Gödel-type universes in $f(R)$ gravity

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The $f(R)$ gravity theories provide an alternative way to explain the current cosmic acceleration without a dark energy matter component. If gravity is governed by a $f(R)$ theory, a number of issues should be reexamined in this framework, including the violation of causality problem on nonlocal scale. We examine the question as to whether the $f(R)$ gravity theories permit space-times in which the causality is violated. We show that the field equations of these $f(R)$ gravity theories do not exclude solutions with breakdown of causality for a physically well-motivated perfect-fluid matter content. We demonstrate that every perfect-fluid Gödel-type solution of a generic $f(R)$ gravity satisfying the condition $df/dR > 0$ is necessarily isometric to the Gödel geometry, and therefore presents violation of causality. This result extends a theorem on Gödel-type models, which has been established in the context of general relativity. We also derive an expression for the critical radius r_c (beyond which the causality is violated) for an arbitrary $f(R)$ theory, making apparent that the violation of causality depends on both the $f(R)$ gravity theory and the matter content. As an illustration, we concretely take a recent $f(R)$ gravity theory that is free from singularities of the Ricci scalar and is cosmologically viable, and show that this theory accommodates noncausal as well as causal Gödel-type solutions.

I. INTRODUCTION

The possibility of modifying Einstein's theory of gravitation by adding terms proportional to powers of the Ricci scalar R to the Einstein-Hilbert Lagrangian, presently known as $f(R)$ gravity, has a long history (see, e.g., [1]) and received the attention of many researchers (see, e.g., Ref. [2] for historical reviews). Quadratic corrections were used to construct a renormalizable gravity action [3] and to fuel inflation [4]. Modifications with negative power of R motivated by string/M theory [5], were also proposed in the scientific literature. Many of these works were motivated by quantum corrections, which are important close to the Planck scale. More recently, due to the impressive amount of astrophysical data pointing to a phase of accelerated expansion of the Universe [6], $f(R)$ gravity had a revival, motivated by the fact that these theories can be used to explain the observed accelerating late expansion with no need of a dark energy component. This has given birth to a great number of papers [7] on $f(R)$ gravity (see also Ref. [8] for recent reviews). Several features of these theories, including solar system tests [9], Newtonian limit [10], gravitational stability [11] and singularities [12], have been exhaustively discussed. General principles such as the so-called energy conditions have also been used to place constraints on $f(R)$ theory [13]. As a result, a number of $f(R)$ theories have been suggested to describe the evolution of the Universe, retaining the standard local gravity constraints (see, for example, Refs. [14–16]).

If gravitation can be described by a $f(R)$ theory instead of general relativity, there are a number of issues that ought to be reexamined in the $f(R)$ gravity framework, including the question as to whether these theories permit space-time solutions in which the causality is violated. To tackle this problem in the $f(R)$ gravity framework, we first recall that there are solutions to the Einstein field equations that possess causal anomalies in the form of closed timelike curves. The famous solution found by Gödel [17] 60 years ago is the best known example of a model that makes it apparent that the general relativity theory does not exclude the existence of closed timelike world lines, despite its Lorentzian character, which leads to the local validity of the causality principle. The Gödel model is a solution of Einstein's equations with cosmological constant Λ for dust of density ρ, but it can also be interpreted as perfect-fluid solution (with pressure $p = \rho$) without cosmological constant. In this regard, it was shown by Bampi and Zordan [18] (for a generalization see Ref. [19]) that every Gödel-type solution of Einstein's equations with a perfect-fluid energy-momentum tensor is necessarily isometric to the Gödel space-time. Owing to its unexpected properties, Gödel's model has a well-recognized importance and has motivated a number of investigations on rotating Gödel-type models as well as on causal anomalies not only in the context of general relativity (see, e.g. Refs. [20]) but also in the framework of other theories of gravitation (see, for example, Refs. [21]).

Gödel-type universes in gravity theories whose Lagrangian is an arbitrary function of the curvature invariants R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ were recently examined by Clifton and Barrow [22]. In particular, they have shown

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that any $f(R)$ gravity theory in which $df/dR \neq 0$, admits a perfect-fluid Gödel-type solution with closed timelike curves.[1]

In this article, to proceed further with the investigation of Gödel-type universes along with the question of breakdown of causality in $f(R)$ gravity, we extend the results of Refs. [22,23] in four different ways. First, we examine the dependence of the critical radius r_c (beyond which the causality is violated) with the $f(R)$ gravity theory, and derive an expression for the critical radius of Gödel-type perfect-fluid solutions of any $f(R)$ gravity theory. Second, we demonstrate that every perfect-fluid Gödel-type solution of a generic $f(R)$ gravity satisfying the condition[2] $f_R \equiv df/dR > 0$ is necessarily isometric to the Gödel geometry, and hence any $f(R)$ gravity exhibits violation of causality. This extends to the context of $f(R)$ gravity a theorem on Gödel-type models, which has been established in the framework of general relativity. Third, given the inevitable breakdown of causality for any perfect-fluid Gödel-type solution, we reexamine the violation of causality by considering two other matter sources, namely, combination of a perfect fluid with a scalar field, and a single scalar field. For both cases we show that $f(R)$ gravity permit solutions without violation of causality. Fourth, we concretely illustrate our general results by taking a recent $f(R)$ gravity theory that is free from singularities of the Ricci scalar and is cosmologically viable [16], and show that this theory accommodates both causal and noncausal solutions.

II. $f(R)$ GRAVITY AND GÖDEL-TYPE UNIVERSES

The causality problem in $f(R)$ theories can be looked upon as having two interconnected physically relevant ingredients, namely, the gravity theory (which involves the matter source) and the space-time geometry. Regarding the former, we begin by recalling that the action that defines an $f(R)$ gravity is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{f(R)}{2\kappa^2} + \mathcal{L}_m \right], \quad (1)$$

where $\kappa^2 \equiv 8\pi G$, g is the determinant of the metric $g_{\mu\nu}$, $f(R)$ is a function of the Ricci scalar R, and $\mathcal{L}_m$ the Lagrangian density for the matter fields. Varying this action with respect to the metric we obtain the field equations

$$f_R R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)f_R = \kappa^2 T_{\mu\nu}, \quad (2)$$

where $f_R \equiv df/dR$, $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, ∇_μ denotes the covariant derivative, and $T_{\mu\nu} = -(2/\sqrt{-g}) \times \delta(\sqrt{-g}\mathcal{L}_m)/\delta g^{\mu\nu}$ is the matter energy-momentum tensor. Clearly, for $f(R) = R$ these field equations reduce to the Einstein equations. An important constraint, often used to simplify the field equations, comes from the trace of Eq. (2), which is given by

$$3\Box f_R + f_R R - 2f = \kappa^2 T, \quad (3)$$

where $T \equiv T^\mu_{\ \mu}$ is the trace of the energy-momentum tensor.

The second important ingredient of the above mentioned causality problem is related to the space-time geometry. In this regard, we recall that the Gödel-type space-time-homogeneous metrics that we focus our attention on in this paper is given, in cylindrical coordinates [(r, ϕ, z)], by [23]

$$ds^2 = [dt + H(r)d\phi]^2 - D^2(r)d\phi^2 - dr^2 - dz^2, \quad (4)$$

where

$$H(r) = \frac{4\omega}{m^2} \sinh^2\left(\frac{mr}{2}\right), \quad (5)$$

$$D(r) = \frac{1}{m} \sinh(mr), \quad (6)$$

with ω and m being parameters such that $\omega^2 > 0$ and $-\infty \leq m^2 \leq +\infty$.[3] All Gödel-type metrics are characterized by the two parameters m and ω: identical pairs (m^2, ω^2) specify isometric space-times [23–25]. Gödel solution is a particular case of the $m^2 > 0$ class of space-times in which $m^2 = 2\omega^2$.

The line element of Gödel-type metrics can also be written as

$$ds^2 = dt^2 + 2H(r)dtd\phi - dr^2 - G(r)d\phi^2 - dz^2, \quad (7)$$

where $G(r) = D^2 - H^2$. In this form it is clear that the existence of closed timelike curves of Gödel-type, i.e., circles defined by $t, z, r = \text{const}$, depend on the behavior of the function $G(r)$. If $G(r) < 0$ for a certain range of r ($r_1 < r < r_2$, say) *Gödel's circles* defined by $t, z, r = \text{const}$ are closed timelike curves. In this regard, it is easy to show that the causality features of the Gödel-type space-times depend upon the two independent parameters m and ω [23]. For $m = 0$ there is a critical radius, defined by $\omega r_c = 1$, such that for all $r > r_c$ there are noncausal Gödel's circles. For $m^2 = -\mu^2 < 0$ there is an infinite sequence

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[1]For $f(R)$ gravity with $df/dR = 0$, the existence of these curves depends on the functional form of $f(R)$, i.e., the violation of causality may or may not occur [22]. These theories, however, do not fulfill the conditions to avoid instabilities and to ensure agreement with local tests of gravity.

[2]Classically, this condition is necessary to ensure that the effective Newton constant $G_{\text{eff}} = G/f_R$ does not change its sign. At a quantum level, it prevents the graviton from becoming ghostlike (see, e.g., Refs. [15] for details).

[3]Clearly, for $m^2 = -\mu^2 < 0$ the metric functions $H(r)$ and $D(r)$ become circular functions $H(r) = (4\omega/\mu^2)\sin^2(\mu r/2)$ and $D(r) = \mu^{-1}\sin(\mu r)$, while in the limiting case $m = 0$, they become $H = \omega r^2$ and $D = r$.

of alternating causal and noncausal t, z, $r =$ const regions without and with Gödel's circles. For $0 < m^2 < 4\omega^2$ noncausal Gödel's circles occur for $r > r_c$ such that

$$\sinh^2 \frac{mr_c}{2} = \left[\frac{4\omega^2}{m^2} - 1\right]^{-1}. \tag{8}$$

When $m^2 = 4\omega^2$ the critical radius $r_c \to \infty$. Thus, for $m^2 \geq 4\omega^2$ there are no Gödel's circles, and hence the breakdown of causality of Gödel type is avoided.

From Eqs. (4)–(6) it is straightforward to show that the Ricci scalar for the Gödel-type metrics takes a constant value $R = 2(m^2 - \omega^2)$, hence the third term on the left-hand side of Eqs. (2) vanishes. A further simplification comes about by the following choice of basis:

$$\theta^0 = dt + H(r)d\phi, \qquad \theta^1 = dr, \tag{9}$$

$$\theta^2 = D(r)d\phi, \qquad \theta^3 = dz, \tag{10}$$

relative to which the Gödel-type line element (4) takes the form

$$ds^2 = \eta_{AB}\theta^A\theta^B = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2, \tag{11}$$

where clearly $\eta_{AB} = \text{diag}(1, -1, -1, -1)$. Indeed, taking into account the constraint Eq. (3), the field Eqs. (2) take the form

$$f_R G_{AB} = \kappa^2 T_{AB} - \tfrac{1}{2}(f + \kappa^2 T)\eta_{AB}, \tag{12}$$

where the nonvanishing components of the Einstein tensor G_{AB} take the quite simple form

$$G_{00} = 3\omega^2 - m^2, \qquad G_{11} = G_{22} = \omega^2,$$
$$G_{33} = m^2 - \omega^2. \tag{13}$$

Having set up the basic ingredients of the causality problem in $f(R)$ gravity, in the next sections we shall examine whether these theories permit causal and non-causal solutions.

III. NONCAUSAL GÖDEL-TYPE SOLUTION

An important component of the above gravitational ingredient of the causality problem is the matter source. In this regard, we first consider a physically well-motivated perfect fluid of density ρ and pressure p, whose energy-momentum tensor in the basis (9) and (10) is clearly given by

$$T_{AB}^{(M)} = (\rho + p)u_A u_B - p\eta_{AB}. \tag{14}$$

For this matter source, the field Eqs. (12) reduce to

$$2(3\omega^2 - m^2)f_R + f = \kappa^2(\rho + 3p), \tag{15}$$

$$2\omega^2 f_R - f = \kappa^2(\rho - p), \tag{16}$$

$$2(m^2 - \omega^2)f_R - f = \kappa^2(\rho - p), \tag{17}$$

where we have used Eq. (13). Equations (16) and (17) give

$$(2\omega^2 - m^2)f_R = 0. \tag{18}$$

Thus, for $f(R)$ theories that satisfy the condition to keep unaltered the sign of the effective Newton constant as well as to avoid graviton from becoming ghostlike [15], i.e., $f_R > 0$, Eq. (18) gives $m^2 = 2\omega^2$, which defines the Gödel metric, and the remaining field equations reduce to

$$\kappa^2 p = \frac{f}{2}, \tag{19}$$

$$\kappa^2 \rho = m^2 f_R - \frac{f}{2}, \tag{20}$$

where f is an arbitrary function of the Ricci scalar (with $f_R \neq 0$), and both f and f_R are evaluated at $R = m^2 = 2\omega^2$. This result can be seen as an extension of Bampi and Zordan [18] result (obtained in the framework of general relativity) to the context of $f(R)$ gravity in the sense that for arbitrary ρ and p (with $p \neq -\rho$) perfect-fluid solution of every $f(R)$ gravity, which satisfies the condition $f_R > 0$, is necessarily isometric to the Gödel geometry.[4] Concerning the causality features of these solutions we first note that since they are isometric to Gödel geometry they unavoidably exhibit closed timelike curves, i.e., noncausal Gödel's circles whose critical radius r_c is given by Eq. (8). But, taking into account Eqs. (19) and (20) we have that, in the framework of $f(R)$ gravity, r_c is given by

$$r_c = \frac{2}{m}\sinh^{-1}(1) = 2\sinh^{-1}(1)\sqrt{\frac{2f_R}{2\kappa^2\rho + f}}, \tag{21}$$

making apparent that the critical radius, beyond which there exist noncausal Gödel's circles, depends on both the gravity theory and the matter content. We emphasize this expression (21) for the critical radius holds for any $f(R)$ gravity which satisfies the condition $f_R > 0$.

Despite this inescapable breakdown of causality for any perfect-fluid Gödel-type $f(R)$ solution, to concretely illustrate an estimation of the bounds on r_c for a specific theory, let us consider the recently proposed $f(R)$ theory described by [16]

$$f(R) = R - \alpha R_* \ln\left(1 + \frac{R}{R_*}\right), \tag{22}$$

which is free from singularities of the Ricci scalar, cosmologically viable and satisfies the existence of relativistic stars for positive parameters α and R_*. To this end, we use the positivity of the density ρ and Eq. (20) to obtain

$$m^2 f_R - \frac{f}{2} \geq 0, \tag{23}$$

where f is an arbitrary function of the R (with $f_R \neq 0$), and

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[4]We note that this extension is contained in Ref. [22] but it has not been explicitly stated.

both f and f_R are evaluated at $R = m^2$. By using (22) for $\alpha = 2$ (see Ref. [16]) it is easy to show that the inequality (23) holds for all m such that $m^2 \gtrsim 0.55 R_*$, making therefore explicit the lower bound on m^2 and therefore on the critical radius r_c for this theory.

IV. CAUSAL GÖDEL-TYPE SOLUTION

Since any perfect-fluid Gödel-type solution of $f(R)$ gravity is inevitably noncausal, the question as to whether other matter sources could generate Gödel-type causal solutions naturally arises at this point. In this section we shall examine this problem by considering two different matter sources, namely, a combination of a perfect fluid with a scalar field, and a single scalar field.[5]

A. Perfect fluid plus scalar field

The combined energy-momentum tensor we consider is given by

$$T_{AB} = T_{AB}^{(M)} + T_{AB}^{(S)}, \tag{24}$$

where $T_{AB}^{(M)}$ corresponds to a perfect fluid [Eq. (14)] and $T_{AB}^{(S)}$ is energy-momentum tensor of a scalar field, i.e.,

$$T_{AB}^{(S)} = \Phi_{|A}\Phi_{|B} - \frac{1}{2}\eta_{AB}\Phi_M\Phi_N\eta^{MN}, \tag{25}$$

where a vertical bar denotes components of covariant derivatives relative to the local basis $\theta^A = e_\alpha^{(A)}dx^\alpha$ [see Eqs. (9) and (10)]. Following Ref. [23] it is straightforward to show that a scalar field of the form $\Phi(z) = ez + \text{const}$ satisfies the scalar field equation $\Box\Phi = \eta^{AB}\nabla_A\nabla_B\Phi = 0$ for a constant amplitude e of $\Phi(z)$. Thus, the nonvanishing components of energy-moment tensor for this scalar field are

$$T_{00}^{(S)} = -T_{11}^{(S)} = -T_{22}^{(S)} = T_{33}^{(S)} = \frac{e^2}{2}, \tag{26}$$

and the field Eqs. (12) for the combined matter source (24) can be written in the form

$$\kappa^2 e^2 = (m^2 - 2\omega^2)f_R, \tag{27}$$

$$\kappa^2 p = \frac{1}{2}(2\omega^2 - m^2)f_R + \frac{f}{2}, \tag{28}$$

$$\kappa^2 \rho = \frac{1}{2}(6\omega^2 - m^2)f_R - \frac{f}{2}, \tag{29}$$

where f is an arbitrary function of the Ricci scalar (with

$f_R \neq 0$), and both f and f_R are evaluated at $R = 2(m^2 - \omega^2)$. A causal Gödel-type class of solutions of these equations that satisfies the condition $f_R > 0$ is given by

$$m^2 = 4\omega^2, \tag{30}$$

$$f_R = \frac{\kappa^2 e^2}{2\omega^2}, \tag{31}$$

$$\kappa^2 p = -\kappa^2 \rho = -\omega^2 f_R + \frac{f}{2}, \tag{32}$$

where from Eqs. (8) and (30) one clearly has that the critical radius $r_c \to \infty$. Hence, for this combination of matter fields, there is no violation of causality of Gödel type (Gödel's circles) for any $f(R)$ gravity that satisfies the conditions $f_R > 0$.

As an illustration, we shall now concretely examine whether the theory described by (22) admits this type of causal solution. For this theory, Eq. (31) gives rise to a quadratic equation in the variable m^2/R_* whose roots are given in terms of e^2/R_* by

$$\frac{m_\pm^2}{R_*} = \frac{1}{3}\left[1 + 3\frac{\kappa^2 e^2}{R_*} \pm \sqrt{1 + 18\frac{\kappa^2 e^2}{R_*} + 9\left(\frac{\kappa^2 e^2}{R_*}\right)^2}\right], \tag{33}$$

where we have taken $\alpha = 2$ (see Ref. [16] for details). Clearly, the positivity of the density parameter ρ [as given by (32) for f evaluated at $R = 6\omega^2 = 3m^2/2$ and f_R given by (31)] is assured by $\kappa^2 e^2 - f \geq 0$ for each root of Eq. (33). Regarding the first root m_+^2/R_* the positivity of ρ gives

$$0 \leq \frac{\kappa^2 e^2}{R_*} \lesssim 0.8 \quad \text{and} \quad 0.7 \lesssim \frac{m^2}{R_*} \lesssim 2.7. \tag{34}$$

Thus, for values $\kappa^2 e^2/R_*$ and m^2/R_* within these intervals there are causal solutions of the $f(R)$ gravity of Ref. [16] generated by the combination of a perfect fluid with a scalar field such that $\rho \geq 0$.[6]

B. Scalar field

For the scalar field $\Phi(z)$ as the single source component, and $f_R \neq 0$, the field Eqs. (27)–(29) give rise to the unique class of Gödel-type solutions

$$m^2 = 4\omega^2, \tag{35}$$

$$f_R = \frac{\kappa^2 e^2}{2\omega^2}, \tag{36}$$

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[5]We note that the presence of a single closed timelike curve as, for example, a Gödel's circle, is an unequivocal manifestation of violation of causality. However, a space-time may admit noncausal curves other than Gödel's circles. Therefore, throughout this paper by causal solutions we mean solutions with no violation of causality of Gödel-type, i.e., no Gödel's circles.

[6]For completeness, we mention that the second root of (33), i.e., m_-^2/R_* along with the positivity of ρ furnishes $\kappa^2 e^2/R_* \gtrsim 2.5$ and $m^2/R_* < 0$. Negative values of m^2 are known to lead to violation of causality with alternating causal and noncausal Gödel's circles [23,26].

$$f = k^2 e^2, \qquad (37)$$

where f is an arbitrary function of R (with $f_R \neq 0$), and both f and f_R are evaluated at $R = 2(m^2 - \omega^2)$. This clearly defines a class of solutions with no violation of causality of Gödel type ($r_c \rightarrow \infty$) for an arbitrary $f(R)$ with $f_R \neq 0$.

As an illustration, we note that for this source, the $f(R)$ theory described by (22) also permits a causal solution. Indeed, as Eq. (36) is identical to Eq. (31) it clearly has two roots given by Eq. (33). Inserting the first root, m_+^2/R_*, into (37) one finds the following values:

$$\frac{\kappa^2 e^2}{R_*} \approx 0.82 \quad \text{and} \quad \frac{m^2}{R_*} \approx 2.7, \qquad (38)$$

making apparent that the theory of Ref. [16] accommodates the solution given by (35)–(38), which has no violation of causality of Gödel type.[7]

V. CONCLUDING REMARKS

The so-called $f(R)$ gravity theory provides an alternative way to explain the current cosmic acceleration with no need of invoking either the existence of an extra spatial dimension or a dark energy component. If gravity is governed by a $f(R)$ theory instead of general relativity, various issues should be reexamined in the $f(R)$ framework. This includes the breakdown of causality. In $f(R)$ gravity theories the causal structure of four-dimensional space-time has locally the same qualitative nature as the flat space-time of special relativity—causality holds locally. The nonlocal question, however, is left open, and violation of causality can occur.

In this paper, we have examined the question as to whether the $f(R)$ gravity theories permit space-times in which the causality is violated or not, and generalize the results of Refs. [22,23]. For physically well-motivated perfect-fluid matter sources, we showed that every perfect fluid (with $p \neq \rho$) Gödel-type solution of an arbitrary $f(R)$ gravity that satisfies the the condition $f_R > 0$ is necessarily isometric to the Gödel geometry, making explicit that the violation of causality is unavoidable feature of any $f(R)$ gravity. This results is a generalization of the Bampi-Zordan theorem [18], which has been established in the context of Einstein's theory of gravitation. We have derived an expression for the critical radius r_c (beyond which the causality is violated) for an arbitrary $f(R)$ theory (with $f_R \neq 0$), making apparent that the functional character of the violation of causality depends on both the $f(R)$ gravity theory and the matter content.

We have also examined the question as to whether other matter sources could give rise to Gödel-type causal solutions by considering a combination of perfect fluid with a scalar field, and simply a single scalar field. We have shown that in both cases Gödel-type solutions of an arbitrary $f(R)$ theory (with $f_R \neq 0$) with no violation of causality are permitted. We have also found a general class of such causal solution for an arbitrary $f(R)$ theory that satisfies the condition $f_R > 0$. As an illustration, we have concretely considered a recent $f(R)$ gravity theory that is free from singularities of the Ricci scalar and is cosmologically viable [16], and showed that this theory accommodates both noncausal and causal Gödel-type solutions.

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[7]The second root of (33) gives $\kappa^2 e^2/R_* \approx 2.44$ and $m^2/R_* \approx -0.53$, which leads again to violation of causality with alternating causal and noncausal circles [23,26].

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