Bose-Einstein condensation of strongly interacting bosons: From liquid ^{4}He to QCD monopoles

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Starting from the classic work of Feynman on the λ point of liquid helium, we show that his idea of universal action per particle at the Bose-Einstein condensation (BEC) transition point is much more robust that it was known before. Using a simple "moving string model" for supercurrent and calculating the action, both semiclassically and numerically, we show that the critical action is the same for non-interacting and strongly interacting systems such as liquid ^{4}He. Inversely, one can obtain an accurate dependence of critical temperature on density: one important consequence is that high density (solid) He cannot be a BEC state of He atoms, with upper density accurately matching the observations. We then use this model for the deconfinement phase transition of QCD-like gauge theories, treated as BEC of (color-)magnetic monopoles. We start with a Feynman-like approach without interaction, estimating the monopole mass at T_c. Then we include the monopole's Coulomb repulsion, and formulate a relation between the mass, density and coupling which should be fulfilled at the deconfinement point. We end up proposing various ways to test on the lattice whether it is indeed the BEC point for monopoles.

I. INTRODUCTION

The goals of this paper are twofold. The *first goal* is rather general: to get better qualitative understanding of the parameters controlling the transition between the "normal" matter and its Bose-condensed versions, for strongly interacting bosons which may be in the form of a liquid (fluid-superfluid transition well known for ^{4}He) or solid (solid-supersolid transition yet to be found). Although there are high quality Monte Carlo numerical results for ^{4}He and many other systems, we think the universal condensation criterion is still very much needed, as performing numerical simulations is not trivial in each new setting.

As it will be explained in detail below, we will follow the 50-year-old Feynman theory of Bose condensation [1,2], in which he introduced the notion of the critical value for the jump amplitude y_c or the critical action $y_c = \exp(-S_c)$. When Feynman realized that his simple treatment (evaluation of only the kinetic energy part of the action) needed correction, he simply introduced an "effective mass" thinking that some extra matter is incorporated into exchange motion. We think instead that other particles (except the ones in the exchanged polygon) have very little chance to move. Instead, the jump amplitude should be correctly evaluated, with the interaction term included. We thus revive Feynman's idea, using a simple model of particle motion –the "moving string model"—which can be studied either semiclassically or numerically of the smallest-action paths which particles should follow during their exchanges.

Our *second goal* is very far from atomic systems: it is related to the *deconfinement phase transition in QCD* and related gauge theories. "Dual superconductivity" or Bose-Einstein condensation (BEC) of certain magnetic objects in the vacuum of these theories were proposed to be responsible for confinement by t 'Hooft and Mandelstam [3]. A number of authors [4,5] have used effective dual models to describe the structure of the QCD confining strings (electric flux tubes), as dual to the Abrikosov flux tubes in superconductors. These ideas have been extensively studied in lattice gauge theory calculations, in which a flux tube is induced by two static charges and monopoles are found in the maximal Abelian gauge; see [6,7]. Quite good correspondence between these numerical data and the effective dual models has been found.

As Feynman did in the 1950s, we would however approach the problem from the normal phase above the deconfinement phase transition called "quark gluon plasma" (QGP). At very high T one can view QGP as a plasma made of "electric" quasiparticles, quarks and gluons. However at lower T it becomes a "dual plasma" containing not only electrically charged quasiparticles but also magnetically charged objects—monopoles and dyons. Apart from lattice studies, in which those have been observed in certain gauges above deconfinement (see e.g. the discussion in [8]), there were attempts to understand the magnetic sector of the plasma phase in a framework of "monopole gas" models, e.g. [9,10].

A much more specific dynamical model has been proposed by Liao and Shuryak [11], who used a strongly coupled classical plasma description for the ensemble of monopoles and calculated such transport properties as viscosity and diffusion constant. Furthermore, it was pointed in this work that the magnetic coupling should run opposite to the electric one. As a result, a decrease of the temperature toward the critical value T_c should lead to a gradual shift from electric to magnetic dominance, with a significant increase of the density of (color-magnetic)

monopoles near T_c. Recent lattice studies, in particular, Ref. [12], have discovered many important details about such monopoles. They indeed found rather high densities of them close to T_c, as well as clustering behavior similar to BEC. They also measured equal-time density correlators of like and unlike monopoles (Fig. 5 of [12]) which clearly show the peaks characteristic for strongly coupled ionic liquids. The estimated magnetic Coulomb coupling does indeed show running opposite to electric coupling predicted in [11]: see discussion in a review [13].

With the QCD deconfinement transition interpreted as BEC of monopoles, one should be able to apply Feynman's universal action idea to constrain the properties of these monopoles (mass, density and coupling constant). This is done in the second part of the paper.

II. REVIVING FEYNMAN'S THEORY OF THE HELIUM λ POINT, 55 YEARS LATER

A. Feynman's theory

Feynman's starting point is the thermal partition function in the form

$$Z = e^{-(F/T)} = \frac{1}{N} \sum_P \int \left(\frac{m^* T}{2\pi\hbar}\right)^{3N/2}$$
$$\times \exp\left[-\frac{m^* T}{\hbar^2} \sum_i (\underline{R}_i - P\underline{R}_i)^2\right]$$
$$\times \rho(\underline{R}_1 \ldots \underline{R}_N)d^3\underline{R}_1 \ldots d^3\underline{R}_N, \qquad (1)$$

where m^* is the effective mass of a He atom, the sum is done over permutation P of the particle coordinates, the function ρ includes effects of the interparticle interactions and the exponent contains additional action due to kinetic energy. His main idea is that the function ρ can be inferred from general properties of the liquid, its quasiordered local structure with peaked distribution over interparticle distances at some nearest-neighbor value d (in the case of a cubic lattice d is the lattice spacing). Then the relative magnitude of a term with a permutation of n atoms would be proportional to the nth power of the "jump amplitude" which is approximated as

$$y_F = \exp\left[-\frac{m^* T d^2}{2\hbar^2}\right] \qquad (2)$$

with some combinatorial prefactors, describing a number corresponding to noncrossing polygons in the $3d$ lattice, such as that shown in Fig. 1. The divergence of this sum at the parameter approaching some critical value $y_F \to y_c$ should indicate the presence of an "infinite cluster," the signature of Bose condensation. The condition for this, Feynman argues, is $y_c \approx 1/s$, where s is the growth factor in the number of polygons when n is increased by one unit. Feynman mentioned that he expected $y_c = 1/4 - 1/3$.

The combinatoric problem to find the correct critical value of the Feynman parameter was studied in detail by
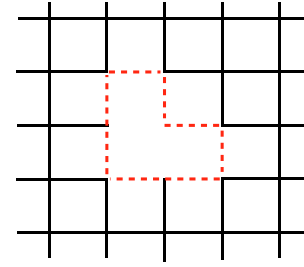

FIG. 1 (color online). Example of a polygon due to particle exchange.

Kikuchi *et al.* [14]. They found a critical action $S_c \approx 1.9$. We can consider the distance d of a "jump" fixed to the position of the nearest-neighbor maximum in the static correlation function $g(r)$ for liquid He, which is $d \approx 3.5$ Å. If this value is used and the critical action is taken to be the one obtained by Kikuchi, in order to recover the correct position of the λ point $T_c = 2.17$, the helium mass is changed to an effective value $m^* = 1.64m_{\mathrm{He}}$. Otherwise, using the physical helium mass we obtain $T_c = 3.57$ which is, of course, 64% bigger than the real critical temperature. In the 1980s Elser [15] determined numerically that the critical action should be smaller than the values obtained in [14], $S_c \approx 1.44$. Using this new parameter the predicted critical temperature reduces to $T_c = 2.72$, closer to the physical temperature.

In the Feynman picture all the effects of the potential are absorbed into an effective mass m^* that appears in the kinetic part of the partition function. In this paper we try to study the He λ point, starting from the idea of Feynman, but including the effects due to the potential at a mean field level. In our calculations we have considered the Aziz potential HFDHE2 [16]. We will show that in this context an estimate for the helium critical temperature can be obtained within an error of 5% if we assume that the critical action will be the same as the free case, which, as we will show, can be estimated to $S_c = 1.655$ for a cubic lattice.

Following Feynman we put the helium atoms filling a cubic lattice and consider two different states for the system: the first where the atoms are static on the lattice sites, and the second where the atoms on a line of the lattice are moving coherently in a given direction, for example, along x; see Fig. 2. If we consider an infinite system, we can consider that this line is a side of a polygon with an infinite number of atoms. Another difference with Feynman's work is that in our case the atoms lying on the moving line are not restricted to move on the lattice link: what they must do is to jump from one site to the following on the same line, but in between they move following the three-dimensional equation of motion imposed by the geometry of the selected configuration.

The appearance of such a kind of particle cluster, as proposed by Feynman, is connected to the transition into
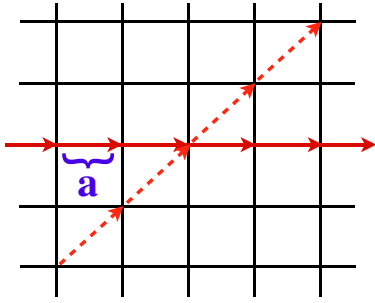
FIG. 2 (color online). $z = 0$ of the three-dimensional lattice: the red continuous arrows identify the moving atoms. The dashed arrows represents the diagonal trajectory of monopoles discussed in Sec. III.

FIG. 3 (color online).   Projection on the $x - y$ plane of the potential for a particle on the moving line.

the superfluid phase. Confirmation of this hypothesis comes also from path integral Monte Carlo (PIMC) calculations, which, today, represent the only way to determine without any approximation the helium critical temperature. In this framework, the transition point is fixed exactly from the appearance of atoms involved in an exchange which winds around the simulation box.

The idea of Feynman was to consider the ratio between the partition function of the system where permutations are possible with respect to the "Boltzmann" case where particles do not permute. In our case we are interested to understand the properties of the particular system where there is exactly one line involved in permutation with respect to the Boltzmann system.

Systems at finite temperature can be studied using path integral techniques in the Euclidean space, where $t \rightarrow -i\tau$.

In general the partition function for bosonic interacting particles in Euclidean space can be written as

$$\mathcal{Z}(x_1, \ldots x_N; \beta) = \frac{1}{\mathcal{N}} \sum_P \int \mathcal{D}x_1(\tau) \ldots \mathcal{D}x_N(\tau)$$

$$\times e^{-\int_0^\beta S_E[x_1, \ldots, x_N; \tau] d\tau}, \qquad (3)$$

where $\beta = \frac{1}{k_B T}$ and the Euclidean action is

$$S_E[x_1, \ldots, x_N; \tau] = \sum_{i=1}^N \left(\frac{m}{2} \dot{x}_1^2(\tau) + \sum_{j=1}^N \frac{V(x_i, x_j)}{2} \right), \qquad (4)$$

and only two-body interactions are taken into account.

Now we consider the case where K particles $(x_k, \ldots x_{k+K})$ on a line move coherently, that is, $x_{k+j}(\tau) = x_k(\tau) + jd$. This hypothesis implies that we have chosen the particular permutation where $x_{i \neq k, \ldots, k+K}(\tau) = x_i$, $x_k(\tau) = x_{k+1}(0), \ldots, x_K(\tau) = x_k(0)$. Moreover we impose that the position of all other particles is constant.

B. The moving string model

Having selected this particular configuration, we can evaluate the potential acting on a particle lying on the line, due to all the other particles.
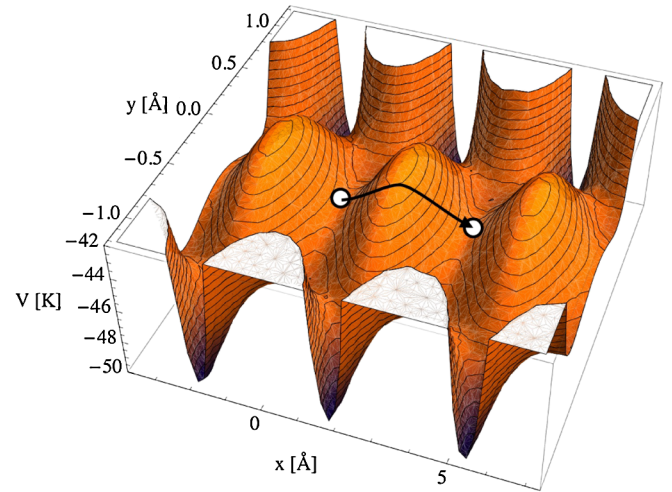
In Fig. 3 we can see the projection on the $x - y$ plane of the resulting mean field potential and one of the possible paths followed by the atoms on the moving line.

In particular, we consider only the x coordinate and study the dependence from the density of the mean field potential. In Fig. 4 we can see how to change the behavior of the potential considering the atoms disposed on a cubic lattice and changing the density. In Fig. 5 we do the same for an hcp crystal. We can see that there is a range of density in both cases where the origin is no longer a minimum, which means that the lattice considered is not the correct one to study the system. Therefore we need to pay attention to which configuration we choose depending on the density.

Another fact related to the Aziz potential is represented in Fig. 6, where we see that the amplitude of the sinusoidal potential does not increase continuously when the density grows. We can think that such a behavior will be connected with a wrong description of the atoms' distribution in this range of densities.
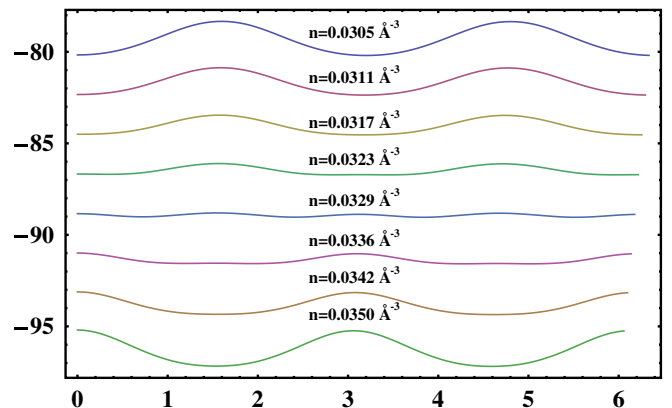

FIG. 4 (color online).   Density dependence of the potential when we consider the atoms on a cubic lattice.
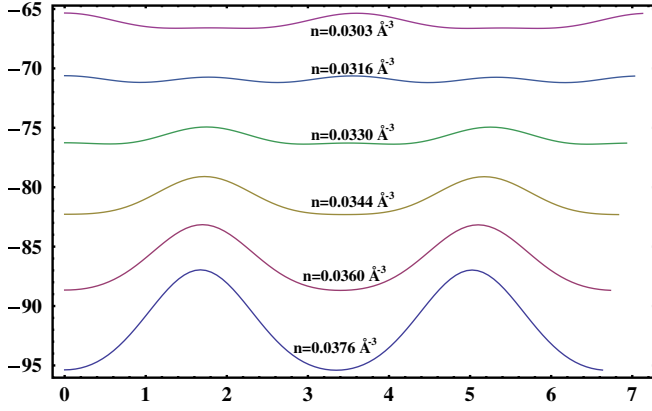
FIG. 5 (color online). Density dependence of the potential when we consider the atoms on a hcp lattice.

FIG. 6 (color online). Density dependence of the amplitude of the sinusoidal potential.

This is indeed peculiar to the Aziz potential where we have both attractive and repulsive terms. If, for example, we consider a simple negative power potential, such a behavior not only disappears but we can find a simple relation between the amplitude V_0 and the density.

Define the negative power (NP) potential

$$V^{\mathrm{NP}} = \frac{\alpha_p}{r^p} \tag{5}$$

with $\alpha_p > 0$. The scattering length can be defined for this potential when $p > 3$ and is given by

$$a = \left(\frac{2m\alpha_p/\hbar^2}{(p-2)^2}\right)^{1/(p-2)} \frac{\Gamma[(p-3)/(p-2)]}{\Gamma[(p-1)/(p-2)]}, \tag{6}$$

where $\Gamma[x]$ is the Gamma function. We consider $p = 1, 4, 6, 9, 12$ and choose to keep fixed the value $a = 1$ for all the potentials, changing accordingly α_p. For $p = 1$ we arbitrarily fixed $\alpha_1 = \alpha_4$. For this potential we compute V_0 for different density, taking care that for such a NP poten-
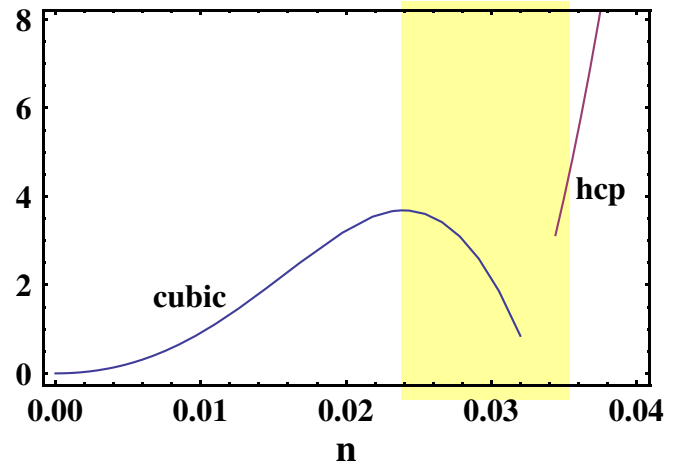
tial the atom disposition must be a bcc crystal. The result are presented in Fig. 7: the data can be fitted with extraordinary accuracy using the function

$$V_0(n) = a n^{p/3}. \tag{7}$$

We can try to use the same function also to fit the sinusoidal amplitude obtained by the Aziz potential; in this case we can use $V_0(n) = a n^{6/3}$, where $p = 6$ is the dominating term for large distances that appear in the Aziz potential. The result is in Fig. 8, we can see that the fit is quite good until, as expected, we enter in the yellow band described before.

Nevertheless we decided to use this simple scaling relation for our calculations.

Because of the particular configuration we have chosen, the partition function for the Bose system can be rewritten as

$$Z(x_1, \ldots x_N; \beta) = \frac{1}{\mathcal{N}} \left[\int \mathcal{D}x_k(\tau) e^{-\int_0^\beta ((m/2)\dot{x}_k^2(\tau) + C_0(x)) d\tau} \right]^{(N-K)} \cdot \left[\int \mathcal{D}x_n(\tau) e^{-\int_0^\beta ((m/2)\dot{x}_n^2(\tau) + V_0(x)) d\tau} \right]^K, \tag{8}$$

where $C_0(x)$ is the potential acting on the particle at rest and which is exactly the potential that also appears in the Boltzmann partition function.

For this reason the ratio between the Bose (Z) and the Boltzmann (Z_B) partition functions reduces to

$$\frac{Z}{Z_B} = \left[\frac{\int \mathcal{D}x_k(\tau) e^{-\int_0^\beta ((m/2)\dot{x}_k^2(\tau) + V_0(x)) d\tau}}{\int \mathcal{D}x_k(\tau) e^{-\int_0^\beta ((m/2)\dot{x}_k^2(\tau) + C_0(x)) d\tau}} \right]^K, \tag{9}$$

and the Feynman parameter is given by

$$y_F = \frac{\int \mathcal{D}x_k(\tau) e^{-\int_0^\beta ((m/2)\dot{x}_k^2(\tau) + V_0(x)) d\tau}}{\int \mathcal{D}x_k(\tau) e^{-\int_0^\beta ((m/2)\dot{x}_k^2(\tau) + C_0(x)) d\tau}}. \tag{10}$$

C. The semiclassical calculation

The simplest case is that of noninteracting particles, for which the optimal path ("Feynman's instanton") is just the straight line starting at the initial position of a particle and ending at the previous position of its neighbor (as indicated by lines with arrows in Fig. 2). Since velocity is constant, the extra action per particle needed for its jump is nothing but just the kinetic energy times the Matsubara time $\beta = \hbar/T$ available for the interchange, with an obvious velocity on the segment being d/β

$$S_{\mathrm{ideal}} = \frac{m}{2}\left(\frac{d}{\beta}\right)^2 \beta \tag{11}$$

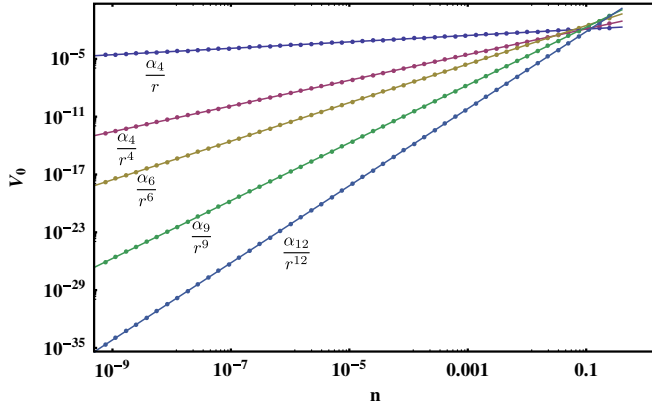per particle, for each of the diagrams. Let us see what is this

FIG. 7 (color online). Density dependence of the NP potential for five different values of p. The point can be successfully fitted by the function $V_0(n) = an^{p/3}$.
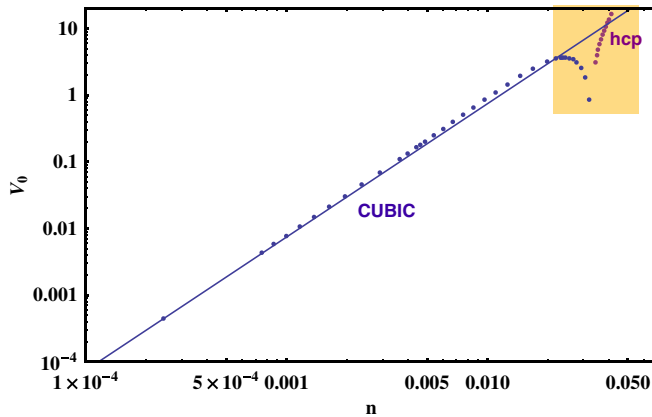

FIG. 8 (color online). Density dependence of the Aziz potential fitted by the function $V_0(n) = an^{6/3}$.

action at the BEC line, for the ideal gas. Using

$$T_c = 3.31\hbar^2 n^{2/3}/m \qquad (12)$$

and relating the distance of the jump to the density by $d = \kappa n^{-1/3}$, with some coefficient $\kappa = O(1)$ one finds that the critical action for the Feynman instanton is

$$S_{\text{BEC}}/\hbar = \frac{3.31\kappa^2}{2} = \kappa^2 1.655. \qquad (13)$$

The precise value depends on the numerical constant κ, which in turn depends how exactly particles are correlated in space. It is unclear what should it be for ideal gas, but for strongly coupled systems we are mostly interested in it is uniquely determined by the type of local crystal structure developing; e.g. it is exactly $\kappa = 1$ for cubic crystal. We will use below this value, although there is some uncertainty here for weakly coupled systems.

When density is too low or T is too high, so that $S > S_{\text{BEC}}$, the sum over the polygons is exponentially convergent and thus the gas remains in normal phase.

Now we switch on the potential and impose our atoms to be displaced on a cubic lattice. As a first analytic approach to the problem we look for the semiclassical tunneling path—known as the periodic instanton, or caloron. It corresponds to a solution of the classical equation of motion in Euclidean time, now including the potential. For all cases we are interested in, this periodic potential on various lattices and forces happens to be well described by its first harmonics, the sinusoidal potential:

$$V(x) = V_0\sin^2(x\pi/d) \qquad (14)$$

which interpolates between the minima of V

$$x_{\text{cl}}(0) = 0, \qquad x_{\text{cl}}(\beta) = d.$$

Introducing conserved Euclidean energy E_E

$$E_E = \frac{m}{2}\dot{x}^2 - V(x) \rightarrow \dot{x} = \sqrt{\frac{2}{m}(V(x) + E_E)} \qquad (15)$$

after separation of variables we have the solution

$$\frac{dx}{\sqrt{\frac{2}{m}(E_E + V(x))}} = d\tau \qquad (16)$$

which can be easily integrated

$$\sqrt{\frac{m}{2}}\int_{x_{\text{cl}}(0)}^{x_{\text{cl}}(\tau)}\frac{dx}{\sqrt{E_E + V(x)}} = \sqrt{\frac{m}{2}}d\frac{\text{F}[x\pi/d, -V_0/E_E]}{\pi\sqrt{E_E}}\Big|_{x_{\text{cl}}(0)}^{x_{\text{cl}}(\tau)}$$

$$= \tau - \tau_0, \qquad (17)$$

where F is the elliptic integral of the first kind. Imposing $x_{\text{cl}}(\tau_0 = 0) = 0$ we have

$$x_{\text{cl}}(\tau) = \frac{d}{\pi}\text{JA}\left[\sqrt{\frac{2E_E}{m}}\frac{\pi}{d}\tau, -\frac{V_0}{E_E}\right] \qquad (18)$$

with JA the Jacobi amplitude for elliptic functions. The particular solution for $E_E = 0$ is called the instanton, and it corresponds to the zero temperature (or $\beta \rightarrow \infty$) limit. In our setting it is not interesting since we only approach the critical point from above and thus $T > T_c$.

For arbitrary temperature the duration of the Matsubara time β is prescribed: thus from Eq. (17) we obtain the equation to fix the energy. One can also read it immediately as the critical temperature as a function of the parameter of our system (d, V_0 and E_E)

$$T_c[d, V_0, E_E] = \left(\sqrt{\frac{m}{2E_E}}\frac{d}{\pi}\text{F}[\pi, -V_0/E_E]\right)^{-1}$$

$$= \left(\sqrt{\frac{m}{2E_E}}\frac{d}{\pi}2\text{K}[-V_0/E_E]\right)^{-1} \qquad (19)$$

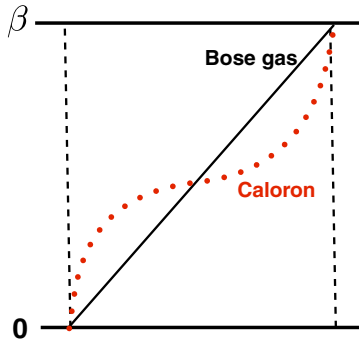provided the energy is substituted from the Feynman condition, which says that the Euclidean action

FIG. 9 (color online). Instanton trajectory we have in the presence of a potential barrier, compared to the free Bose gas case.

$$S_E[d, V_0, E_E] = \int_0^\beta d\tau \left(\frac{m}{2} \dot{x}_{cl}^2 + V(x) \right)$$
$$= \frac{d}{\pi} \sqrt{\frac{mE_E}{2}} 2 \left(2E\left[-\frac{V_0}{E_E} \right] - K\left[-\frac{V_0}{E_E} \right] \right)$$
(20)

is equal to the Feynman critical value.

In the instanton limit $E_E = 0$ we have for the action

$$S_0 = \frac{2d\sqrt{2mV_0}}{\pi},$$
(21)

and the corresponding trajectory is given by

$$x(\tau) = \frac{2}{\pi} d \operatorname{arccot}\left[\exp\left(-\sqrt{\frac{m}{2}} \frac{\pi V_0}{d} \tau \right) \right].$$
(22)

In Fig. 9 we can see the difference between the instanton trajectory and the simple straight line corresponding to the case of a free Bose gas.

More detailed comparison between real He and caloron/instanton approximation is shown in Fig. 10(b). While it is

qualitatively correct, it is not quite accurate. This is by no means surprising: it should be accurate provided the critical action for BEC would happen to be much larger than 1, while unfortunately it is only 1.65 or so. Therefore the semiclassical result should be used for qualitative comparison only.

One of those is that as the matter is further compressed by extra pressure, its density and the amplitude of the potential $V_0(n)$ grow. Respectively the action required for tunneling grows [see Fig. 10(a)], and when the action gets too large $S_0 > S_c$, Feynman's condition could not be fulfilled. Therefore, a sufficiently compressed ^{4}He *cannot* support a supercurrent of the ^{4}He atoms.

(For clarity: we do not make any statements here about possible "supersolid" behavior of the solid ^{4}He induced by supercurrent of some defects/dislocations imposed on it. All we are saying is that the ^{4}He atoms themselves do not create such a supercurrent: this statement is however not new and it has been verified in dedicated numerical Monte Carlo studies. The only new element in our statement is that it follows from Feynman's universal action.)

One may improve the semiclassical expressions for the tunneling action in one- or two-loop quantum corrections. For example, with the one-loop accuracy the Feynman parameter can be written as

$$y_F = \frac{e^{-(d/\pi)\sqrt{(mE_E/2)}2(2E[-(V_0/E_E)] - K[-(V_0/E_E)])}}{(\det \hat{F}[x_{cl}])^{1/2}}.$$
(23)

Expressions for two loops are a bit more involved; see Ref. [17]. However we have not calculated it analytically, using a numerical path integral instead.

D. The 1D quantum path integral

As we have already emphasized above, since the action is not large, one cannot expect the semiclassical theory to
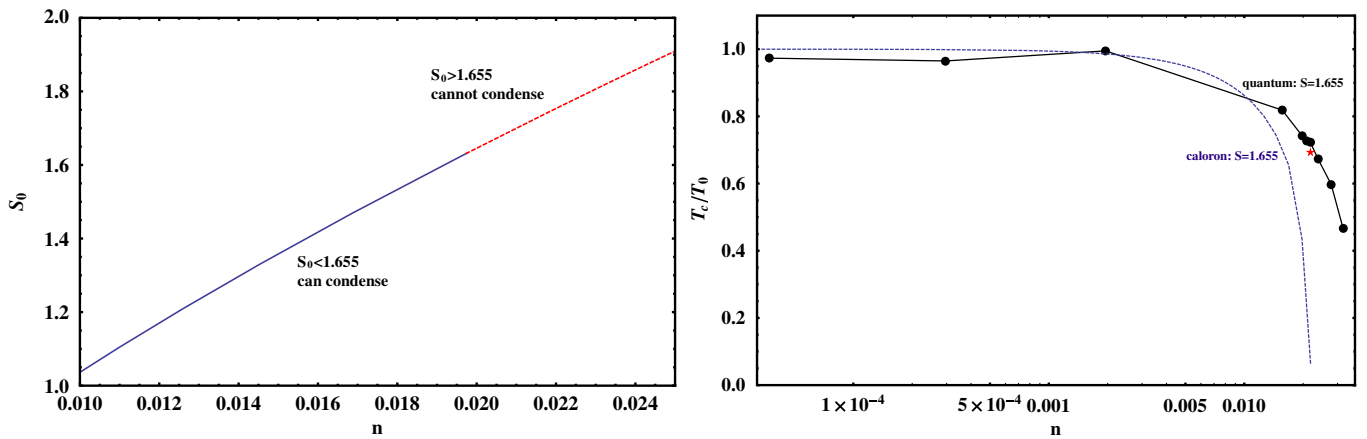

FIG. 10 (color online). (a) The action for the tunneling (caloron) solution as a function of the density. When the action is larger than the critical value $S = 1.655$ (dashed line) the system cannot be a Bose-Einstein condensed phase. (b) The critical temperature obtained from the caloron/instanton solution (dashed line) and from the 1D PIMC simulation (points). The red star shows the physical location of the lambda point.

be really accurate. Fortunately quantum fluctuation around the classical trajectory can be taken into account by numerical evaluation of the right-hand side of (10) that can be simply performed using PIMC code. Note that compared to PIMC simulation of many-body these calculations are very cheap in terms of computational power because we consider a one-dimensional system with only one particle.

In order to find the critical temperature as a function of the density we do the ansatz that the value of the Feynman parameter of our system is more or less the same as for the free Bose gas. Because we know that for the free Bose gas $S_F = -\log(y_F) = 1.655$ independent from the density, we simply computed S for a different temperature at fixed density and when $S(T, n) = 1.655$ we simply called that the critical temperature. In Fig. 11 we plotted the results obtained in the case of the cubic lattice. We can see the good agreement between our calculation and the physical critical point. Also when we use the formula $V_0(n) = an^{6/3}$ for the prediction of the amplitude, our prediction is only 4% larger than the physical value.

It is also interesting to notice that the physical point is the point where the critical temperature is larger; in this sense it looks natural that condensation happens in nature exactly at that point.

Now we plot the ratio T_c/T_0 [Fig. 10(b), black points], where T_0 is the critical temperature for the free Bose gas. In the limit of vanishing density we recover Einstein's critical temperature as it should be. In the region of small density there is a universal theory using small parameter $a^3 n$, with a being the scattering length. As shown e.g. in Ref. [18], there is a nontrivial increase of T_c by several percent there, in agreement with numerical studies. Our model is obviously too crude to reproduce it.

We are mostly interested in the opposite limit of high density: here the considered ratio decreases until a certain point. For larger density we simply cannot have condensa-

FIG. 12 (color online).   Action as a function of the temperature for different densities.

tion in the sense that we cannot find any temperature where $S(T) = 1.655$. This can be seen more clearly by looking at Fig. 12. Starting from around $n \simeq 0.033$ Å^{-3} we cannot find any critical temperature different from zero. As a consequence we have a jump from a minimum of $T_c \simeq 1.7$ K to $T_c = 0$ around that density.

Looking at the ^{4}He phase diagram (Fig. 13) it is also interesting to notice that the λ line starts at $P = 0$, $T = 2.17$ and ends at some pressure with $T = 1.76$ very close to our estimate for the minimal critical temperature. Obviously the density where we observe this behavior is completely different; nevertheless it is interesting to notice that the two minimal critical temperatures coincide. It seems to suggest that it is not possible to have a supersolid phase of helium because, as soon as we enter in the nonliquid phase, the critical temperature jumps from $T_c = 1.76$ to zero.
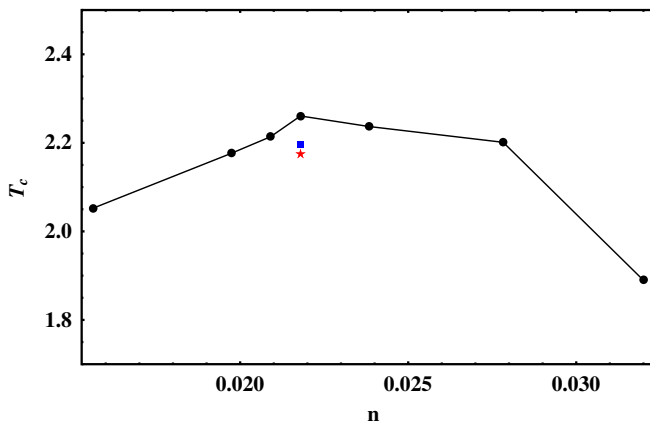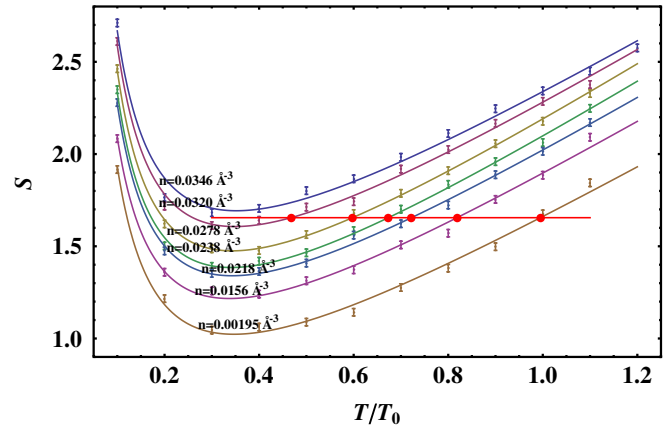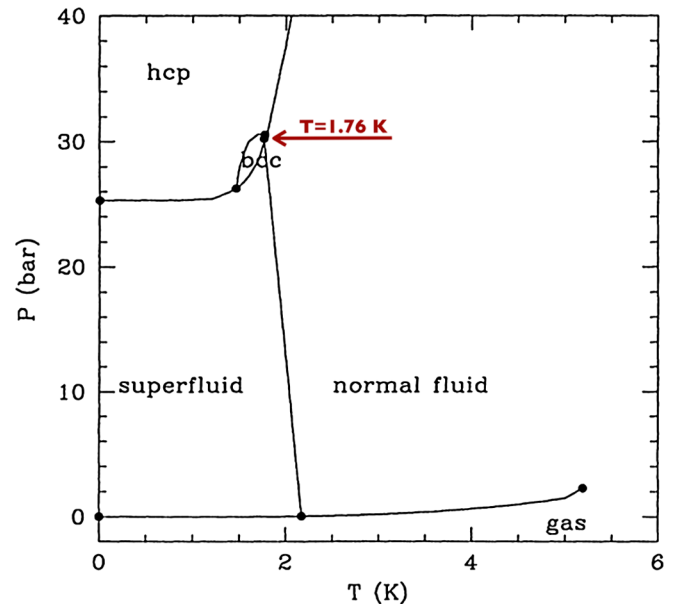

FIG. 11 (color online).   Density dependence of the critical temperature. Black points are obtained using the exponential dependence of $V_0(n)$. The red star is the physical result for ^{4}He. The blue square is obtained using the V_0 as obtained directly from the Aziz potential.

FIG. 13 (color online). ^{4}He phase diagram.

III. CONFINEMENT AS BOSE-EINSTEIN CONDENSATION OF MONOPOLES

First of all, let us explain why we think the Feynman criterion based on particle paths and clusters is especially well suited for this task. One reason is that lattice simulations are able to follow monopole's paths, and thus identify "single" ones, with individually periodic paths, as well as those belonging to k clusters. Therefore, one can see divergence of the cluster expansion at T_c directly, without any calculations.

The second reason is that the excited matter we discuss has no nonzero quantum numbers; it is just an "excited vacuum" produced in high energy collisions. Thus there is not any nonzero conserved charge to which the corresponding chemical potential can be coupled. Neutrality leads to an equal number of electrically charged quarks and antiquarks, as well as an equal number of monopoles and antimonopoles. Thus unfortunately one cannot introduce a chemical potential and use the usual reasoning related to its crossing the lowest level.

Yet, since the density and mass of such monopoles depend on T, only when the Feynman criterion is satisfied may their BEC happen. In this section we estimate what mass and coupling constant the monopole should have in order to condense. We do so in two steps, first for noninteracting monopoles, including relativistic action, and then for interacting ones using nonrelativistic formulas derived above. Again following Feynman, we approach the problem from the high-T phase—the QGP—in which monopoles are not Bose-condensed, calculate the action needed for a jump to the site of the identical neighbor, and compare it to the universal value S_c.

We will only consider pure gauge theories (no quarks) and ignore gluon quasiparticles. The standard units used in this field are based on $\hbar = c = 1$, and thus only one unit (length in fm or energy in GeV, so fm*GeV = 0.1973) needs to be defined. It is standard in lattice works to calculate the string tension σ at zero T and fix its value to be the same as in real QCD, namely, $\sqrt{\sigma} = 0.42$ GeV, which we will follow. The critical temperature for pure SU(3) gauge theory is in such units about 270 MeV.

A. Free monopoles

The work [12] [pure gauge SU(2) theory] has provided measurements for monopole (plus antimonopole) density n at $T = (1-12)T_c$. In this theory, the Higgs mechanism leaves only one massless U(1), and thus there is only one kind of monopole. For orientation, near T_c they find $n/T^3 \approx 0.3$. This distance between identical monopoles is

$$a = (n/2)^{-1/3} = 1/(0.53 * 0.27 \text{ GeV}) \approx 1.4 \text{ fm} \quad (24)$$

(where 2 is because we only need the density of one charge).

We now estimate the action of the monopole which jumps to the position of another identical monopole at distance a away, during Euclidean time duration equal to the critical Matsubara time $\beta = \hbar/T_c = 1/(0.27 \text{ GeV}) = 0.73$ fm. In the Feynman approximation—when only the kinetic part of the action is included—the (Euclidean) velocity on the optimal path is constant $v = a/\beta$. Putting numbers reveals two complications: (i) $v > 1$ and (ii) relativistic monopoles require a relativistic form for Euclidean action. Since we speak about tunneling, having imaginary action and velocity above that of light is in fact appropriate: no negative roots appear. The Euclidean action at the BEC T_c point should be

$$S_E = m \int ds = m\beta\sqrt{1 + a^2/\beta^2} = S_c. \quad (25)$$

Putting numbers into the square root one finds

$$m = T_c * S_c/\sqrt{5} \approx 200 \text{ MeV}. \quad (26)$$

This small value of the mass is highly nontrivial. At large T monopoles are viewed as heavy semiclassical objects, and also in the vacuum $T = 0$ those objects should be massive enough to get comparable to glueball masses, or an order of magnitude heavier. According to the "magnetic scenario" [11] monopoles get light enough and play a significant dynamical role only close to T_c. Still, this small mass—which seems an inevitable consequence of Feynman's universal criterion—is shockingly light.

From existing lattice data on the mass, estimated from the paths themselves by D'Elia [19], one finds a rapid decrease of the monopole mass as $T \to T_c$ to a similar ballpark, but the accuracy of the data is not yet sufficient to tell its numerical value at T_c yet. Before this comparison is made, we should move on into a much more involved estimate for a strongly interacting monopole plasma.

B. Strongly interacting monopoles

Color monopoles interact via Coulomb-like magnetic forces related to their charges. We remind the reader that the Higgs mechanism is assumed to be due to certain distribution (with a nonzero mean) of the so-called Polyakov loop $P = (1/N_c) \text{Tr} P \exp(i \int d\tau A_0)$, described by a certain effective potential. Although the zeroth component of the gauge field A_0 does not have a global nonzero vacuum expectation value, and thus the color group is not broken, locally it has a certain value. It is in respect to such a local field that the Abelian subgroups which remain unbroken are defined, as well as color charges of the monopoles. For the SU(2) color group there is only one diagonal generator τ^3, and the corresponding gluon is left massless. So there is only one U(1) charge and thus we can use the language of electrodynamics, simply calling monopoles to be positive and negative charges. [For physical SU(3) color there are two unbroken Abelian fields, proportional to λ^3, λ^8 Gell-Mann matrices, with two families of monopoles.] One more comment is that in general forces include also Higgs exchange, which in principle can also

be long range if Higgs is massless. However the data about monopole correlations [12] seem to show only pure Coulomb-like forces.

Such charges obviously prefer an alternating cubic lattice. Furthermore, classical molecular dynamics which reproduces their static correlation functions observed on the lattice (see discussion and references in [13]) complement the Coulomb potential by the repulsive core, to ensure classical stability of the lattice. The potential is thus written as

$$V_{ij}(r) = \frac{g_m^2}{4\pi}\left[\frac{(-)^{Q_i+Q_j}}{r} + \frac{1}{br^b}\right], \qquad (27)$$

where charges $Q_i, Q_j = \pm 1$ and additional dimensionless parameter b is the "core power." It is usually selected to be large, to produce small corrections except close to the origin: we use the conventional $b = 9$. The forces are balanced at distance 1, which defines the normalization.

Near T_c the condensate is still a small fraction of all particles, in spite of divergent (or highly peaked) specific heat: thus we may think that only particles of one kind (e.g. only positive ones) are moving along the line. This prompts us to take *one diagonal* of the cubic lattice (rather than a line parallel to axes) to be jumping the distance $\sqrt{2}$, while keeping all other positive and *all* negative ones stationary. The potential corresponding to such forces was calculated and plotted in Fig. 14 (points). As usual, it is well represented by the sinusoidal potential

$$V = \frac{g_m^2}{4\pi d}\left[C + V_0\sin^2\left(\frac{\pi x}{\sqrt{2}d}\right)\right] \qquad (28)$$

and therefore one can use the results obtained for ^{4}He in Sec. II (note that because of the diagonal supercurrent we have substituted $d \rightarrow \sqrt{2}d$ into it). In particular, we will use caloron amplitude (20).

The main equation is that the jump Euclidean action is equal to the Feynman critical value, which for a caloron reads
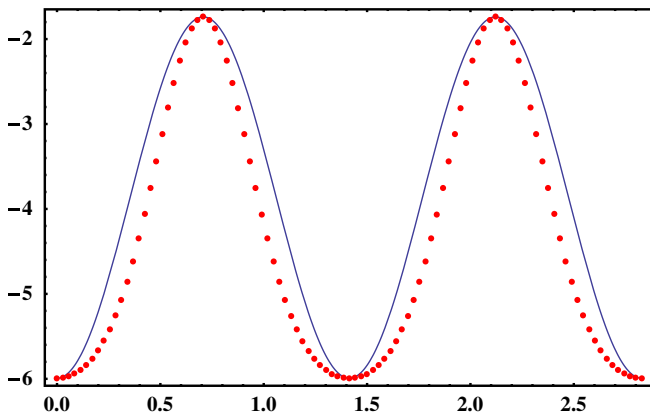
$$S_c = 1.655 = S_E[\sqrt{2}d, V_0, E_E]$$

$$= \frac{\sqrt{2}d}{\pi}\sqrt{\frac{mE_E}{2}}2\left(2E\left[-\frac{V_0}{E_E}\right] - K\left[-\frac{V_0}{E_E}\right]\right). \qquad (29)$$

In Sec. II above we extracted the energy value from it, and then inserting it into the expression for the caloron period we found the BEC critical temperature itself. Now, considering the T_c of QCD known, we can do the inverse: extract energy from the period and put it into (29). After this is done, it becomes one nontrivial condition connecting the values of the monopole mass, coupling and density together. Unfortunately, it is hard to write it down in a simple form.

This formula was derived starting from a nonrelativistic Lagrangian, while we have shown that in this regime monopoles move relativistically. In order to give an estimate of the relativistic correction we can compare the monopole mass predicted using Eq. (25) with the mass obtained from the analogous nonrelativistic formula.

Fixing $T_c = 0.27$ GeV and the jumping distance $\sqrt{2}d = \sqrt{2}n^{-1/3}$ we can extract the monopole mass as a function of the Feynman action. The result is presented in Fig. 15, where we can see that the relativistic correction is very small, on the order of 5%, and therefore we can hope that, also introducing the potential, using the nonrelativistic formula the error in the prediction of the mass remains small.

In order to include the potential we need an estimate of the coupling constant g_m. This estimate has been provided studying lattice correlation function in [12,20] to be such that plasma parameter

$$\Gamma = \frac{g_m^2}{4\pi dT_c} \sim (1-2). \qquad (30)$$

FIG. 14 (color online). The effective potential for a diagonal moving line of charges (points) is compared to a fitted sinusoidal potential (curve).
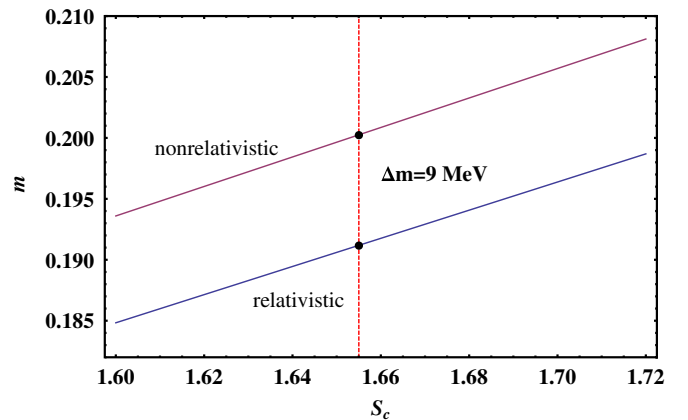

FIG. 15 (color online). Relativistic correction to the prediction of the monopole mass at $T = T_c$ without the inclusion of effects connected with the interparticle Coulomb potential.
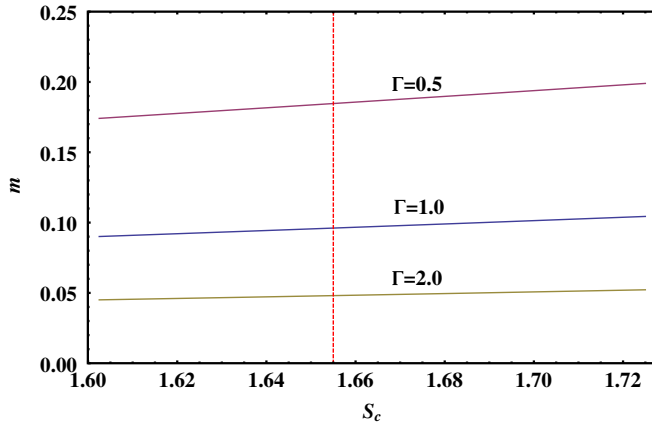
FIG. 16 (color online). Monopole mass prediction at $T = T_c$ using the nonrelativistic formula for the Feynman parameter [Eq. (20)]. The three lines correspond to different strengths of the Coulomb potential.

In Fig. 16 you can see the dependence of the monopole mass from the Feynman parameter for three different choices of Γ: $\Gamma = 0.5, 1.0, 2.0$. Remember that for $\Gamma \sim 1$ the plasma is a liquid (only for $\Gamma \ll 1$ we have a gas and for $\Gamma \sim 100$ a solid), and therefore inside the range considered the state of the system does not change; nevertheless we can see that such a small variation in the plasma parameter is sufficient to produce a large variation in the monopole mass going from 50 MeV for $\Gamma = 2$ until 180 MeV for $\Gamma = 0.5$. On the other hand the dependence on the choice of the Feynman parameter is very small.

In conclusion of this section we could say that in order to study the temperature dependence of the monopole mass it seems to be crucial to fix accurately the strength of the interparticle potential.

C. Discussion

Liquid ^{4}He at the lambda point shows a characteristic infinitely high peak in its specific heat, famously attributed by Feynman to divergence in the sum over the contribution of the k polygons.

The pure gauge theory deconfinement is second order for the SU(2) group; it is the first order transition for SU($N_c > 2$) and so far is designated to be a crossover one in QCD, with dynamical quarks. This means that added quarks somehow tame the peak in the specific heat, to still large but finite height. (It may still be a finite-volume effect.) It is perhaps fair to say that at the moment nobody has a clue why it is so; at least we are not aware of any definite ideas.

The real-world QCD has fundamental quarks, and for those another transition—chiral symmetry restoration—coincides with or is very close to deconfinement. However for different quarks, with adjoint (roughly twice larger) color charge, those are separated and deconfinement is the first order transition.

In this paper we assumed that deconfinement is the BEC of monopoles, and it is crucially important to test on the lattice whether it is indeed true or not. This means in general to look for a monopole condensate at $T < T_c$. We would suggest to follow Feynman and test his predictions: (i) at T_c the probability of polygons of any k becomes comparable; and (ii) the extra action for jumps reaches the same critical value as for all other BECs. We know from simulations for ^{4}He that those were indeed true in this case: why should monopoles be any different?

ACKNOWLEDGMENTS

Note added.—When this paper was completed, we learned about one more interesting application of strongly coupled charged plasma possibly undergoing Bose-Einstein condensation: helium dwarf stars; see [21].

Note added in proof.—Our main result is the prediction of an unexpectedly light monopole mass $m \approx 200$ MeV at T_c, an order of magnitude below the glueball scale at $T = 0$. We have been informed since submitting the manuscript that Chernodub and Ilgenfritz [22] reported a mass of exactly 200 MeV at T_c, in the fit to T dependence of the asymmetry of the so-called A^2 condensate.

[1] R. P. Feynman, Phys. Rev. **90**, 1116 (1953).
[2] R. P. Feynman, Phys. Rev. **91**, 1291 (1953).
[3] S. Mandelstam, Phys. Rep. **23**, 245 (1976); G. 't Hooft, Nucl. Phys. **B190**, 455 (1981).
[4] M. Baker, J. S. Ball, and F. Zachariasen, Phys. Rep. **209**, 73 (1991).
[5] G. Ripka, arXiv:hep-ph/0310102.
[6] V. Singh, D. A. Browne, and R. W. Haymaker, Phys. Lett. B **306**, 115 (1993).
[7] G. S. Bali, C. Schlichter, and K. Schilling, Prog. Theor. Phys. Suppl. **131**, 645 (1998); G. S. Bali, arXiv:hep-ph/9809351.

[8] M. N. Chernodub and V. I. Zakharov, Phys. Rev. Lett. **98**, 082002 (2007).

[9] J. Smit and A. van der Sijs, Nucl. Phys. **B355**, 603 (1991).

[10] P. Giovannangeli and C. P. Korthals Altes, Nucl. Phys. **B608**, 203 (2001).

[11] J. Liao and E. Shuryak, Phys. Rev. C **75**, 054907 (2007).

[12] A. D'Alessandro and M. D'Elia, Nucl. Phys. **B799**, 241 (2008).

[13] E. Shuryak, Prog. Part. Nucl. Phys. **62**, 48 (2009).

[14] R. Kikuchi *et al.*, Phys. Rev. **119**, 1823 (1960).

[15] V. Elser, Ph.D. thesis, U.C. Berkeley (1984).

[16] R. A. Aziz *et al.*, J. Chem. Phys. **70**, 4330 (1979).

[17] C. F. Wohler and E. V. Shuryak, Phys. Lett. B **333**, 467 (1994).

[18] M. Holzmann, G. Baym, J. P. Blaizot, and F. Laloe, Phys. Rev. Lett. **87**, 120403 (2001).

[19] M. D'Elia (private communication).

[20] J. Liao and E. Shuryak, Phys. Rev. Lett. **101**, 162302 (2008).

[21] G. Gabadadze and R. A. Rosen, J. Cosmol. Astropart. Phys. 10 (2008) 030.

[22] M. N. Chernodub and E.-M. Ilgenfritz, Phys. Rev. D **78**, 034036 (2008).