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Distributions of the phase angle of the fermion determinant in QCD

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The distribution of the phase angle and the magnitude of the fermion determinant as well as its correlations with the baryon number and the chiral condensate are studied for QCD at non zero quark chemical potential. Results are derived to one-loop order in chiral perturbation theory. We find that the distribution of the phase angle is Gaussian for small chemical potential and a periodic Lorentzian when the quark mass is inside the support of the Dirac spectrum. The baryon number and chiral condensate are computed as a function of the phase of the fermion determinant and we discuss the severe cancellations which occur upon integration over the angle. We compute the distribution of the magnitude of the fermion determinant as well as the baryon number and chiral condensate at fixed magnitude. Finally, we consider QCD in one Euclidean dimension where it is shown analytically, starting from the fundamental QCD partition function, that the distribution of the phase of the fermion determinant is a periodic Lorentzian when the quark mass is inside the spectral density of the Dirac operator.

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I. INTRODUCTION

The phase diagram of strongly interacting matter is expected to show several phases as a function of the temperature and the baryon chemical potential. Matter in nuclei, in compact stars, and in the early universe are in different parts of the phase diagram and large experimental and theoretical efforts have been invested to understand their properties. Of particular intense interest is the critical end-point. Its existence is expected mainly on the findings of model studies that the baryon density is discontinuous as a function of the chemical potential [1]. Lattice QCD, which has allowed us to determine the nature of the phase transition at zero baryon chemical potential [2], appears to be the natural tool to study the nonperturbative phenomena which take place near the end point. However, probabilistic lattice QCD methods are not directly applicable at nonzero baryon chemical potential: Monte Carlo importance sampling, which is at the core of Lattice QCD computations, requires that the Euclidean action is real. At non zero chemical potential, though, the quark determinant is complex. This severe obstacle is known as *the sign problem*.

Recent numerical progress in understanding the phase diagram of strongly interacting matter at nonzero chemical potential has reopened the field. Not only has it been understood that the location of the end point in the (μ, T)-plane is extremely sensitive to the quark mass [3], it may also be that the dependence of the end point on quark mass is very different from what was commonly accepted [3]. Because of the sign problem these conclusions where reached from analytic continuations of lattice simulations carried out at imaginary values of the chemical potential. Such an extrapolation [4–7] is not without pitfalls. It has recently been demonstrated [8] that utmost care should be taken when attempting to extract information on the critical end point from a Taylor expansion at $\mu = 0$ [9–13]. Moreover, it was demonstrated in [14] that the numerical implementation of the reweighting approach [15–17] is extremely delicate even at small values of the chemical potential.

Lately alternative numerical methods such as the density of states method and the complex Langevin method have been explored. Despite early reports of its failure [18–20], the complex Langevin method has been shown to be able to deal with sign problems in simple models and for a gas of relativistic bosons [21]. On the analytical front, the severity of the sign problem was analyzed for QCD at low energy and for models of the QCD partition function [22–26]. The intricate connections between the sign problem, chiral symmetry, and the Dirac spectrum, have been understood in the ϵ-regime of QCD [27].

In the present work we focus on the density of states method [28–33]. In this approach one evaluates an observable numerically for a fixed given quantity and thereby obtain the distribution of this observable over the fixed quantity. The full expectation value of the observable is then obtained by integration over the fixed quantity. This method has had some success when the baryon number, the average plaquette or the phase of the fermion determinant is kept fixed. In this paper we are particularly interested in the last approach since it goes back to the root of the sign problem. If we would know the exact distribution function of the phase of the fermion determinant as well as its correlations with physical observables, the sign problem would have been solved: the delicate cancellations due to the fluctuations of the phase could be realized exactly by an analytical integration over the phase according to the distribution function and its correlations.

We will use chiral perturbation theory to compute the distribution of the phase of the fermion determinant

$$\langle\delta(\theta-\theta')\rangle_{N_f}d\theta = \frac{\int dA|\det(D+\mu\gamma_0+m)|^{N_f}e^{iN_f\theta'}\delta(\theta-\theta')e^{-S_{\rm YM}}}{\int dA|\det(D+\mu\gamma_0+m)|^{N_f}e^{iN_f\theta'}e^{-S_{\rm YM}}}d\theta. \tag{1}$$

Here θ' refers to the phase of the fermion determinant. It is a function of the gauge field configuration which we average over, i.e. $\exp(2i\theta')=\det(D(A)+\mu\gamma_0+m)/\det^*(D(A)+\mu\gamma_0+m)$. Because of the sign problem the distribution of the phase is not real and positive. The complex nature, however, is of the simplest possible form: Since

$$\langle\delta(\theta-\theta')\rangle_{N_f}=e^{i\theta N_f}\frac{Z_{|N_f|}}{Z_{N_f}}\langle\delta(\theta-\theta')\rangle_{|N_f|}. \tag{2}$$

the θ-distribution factorizes into $\exp(i\theta N_f)$ and a real and positive distribution. Here, Z_{N_f} is the N_f flavor partition function and $Z_{|N_f|}$ is the phase-quenched N_f flavor partition function. The subscripts N_f and $|N_f|$ refer to averages with respect to these two partition functions, in this order. For $N_f=2$ this relation reads

$$\langle\delta(\theta-\theta')\rangle_{1+1}=e^{2i\theta}\frac{Z_{1+1^*}}{Z_{1+1}}\langle\delta(\theta-\theta')\rangle_{1+1^*}, \tag{3}$$

where here and below the subscript $1+1$ refers to QCD with two ordinary flavors whereas the subscript $1+1^*$ refers to QCD with one ordinary flavor and one conjugate flavor. By definition the fermion determinant of a quark and a conjugate quark are each others complex conjugates so that the total measure is real and positive. The θ-distribution of the *phase-quenched* theory, $\langle\delta(\theta-\theta')\rangle_{1+1^*}$, is necessarily real and positive. Moreover it is normalized to one. Also the θ-distribution of the full theory $\langle\delta(\theta-\theta')\rangle_{1+1}$ is normalized to one. On the right-hand side (rhs) of (3), however, the ratio Z_{1+1^*}/Z_{1+1} grows exponentially fast with the volume so that the phase factor, $e^{2i\theta}$, must lead to exponentially large cancellations.

As we shall see it is essential to discuss separately the case when $2\mu/m_\pi<1$ and the case when the quark mass is inside the support of the spectral density of the Dirac operator. (The scale $m_\pi/2$ appears in the spectrum of the Dirac operator because the generating functionals for the eigenvalue density has quarks and conjugate quarks. The link between the Dirac spectrum of the full theory and the phases of the phase-quenched theory are discussed in detail in [34–37]). We will show below that the real and positive part of the θ-distribution becomes a periodic superposition of Gaussians when the quark mass m is outside the support of the Dirac spectrum. When the quark mass is inside the support of the Dirac spectrum the sign problem becomes much more severe [24,27]. Figure 1 gives a schematic picture of the phase diagram of QCD as well as the region where the quark mass is inside the spectral support of the Dirac operator. As we will show below, the θ-distribution in this region is not only very wide/flat, it also changes shape into a periodic superposition of Lorentzians. A hint of this dramatic change is already present in (3). When the quark mass enters the spectral support of the Dirac operator a phase transition occurs in the phase-quenched theory while the full theory remains unaltered [27]. The exponential growth of the ratio Z_{1+1^*}/Z_{1+1} with the volume is thus particularly rapid when the quark mass enters the spectral support of the Dirac operator.

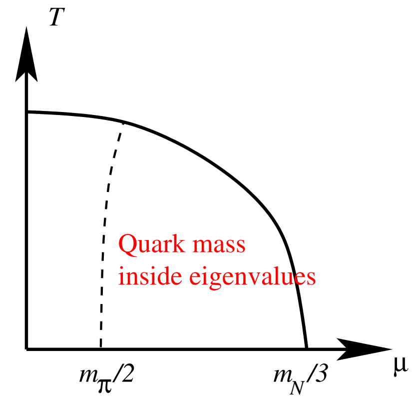

FIG. 1 (color online). A schematic picture of the phase diagram of QCD as a function of the quark chemical potential μ and the temperature T. Chiral symmetry is spontaneously broken below the full curve. The dashed curve indicates where the quark mass enters the Dirac spectrum. As this happens the nature of the sign problem changes. To the left of the dashed curve the distribution of phase of the fermion determinant is a periodic superposition of Gaussians whereas it is a periodic superposition of Lorentzians to the right of the dashed curve. We stress that the dashed curve does *not* indicate a phase transition in QCD.

The Gaussian shape of the θ-distribution for small μ was first observed numerically by Ejiri in [33] where it is also argued that this form is a natural consequence of the central limit theorem. The change from the Gaussian to the Lorentzian form for larger values of μ therefore suggests a breakdown of the conditions for the application of the central limit theorem. To cast further light on this we also compute the distribution of the phase, $\langle\delta(\theta-\theta')\rangle$, for lattice QCD in one Euclidean dimension. As we will show, it is possible to derive the Lorentzian form of the θ-distribution directly from the one-dimensional lattice QCD partition function, when the quark mass is inside the support of the Dirac spectrum.

In addition to the distribution of the phase of the fermion determinant we also consider the direct dependence of observables $\mathcal{O}$ on the phase θ through the distribution-function

$$\langle\mathcal{O}\delta(\theta-\theta')\rangle. \tag{4}$$

The integral over θ obviously gives the full expectation value $\langle\mathcal{O}\rangle$. The θ-dependence of the observable shows if severe cancellations take place in this integral. Furthermore, the distribution of the observable with the phase allows us to address which range of the phase is essential for the full expectation value of $\mathcal{O}$.

We will compute the distribution of the baryon number operator, its square as well as the distribution of the chiral condensate over θ. It is found that the distributions, $\langle\mathcal{O}\delta(\theta-\theta')\rangle$, take complex values and that drastic cancellations occur when integrating over θ.

This paper is organized as follows. In Secs. II and III we briefly recall a few facts about chiral perturbation theory which are relevant for the calculation of the average phase factor and the distribution of the phase angle. Then we turn to the distributions of the baryon number (Sec. IV), the off-diagonal susceptibility (Sec. V) and the chiral condensate (Sec. VI) over the phase angle of the fermion determinant. These one-loop results are all valid for $\mu < m_\pi/2$. Next we discuss the distribution of the phase for an ensemble generated at $\mu = 0$. The difference in the phase distribution for ϵ-counting rather than the p-counting in pointed out in Sec. VIII. In Sec. IX it is shown that the leading order prediction for the θ-distribution takes a Lorentzian shape for $\mu > m_\pi/2$. The Lorentzian form is then obtained as an exact result for lattice QCD in one Euclidean dimension in Sec. XI. The remainder of the paper discusses the radial distribution of the fermion determinant.

II. 1-LOOP CHIRAL PERTURBATION THEORY AND THE AVERAGE PHASE FACTOR

The first step towards obtaining the distribution of the phase is to understand the average of the phase factor. In this section we review the calculation of the average phase factor in chiral perturbation theory.

Chiral perturbation theory [38] is the low energy effective theory of QCD in the phase where chiral symmetry is broken spontaneously. It describes the dynamics of the Goldstone modes, i.e. the pions and the kaons. We shall work in the so called p-expansion of chiral perturbation theory where the small expansion parameter is

$$p \sim m_\pi \sim \mu \sim T \sim \frac{1}{L}. \tag{5}$$

For $\mu < m_\pi/2$ the chemical potential modifies the pion propagator in the standard way for relativistic bosons. The one-loop contribution to the free energy from a pair of charge conjugate pions (the chemical potentials are therefore μ and $-\mu$) is thus given by

$$G_0(\mu,-\mu) \equiv -\sum_{p_{k\alpha}} \log(|\vec{p}_{k\alpha}^{\,2} + m_\pi^2 + (p_{k0} - 2i\mu)^2|^2), \tag{6}$$

where

$$p_{k\alpha} = \frac{2\pi k_\alpha}{L_\alpha}, \qquad k_\alpha \text{ integer.} \tag{7}$$

After a Poisson resummation this can be expressed as [23]

$$G_0(\mu,-\mu) = -V\sum_{l_\alpha}\int\frac{d^dp}{(2\pi)^d}e^{iL_\alpha p_\alpha l_\alpha} \times \log(|\vec{p}^{\,2} + m_\pi^2 + (p_0 - 2i\mu)^2|^2), \tag{8}$$

where the sum is over all integers. The thermodynamic limit is given by the term $l_\alpha = 0$. Here, two facts about this term, which we denote by $G_0|_{V=\infty}$, are essential: *i)* it is independent of μ *ii)* it includes the entire 1-loop divergence (see [39] for a discussion). In dimensional regularization it is given by

$$G_0|_{V=\infty} = \frac{2}{(4\pi)^{d/2}}\Gamma\Big(-\frac{d}{2}\Big)m_\pi^d. \tag{9}$$

The finite part of the 1-loop free energy, denoted by $g_0(\mu)$, contains the sum over the terms with $l_\alpha \neq 0$. This results in the decomposition

$$G_0(\mu,-\mu) = G_0|_{V=\infty} + g_0(\mu,-\mu). \tag{10}$$

If we wish to keep track of the leading $1/V$ corrections to the infinite volume result we have to evaluate the sum over all four components of the momentum. The finite, μ, L and T dependent part then reads [23] (this expression generalizes the result of [40] for $\mu = 0$ to nonzero chemical potential)

$$g_0(\mu,-\mu) = 2\int_0^\infty\frac{d\lambda}{\lambda^3}e^{-m_\pi^2L^2\lambda/4\pi} \times \Big(\prod_{\alpha=0}^{3}\sum_{l_\alpha}e^{-2\mu l_0 L_0\delta_{\alpha 0}}e^{-\pi((l_\alpha^2L_\alpha^2)/(\lambda L^2))} - 1\Big), \tag{11}$$

where l_α runs over all integers and $L \equiv (L_0L_i^3)^{1/4}$.

When the length of the box is considerably larger than the Compton wavelength of the pion the sum over momenta can be replaced by an integral, and the 1-loop contribution to the free energy simplifies to the familiar expression

$$g_0(\mu,-\mu) = \frac{Vm_\pi^2T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\cosh\Big(\frac{2\mu n}{T}\Big). \tag{12}$$

As the simplest relevant example let us now consider the average phase factor for the phase-quenched theory. By definition we have that

$$\langle e^{2i\theta'}\rangle_{1+1^*} = \frac{Z_{1+1}}{Z_{1+1^*}}. \tag{13}$$

The phase-quenched theory in the denominator is identical to QCD at nonzero chemical potential for the third component of isospin [41]. This has an immediate consequence: since the pions carry isospin charge but no baryon charge the free energy of Z_{1+1^*} depends on μ while the usual free energy of Z_{1+1} is independent of μ when evaluated in chiral perturbation theory. It is this dependence on the chemical potential which makes it possible to compute the average phase factor in chiral perturbation theory despite the fact that pions have baryon charge zero.

For small μ the leading (mean field) term in the chiral Lagrangian, $2m\langle\bar{\psi}\psi\rangle V$, is identical in the two cases and hence the phase factor is determined to leading order by the one-loop effect

$$\langle e^{2i\theta'}\rangle_{1+1^*} = \frac{e^{G_0(\mu,\mu)}}{e^{G_0(\mu,-\mu)}} = e^{g_0(\mu=0)-g_0(\mu)}. \tag{14}$$

With p-counting (5) we have that $g_0(\mu) - g_0(\mu = 0) \sim V\mu^2T^2 \sim 1$ as was discussed in detail in [24].

For $\mu > m_\pi/2$ a Bose Einstein condensate of pions forms in the phase-quenched theory and the mean field terms in the chiral Lagrangian contribute to $\langle\exp(2i\theta')\rangle$. These terms are of order $\mu^2F^2V \sim V/L^2 \sim L^2$. Hence, for $\mu > m_\pi/2$, the strength of the sign problem depends on L even if we scale m_π and μ with L according to p-counting.

Since the difference of the finite parts of the one-loop free energy appears repeatedly below, it will be convenient to introduce the notation

$$\begin{aligned}\Delta G_0 &\equiv \Delta G_0(\mu, -\mu, m, m)\\ &\equiv G_0(\mu, -\mu, m, m) - G_0(\mu, \mu, m, m)\\ &= g_0(\mu, -\mu, m, m) - g_0(\mu, \mu, m, m).\end{aligned} \tag{15}$$

Below we will also meet free energies where the chemical potentials are not of opposite sign and where the quark masses are different. To be precise we reserve the notation ΔG_0 as defined in (15), and explicitly write the dependence on the chemical potentials and quark masses when necessary.

III. THE DISTRIBUTION OF THE PHASE ($\mu < m_\pi/2$)

The distribution of the phase angle can be obtained from the moments of the phase factor [24]

$$\langle\delta(\theta - \theta')\rangle_{N_f} = \frac{1}{2\pi}\sum_{p=-\infty}^{\infty} e^{-ip\theta}\langle e^{ip\theta'}\rangle_{N_f}. \tag{16}$$

The even moments are ratios of a partition function with p additional determinants and inverse conjugate determinants and the usual N_f flavor partition function

$$\langle e^{2ip\theta'}\rangle_{N_f} = \frac{1}{Z_{N_f}}\left\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)}\det^{N_f}(D+\mu\gamma_0+m)\right\rangle. \tag{17}$$

Since the number of charged Goldstone modes of the partition function in the numerator is $p(p + N_f)$ whereas the contributions of the neutral Goldstone bosons from the numerator and the denominator cancel, we obtain

$$\langle e^{2ip\theta'}\rangle_{N_f} = e^{-p(N_f+p)\Delta G_0}. \tag{18}$$

When the quark mass is outside the support of the Dirac spectrum, the contribution to the phase angle of individual eigenvalues is in the range $[-\pi/2, \pi/2]$, and we expect half-integer powers of the determinants in (17) are smoothly connected to results obtained for integer powers. In other words, we expect that the replica trick [42,43] can be used to analytically continue the moments to half-integer values of p. We then find

$$\begin{aligned}\langle\delta(\theta - \theta')\rangle_{N_f} &= \frac{1}{2\pi}\sum_{p=-\infty}^{\infty} e^{-ip\theta-(p/2)((p/2)+N_f)\Delta G_0}\\ &= \frac{1}{2\pi}e^{iN_f\theta+(1/4)N_f^2\Delta G_0}\sum_{u=-\infty}^{\infty} e^{-iu\theta-u^2\Delta G_0/4}\\ &= \frac{1}{2\pi}e^{iN_f\theta+(1/4)N_f^2\Delta G_0}\vartheta_3(\theta/(2\pi), e^{-\Delta G_0/4}).\end{aligned} \tag{19}$$

After a Poisson resummation this can be rewritten as [24]

$$\begin{aligned}\langle\delta(\theta - \theta')\rangle_{N_f} &= \frac{1}{\sqrt{\pi\Delta G_0}}e^{iN_f\theta+(1/4)N_f^2\Delta G_0}\\ &\quad\times\sum_{n=-\infty}^{\infty} e^{-(\theta+2n\pi)^2/\Delta G_0},\\ &\quad\theta \in [-\pi, \pi]\end{aligned} \tag{20}$$

valid for a compact phase angle $\theta \in [-\pi, \pi]$.

Notice that

$$\frac{Z_{N_f}}{Z_{|N_f|}} = e^{-(1/4)N_f^2\Delta G_0} \tag{21}$$

so that to be consistent with the general form given in (3), the result (20) shows that the quenched and the phase-quenched θ-distributions are identical. Also note that the θ-distribution depends only on ΔG_0. Plots for $\Delta G_0 = 0.2$ and $\Delta G_0 = 10$ are shown in Fig. 2. Notice the different scales in the two plots. For $\Delta G_0 = 10$, when the sign problem is severe, the normalization to one requires a delicate cancellation.

As long as the contribution to the phase of the fermion determinant from individual eigenvalue pairs does not exceed $\pi/2$ one can unambiguously define the phase of the determinant on $[-\infty, \infty]$ as was done by Ejiri [33]. To obtain this distribution simply interpret the angle in (20) as ranging from $-\infty$ to ∞. This leads to the Gaussian distribution (here for $N_f = 2$)

$$\begin{aligned}\langle\delta(\theta - \theta')\rangle_{1+1} &= \frac{e^{2i\theta}}{\sqrt{\pi\Delta G_0}}e^{-\theta^2/\Delta G_0+\Delta G_0},\\ &\quad\theta \in [-\infty, \infty].\end{aligned} \tag{22}$$

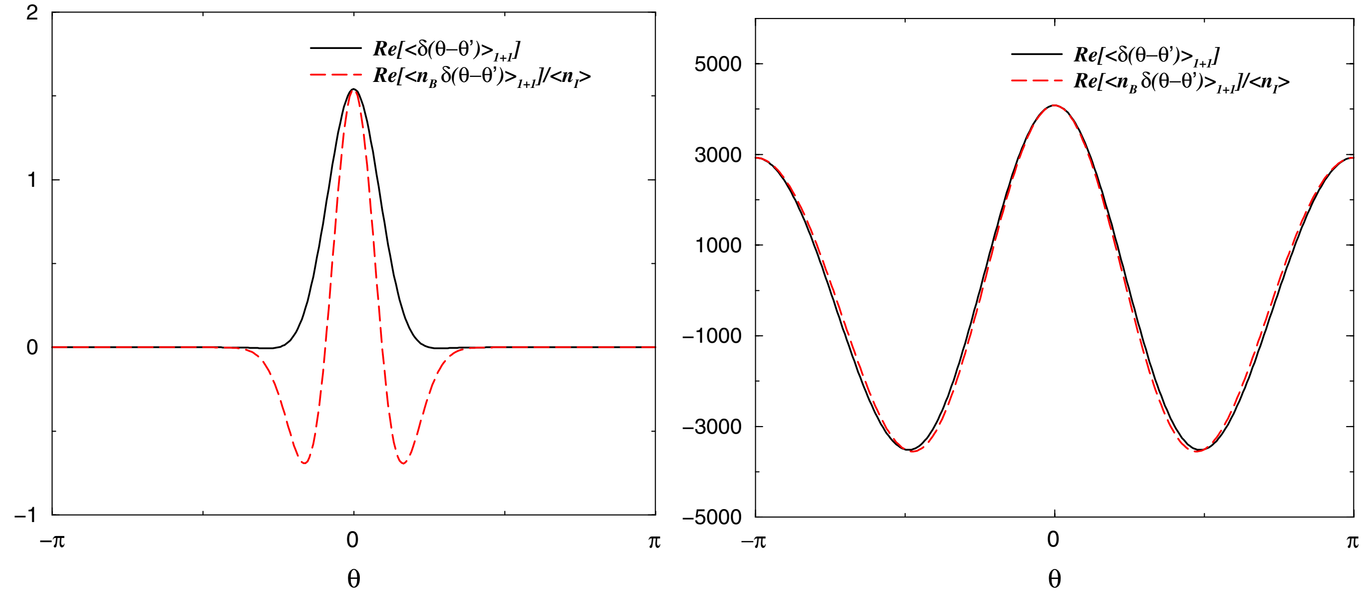

FIG. 2 (color online). The real part of the distribution of the phase $\langle\delta(\theta-\theta')\rangle_{1+1}$ (solid curve) for $\Delta G_0 = 0.2$ left and $\Delta G_0 = 10$ right. Also shown is the real part of the distribution of the baryon number over θ (dashed curve). For better comparison the latter has been rescaled by $(\lim_{\tilde{\mu}\to\mu}\frac{d}{d\tilde{\mu}}\Delta G_0(-\mu,\tilde{\mu}))$. The fact that the θ-distribution is normalized to unity while the distribution of the baryon number over θ integrates to zero is not easy to see when $\Delta G_0 = 10$. This directly illustrates the severity of the sign problem. Note that the phase is constrained to $\theta \in [-\pi, \pi]$.

However, when the quark mass is inside the support of the spectrum of the Dirac operator only the phase restricted $[-\pi, \pi]$ can be defined uniquely. We return to this point in Sec. IX where we derive the θ-distribution for $\mu > m_\pi/2$.

When the angles are noncompact and replica trick can be used, it is useful to represent the δ-function in Eq. (19) by an integral over p instead of a sum over p. Below this will be exploited on several occasions to simplify our expressions.

IV. THE BARYON NUMBER OPERATOR ($\mu < m_\pi/2$)

Since the pions have zero baryon charge the baryon number in chiral perturbation theory is automatically zero. We will see below that the baryon number at fixed θ is a total derivative.

To derive $\langle n_B\delta(\theta-\theta')\rangle$ we first compute the correlation between the baryon number and all moments of the phase factor

$$\langle n_B e^{2ip\theta'}\rangle_{1+1} = \frac{1}{2Z_{1+1}}\lim_{\tilde{\mu}\to\mu}\frac{d}{d\tilde{\mu}}\left\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)}\times\det^2(D+\tilde{\mu}\gamma_0+m)\right\rangle. \quad (23)$$

To one-loop order in chiral perturbation theory we obtain

$$\frac{\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)}\det^2(D+\tilde{\mu}\gamma_0+m)\rangle}{\langle\det^2(D+\mu\gamma_0+m)\rangle} = e^{-2p(\Delta G_0(-\mu,\tilde{\mu})-\Delta G_0(\mu,\tilde{\mu}))-p^2\Delta G_0(-\mu,\mu)}. \quad (24)$$

To keep track of the combinatorics it is essential to recall that the one-loop free energy does not depend on the baryon chemical potential, that is $G_0(\mu,\mu) = G_0(\mu = 0)$. We conclude that

$$\langle n_B e^{2ip\theta'}\rangle_{1+1} = -\Big(\lim_{\tilde{\mu}\to\mu}\frac{d}{d\tilde{\mu}}p\Delta G_0(-\mu,\tilde{\mu})\Big)\times e^{-p(2+p)\Delta G_0(-\mu,\mu)}. \quad (25)$$

The delta function $\delta(\theta-\theta')$ is obtained after summing over p. Interpreting the phase angle on $\langle-\infty,\infty\rangle$ and proceeding in the same way as for the distribution of θ we obtain

$$\langle n_B\delta(\theta-\theta')\rangle_{1+1} = \Big(\lim_{\tilde{\mu}\to\mu}\frac{d}{d\tilde{\mu}}\Delta G_0(-\mu,\tilde{\mu})\Big)\Big(1+i\frac{\theta}{\Delta G_0}\Big)\times\frac{e^{2i\theta}}{\sqrt{\pi\Delta G_0}}e^{-\theta^2/\Delta G_0+\Delta G_0}. \quad (26)$$

The total baryon number density should vanish because chiral perturbation theory does not include baryonic degrees of freedom. This can be seen simply by writing the above expression as a total derivative

$$\langle n_B\rangle_{1+1} = \Big(\lim_{\tilde{\mu}\to\mu}\frac{d}{d\tilde{\mu}}\Delta G_0(-\mu,\tilde{\mu})\Big)\times\frac{1}{\sqrt{\pi\Delta G_0}}\int_{-\infty}^{\infty}d\theta\frac{1}{2i}\frac{d}{d\theta}e^{2i\theta}e^{-\theta^2/\Delta G_0+\Delta G_0} = 0. \quad (27)$$

The total derivative appears because all one-loop contributions to the $2p$'th moment of the phase factor are proportional to p or p^2 so that the differentiation to obtain the baryon density leads to an overall factor p. This factor can be expressed as a total derivative with respect to θ. Notice that when $\Delta G_0 \gg 1$ the extreme tail of the distribution over θ can contribute significantly to the cancellation of the total baryon number.

If we, as is usually the case, consider the phase on $[-\pi, \pi]$ we get instead

$$\langle n_B \delta(\theta - \theta')\rangle_{1+1} = \left(\lim_{\tilde{\mu}\to\mu} \frac{d}{d\tilde{\mu}} \Delta G_0(-\mu, \tilde{\mu})\right) \times \sum_{n=-\infty}^{\infty} \left(1 + i\frac{\theta + 2\pi n}{\Delta G_0}\right) \times \frac{e^{2i\theta}}{\sqrt{\pi \Delta G_0}} e^{-(\theta+2\pi n)^2/\Delta G_0 + \Delta G_0}. \quad (28)$$

An illustration of $\langle n_B \delta(\theta - \theta')\rangle_{1+1}$ is given in Fig. 2. For small ΔG_0, a small phase angle gives an excess of baryons over antibaryons, which is cancelled by the opposite effect at larger phase angle, resulting in $n_B = 0$. For large ΔG_0 the plot is quite similar to the θ-distribution which is also shown in this figure. There is however an important difference: The integral over θ of the θ-distribution is unity while the total baryon number is zero.

The importance of the tail for the cancellation of the total baryon number translates into the importance of the terms with large values of $|n|$.

V. THE OFF-DIAGONAL SUSCEPTIBILITY ($\mu < m_\pi/2$)

Even though pions have zero baryon charge chiral perturbation theory gives a nontrivial prediction for the off-diagonal quark number susceptibility. To compute this expectation value we start from

$$Z_{1+1}(\mu, \mu_a) = \langle \det(D + \mu\gamma_0 + m) \det(D + \mu_a\gamma_0 + m)\rangle. \quad (29)$$

where $\langle \ldots \rangle$ is the quenched average. The average of the off-diagonal susceptibility is then given by

$$\langle \chi \rangle_{1+1} = \frac{1}{Z_{1+1}(\mu, \mu)} \lim_{\mu_a\to\mu} \frac{d}{d\mu} \frac{d}{d\mu_a} Z(\mu, \mu_a). \quad (30)$$

To one-loop order in chiral perturbation theory we find

$$\frac{Z_{1+1}(\mu_a, \mu_b)}{Z_{1+1}(\mu, \mu)} = e^{G_0(\mu_a,\mu_b) - G_0(\mu=0)}. \quad (31)$$

The one-loop contribution $G_0(\mu_a, \mu_b)$ to the free energy from a charged pion pair made out of quarks with chemical potentials μ_a and μ_b only depends on the absolute value of the difference $\mu_a - \mu_b$. Moreover, since $\lim_{\mu_a\to\mu} d/d\mu G_0(\mu, \mu_a) = 0$ we immediately get

$$\langle \chi \rangle_{1+1} = \lim_{\mu_a\to\mu} \frac{d}{d\mu} \frac{d}{d\mu_a} \Delta G_0(\mu, \mu_a). \quad (32)$$

A. The distribution

To compute the contribution of configurations with a specific phase to the off-diagonal susceptibility we first compute the moments $\langle \chi e^{2ip\theta'}\rangle_{1+1}$. We start from

$$Z_{1+1+p|p^*}(\mu_a, \mu_b, \mu|\mu) = \left\langle \frac{\det^p(D + \mu\gamma_0 + m)}{\det^p(D - \mu\gamma_0 + m)} \times \det(D + \mu_a\gamma_0 + m) \times \det(D + \mu_b\gamma_0 + m) \right\rangle, \quad (33)$$

and evaluate the limit

$$\langle \chi e^{2ip\theta'}\rangle_{1+1} = \frac{1}{Z_{1+1}(\mu, \mu)} \lim_{\mu_a,\mu_b\to\mu} \frac{d}{d\mu_a} \times \frac{d}{d\mu_b} Z_{1+1+p|p^*}(\mu_a, \mu_b, \mu|\mu). \quad (34)$$

For the fermionic Goldstone modes we have an additional minus sign leading to

$$\frac{Z_{1+1+p|p^*}(\mu_a, \mu_b, \mu|\mu)}{Z_{1+1}(\mu, \mu)} = e^{-p\Delta G_0(\mu_a,-\mu) - p\Delta G_0(\mu_b,-\mu) - p^2\Delta G_0(\mu,-\mu) + p\Delta G_0(\mu_a,\mu) + p\Delta G_0(\mu_b,\mu) + \Delta G_0(\mu_a,\mu_b)}. \quad (35)$$

Keeping track of p we find

$$\langle \chi e^{2ip\theta'}\rangle_{1+1} = \lim_{\mu_a\to\mu} \left[p^2 \left(\frac{d}{d\mu} \Delta G_0(\mu_a, -\mu)\right)^2 + \frac{d}{d\mu} \frac{d}{d\mu_a} \times \Delta G_0(\mu_a, \mu) \right] e^{-p(2+p)\Delta G_0(\mu,-\mu)}. \quad (36)$$

For a noncompact phase angle $\theta \in [-\infty, \infty]$ we obtain a δ-function in the left-hand side (lhs) after integrating over p. Proceeding in the same way as for the distribution function of the phase we find

$$\langle \chi \delta(\theta - \theta')\rangle_{1+1} = \left(\left[\left(1 + i\frac{\theta}{\Delta G_0}\right)^2 + \frac{1}{2\Delta G_0} \right] \times \left[\frac{d}{d\mu} \Delta G_0(\mu_a, -\mu) \right]^2_{\mu_a=\mu} + \frac{d}{d\mu} \frac{d}{d\mu_a} \Delta G_0(\mu_a, \mu)_{\mu_a=\mu} \right) \times \frac{e^{2i\theta}}{\sqrt{\pi \Delta G_0}} e^{-\theta^2/\Delta G_0 + \Delta G_0}. \quad (37)$$

The first term between round brackets results from the term $\sim p^2$ in Eq. (36) which, before summing over p can be simply rewritten as second derivative with respect to θ,

$$\left[\frac{d}{d\mu}\Delta G_0(\mu_a, -\mu)\right]^2_{\mu_a=\mu}\frac{1}{(2i)^2}\frac{d^2}{d\theta^2} \times \frac{e^{2i\theta}}{\sqrt{\pi\Delta G_0}}e^{-\theta^2/\Delta G_0+\Delta G_0}. \tag{38}$$

and vanishes upon integration over θ. The θ dependence of the second term is the same as for the θ-distribution which is normalized to 1. Upon integration over the angle θ we thus recover the expectation value of the susceptibility (32). Again we emphasize that for $\Delta G_0 \gg 1$ contributions from the extreme tail may give important contributions to the off-diagonal quark number susceptibility.

VI. THE CHIRAL CONDENSATE ($\mu < m_\pi/2$)

In this section we compute the chiral condensate when the phase angle of the fermion determinant is constrained to θ. This quantity is defined as

$$\langle\bar{\psi}\psi\delta(\theta-\theta')\rangle. \tag{39}$$

Since the chiral condensate is nonzero for $T = 0$ and $\mu = 0$ this derivation requires also the divergent part of the free energy. The required generating functional has different masses

$$\frac{\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)}\det(D+\mu\gamma_0+\tilde{m})^2\rangle}{\langle\det(D+\mu\gamma_0+m)^2\rangle}. \tag{40}$$

The desired expectation value $\langle e^{2ip\theta'}\bar{\psi}\psi\rangle_{1+1}$ is obtained by taking the derivative with respect to $\tilde{m}$ and subsequently the limit $\tilde{m}\to m$. The combinatorics is much like for the baryon number, but here we have to keep track of both mass derivatives and the chemical potentials. This leads to

$$\begin{aligned}\langle\bar{\psi}\psi e^{2ip\theta'}\rangle_{1+1} &= \frac{1}{2}\lim_{\tilde{m}\to m}\frac{d}{d\tilde{m}} \times \frac{\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)}\det^2(D+\mu\gamma_0+\tilde{m})\rangle}{\langle\det^2(D+\mu\gamma_0+m)\rangle}\\ &= \Big(\langle\bar{\psi}\psi\rangle^0_{1+1} + \frac{d}{d\tilde{m}}[-p(G_0(\mu,-\mu,\tilde{m},m) - G_0(\mu,\mu,\tilde{m},m)) + (G_0(\mu,\mu,\tilde{m},\tilde{m}) - G_0(\mu,\mu,m,m))]_{\tilde{m}=m}\Big)e^{-p(2+p)\Delta G_0}.\end{aligned} \tag{41}$$

Where $\langle\bar{\psi}\psi\rangle^0_{1+1}$ is the one-loop renormalized chiral condensate at zero temperature and zero chemical potential. For $p = 0$ we obtain the one-loop renormalized chiral condensate at nonzero temperature and nonzero chemical potential

$$\langle\bar{\psi}\psi\rangle_{1+1} = \langle\bar{\psi}\psi\rangle^0_{1+1} + \frac{d}{d\tilde{m}}[(G_0(\mu,\mu,\tilde{m},\tilde{m}) - G_0(\mu,\mu,m,m))]_{\tilde{m}=m}, \tag{42}$$

which is independent of the chemical potential.

The distribution of the chiral condensate over the phase θ is obtained after multiplication by $\exp(-ip\theta)$ and integrating over p

$$\begin{aligned}\langle\bar{\psi}\psi\delta(\theta-\theta')\rangle_{1+1} = \Big(&\langle\bar{\psi}\psi\rangle^0_{1+1} + \lim_{\tilde{m}\to m}\frac{d}{d\tilde{m}}\Big[\Big(1+i\frac{\theta}{\Delta G_0}\Big) \times (G_0(\mu,-\mu,\tilde{m},m) - G_0(\mu,\mu,\tilde{m},m)) + (G_0(\mu,\mu,\tilde{m},\tilde{m}) - G_0(\mu,\mu,m,m))\Big]\Big)\\ &\times\frac{e^{2i\theta}}{\sqrt{\pi\Delta G_0}}e^{-\theta^2/\Delta G_0+\Delta G_0}.\end{aligned} \tag{43}$$

The factor $1 + i\theta/\Delta G_0$ can again be written as a total derivative of the exponential factors. Upon integration the contribution from this term vanishes, and we recover the full condensate (42)

$$\langle\bar{\psi}\psi\rangle_{1+1} = \int_{-\infty}^{\infty}d\theta\langle\bar{\psi}\psi\delta(\theta-\theta')\rangle_{1+1}. \tag{44}$$

Again important tail contributions arise for $\Delta G_0 \gg 1$.

VII. THE θ-DISTRIBUTION FOR AN ENSEMBLE GENERATED AT $\mu = 0$

In the method of Ejiri [33] one evaluates the θ-distribution as a function of the chemical potential for an ensemble generated at zero chemical potential. Here we compute this partially quenched θ-distribution within one-loop chiral perturbation theory.

We start out evaluating the moments of the phase factor for an ensemble generated at zero chemical potential

$$\frac{1}{Z_{1+1}(\mu=0)}\Big\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)}\det^2(D+m)\Big\rangle = e^{-p^2\Delta G_0(\mu,-\mu)}. \tag{45}$$

We then obtain the distribution (for $\mu < m_\pi/2$)

$$\langle\delta(\theta-\theta')\rangle_{\mu=0} = \frac{1}{\sqrt{\pi\Delta G_0}}e^{-\theta^2/\Delta G_0}. \tag{46}$$

This one-loop prediction is identical to that for the quenched and phase-quenched ensemble: Whether we compute the width of the Gaussian for the θ-distribution in the full ensemble generated at μ, or the full ensemble generated at $\mu = 0$, or in the quenched ensemble, or the phase-quenched ensemble, we find exactly the same result.

Ejiri also has studied distributions of $F = |\det(D + \mu\gamma_0 + m)|/\det(D + m)$. His assumption is that the θ distribution remains Gaussian even for a fixed value of F. As we shall see below, this assumption is justified for $\mu < m_\pi/2$ to one-loop order in chiral perturbation theory.

VIII. THE ϵ-REGIME

The above analysis suggests that the chemical potential has to be of the order of $1/\sqrt{V}$ to suppress the correlation between the phase and the chiral condensate or baryon density. Such a scaling corresponds to the ϵ-regime [44,45] where the dimensionless quantities

$$\hat{m} \equiv m\Sigma V \quad \text{and} \quad \hat{\mu}^2 \equiv \mu^2 F^2 V, \tag{47}$$

are kept fixed for $V \to \infty$. Here and below Σ and F are the chiral condensate and the pion decay constant as they appear in the chiral Lagrangian. Note that it is possible to go smoothly between the ϵ- and p-regime see [46].

In the ϵ-regime the moments of the phase factor remain finite for $V \to \infty$ [22,23]

$$\langle e^{2ip\theta'} \rangle_{N_f} = (1 - 2\hat{\mu}^2/\hat{m})^{p(p+N_f)}, \tag{48}$$

where we quote the result valid for $\hat{m}$, $\hat{\mu} \gg 1$ and $2\hat{\mu}^2 < \hat{m}$. To obtain the distribution of the phase in the ϵ-regime let us rewrite this as

$$\langle e^{2ip\theta'} \rangle_{N_f} = e^{-p(p+N_f)\Delta\hat{G}_0}, \tag{49}$$

with

$$\Delta\hat{G}_0 = -\log(1 - 2\hat{\mu}^2/\hat{m}). \tag{50}$$

It is of exactly the same form as Eq. (18). So we again find the distribution (22) but now with $\Delta\hat{G}_0$ instead of ΔG_0

$$\langle \delta(\theta - \theta') \rangle_{1+1} = \frac{e^{2i\theta}}{\sqrt{\pi\Delta\hat{G}_0}} e^{-\theta^2/\Delta\hat{G}_0 + \Delta\hat{G}_0}. \tag{51}$$

The variance of the Gaussian envelope starts out at zero for small μ (i.e. for $2\hat{\mu}^2 \ll \hat{m}$) and approaches infinity as $\log(1 - x^2)$ for $x \to 1$.

IX. THE θ-DISTRIBUTION FOR $\mu > m_\pi/2$

We now turn to the distribution of the phase of the fermion determinant when the quark mass is inside the support of the Dirac operator. For low T this means that $\mu > m_\pi/2$.

Because the Dirac operator is only determined up to an operator with determinant equal to unity,

$$\det[D + \mu\gamma_0 + m] = \det[A(D + \mu\gamma_0 + m)] \quad \text{with} \quad \det A = 1, \tag{52}$$

the sum of the phases of individual eigenvalues of the Dirac operator may differ by multiples or 2π depending on the choice of A. As will be illustrated below the occurrence of jumps by 2π will be qualitatively different whether or not the quark mass is inside the support of the Dirac spectrum. We now consider a family of matrices A that depend continuously on a parameter ϕ such that for $\phi = 0$ the matrix A is equal to the identity and for $\phi = 1$ the matrix A is far from the identity. The spectral domain is then deformed continuously as a function of ϕ, and as long a no eigenvalues cross the negative real axis, the sum of the phases of the individual eigenvalues of the Dirac operator is defined uniquely. When the quark mass is outside the Dirac spectrum ($\mu < m_\pi/2$) the cloud of eigenvalues will not enclose the negative real axis for a finite range of ϕ. This is not the case when the quark mass is inside the cloud of eigenvalues and the sum of the individual phases is only defined up to a multiple of 2π.

As example we consider

$$A = \cos\phi + i\gamma_0 \sin\phi \tag{53}$$

and a Dirac operator D with matrix elements given by the chiral random matrix model [44]. In Figs. 3 and 4 we show the spectrum of the operator $A(D + m + \mu\gamma_0)$ as a function of ϕ for an ensemble of 4 1600×1600 matrices and parameters as indicated in the caption of the figures. In Fig. 3 the quark mass is just outside the spectral support ($\mu < m_\pi/2$), and in Fig. 4 the quark mass is just inside the spectral support ($\mu > m_\pi/2$). In the upper right corner of each figure we give the overall phase of the fermion determinant for each of the 4 configurations. The origin is denoted by a red dot. We observe that in Fig. 3 the overall phase does not change until the cloud of eigenvalues crosses the negative real axis which happens close to $\phi = 1$. In Fig. 4, on the other hand, the phase jumps by multiples of 2π for all values of ϕ. We also have considered other choices for A such as a an random unimodular complex matrix and similar behavior has been found.

For $\mu < m_\pi/2$ it therefore makes sense to extend the total phase to $\langle -\infty, \infty \rangle$. For $\mu > m_\pi/2$, the origin is inside the cloud of eigenvalues and the phase of the determinant can differ by multiples of 2π for any choice of A. Therefore, when $\mu > m_\pi/2$, it only makes sense to define the phase modulo 2π.

As before, the δ-function, $\delta(\theta - \theta')$, will be obtained from the moments of the phase factor which now are dominated by the leading order term in the chiral expansion. Not surprisingly, this leads to a much wider θ-distribution. What is perhaps somewhat surprising is that, as will be shown below, the distribution now takes a Lorentzian shape. Because of ambiguities in the phase angle we do not expect that we can use the replica trick to calculate half-integer moments of the phase factor. Therefore we will only evaluate the even moments, $\langle \exp(2ip\theta') \rangle$, with integer values of p. This is sufficient to obtain the full distribution of the total phase angle, $2\theta \in [-\pi, \pi]$, of $\det(D + \mu\gamma_0 + m)^2$ relevant for the two flavor theory

$$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \sum_{p=-\infty}^{\infty} e^{-2ip\theta} \langle e^{2ip\theta'} \rangle. \tag{54}$$

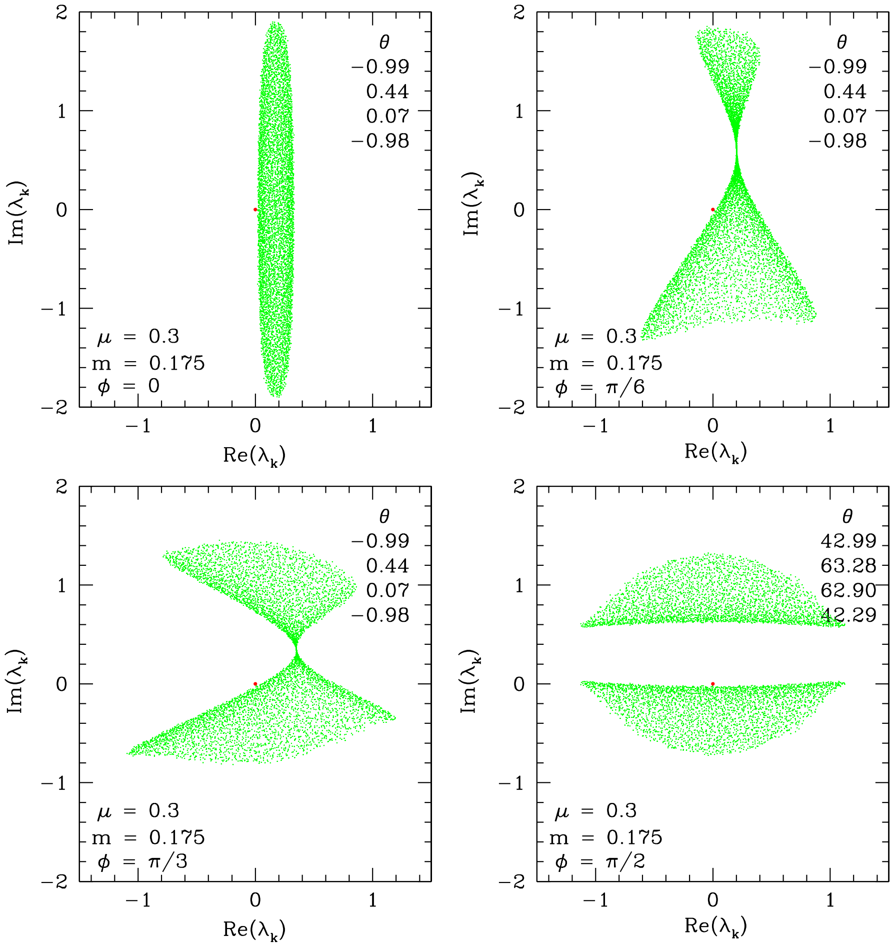

FIG. 3 (color online). Scatter plot of the spectrum of the Dirac operator that interpolates between a basis where the mass matrix is diagonal and a basis where $\mu\gamma_0$ is diagonal. Results are shown for an ensemble of 4 1600×1600 matrices with mass, chemical potential and interpolation parameter as shown in the caption of the figure. The origin is indicated by a red dot. The sum of the phases of the eigenvalues of the fermion determinant denoted by θ and is also shown in the figure (the values correspond to each of the 4 configurations).

Alternatively this can be seen as the combination $\frac{1}{2}[\langle\delta(\theta - \theta')\rangle + \langle\delta(\theta - \theta' + \pi)\rangle]$ of the distribution of the phase angle, θ, of $\det(D + \mu\gamma_0 + m)$.

A. Bosonic partition function

The moments of the phase factor involve inverse powers of determinants, c.f. Eq. (17). As was realized when investigating the partition function with one bosonic flavor such inverse determinants lead to a phase transition at $\mu = m_\pi/2$. In order to compute the moments of the phase factor for $\mu > m_\pi/2$ to leading order in chiral perturbation theory we therefore first recall the explanation of the exact results for the bosonic partition function (obtained by integration over the Goldstone manifold [47] or from the Cauchy transform of the fermionic partition function [48–50]) in terms of a mean field argument.

The observation of [48] is that the bosonic partition function

$$\left\langle\frac{1}{\det(D + \mu\gamma_0 + m)}\right\rangle = \left\langle\frac{\det(D - \mu\gamma_0 + m)}{\det(D + \mu\gamma_0 + m)(D - \mu\gamma_0 + m)}\right\rangle \tag{55}$$

at a mean field level behaves like

$$\frac{\langle\det(D - \mu\gamma_0 + m)\rangle}{\langle\det(D + \mu\gamma_0 + m)(D - \mu\gamma_0 + m)\rangle}. \tag{56}$$

The reason is loosely speaking that the inverse determinant must be regularized in order to be convergent and that Grassmannian mean field terms are absent.

The denominator of Eq. (56) is the phase-quenched theory which has a phase transition at $\mu = m_\pi/2$. The mean field result for the phase-quenched theory is given by

$$\langle\det(D + \mu\gamma_0 + m)(D - \mu\gamma_0 + m)\rangle = e^{-VL_I}, \tag{57}$$

where [51,52]

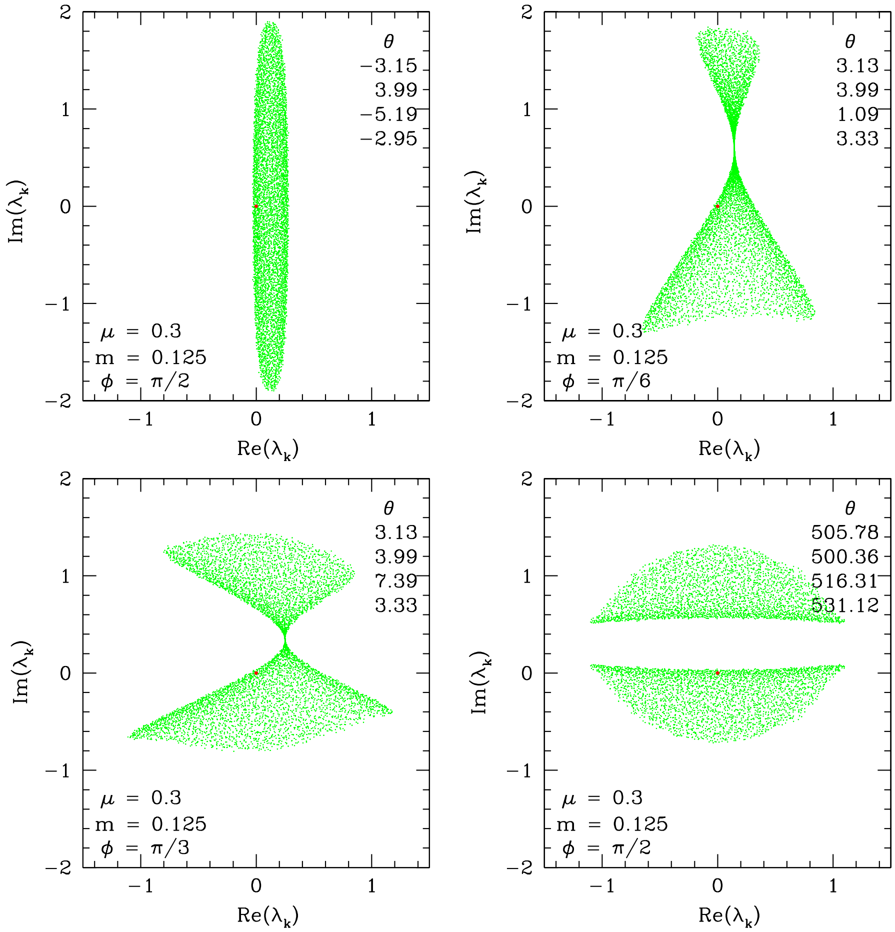

FIG. 4 (color online). Scatter plot of the spectrum of the Dirac operator that interpolates between a basis where the mass matrix is diagonal and a basis where $\mu\gamma_0$ is diagonal. The mass is such that the origin is inside the cloud of eigenvalues but otherwise the parameters are the same as in Fig. 3. The overall phase changes by multiples of 2π as ϕ varies. For further explanation, see the caption of Fig. 3.

$$L_I = -2\mu^2F^2 - \frac{\Sigma^2 m^2}{2\mu^2 F^2} \tag{58}$$

is the static Lagrangian for $\mu > m_\pi/2$. The average of the determinant in the numerator of Eq. (56) is the familiar one flavor partition function (which is independent of μ in chiral perturbation theory)

$$\langle \det(D - \mu\gamma_0 + m)\rangle = e^{-VL_0/2}, \tag{59}$$

where

$$L_0 = -2m\Sigma \tag{60}$$

is the mean field Lagrangian at $\mu = 0$. In conclusion, the mean field result for the bosonic partition function is given by

$$\left\langle \frac{1}{\det(D + \mu\gamma_0 + m)} \right\rangle = e^{-VL_0/2 + VL_I}. \tag{61}$$

As shown in detail in [47,48] this gives the correct mean field physics. Note the striking difference with the fermionic partition function (59) which is independent of the chemical potential.

B. The quenched θ-distribution

Let us now use what we learned from the bosonic case to compute the quenched distribution of the phase of the fermion determinant for $\mu > m_\pi/2$: We will first show that

$$\langle e^{2ip\theta'}\rangle = e^{-VL_B|p|}, \tag{62}$$

where $L_B = L_0 - L_I$ with L_0 and L_I given above (note that $L_B \geq 0$).

Since by charge conjugation symmetry $\langle \exp(2ip\theta')\rangle = \langle \exp(-2ip\theta')\rangle$ this expectation value only depends on the absolute value of p, and we only need to consider $p > 0$. First, we rewrite the moments as

$$\langle e^{2ip\theta'}\rangle = \left\langle \frac{(\det(D + \mu\gamma_0 + m)\det(D + \mu\gamma_0 + m))^p}{(\det(D + \mu\gamma_0 + m)\det(D - \mu\gamma_0 + m))^p} \right\rangle. \tag{63}$$

Now the contribution from the denominator is the inverse of the replicated phase-quenched theory. This was worked out in [37]

$$\frac{1}{\langle(\det(D+\mu\gamma_0+m)\det(D-\mu\gamma_0+m))^p\rangle}=e^{pVL_I}. \tag{64}$$

The contribution from the numerator is just

$$\langle(\det(D+\mu\gamma_0+m)\det(D+\mu\gamma_0+m))^p\rangle=e^{-pVL_0}, \tag{65}$$

which together with the previous result reproduces Eq. (62). The sum over p results in

$$\begin{aligned}\langle\delta(2\theta-2\theta')\rangle&=\frac{1}{\pi}\sum_{p=-\infty}^{\infty}e^{-2i\theta p}e^{-VL_B|p|}\\&=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{2VL_B}{(VL_B)^2+(2\theta+2\pi n)^2}.\end{aligned} \tag{66}$$

The sum over n can be evaluated as

$$\langle\delta(2\theta-2\theta')\rangle=\frac{1}{\pi}\frac{\sinh(VL_B)}{\cosh(VL_B)-\cos(2\theta)}. \tag{67}$$

This is a compactified Lorentzian, centered at zero. We recall that $2\theta\in[-\pi,\pi]$ is the phase of $\det(D+\mu\gamma_0+m)^2$.

C. The unquenched θ-distribution

To calculate the unquenched θ-distribution function we again consider the moments $\langle\exp(2ip\theta')\rangle_{N_f}$. They can be rewritten as

$$\begin{aligned}\langle e^{2ip\theta'}\rangle_{N_f}=&\frac{1}{Z_{N_f}}\Big\langle\frac{\det^u(D+\mu\gamma_0+m)}{\det^{*u}(D+\mu\gamma_0+m)}\\&\times(\det^*(D+\mu\gamma_0+m)\det(D+\mu\gamma_0+m))^{N_f/2}\Big\rangle,\end{aligned} \tag{68}$$

where we have introduced $u=p+N_f/2$. By charge conjugation, this expectation values does not depend on the sign of $u=p+N_f/2$, i.e. it only depends on $|u|$, and it only has to be calculated for $p\geq -N_f/2$. We separately consider the cases $p>0$ and $0\geq p\geq -N_f/2$.

For $p>0$ there are inverse powers of $\det^*$, and we apply the rules of Sec. IX A

$$\begin{aligned}&\frac{1}{Z_{N_f}}\langle e^{2ip\theta'}\det^{N_f}(D+\mu\gamma_0+m)\rangle\\&=\frac{1}{Z_{N_f}}\Big\langle\frac{\det^{2p+N_f}(D+\mu\gamma_0+m)}{(\det(D+\mu\gamma_0+m)\det(D-\mu\gamma_0+m))^p}\Big\rangle\\&\simeq\frac{1}{Z_{N_f}}\frac{\langle\det^{2p+N_f}(D+\mu\gamma_0+m)\rangle}{\langle(\det(D+\mu\gamma_0+m)\det(D-\mu\gamma_0+m))^p\rangle},\end{aligned} \tag{69}$$

where the final equality holds at the mean field level. The contribution from the denominator follows again from the result of the replicated fermionic theory $\langle(\det\det^*)^p\rangle$, see Eq. (65). The numerator is equal to $\exp(-(p+N_f/2)L_0)$ and the normalization, $1/Z_{N_f}$, gives $\exp(N_f/2L_0)$. Therefore the N_f dependence cancels, and we find the quenched result for $p>0$

$$\langle e^{2ip\theta'}\rangle_{N_f}=e^{-pVL_B},\qquad p\geq 0. \tag{70}$$

Here, we extended the equality to $p=0$ which is satisfied trivially.

Now, let us look at negative values of p. This means that $\det^*$ is in the numerator and det is in the denominator. For $-N_f/2\leq p\leq 0$ the moments can be rewritten as

$$\begin{aligned}&\frac{1}{Z_{N_f}}\Big\langle\frac{\det^p(D+\mu\gamma_0+m)}{\det^{*p}(D+\mu\gamma_0+m)}\det^{N_f}(D+\mu\gamma_0+m)\Big\rangle\\&=\frac{1}{Z_{N_f}}\Big\langle\frac{\det^{*|p|}(D+\mu\gamma_0+m)}{\det^{|p|}(D+\mu\gamma_0+m)}\det^{N_f}(D+\mu\gamma_0+m)\Big\rangle,\\&=\frac{1}{Z_{N_f}}\langle(\det(D+\mu\gamma_0+m)\det^*(D+\mu\gamma_0+m))^{|p|}\\&\quad\times\det^{N_f-2|p|}(D+\mu\gamma_0+m)\rangle.\end{aligned} \tag{71}$$

Note that both exponents are positive. The $|p|$ pairs of conjugate quarks form a pion condensate while, at the mean field level, the $N_f-|p|$ quarks are passive spectators resulting in the average phase factor

$$\begin{aligned}\langle e^{2ip\theta'}\rangle_{N_f}&=e^{VN_f/2L_0-V|p|L_I-V(N_f/2-|p|)L_0},\\&=e^{V|p|(L_0-L_I)}=e^{V|p|L_B}\\&-N_f/2\leq p\leq 0.\end{aligned} \tag{72}$$

Note that it smoothly connects to the $p\geq 0$ result (70).

Combining the above results we find

$$\langle e^{2ip\theta'}\rangle_{N_f}=e^{-VL_B(|p+N_f/2|-N_f/2)} \tag{73}$$

for any integer values of p.

The general result (73) implies that the θ-distribution is a Lorentzian times the phase factor:

$$\begin{aligned}\langle\delta(2\theta-2\theta')\rangle_{1+1}&=\frac{1}{\pi}\sum_{p=-\infty}^{\infty}e^{-2i\theta p}e^{-VL_B(|p+1|-1)}\\&=e^{2i\theta}\frac{e^{VL_B}}{\pi}\frac{\sinh(VL_B)}{\cosh(VL_B)-\cos(2\theta)}.\end{aligned} \tag{74}$$

As we have seen previously, the unquenched 1 + 1 distribution is related to the phase-quenched $1+1^*$ distribution

$$\langle\delta(2\theta-2\theta')\rangle_{1+1}=e^{2i\theta}\frac{Z_{1+1^*}}{Z_{1+1}}\langle\delta(2\theta-2\theta')\rangle_{1+1^*}. \tag{75}$$

Comparing this with Eq. (67) and (74) we see that the quenched and phase-quenched θ-distributions are identical also for $\mu > m_\pi/2$.

In conclusion, we have shown that the θ-distribution is nonanalytic at the point where the quark mass enters the support of the Dirac spectrum. This implies, for example, that the distribution of the phase in this regime cannot be obtained by analytic continuation from imaginary values of μ (see [53,54] for a discussion of the analytic continuation of the phase factor to imaginary values of the chemical potential).

X. THE DISTRIBUTION OF THE BARYON NUMBER AND THE CHIRAL CONDENSATE ($\mu > m_\pi/2$)

In this section we compute the distribution of the baryon number and the chiral condensate over the phase angle. As for the distribution of the angle itself we will work to leading order in chiral perturbation theory which is the mean field result for $\mu > m_\pi/2$.

A. The baryon number

In order to work out $\langle n_B \delta(2\theta - 2\theta')\rangle_{N_f}$ we need the moments

$$\frac{1}{Z_{N_f}}\left\langle \frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)} \det(D+\tilde{\mu}\gamma_0+m)^{N_f}\right\rangle, \quad (76)$$

where the chemical potential for the N_f quarks is denoted by $\tilde{\mu}$. The distribution is then obtained after differentiation w.r.t. $\tilde{\mu}$ at $\tilde{\mu} = \mu$, multiplication by $\exp(-2ip\theta)$ and summation over p.

For $p \geq 0$ the bosonic mean field rules discussed in previous section lead to a factorization of the moments as follows

$$\frac{1}{Z_{N_f}} \frac{\langle \det^{2p}(D+\mu\gamma_0+m)\det(D+\tilde{\mu}\gamma_0+m)^{N_f}\rangle}{\langle \det^p(D-\mu\gamma_0+m)\det^p(D+\mu\gamma_0+m)\rangle}. \quad (77)$$

Since $|\tilde{\mu}-\mu| < m_\pi$ there is no condensation of pions for the partition function in the numerator. It follows that there is no dependence on $\tilde{\mu}$ and hence all terms with $p \geq 0$ vanish after differentiation w.r.t. $\tilde{\mu}$.

When p is negative the $\det^{|p|}(D-\mu\gamma_0+m)$ is in the numerator and condensation of pions occurs. This leads to a dependence on $\tilde{\mu}$ through the mean field Lagrangian

$$L_I(-\mu,\tilde{\mu}) = -2F^2(\mu+\tilde{\mu})^2/4 - \frac{2\Sigma^2 m^2}{(\mu+\tilde{\mu})^2F^2}. \quad (78)$$

Note that this reduces to L_I given in (58) for $\tilde{\mu} = \mu$.

As in the previous section we must consider separately the cases $-N_f/2 \leq p < 0$ and $p < -N_f/2$. For $-N_f/2 \leq p < 0$ the moments are given by

$$\frac{1}{Z_{N_f}}\langle e^{2pi\theta'(\mu)}\det^{N_f}(D+\tilde{\mu}\gamma_0+m)\rangle = e^{-2|p|VL_I(-\mu,\tilde{\mu})+|p|VL_I(-\mu,\mu)+|p|VL_0} \quad (79)$$

at mean field level. While for $p < -N_f/2$ we find

$$\frac{1}{Z_{N_f}}\langle e^{2pi\theta'(\mu)}\det^{N_f}(D+\tilde{\mu}\gamma_0+m)\rangle = e^{-N_fVL_I(-\mu,\tilde{\mu})+|p|VL_I(-\mu,\mu)-(|p|-N_f)VL_0}. \quad (80)$$

In both cases the derivative w.r.t. $\tilde{\mu}$ pulls down the prefactor $V[d/d\tilde{\mu}]L_I(-\mu,\tilde{\mu})$ but multiplied with a different numerical factor. This leads to

$$\langle n_B\delta(2\theta-2\theta')\rangle_{N_f} = \frac{1}{\pi}\left[\frac{d}{d\tilde{\mu}}L_I(-\mu,\tilde{\mu})\right]_{\tilde{\mu}=\mu} \times\left(\sum_{-N_f/2\leq p<0}(-2)|p|e^{-2ip\theta}e^{-VL_B(|p+N_f/2|-N_f/2)} + \sum_{p<-N_f/2}(-N_f)e^{-2ip\theta}e^{-VL_B(|p+N_f/2|-N_f/2)}\right). \quad (81)$$

For $N_f = 2$ there is only one term in the first sum, namely $p = -1$, and it can be included in the second sum

$$\langle n_B\delta(2\theta-2\theta')\rangle_{1+1} = \frac{1}{\pi}\left[\frac{VL_I}{\mu}\right] \times(-2)\sum_{p\leq -1}e^{-2ip\theta}e^{-VL_B(|p+1|-1)}. \quad (82)$$

The sum can be performed analytically,

$$\langle n_B\delta(2\theta-2\theta')\rangle_{1+1} = -\frac{2}{\pi}\left[\frac{VL_I}{\mu}\right]e^{2i\theta}e^{2VL_B}\frac{1}{e^{VL_B}-e^{2i\theta}} = -\frac{2}{\pi}\left[\frac{VL_I}{\mu}\right]e^{2VL_B}\frac{-1}{2i} \times\frac{d}{d\theta}\log(e^{VL_B}-e^{2i\theta}). \quad (83)$$

The total baryon number density is given by the integral over the distribution (recall that $2\theta \in [-\pi, \pi]$) and vanishes. The distribution of the baryon number over the phase angle is proportional to a total derivative but not of the distribution of the phase as was the case for $\mu < m_\pi/2$.

B. The chiral condensate

As in Sec. VI we now denote the mass of the N_f quarks by $\tilde{m}$ and differentiate with respect to this mass. The computation is somewhat analogous to the one given in the previous section except that the terms with positive p also contribute. For $N_f = 2$ we find

$$\langle \bar{\psi}\psi\delta(2\theta - 2\theta')\rangle_{1+1} = \frac{2\Sigma V}{\pi}\sum_{p=0}^{\infty} e^{-2ip\theta}e^{-VL_B(|p+1|-1)} - \frac{2}{\pi}\left[\frac{d}{d\tilde{m}}L_I(m,\tilde{m})\right]_{\tilde{m}=m} \times \sum_{p=-\infty}^{-1} e^{-2ip\theta}e^{-VL_B(|p+1|-1)} \quad (84)$$

where

$$L_I(m,\tilde{m}) = -2\mu^2F^2 - \frac{\Sigma^2(m+\tilde{m})^2}{8\mu^2F^2}. \quad (85)$$

The sums can be rewritten as

$$\langle \bar{\psi}\psi\delta(2\theta - 2\theta')\rangle_{1+1} = \frac{2\Sigma V}{\pi}\frac{e^{-VL_B}}{e^{2i\theta} - e^{-VL_B}} + \frac{2\Sigma V}{\pi} + \frac{2}{\pi}\frac{\Sigma^2 m}{\mu^2F^2}e^{2i\theta}e^{VL_B}\frac{e^{VL_B}}{e^{VL_B} - e^{2i\theta}}. \quad (86)$$

The first and the last term both vanish upon integration over $\theta \in [-\pi/2, \pi/2]$ and leaves, $2V\Sigma$, which, after dividing by the volume, is the expected mean field value of the chiral condensate for $N_f = 2$. Note that the amplitude of the first term is exponentially small while that of the last term is exponentially big. The severe cancellations which take place upon integration of the last term over θ are just like those for the baryon number.

XI. QCD IN ONE EUCLIDEAN DIMENSION

In this section we will show that for one-dimensional QCD the distribution of the phase of the fermion determinant changes from Gaussian to Lorentzian shape when the quark mass enters the Dirac spectrum.

Lattice QCD in one Euclidean dimension (time only) with gauge group $U(N_c)$ is sufficiently simple that we can solve the partition function and moments of the phase factor analytically starting from the fundamental partition function. The reason is twofold. First, there is no Yang-Mills action, and second, the staggered Dirac operator, M, can be reduced to the determinant a $N_c \times N_c$ matrix [55]

$$\det M = 2^{-nN_c}\det[e^{n\mu_c} + e^{-n\mu_c} + e^{n\mu}U + e^{-n\mu}U^\dagger], \quad (87)$$

where $U \in U(N_c)$. The analogue of $m_\pi/2$ or m_N/N_c is $\mu_c = \sinh^{-1}m$ and n is the number of lattice points. The eigenvalues of M are located on an ellipse of width $\sinh\mu$ along the real axis. This means that the quark mass is inside the eigenvalue domain when $\mu > \mu_c$. In the limit $nN_c \to \infty$ the ratio of the full partition function and the phase-quenched partition function approaches one for the $SU(N_c)$ theory whereas this ratio the $U(N_c)$ partition functions shows a phase transition when the quark mass enters the Dirac spectrum exactly as in QCD [56]. For this reason we study the $U(N_c)$ lattice model rather than the $SU(N_c)$ lattice model. In adddtion, the $U(N_c)$ model is mathematically simpler than the $SU(N_c)$ lattice model. The partition function is defined by

$$Z_{N_f}(\mu_c, \mu) = \int_{U(N_c)} dU \det M. \quad (88)$$

Despite its simplicity many interesting things can be learned from QCD in one dimension. For example, in [56] it was found that spectral density of the Dirac operator is a highly oscillatory function when the quark mass is inside the ellipse of eigenvalues, and that the link between these oscillations and the chiral condensate is exactly the same as what was found for 4d QCD with dynamical quarks [27].

Below we will show that the distribution of the phase of the fermion determinant in one-dimensional QCD also undergoes a transition from a Gaussian to a Lorentzian shape when the quark mass enters the eigenvalue spectrum. For simplicity we only work out the quenched distribution. As above we start from the moments of the phase factor

$$\langle e^{2ip\theta'}\rangle = \int_{U(N_c)} dU \frac{\det^p M}{\det^p M^\dagger}. \quad (89)$$

Notice that the expectation value only depends on $|p|$. Following [56], where the first moment ($p = 1$) was worked out, we rewrite the U-integral as

$$\langle e^{2ip\theta'}\rangle = \int_{U(N_c)} dU \frac{\det^p(1 - Ue^{n\mu - n\mu_c})\det^p(1 - U^\dagger e^{-n\mu - n\mu_c})}{\det^p(1 - Ue^{-n\mu - n\mu_c})\det^p(1 - U^\dagger e^{n\mu - n\mu_c})}. \quad (90)$$

In this form the Conrey-Farmer-Zirnbauer formula [57] can be applied directly for $\mu < \mu_c$. In the large N_c limit the result simplifies to

$$\langle e^{2ip\theta'}\rangle = \langle e^{2i\theta'}\rangle^{p^2} = \left(1 - \frac{\mu^2}{\mu_c^2}\right)^{p^2}. \quad (91)$$

If μ_c is interpreted as the chemical potential for which the quark mass enters the eigenvalue domain, this is exactly the same form as we obtained for the ϵ-regime of QCD in Sec. VIII. Hence we find the expected Gaussian form for the distribution of the phase of the fermion determinant,

$$\langle \delta(\theta - \theta')\rangle = \frac{1}{\sqrt{\pi\Omega}}e^{-\theta^2/\Omega} \quad \text{for } \mu < \mu_c, \qquad N_c \to \infty, \quad (92)$$

where $\Omega \equiv -\log(1 - \mu^2/\mu_c^2)$.

For $\mu > \mu_c$ the conditions for applying the Conrey-Farmer-Zirnbauer formula directly are violated. Now, however, we instead can rewrite the determinants containing $U^\dagger$ as

$$\frac{\det^p(1 - U^\dagger e^{-n\mu - n\mu_c})}{\det^p(1 - U^\dagger e^{n\mu - n\mu_c})} = e^{-2pnN_c\mu} \frac{\det^p(1 - Ue^{n\mu + n\mu_c})}{\det^p(1 - Ue^{-n\mu + n\mu_c})}, \tag{93}$$

so that the entire integrand in Eq. (90) only depends on U. This implies that when we expand the denominator in U (which is allowed for $\mu > \mu_c$) only the constant term is nonzero upon integration over U. Using that the moments of the phase factor only depend on $|p|$ we obtain the exact result

$$\langle e^{2ip\theta'} \rangle = e^{-2n|p|N_c\mu}, \quad \text{for } \mu > \mu_c. \tag{94}$$

Summing over p in Eq. (54) results in the compact Lorentzian c.f. Eq. (66)

$$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \frac{\sinh(2nN_c\mu)}{\cosh(2nN_c\mu) - \cos(2\theta)} \quad \text{for } \mu > \mu_c, \qquad 2\theta \in [-\pi, \pi]. \tag{95}$$

We stress that this exact result is valid for any value of N_c.

Note that we have computed the distribution of the phase angle of the square of the fermion determinant, i.e. of 2θ. The reason is that this does not require the use of the replica trick. By comparing the numerical result for half-integer moments with the analytical result (94) obtained for integer moments one finds that the replica trick does not work when quark mass is inside the eigenvalues. See Fig. 5. The only exception is the case $\mu_c = 0$: Then the rewriting in Eq. (93) results in $2p$ powers of the determinants [58] which are then well-defined for half-integer p. The expression for the odd moment when $\mu_c = 0$ is therefore also given by (94).

XII. DISTRIBUTION OF $f = \log|\det(D + \mu\gamma_0 + m)|/\det(D + m)$ FOR $\mu < m_\pi/2$

So far we have considered distributions of the phase of the fermion determinant. As we now show it is also possible to compute the distributions as a function of the absolute value of the fermion determinant. We will do this to one-loop order in chiral perturbation theory using the replica trick. Since only the case $\mu < m_\pi/2$ will be considered there are no issues with the use of the replica trick.

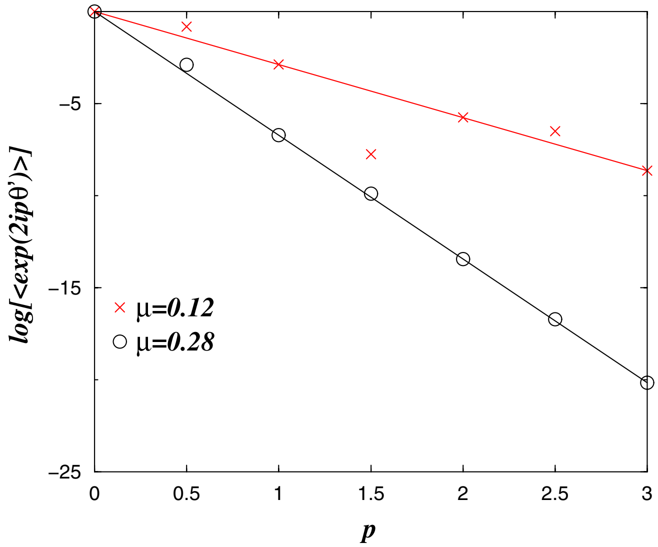

FIG. 5 (color online). Numerical evaluation of the quenched moments of the average phase factor in one-dimensional QCD versus p for $\mu_c = 0.1$, $n = 4$ and $N_c = 3$. As indicated by the lines the even moments join smoothly in accordance with (94). However, the even and odd moments are not smoothly connected for $0 < \mu_c < \mu$. Since the replica trick works when $\mu_c = 0$ the odd moments at the largest value of μ fall closer to the line than those at the smaller values of μ.

Since the absolute value of the fermion determinant depends on the large eigenvalues of the Dirac operator we analyze the distribution of $f \equiv \log[|\det(D + \mu\gamma_0 + m)|/\det(D + m)]$ which, as we shall see below, depends only on the finite difference of the one-loop free energy at μ and at $\mu = 0$. In [33] the distribution of $F \equiv \exp(f)$ was studied in lattice QCD using the Taylor expansion method. Since determinants fluctuate by many orders of magnitude we feel that it is more appropriate to analyze the distribution of the logarithm of their magnitude instead. The two distributions are related by a simple transformation

$$\langle \delta(f - f') \rangle = F\langle \delta(F - F') \rangle, \tag{96}$$

where f' is the magnitude of the logarithm of the ratio of the determinants—its fluctuations are induced by the gauge field fluctuations.

To compute the distribution of the magnitude of the logarithm of the determinants we rewrite the δ-function as

$$\langle \delta(f - f') \rangle_{1+1} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle e^{-ip(f-f')} \rangle_{1+1} = \frac{1}{Z_{1+1}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ipf} \left\langle \left(\frac{\det(D + \mu\gamma_0 + m)\det(D - \mu\gamma_0 + m)}{\det^2(D + m)} \right)^{ip/2} \det^2(D + \mu\gamma_0 + m) \right\rangle. \tag{97}$$

For even ip the average can be interpreted as a partition function with bosonic and fermionic flavors. We will calculate this partition function to one-loop order in chiral perturbation theory. Since we consider the magnitude of the determinant we expect that the moments will be analytic in p and can be analytically continued to imaginary ip As far as we know this is the first case where the replica trick is used this way.

Using the same one-loop combinatorics as before, we find after analytical continuation to imaginary ip,

$$\langle\delta(f-f')\rangle_{1+1} = \int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-ip(f-E_f)-(1/2)\sigma_f^2p^2}, \quad (98)$$

where

$$E_f = 2\Delta G_0(\mu) - 4\Delta G_0(\mu/2), \quad (99)$$

$$\sigma_f^2 = \frac{1}{2}\Delta G_0(\mu) - 2\Delta G_0(\mu/2). \quad (100)$$

The integral over p is Gaussian and can be evaluated by completing squares. This results in

$$\langle\delta(f-f')\rangle_{1+1} = \frac{1}{\sigma_f\sqrt{2\pi}}e^{-(f-E_f)^2/(2\sigma_f^2)}. \quad (101)$$

Both E_f and σ_f^2 are positive. In the thermodynamic limit at nonzero T and μ we can see this using Eq. (12)

$$\sigma_f^2 = \frac{Vm_\pi^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\left[\cosh\left(\frac{2\mu n}{T}\right) - 4\cosh\left(\frac{\mu n}{T}\right)+3\right],$$
$$= \frac{Vm_\pi^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}8\sinh^4\left(\frac{\mu n}{2T}\right). \quad (102)$$

Similarly we can write E_f as

$$E_f = \frac{Vm_\pi^2T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\left[\cosh\left(\frac{2\mu n}{T}\right) - 2\cosh\left(\frac{\mu n}{T}\right)+1\right],$$
$$= 2\frac{Vm_\pi^2T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\cosh\left(\frac{\mu n}{T}\right)\left(\cosh\left(\frac{\mu n}{T}\right)-1\right). \quad (103)$$

Exactly the same combinatorics can be applied to the finite L expressions for σ_f^2 and E_f (see (11)) resulting in the positivity of these quantities at finite L and L_0.

Let us make a simple cross check of the formula for E_f and σ_f^2. Since $f=0$ for $\mu=0$ the expectation value and the variance of the f-distribution must vanish in the limit $\mu\to 0$ which is indeed the case (see Fig. 6).

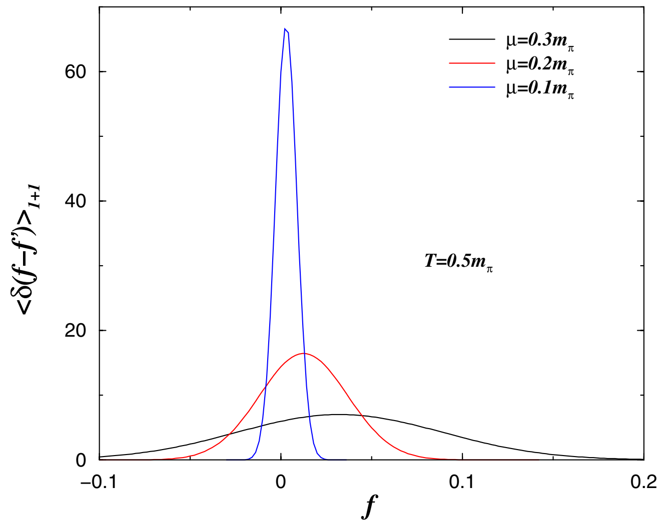

FIG. 6 (color online). *The f-distribution in 1-loop chiral perturbation theory.* The distribution of the partition function with $f=\log(|\det(D+\mu\gamma_0+m)|/\det(D+m))$ in a box with $Vm_\pi^4=10$. The temperature is fixed and as μ increases the distribution becomes broader and moves away from zero.

In order to better understand the structure of the result it is useful to work out the combinatorics for an arbitrary number of flavors N_f

$$\langle\delta(f-f')\rangle_{N_f} = \frac{1}{\sigma_f\sqrt{2\pi}}e^{-(f-N_fE_f/2)^2/(2\sigma_f^2)}. \quad (104)$$

We note that, an increasing number of flavors simply shifts the average value of f.

A. The distribution of the baryon number over f

Even though the baryon number is zero when evaluated in chiral perturbation theory it does not necessarily vanish when evaluated for a constrained fermion determinant. In Sec. IV we derived the distribution of the baryon number for fixed phase. Here we compute the distribution of the baryon number as a function of f.

In order to compute $\langle n_B\delta(f-f')\rangle_{1+1}$ we denote the chemical potential in the usual two flavor determinant by $\tilde\mu$ instead of μ, then differentiate with respect to $\tilde\mu$ and finally take the limit $\tilde\mu\to\mu$. The δ function is represented as in the previous sections

$$\langle n_B\delta(f-f')\rangle_{1+1} = \frac{1}{2Z_{1+1}}\lim_{\tilde\mu\to\mu}\frac{d}{d\tilde\mu}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-ipf}\left\langle\left(\frac{\det(D+\mu\gamma_0+m)\det(D-\mu\gamma_0+m)}{\det^2(D+m)}\right)^{ip/2}\det^2(D+\tilde\mu\gamma_0+m)\right\rangle. \quad (105)$$

To one-loop order in chiral perturbation theory this becomes

$$\langle n_B\delta(f-f')\rangle_{1+1} = \left[\frac{d}{d\tilde\mu}(G_0(-\mu,\tilde\mu)-2G_0(0,\tilde\mu))\right]_{\tilde\mu=\mu}\int_{-\infty}^{\infty}\frac{dp}{2\pi}ipe^{-ip(f-E_f)-(1/2)\sigma_f^2p^2}. \quad (106)$$

The Gaussian integral over p results in

$$\langle n_B\delta(f-f')\rangle_{1+1} = -\left[\frac{d}{d\tilde{\mu}}(G_0(-\mu,\tilde{\mu}) - 2G_0(0,\tilde{\mu}))\right]_{\tilde{\mu}=\mu} \times \frac{E_f - f}{\sigma_f^2}\langle\delta(f-f')\rangle_{1+1}. \quad (107)$$

The baryon number operator is not positive definite and neither is its distribution over f. It changes sign at the expectation value of the Gaussian distribution so that the total baryon density vanishes

$$\langle n_B\rangle_{1+1} = \int_{-\infty}^{\infty} df \langle n_B\delta(f-f')\rangle_{1+1} = 0. \quad (108)$$

As is the case for the distribution of the baryon number over θ the zero value can also be obtained by noting that the integrand is a total derivative.

B. The distribution of the chiral condensate over f

In this section we derive the distribution of the chiral condensate over f. As above we represent $\delta(f-f')$ by an integral over the moments so that

$$\langle\bar{\psi}\psi\delta(f-f')\rangle_{1+1} = \frac{1}{Z_{1+1}}\lim_{\tilde{m}\to m}\frac{d}{d\tilde{m}}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-ipf}\left\langle\left(\frac{\det(D+\mu\gamma_0+m)\det(D-\mu\gamma_0+m)}{\det^2(D+m)}\right)^{ip/2}\det^2(D+\mu\gamma_0+\tilde{m})\right\rangle. \quad (109)$$

The combinatorics of possible one-loop contributions of Goldstone bosons leads to

$$\begin{aligned}\langle\bar{\psi}\psi\delta(f-f')\rangle_{1+1} &= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\bigg[ip\frac{d}{d\tilde{m}}(G_0(\mu,\mu,m,\tilde{m}) + G_0(-\mu,\mu,m,\tilde{m}) - 2G_0(0,\mu,m,\tilde{m}))\\ &\quad + 4\frac{d}{d\tilde{m}}(G_0(0,\tilde{m}) - G_0(0,m))\bigg]_{\tilde{m}=m} e^{-(1/2)\sigma_f^2p^2 + ip(E_f-f)},\\ &= \langle\bar{\psi}\psi\rangle_{1+1}\langle\delta(f-f')\rangle_{1+1} - \frac{d}{d\tilde{m}}[G_0(\mu,\mu,m,\tilde{m}) + G_0(-\mu,\mu,m,\tilde{m}) - 2G_0(0,\mu,m,\tilde{m})]_{\tilde{m}=m}\\ &\quad\times\frac{E_f-f}{\sigma_f^2}\langle\delta(f-f')\rangle_{1+1}.\end{aligned} \quad (110)$$

The first term in the last line gives the chiral condensate upon integration over f while the second term in the final line integrates to zero in precisely the same way as in the case of the baryon density.

C. The f-distribution evaluated in an ensemble generated at $\mu = 0$

In [33] the distribution of $F = |\det(D+\mu\gamma_0+m)|/\det(D+m)$ is studied in lattice QCD for an ensemble generated at $\mu = 0$. We will again study the distribution of $f \equiv \log F$ for this case. It is given by

$$\frac{1}{Z_{1+1}(\mu=0)}\langle\delta(f-f')\det^2(D+m)\rangle = \frac{1}{Z_{1+1}(\mu=0)}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-ipf} \times\left\langle\left(\frac{\det(D+\mu\gamma_0+m)\det(D-\mu\gamma_0+m)}{\det^2(D+m)}\right)^{ip/2}\det^2(D+m)\right\rangle. \quad (111)$$

When evaluated to one-loop order in chiral perturbation theory we find

$$\begin{aligned}&\frac{1}{Z_{1+1}}\langle\delta(f-f')\det^2(D+m)\rangle\\ &= \int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-ip(f-\tilde{E}_f)-(1/2)\sigma_f^2p^2}\\ &= \frac{1}{\sigma_f\sqrt{2\pi}}e^{-(f-\tilde{E}_f)^2/(2\sigma_f^2)},\end{aligned} \quad (112)$$

where

$$\tilde{E}_f = 2\Delta G_0(\mu/2). \quad (113)$$

In comparison to (101) we see that only the expectation value of f has changed whereas the variance takes the same value as in previous sections.

XIII. CONSTRAINING BOTH θ AND F FOR ($\mu < m_\pi/2$)

In order to understand what happens if both the phase and the magnitude of the fermion determinant are fixed we need to compute the correlation between the moments of the phase factor and F.

Let us consider the correlation of any moment of the phase factor and F

$$\left\langle \frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)} \frac{\det^q(D+\mu\gamma_0+m)\det^q(D-\mu\gamma_0+m)}{\det^q(D+m)\det^q(D+m)} \right\rangle - \left\langle \frac{\det^p(D+\mu\gamma_0+m)}{\det^p(D-\mu\gamma_0+m)} \right\rangle \times \left\langle \frac{\det^q(D+\mu\gamma_0+m)\det^q(D-\mu\gamma_0+m)}{\det^q(D+m)\det^q(D+m)} \right\rangle = e^{-p^2\Delta G_0 + q^2\Delta G_0 - 4q^2\Delta G_0(\mu/2)} - e^{-p^2\Delta G_0} e^{q^2\Delta G_0 - 4q^2\Delta G_0(\mu/2)} = 0. \tag{114}$$

The reason is that terms linear in p in the first exponent cancel completely. In other words, even though there are bound states (Goldstone bosons) with non zero charge which potentially can couple the phase factor to the absolute value of the determinant, their contributions exactly cancel each other. This is also the case if the average is calculated for N_f dynamical flavors.

We have thus shown there are no correlations between the absolute value of the fermion determinant and the phase to one-loop order in chiral perturbation theory (for $\mu < m_\pi/2$). Hence we automatically find

$$\langle \delta(f-f')\delta(\theta-\theta')\rangle_{1+1} = \langle \delta(f-f')\rangle_{1+1}\langle \delta(\theta-\theta')\rangle_{1+1} \tag{115}$$

and

$$\langle n_B \delta(f-f')\delta(\theta-\theta')\rangle_{1+1} = \langle n_B\delta(f-f')\rangle_{1+1}\langle\delta(\theta-\theta')\rangle_{1+1} + \langle\delta(f-f')\rangle_{1+1}\langle n_B\delta(\theta-\theta')\rangle_{1+1}. \tag{116}$$

One can convince oneself that this factorization does not hold for $\mu > m_\pi/2$.

XIV. CONCLUSIONS

The distribution of the phase of the fermion determinant for QCD with nonzero quark chemical potential has been computed to leading order in chiral perturbation theory. When the quark mass is outside the support of the Dirac spectrum (small μ) the distribution becomes Gaussian whereas the distribution is Lorentzian (modulo 2π) when the quark mass is inside the support. This nonanalytic behavior is also found for QCD in one Euclidean dimension by a direct evaluation of the involved partition functions.

The distribution of the baryon number and the chiral condensate as a function over the phase angle has also been computed in chiral perturbation theory. The results show analytically that extreme cancellations are essential for the vacuum expectation values of these fundamental quantities.

The ratio of the magnitude of the fermion determinant to its value at $\mu = 0$ is ultraviolet finite and can be studied within chiral perturbation theory. We have computed the distribution of the logarithm of this ratio, f, as well as the distribution of the baryon number and the chiral condensate over f. Contrary to the θ-distribution the distribution of f is real and positive. In fact, within one-loop chiral perturbation theory for $\mu < m_\pi/2$ there are no correlations between the phase and the absolute value of the fermion determinant.

The results obtained here are complementary to lattice results obtained by Ejiri [33]. The results for one-loop chiral perturbation theory when the quark mass is outside the eigenvalue distribution of the Dirac operator, confirms the Gaussian shape of the θ-distribution first found in lattice simulations [33]. The analytical results, however, also show that exponentially large cancellations may take place when integrating over θ. Not only are these cancellations essential in order to measure the baryon number and the chiral condensate correctly, the extreme tail of the distribution may contribute significantly to the final result. A small non Gaussian term in the tail of the θ-distribution therefore could be the dominant term after integration over θ. The precise form of this tail is of course difficult to access numerically.

The Lorentzian shape of the distribution of the phase valid for larger values of the chemical potentially shows that one should not take for granted that the conditions for the central limit theorem are satisfied. The nonanalyticity means that the Lorentzian shape cannot be obtained by analytic continuation from imaginary values of the chemical potential. Since the Lorentzian form is present also for quenched QCD this prediction can be tested in lattice QCD without worrying about the sign problem. Numerical convergence is however expected to slow because of the large fluctuations of the phase. If staggered fermions are used one also has to address the issues raised in [59].

Finally let us stress that both the Gaussian and the Lorentzian forms for the θ-distributions found here are leading order predictions of chiral perturbation theory. It would be of considerable interest to work out the next to leading order corrections. It seems natural that these terms will give corrections to both the shape of the θ-distribution and to its width.

The analytical work of this paper was inspired by new developments in the numerical density of states method. Such interplay between numerical lattice QCD and analytical methods, is essential for progress towards our understanding of strongly interacting matter. In this paper this was illustrated by simulations of one-dimensional lattice QCD. Even if the analytical results do not yet offer a direct solution of the sign problem, they allow us to better understand the regions where current numerical methods can be applied [60].

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