding to small q and r , must stay bounded during the cosmological history. In fact, it is desirable that they diminish, tending towards an isotropic situation. However, this does not happen in general if $F(R, \phi, X) = 0$ in (25) and (26). Let us assume that $\dot{F} \neq 0$ on the hypersurface \mathcal{F} . (We will return to this point later.) In this case, if any anisotropic solution crosses \mathcal{F} , necessarily \dot{q} and \dot{r} will diverge, corresponding to a real spacetime geometrical singularity, as one can check by considering the Kretschman invariant $I = R_{abcd}R^{abcd}$, which for the metric (5) is given by

$$\begin{aligned} \frac{1}{4}I &= (\dot{H}_1 + H_1^2)^2 + (\dot{H}_2 + H_2^2)^2 + (\dot{H}_3 + H_3^2)^2 + H_1^2 H_2^2 \\ &\quad + H_1^2 H_3^2 + H_2^2 H_3^2. \end{aligned} \quad (27)$$

As one can see, I is the sum of non-negative terms. Moreover, any divergence of the variables H_1, H_2, H_3 , or

of their time derivatives, would suppose a divergence in I , characterizing a real geometrical singularity. Since the relation between the variables p, q, r , and H_1, H_2, H_3 is linear, any divergence of the first, or of their time derivative, will suppose a divergence in I .

There are two basically distinct situations where the singularity corresponding to the hypersurface \mathcal{F} could be evaded dynamically. We will show that both are very unlikely to occur in physical situations. The first one corresponds to the case when the hypersurface \mathcal{F} belongs to some dynamically inaccessible region. In such a case we, of course, do not face any singularity, since $F(R, \phi, X)$ will never vanish along a solution of the system. This would be equivalent to stating that $\mathcal{F} \cap \mathcal{E} \cap \mathcal{Q} = \emptyset$. In our case, it would imply, from (22), that the equation

$$p\dot{F} = Xf_{,X} - \frac{1}{2}f \quad (28)$$

has no solution in \mathcal{M} . This would correspond to a quite concocted and artificial function f . In particular, for all models we could find in the literature having the hypersurface \mathcal{F} , the Eq. (28) has solutions.

The second situation corresponds to the already mentioned case where $\dot{F} = 0$ on the hypersurface \mathcal{F} . From (28), we see that this requires necessarily that the function f be homogeneous of degree $\frac{1}{2}$ in the variable X on \mathcal{F} . Again, a highly artificial situation.

Any point on the energy constraint hypersurface \mathcal{E} is, in principle, a dynamically possible point. Moreover, it is desirable for any cosmological model free of finely-tuned parameters that any point or, at least, a large region of \mathcal{E} could be chosen as the initial condition for a cosmological evolution. This, of course, includes also the neighborhood of the hypersurface \mathcal{F} provided that $\mathcal{F} \cap \mathcal{E} \neq \emptyset$.

III. AN EXPLICIT EXAMPLE

The singularities described in the preceding section imply that any model governed by an action of the type (1) having a hypersurface \mathcal{F} will certainly present severe anisotropic instabilities that will render it physically non-viable. Let us work out an explicit example in order to illustrate the dynamical role of such anisotropic instabilities. The pure modified gravity model

$$f(R) = R - \alpha R_* \ln\left(1 + \frac{R}{R_*}\right), \quad (29)$$

where α and R_* are free positive parameters, was recently proposed [16] as a viable model to describe the recent cosmic acceleration. Such a model has a hypersurface \mathcal{F} corresponding to $f'(R) = F(R) = 0$, where

$$F(R) = 1 - \frac{\alpha R_*}{R + R_*}. \quad (30)$$

In [16], it is assumed a universe filled with radiation and dark matter, but, for our purposes here, it is enough to consider the pure geometrical Lagrangian given by (29).

Let us start, as in [16], assuming a homogeneous and isotropic universe $H_1 = H_2 = H_3 = H$. Einstein Eqs. (7)–(10) for this case would correspond simply to the energy constraint

$$6H\dot{R}F'(R) + f(R) - RF(R) + 6H^2F(R) = 0, \quad (31)$$

and to the generalized Friedman equation

$$\begin{aligned} \ddot{R}F'(R) + (2HF'(R) + \dot{R}F''(R))\dot{R} + \frac{1}{2}f(R) \\ - (\dot{H} + 3H^2)F(R) = 0, \end{aligned} \quad (32)$$

where $R = 6\dot{H} + 12H^2$ in this homogeneous and isotropic case. Note that

$$F'(R) = \frac{\alpha R_*}{(R + R_*)^2} > 0, \quad (33)$$

for $R + R_* \neq 0$. Equation (32) is a third order differential equation for H . Hence, the relevant phase space is 3-dimensional and spanned by the variables (H, \dot{H}, \ddot{H}) , but the solutions are in fact constrained to the 2-dimensional manifold \mathcal{E} corresponding to the energy constraint (31). The manifold \mathcal{E} is an ordinary smooth surface, with a single value of \dot{H} assigned to each pair (H, \ddot{H}) , provided $H \neq 0$ and $R + R_* \neq 0$. Thus, the solutions of (32) can be conveniently projected on the plane (H, \dot{H}) , without any loss of dynamical information.

It is convenient to work with the dimensionless quantities $H = \sqrt{R_*}h$, $\sqrt{R_*}\tau = t$, $R = \rho R_*$. The phase space for this model is quite simple. There are only two fixed points corresponding to $h = \pm\sqrt{\tilde{\rho}/12}$, where $\tilde{\rho}$ is the positive solution of the equation

$$2\alpha \ln(1 + \rho) = \rho + \alpha \frac{\rho}{1 + \rho}. \quad (34)$$

This solution exists and is unique provided that $\alpha > 1$. Both Eqs. (31) and (32) are invariant under the transformation $\tau \rightarrow -\tau$ and $h \rightarrow -h$, implying that the h negative portion of the phase space can be obtained from the positive one by means of a time reversal operation. Typical trajectories projected on the (h, \dot{h}) plane of the phase space are depicted in Fig. 1. Notice that the surfaces corresponding to $\rho = R/R_*$ constant are simple parabolas $6\dot{h} + 12h^2 = \rho$ in the (h, \dot{h}) plane. For the model in question, the \mathcal{F} surface corresponds to one of these parabolas, namely $\rho = \alpha - 1$. The point here is that the existence of such a surface does not imply any singular behavior for the Eq. (32). For instance, the solutions crossing the surface $F(R) = 0$ depicted in 1 are perfectly regular. Hence, homogeneous and isotropic solutions can cross without problem the singular hypersurface.

Suppose now the system has a small amount of anisotropy, *i.e.*, $|q| \ll |p|$ and $|r| \ll |p|$. In this case, we have from (18) $R \approx 2\dot{p} + \frac{4}{3}p^2$ and the Eq. (19) for p will be essentially the same (32) obeyed by H in the isotropic case, provided the anisotropy is indeed kept small along the solutions. For the amounts of anisotropy q and r , however, the relevant equations will be (25) and (26). Since we know

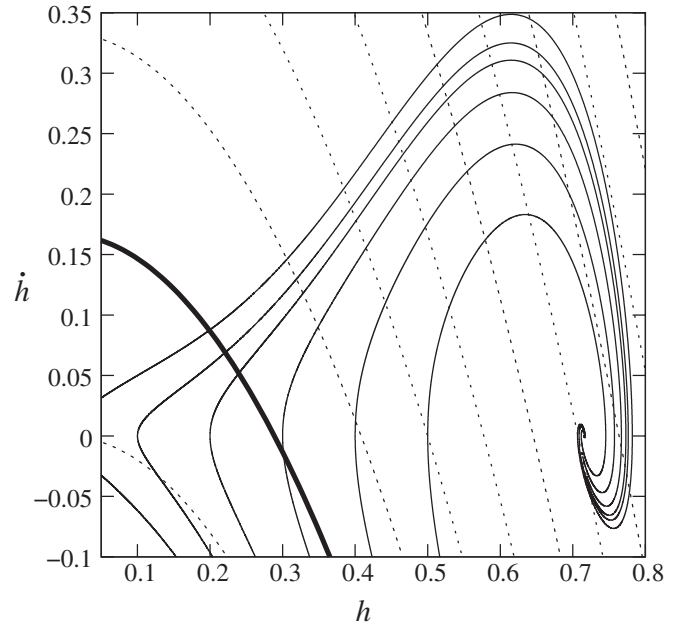


FIG. 1. Typical trajectories for the model (29) projected on the (h, \dot{h}) plane, with $\alpha = 2$ (the same value adopted in the analysis of [16]). The (attractive de Sitter) fixed point in this case corresponds to $h \approx 0.7157$. The ρ -constant parabolas are shown. The solid one, in particular, correspond to $\rho = \alpha - 1$ (the singular \mathcal{F} surface, defined by $F(R) = 0$). The regions corresponding to $F(R) > 0$ and $F(R) < 0$ are, respectively, the region above and the region below the parabola $\rho = \alpha - 1$. As one can see, homogeneous and isotropic solutions can cross without problems the singular surface.

from (24) that $r(t) = \gamma q(t)$, we can consider here only the variable q

$$\dot{q} = -\left(p + \frac{F'(R)}{F(R)}\dot{R}\right)q. \quad (35)$$

In any region of the phase space far from the surface $F(R) = 0$, the right-handed side of (35) is well behaved. Moreover, from the energy constraint (22), we have

$$p + \frac{F'(R)}{F(R)}\dot{R} = \frac{2}{3}p + \frac{\gamma^2 - \gamma + 1}{3} \frac{q^2}{p} + \frac{1}{2p} \left(R - \frac{f(R)}{F(R)}\right). \quad (36)$$

A closer analysis reveals that

$$RF(R) - f(R) = \alpha R_* \left(\ln(1 + \rho) - \frac{\rho}{1 + \rho} \right) \geq 0, \quad (37)$$

with the equality holding only for $\rho = 0$, implying that for the region $F(R) > 0$, at least, the quantity between parenthesis in (35) is positive, leading indeed to an isotropization of the solutions. For regions close to the surface $F(R) = 0$, on the other hand, the situation is qualitatively different. From (36), we have that the right-handed side of (35) diverges on the surface $F(R) = 0$. If an anisotropic solution reaches such surface, we have from Eq. (35) that \dot{q} diverges, implying that this model does not admit any amount of anisotropy at all, precluding any possibility of constructing a realistic model based solely in the geometric Lagrangian (29). Similar results hold also for the other

functions $f(R)$ discussed in [16], namely

$$f(R) = R - \alpha R_* \left(1 + \frac{R}{R_*}\right)^\beta, \quad (38)$$

with $\beta \in (0, 1)$.

IV. FINAL REMARKS

The singularities associated with the hypersurface $F(R, \phi, X) = 0$ described here are not new. They have been discovered and rediscovered many times for many different models in rather complex and cumbersome inhomogeneous perturbation analysis around a given well behaved background solution. Our results, however, enlighten the physical origin of such singularities. They arise already in the background level and are related to anisotropic expansion rates. Any solution crossing the hypersurface $F(R, \phi, X) = 0$ will not admit, in general, any amount of anisotropy, otherwise it will certainly develop a cata-

strophic geometrical singularity with, for instance, the blowing up of the Kretschman invariant (27). This, in fact, precludes the possibility of constructing a realistic model with solutions crossing the hypersurface $F(R, \phi, X) = 0$ since we would have qualitatively distinct behavior for arbitrarily close homogeneous solutions: a perfect isotropic and a slightly anisotropic one.

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