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Isgur-Wise functions and unitary representations of the Lorentz group: The baryon case $j = 0$

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We propose a group theoretical method to study Isgur-Wise (IW) functions. A current matrix element splits into a heavy quark matrix element and an overlap of the initial and final clouds, related to the IW functions, that contain the long distance physics. The light cloud belongs to the Hilbert space of a unitary representation of the Lorentz group. Decomposing into irreducible representations one obtains the IW function as an integral formula, superposition of *irreducible* IW functions with *positive measures*, providing positivity bounds on its derivatives. Our method is equivalent to the sum rule approach, but sheds another light on the physics and summarizes and gives all its possible constraints. We expose the general formalism, thoroughly applying it to the case $j = 0$ for the light cloud, relevant to the semi-leptonic decay $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$. In this case, the principal series of the representations contribute, and also the supplementary series. We recover the bound for the curvature of the $j = 0$ IW function $\xi_\Lambda(w)$ that we did obtain from the sum rule method, and we get new bounds for higher derivatives. We demonstrate also that if the lower bound for the curvature is saturated, then $\xi_\Lambda(w)$ is completely determined, given by an explicit elementary function. We give criteria to decide if any *Ansatz* for the Isgur-Wise function is compatible or not with the sum rules. We apply the method to some simple model forms proposed in the literature. Dealing with a Hilbert space, the sum rules are *convergent*, but this feature does not survive hard gluon radiative corrections.

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I. INTRODUCTION

The heavy quark limit of QCD and, more generally, heavy quark effective theory, has aroused an enormous interest in the decade of the 1990s, starting from the formulation of heavy quark symmetry by Isgur and Wise [1].

Then, the theoretical study of the properties of this limit did slow down, due essentially to the fact that flavor physics had more urgent domains to explore: the determination of $|V_{ub}|$, the study of rare decays like $B \to X_s \gamma$, and the comparison of the many *CP* violation observables with the predictions of the standard model. Presently, the main interest is focused on the search of new physics, in view of the possibilities of the future experimental projects: LHCb, Super-Belle and the Super *B* Factory.

The present paper shows that the richness of the heavy quark limit of QCD had not been explored in the past in all its depth. The method proposed here allows one to obtain important constraints on the Isgur-Wise (IW) functions, that carry the long distance QCD physics in the heavy quark limit. These constraints turn out to have simple and explicit phenomenological applications that can be tested at present or in the future.

As it is well known, in the heavy quark limit, QCD possesses new global symmetries, namely, the spin-flavor symmetry $SU(2N_f)$, where N_f is the number of heavy quark flavors, in practice the b and c quarks.

Hadrons with one heavy quark such that $m_Q \gg \Lambda_{\rm QCD}$ can be thought as a bound state of a light cloud in the color source of the heavy quark. Because of its heavy mass, the latter is unaffected by the interaction with soft gluons.

In this approximation, the decay of a heavy hadron with four-velocity v into another hadron with velocity v', for example, the semileptonic decay $\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell$ or $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$, occurs just by free heavy quark decay produced by a current, and the rearrangement of the light cloud or "brown muck" to follow the heavy quark in the final state and constitute the final heavy hadron.

The dynamics is contained in the complicated light cloud, that concerns long distance QCD and is not calculable from first principles. Therefore, one needs to parametrize this physics through form factors, the IW functions.

The matrix element of a current between heavy hadrons containing heavy quarks Q and Q' can thus be factorized as follows [2]:

$$\langle H'(v')|J^{Q'Q}(q)|H(v)\rangle = \langle Q'(v'), \pm\tfrac{1}{2}|J^{Q'Q}(q)|Q(v), \pm\tfrac{1}{2}\rangle \langle \text{light}, v', j', M'|\text{light}, v, j, M\rangle \quad (1)$$

where v and v' are the initial and final four-velocities, respectively, and j, j', M, and M' are the angular momenta and corresponding projections of the initial and final light clouds, respectively.

The current affects only the heavy quark, and all the soft dynamics is contained in the *overlap* between the initial and final light clouds $\langle v', j', M'|v, j, M\rangle$, that follow the heavy quarks with the same four-velocity. From now on it would be understood that this scalar product concerns the

*Unité Mixte de Recherche UMR 8627 - CNRS light cloud. This overlap is independent of the current heavy quark matrix element, but depends on the four-velocities v and v'. The IW functions are precisely given by these light cloud overlaps.

An important hypothesis has been done in writing the previous expression, namely, neglecting *hard gluon radiative corrections*, that we will assume from now on.

As we will make explicit below, the light cloud belongs to a Hilbert space, and transforms according to a unitary representation of the Lorentz group. Then, as we will show, the whole problem of getting rigorous constraints on the IW functions amounts to decomposing unitary representations of the Lorentz group into irreducible ones. This will allow one to obtain for the IW functions general integral formulas in which the crucial point is that *the measures are positive*.

In [3], for bound states made up of a heavy quark and a nonrelativistic light quark, we did already exploit the *positivity* of matrices of moments of the ground state wave function, that allowed us to bound the derivatives of the corresponding IW function $\xi_{\rm NR}(w)$, where NR stands for the nonrelativistic approximation for the light quark. The present paper extends this method to the nontrivial case of the true QCD in the heavy quark limit.

We treat here the case of a light cloud with angular momentum $j = 0$ in the initial and final states, as happens in the baryon semileptonic decay $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$ where, in the quark model, the light diquark system has $S = 0$ with orbital angular momentum $L = 0$ relative to the heavy quark, and therefore $j = 0$ in the relativistic language. The whole spin of the baryon is carried by the heavy active quark.

A different but, as we will show below, equivalent method to the one of the present paper was developed in a number of articles using *sum rules* in the heavy quark limit, like the famous Bjorken sum rule and its generalizations [4–7].

Although in our previous papers and also in most work by other authors, the sum rules are formulated using the heavy *hadron* states, they could be formulated in an equivalent way using only the light cloud, the reason being that the heavy quark spin decouples from the soft QCD physics.

The sum rule method is completely equivalent to the method of the present paper. Indeed, starting from the sum rules one can demonstrate that an IW function, say, $\xi(v.v') = \langle v'|v\rangle$ in a simplified notation, is a function of *positive type*, and that one can construct a unitary representation of the Lorentz group $U(\Lambda)$ and a vector state $|\phi_0\rangle$ representing the light cloud at rest. The IW function writes then simply (e.g. in the special case $j = 0$)

$$\xi(v.v') = \langle U(B_{v'})\phi_0|U(B_v)\phi_0\rangle \qquad (2)$$

where B_v and $B_{v'}$ are the corresponding boosts. Notice that we are dealing precisely with the Lorentz group and not with the usual Poincaré group. This is due to the fact that we are working within the heavy quark limit of QCD.

Another important aspect worth underlining is that the light cloud belongs to a Hilbert space and that therefore the corresponding sum rules are *convergent*, although this feature does not survive the inclusion of radiative corrections involving *hard gluons*.

Let us now go back to previous work on the sum rule method. In the meson case $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell$, in the leading order of the heavy quark expansion, the Bjorken sum rule (SR) [4,5] gives the lower bound for the derivative of the meson elastic IW function at zero recoil $\rho^2 = -\xi'(1) \geq \frac{1}{4}$. A new SR was formulated by Uraltsev in the heavy quark limit [6] that, combined with Bjorken's, gave the much stronger lower bound $\rho^2 \geq \frac{3}{4}$. A basic ingredient in deriving this bound was the consideration of the nonforward amplitude $\bar{B}(v_i) \to D^{(n)}(v') \to \bar{B}(v_f)$, allowing for general four-velocities v_i, v_f, and v'.

In [7] we did develop a manifestly covariant formalism within the operator product expansion (OPE) and the nonforward amplitude, using the whole tower of heavy meson states [2]. We did recover the Uraltsev SR plus a general class of SR that allow one to bound also higher derivatives of the IW function. In particular, we found a bound on the curvature in terms of the slope ρ^2, namely, $\xi''(1) \geq \frac{1}{5} \times [4\rho^2 + 3(\rho^2)^2]$.

Recently, we have extended the sum rule method to the baryon IW function $\xi_\Lambda(w)$ of the transition $\Lambda_b \to \Lambda_c\ell\bar{\nu}_\ell$ [8]. We have recovered the lower bound for the slope $\rho_\Lambda^2 = -\xi'_\Lambda(1) \geq 0$ [9], and we have generalized it by demonstrating $(-1)^n \xi_\Lambda^{(n)}(1) \geq 0$. Moreover, exploiting systematically the sum rules, we got an improved lower bound for the curvature in terms of the slope:

$$\sigma_\Lambda^2 = \xi''_\Lambda(1) \geq \tfrac{3}{5}[\rho_\Lambda^2 + (\rho_\Lambda^2)^2] \qquad (3)$$

This bound can be useful to constrain the shape of the differential spectrum of future precise data, hopefully at LHCb, on the baryon semileptonic decay $\Lambda_b \to \Lambda_c\ell\bar{\nu}_\ell$, that has a large measured branching ratio of about 5%.

To simplify the notation of the present paper, that is restricted to the baryon $j = 0$ case, we replace from now on the IW function $\xi_\Lambda(w)$ of [8] by $\xi(w)$. However for the slope ρ_Λ^2 and the curvature σ_Λ^2 we still keep this notation, that is used below only in Sects. VI and X. Indeed, there could be an ambiguity in what follows (ρ labels also the irreducible representations of the Lorentz group).

The much more powerful method of the present paper will provide a new insight on the physics of QCD in the heavy quark limit and on its Lorentz group structure.

We will see that we recover the bound (3) and this systematic method will allow us to find bounds for higher derivatives. We will also demonstrate that if, for example, the bound (3) is saturated, then the IW function $\xi(w)$ is completely determined and given by an explicit elementary function, dependent on a single parameter. There is a simple group theoretical argument that explains this feature.

We restrict here to the case $j = 0$ that, interestingly, turns out to be more involved than the meson case $j = \frac{1}{2}$ from the point of view of the decomposition of the corresponding unitary representation of the Lorentz group into irreducible ones. The study of the $j = \frac{1}{2}$ case, that is more complicated from the spin point of view, is postponed to future work.

II. THE LORENTZ GROUP AND THE HEAVY QUARK LIMIT OF QCD

In the heavy mass limit, the states of a heavy hadron H containing a heavy quark Q are described as follows [2]:

$$|H(v), \mu, M\rangle = |Q(v), \mu\rangle \otimes |v, j, M\rangle \tag{4}$$

where there is factorization into the heavy quark state factor $|Q(v), \mu\rangle$ and a light cloud component $|v, j, M\rangle$ (also called "brown muck"). The velocity v of the heavy hadron H is the same as the velocity of the heavy quark Q, and is unquantized (this is the superselection rule of [10]). The heavy quark Q state depends only on a spin $\mu = \pm\frac{1}{2}$ quantum number, and so belongs to a two-dimensional Hilbert space. The light component is the complicated thing, but it does not depend on the spin state μ of the heavy quark Q, nor on its mass, and this gives rise to the symmetries of the heavy quark theory.

As advanced in the introduction, the matrix element of a heavy-heavy current J (acting only on the heavy quark) writes then

$$\langle H'(v'), \mu', M'|J|H(v), \mu, M\rangle$$
$$= \langle Q'(v'), \mu'|J|Q(v), \mu\rangle\langle v', j', M'|v, j, M\rangle \tag{5}$$

and the IW functions are defined as the coefficients, depending only on $v.v'$, in the expansion of the unknown scalar products $\langle v', j', M'|v, j, M\rangle$ into independent scalars constructed from v, v' and the polarization tensors describing the spin states of the light components.

Now, the crucial point in the present work is that *the states of the light components make up a Hilbert space in which acts a unitary representation of the Lorentz group*. In fact, this is more or less implicitly stated, and used in the literature [2].

A. Physical picture of a heavy quark

To see the point more clearly, let us go into the physical picture which is at the basis of (4). Considering first a heavy hadron *at rest*, with velocity

$$v_0 = (1, 0, 0, 0) \tag{6}$$

its light component is submitted to the interactions among the light particles, light quarks, light antiquarks and gluons, and to the external chromo-electric field generated by the heavy quarks at rest. This chromo-electric field does not depend on the spin μ of the heavy quark nor on its mass. We shall then have a complete orthonormal system of energy eigenstates $|v_0, j, M, \alpha\rangle$ of the light component, where j and M are the angular momentum quantum numbers, and α is any needed additional quantum number:

$$\langle v_0, j', M', \alpha'|v_0, j, M, \alpha\rangle = \delta_{j,j'}\delta_{M,M'}\delta_{\alpha,\alpha'} \tag{7}$$

Now, for a heavy hadron moving with a velocity v, the only thing which changes for the light component is that the external chromo-electric field generated by the heavy quark at rest is replaced by the external chromo-electromagnetic field generated by the heavy quark moving with the velocity v. Neither the Hilbert space describing the possible states of the light component, nor the interactions between the light particles, are changed. We shall then have *a new complete orthonormal system* of energy eigenstates $|v, j, M, \alpha\rangle$, in the same Hilbert space. Then, because the color fields generated by a heavy quark for different velocities are related by Lorentz transformations, we may expect that the energy eigenstates of the light component will, for various velocities, be themselves related by Lorentz transformations acting in their Hilbert space.

B. Lorentz representation from covariant overlaps

Let us now show that such a representation of the Lorentz group does in fact underlie the work of Ref. [2].

For integer spin j, the spin state of the light component is described by a polarization tensor $\epsilon^{\mu_1,\ldots,\mu_j}$ subject to the constraints of symmetry, transversality and tracelessness

$$v_{\mu_1}\epsilon^{\mu_1,\ldots,\mu_j} = 0 \qquad g_{\mu\nu}\epsilon^{\mu,\nu,\mu_3,\ldots,\mu_j} = 0 \tag{8}$$

For half-integer spin j, the polarization tensor becomes a Rarita-Schwinger tensor-spinor $\epsilon_\alpha^{\mu_1,\ldots,\mu_{j-1/2}}$ subject to the constraints of symmetry, transversality and tracelessness as above, and

$$(1 - \not{v})_{\alpha\beta}\epsilon_\beta^{\mu_1,\ldots,\mu_{j-1/2}} = 0 \qquad (\gamma_{\mu_1})_{\alpha\beta}\epsilon_\beta^{\mu_1,\ldots,\mu_{j-1/2}} = 0 \tag{9}$$

Then a scalar product $\langle v', j', \epsilon', \alpha'|v, j, \epsilon, \alpha\rangle$ is a covariant function of the vectors v and v' and of the tensors (or tensor-spinors) ϵ'^* and ϵ, bilinear with respect to ϵ'^* and ϵ, and the IW functions, functions of the scalar $v.v'$, are introduced accordingly.

Now, the covariance property of the scalar products is explicitly expressed by the equality

$$\langle \Lambda v', j', \Lambda\epsilon', \alpha'|\Lambda v, j, \Lambda\epsilon, \alpha\rangle = \langle v', j', \epsilon', \alpha'|v, j, \epsilon, \alpha\rangle \tag{10}$$

valid for any Lorentz transformation Λ, with the transformation of a tensor (or tensor-spinor) ϵ given by

$$(\Lambda\epsilon)^{\mu_1,\ldots,\mu_j} = \Lambda^{\mu_1}_{\nu_1}\ldots\Lambda^{\mu_j}_{\nu_j}\epsilon^{\nu_1,\ldots,\nu_j} \tag{11}$$

$$(\Lambda\epsilon)_\alpha^{\mu_1,\ldots,\mu_{j-1/2}} = \Lambda_{\nu_1}^{\mu_1}\ldots\Lambda_{\nu_{j-1/2}}^{\mu_{j-1/2}} D(\Lambda)_{\alpha\beta}\epsilon_\beta^{\nu_1,\ldots,\nu_{j-1/2}} \quad (12)$$

Then, let us *define* the operator $U(\Lambda)$, in the space of the light component states, by

$$U(\Lambda)|v_0, j, \epsilon, \alpha\rangle = |\Lambda v_0, j, \Lambda\epsilon, \alpha\rangle \quad (13)$$

where here v_0 is a fixed, arbitrarily chosen velocity. Let us show that $U(\Lambda)$ gives a unitary representation of the Lorentz group.

Equation (10) implies that $U(\Lambda)$ is a *unitary operator*. Let us find the action of $U(\Lambda)$ on the state $|v, j, \epsilon, \alpha\rangle$. Using a complete orthonormal set $|v_0, j, \epsilon^{(M)}, \alpha\rangle$, where $\epsilon^{(M)}$ for $-j \leq M \leq j$ is a basis for the polarization tensors, we have

$$\begin{aligned} U(\Lambda)|v, j, \epsilon, \alpha\rangle &= \sum_{j',M,\beta} \langle v_0, j', \epsilon^{(M)}, \beta|v, j, \epsilon, \alpha\rangle \\ &\quad \times U(\Lambda)|v_0, j', \epsilon^{(M)}, \beta\rangle \\ &= \sum_{j',M,\beta} \langle v_0, j', \epsilon^{(M)}, \beta|v, j, \epsilon, \alpha\rangle \\ &\quad \times |\Lambda v_0, j', \Lambda\epsilon^{(M)}, \beta\rangle \end{aligned}$$

Using (10), one gets

$$\begin{aligned} U(\Lambda)|v, j, \epsilon, \alpha\rangle = \sum_{j',M,\beta} &\langle \Lambda v_0, j', \Lambda\epsilon^{(M)}, \beta|\Lambda v, j, \Lambda\epsilon, \alpha\rangle \\ &\times |\Lambda v_0, j', \Lambda\epsilon^{(M)}, \beta\rangle \end{aligned}$$

and using the fact that the set $|\Lambda v_0, j, \Lambda\epsilon^{(M)}, \alpha\rangle$ is orthonormal [see (10) again], and complete as well, we finally obtain

$$U(\Lambda)|v, j, \epsilon, \alpha\rangle = |\Lambda v, j, \Lambda\epsilon, \alpha\rangle \quad (14)$$

From (14), it is easy to see that the group property $U(\Lambda')U(\Lambda) = U(\Lambda'.\Lambda)$ holds. Therefore, we have indeed a unitary representation of the Lorentz group in the Hilbert space of the light component states.

C. From a Lorentz representation to Isgur-Wise functions

We have shown above how a unitary representation of the Lorentz group emerges from the usual treatment of heavy hadrons in the heavy quark theory. For the present purpose, we need to go in the opposite way, namely, to show how, starting from a unitary representation of the Lorentz group, the usual treatment of heavy hadrons and the introduction of the IW functions emerges. What follows is not restricted to the $j = 0$ case, but concerns any IW function.

So, let us consider some unitary representation $\Lambda \to U(\Lambda)$ of the Lorentz group, or more precisely of the group $SL(2, C)$, in a Hilbert space $\mathcal{H}$. We have to identify states in $\mathcal{H}$, depending on a velocity v, and which are transformed according to (14). The difficulty is that, in the present abstract setting, we apparently have nothing like a velocity v in sight. But we have in $\mathcal{H}$ an additional structure, namely, the energy operator of the light component *for a heavy quark at rest*. Since this energy operator is invariant under rotations, we have to consider the subgroup $SU(2)$ of $SL(2, C)$. By restriction, the representation in $\mathcal{H}$ of $SL(2, C)$ gives a representation $R \to U(R)$ of $SU(2)$, and the decomposition of $\mathcal{H}$ into irreducible representations of $SU(2)$ comes into play. We then have the eigenstates $|v_0, j, M, \alpha\rangle$ of the energy operator, classified by the angular momentum number j of the irreducible representations of $SU(2)$, and *associated with the rest velocity* v_0, since their physical meaning is to describe the energy eigenstates of the light component for a heavy hadron at rest.

Let us pursue our task, which is to express the states $|v, j, \epsilon, \alpha\rangle$ in terms of the states $|v_0, j, M, \alpha\rangle$. We begin with $v = v_0$. For fixed j and α, the states $|v_0, j, M, \alpha\rangle$ constitute, for $-j \leq M \leq j$, a standard basis of a representation j of $SU(2)$:

$$U(R)|v_0, j, M, \alpha\rangle = \sum_{M'} D^j_{M',M}(R)|v_0, j, M', \alpha\rangle \quad (15)$$

where the rotation matrix elements $D^j_{M',M}$ are defined by

$$D^j_{M',M} = \langle j, M'|U_j(R)|j, M\rangle \qquad R \in SU(2) \quad (16)$$

On the other hand, the states $|v_0, j, \epsilon, \alpha\rangle$ constitute, when ϵ goes over all polarization tensors (or tensor-spinors), the whole space of a representation j of $SU(2)$. In fact, when $v = v_0$, the first constraint in (8) and (9) means the vanishing of any component of ϵ if some μ_i is 0 (or if $\alpha = 3$ or 4, in the case of a tensor-spinor), so that ϵ can be considered as a tensor (respectively, tensor-spinor) constructed from ordinary three-dimensional space (respectively, and from two-dimensional spinor space). The group $SU(2)$ acts on these tensors (or tensor-spinors) in the usual way, through rotations in three-dimensional space and through the spin $1/2$ representation in the spinor space. This representation of $SU(2)$ in the space of 3-tensors (or 3-tensor-spinors) is not irreducible, but contains an irreducible subspace of spin j, which is precisely the polarization 3-tensor (or 3-tensor-spinor) space selected by the other constraints, symmetry and the second constraint in (8) and (9).

The conclusion of this rather long description is that we may introduce a standard basis $\epsilon^{(M)}$, $-j \leq M \leq j$, for the $SU(2)$ representation of spin j in the space of polarization 3-tensors (or 3-tensor-spinors). This basis will be used here to demonstrate Eq. (22) below. Then, using the notation (11) and (12) for $R\epsilon$, one has

$$R\epsilon^{(M)} = \sum_{M'} D^j_{M',M}(R)\epsilon^{(M')} \quad (17)$$

and since, according to (14), we want $U(R)|v_0, j, \epsilon, \alpha\rangle = |v_0, j, R\epsilon, \alpha\rangle$, the states $|v_0, j, \epsilon^{(M)}, \alpha\rangle$ are readily identified:

$$|v_0, j, \epsilon^{(M)}, \alpha\rangle = |v_0, j, M, \alpha\rangle \tag{18}$$

Notice that this is an identity between states described by polarization tensors $\epsilon^{(M)}$ and states belonging to a basis standard under rotations.

Next, denoting by ϵ_M the standard components of an arbitrary ϵ, which are just the components with respect to the basis $\epsilon^{(M)}$:

$$\epsilon = \sum_M \epsilon_M \epsilon^{(M)} \tag{19}$$

and, from (18) and (19), the state $|v_0, j, \epsilon, \alpha\rangle$ for an arbitrary ϵ is obtained:

$$|v_0, j, \epsilon, \alpha\rangle = \sum_M \epsilon_M |v_0, j, M, \alpha\rangle \tag{20}$$

Finally, we have to find the states $|v, j, \epsilon, \alpha\rangle$ associated to an arbitrary velocity v. Wanting (14) to be satisfied, we have no choice. Indeed, (14) gives

$$U(\Lambda)|v_0, j, \epsilon, \alpha\rangle = |\Lambda v_0, j, \Lambda\epsilon, \alpha\rangle \tag{21}$$

for any Lorentz transformation Λ, so that we must have

$$|v, j, \epsilon, \alpha\rangle = \sum_M (\Lambda^{-1}\epsilon)_M U(\Lambda)|v_0, j, M, \alpha\rangle \tag{22}$$

for any Λ such that $\Lambda.v_0 = v$, with v_0 given by (6).

Equation (22) is *our final result* here, defining, in the Hilbert space $\mathcal{H}$ of a unitary representation of $SL(2, C)$, the states $|v, j, \epsilon, \alpha\rangle$ whose scalar products define the IW functions, in terms of $|v_0, j, M, \alpha\rangle$ which occur as $SU(2)$ multiplets in the restriction to $SU(2)$ of the $SL(2, C)$ representation.

However, in order that (22) be really a definition of $|v, j, \epsilon, \alpha\rangle$, there is still something to be verified, namely, that $|v, j, \epsilon, \alpha\rangle$ does not depend on the choice of the Lorentz transformation Λ such that $\Lambda.v_0 = v$. So, let Λ' be another Lorentz transformation such that $\Lambda'.v_0 = v$. Since $\Lambda^{-1}\Lambda'.v_0 = v_0$, $\Lambda^{-1}\Lambda'$ is a rotation $R \in SU(2)$, and we have $\Lambda' = \Lambda R$. Then,

$$\begin{aligned}\sum_M (\Lambda'^{-1}\epsilon)_M U(\Lambda')|v_0, j, M, \alpha\rangle &= \sum_M (R^{-1}\Lambda^{-1}\epsilon)_M U(\Lambda R)|v_0, j, M, \alpha\rangle \\ &= \sum_M (R^{-1}(\Lambda^{-1}\epsilon))_M U(\Lambda)U(R)|v_0, j, M, \alpha\rangle\end{aligned}$$

Using (15) and (17) to expand $U(R)|v_0, j, M, \alpha\rangle$ and $(R^{-1}(\Lambda^{-1}\epsilon))_M$, one obtains

$$\begin{aligned}\sum_M (\Lambda'^{-1}\epsilon)_M U(\Lambda')|v_0, j, M, \alpha\rangle &= \sum_{M,M',M''} D^j_{M,M'}(R^{-1}) \\ &\quad \times (\Lambda^{-1}\epsilon)_{M'} D^j_{M'',M}(R) U(\Lambda)|v_0, j, M'', \alpha\rangle\end{aligned}$$

The sum over M is done using the group property of the $D^j_{M,M'}$:

$$\sum_M D^j_{M,M'}(R^{-1}) D^j_{M'',M}(R) = \delta_{M'',M'}$$

and one obtains

$$\begin{aligned}\sum_M (\Lambda'^{-1}\epsilon)_M U(\Lambda')|v_0, j, M, \alpha\rangle &= \sum_{M'} (\Lambda^{-1}\epsilon)_{M'} U(\Lambda)|v_0, j, M', \alpha\rangle = |v, j, \epsilon, \alpha\rangle\end{aligned}$$

proving that the state $|v, j, \epsilon, \alpha\rangle$ defined by (22) does not depend on the choice of Λ (as long as $\Lambda.v_0 = v$).

To be complete, we have still to show that Eqs. (10) and (14) hold for these states $|v, j, \epsilon, \alpha\rangle$. The proof of (14) is straightforward. Let Λ be any Lorentz transformation. From the definition (22), one has

$$\begin{aligned}U(\Lambda)|v, j, \epsilon, \alpha\rangle &= \sum_M (\Lambda'^{-1}\epsilon)_M U(\Lambda)U(\Lambda')|v_0, j, M, \alpha\rangle \\ &= \sum_M ((\Lambda\Lambda')^{-1}\Lambda\epsilon)_M U(\Lambda\Lambda')|v_0, j, M, \alpha\rangle\end{aligned}$$

where Λ' is some Lorentz transformation such that $\Lambda'.v_0 = v$. Then, since the Lorentz transformation $\Lambda\Lambda'$ satisfies $\Lambda\Lambda'.v_0 = \Lambda v$, using again the definition (22), one has

$$U(\Lambda)|v, j, \epsilon, \alpha\rangle = |\Lambda v, j, \Lambda\epsilon, \alpha\rangle \tag{23}$$

Finally, Eq. (10), which is crucial for the definition of the IW functions, that writes here

$$\langle \Lambda v', j', \Lambda\epsilon', \alpha' | \Lambda v, j, \Lambda\epsilon, \alpha\rangle = \langle v', j', \epsilon', \alpha' | v, j, \epsilon, \alpha\rangle \tag{24}$$

is an immediate consequence of (14) and of the unitarity of $U(\Lambda)$, here assumed from the start.

III. DECOMPOSITION INTO IRREDUCIBLE REPRESENTATIONS AND INTEGRAL FORMULA FOR THE ISGUR-WISE FUNCTION (IN THE CASE $j = 0$)

Let us first explain, in general terms, the decomposition of unitary representations into irreducible ones, and how this gives a general integral formula for the IW functions. As it is well known, in the case of a compact group [as $SU(2)$], any unitary representation can be written as a direct sum of irreducible ones. In the present case of $SL(2, C)$ (a noncompact group), the more general notion of a direct integral is required [11].

Let us denote by X the set of (equivalence classes of) irreducible unitary representations of $SL(2, C)$, by $\mathcal{H}_\chi$ the Hilbert space of a representation $\chi \in X$, and by $U_\chi(\Lambda)$ the unitary operator acting in $\mathcal{H}_\chi$ which corresponds to any $\Lambda \in SL(2, C)$. Then, for any unitary representation of $SL(2, C)$, the Hilbert space $\mathcal{H}$ can be written in the form

$$\mathcal{H} = \int_X^{\oplus} \oplus_{n_\chi} \mathcal{H}_\chi d\mu(\chi) \tag{25}$$

where ⊕ on the integral sign indicates a direct integral of Hilbert spaces, μ is an arbitrary *positive* measure on the set X and n_χ is a function on X with ≥ 1 integer values or possibly ∞. This is a rather symbolic formula. Explicitly, an element $\psi \in \mathcal{H}$ is a function

$$\psi\colon \chi \in X \to \psi_\chi = (\psi_{1,\chi}, \ldots, \psi_{n_\chi,\chi}) \in \oplus_{n_\chi} \mathcal{H}_\chi \tag{26}$$

which assigns to each $\chi \in X$ an element $\psi_\chi \in \oplus_{n_\chi} \mathcal{H}_\chi$, and which is μ-measurable and square μ-integrable. The scalar product in $\mathcal{H}$ is given by

$$\langle \psi' | \psi \rangle = \int_X \langle \psi'_\chi | \psi_\chi \rangle d\mu(\chi) \tag{27}$$

and the operator $U(\Lambda)$ of the representation in the space $\mathcal{H}$ is given by

$$(U(\Lambda)\psi)_{k,\chi} = U_\chi(\Lambda)\psi_{k,\chi} \tag{28}$$

The more familiar notion of a direct sum is the particular case of a direct integral when the measure μ is a sum of Dirac δ functions.

Let us see now the consequences for the IW functions. For simplicity, we take here the case of a scalar ($j = 0$) light component. For the hadron at rest, the light component will be described by *some* element $\psi_0 \in \mathcal{H}$ *which is scalar* for the subgroup $SU(2)$ of $SL(2, C)$. Then, according to the law of transformation (28), requiring that ψ_0 is a scalar under rotations is the same as requiring that $\psi_{0,k,\chi}$ is a scalar under rotations for all χ's (technically, for μ-almost all χ's) and all $k = 1, \ldots, n_\chi$ (all $k \geq 1$ if $n_\chi = \infty$, we omit hereafter to specify this case). We therefore have to look at the $SU(2)$ scalars in each irreducible representation of $SL(2, C)$. More generally, the decomposition of the irreducible representations of $SL(2, C)$ into irreducible representations of $SU(2)$ is known (see the section below). The decomposition is by a direct sum [since $SU(2)$ is compact], and therefore each $\mathcal{H}_\chi$ admits an orthonormal basis adapted to $SU(2)$. Moreover, it turns out that each representation j of $SU(2)$ appears with multiplicity 0 or 1. Then, there is a subset $X_0 \subset X$ of irreducible representations of $SL(2, C)$ containing a nonzero $SU(2)$ scalar subspace and, for $\chi \in X_0$, there is a unique (up to a phase) normalized $SU(2)$ scalar element in $\mathcal{H}_\chi$, which we denote $\phi_{0,\chi}$. Each scalar element in $\mathcal{H}_\chi$ is then proportional to $\phi_{0,\chi}$. So, one has

$$\psi_{0,\chi} = (c_{1,\chi}\phi_{0,\chi}, \ldots, c_{n_\chi,\chi}\phi_{0,\chi}) \tag{29}$$

with some coefficients $c_{1,\chi}, \ldots, c_{n_\chi,\chi}$. From the scalar product (27) in $\mathcal{H}$, one sees that the normalization $\langle \psi_0 | \psi_0 \rangle = 1$ of the light component amounts to

$$\int_{X_0} \sum_{k=1}^{n_\chi} |c_{k,\chi}|^2 d\mu(\chi) = 1 \tag{30}$$

Now, particularizing (22) for $j = 0$, the IW function $\xi(w)$ is given by

$$\xi(w) = \langle \psi_0 | U(\Lambda) \psi_0 \rangle, \tag{31}$$

where here (case $j = 0$) $\Lambda \in SL(2, C)$ is any transformation converting the rest velocity v_0 into a velocity v with $v^0 = w$, for instance, the boost along Oz

$$\Lambda_\tau = \begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{pmatrix} \qquad w = \mathrm{ch}(\tau).$$

From (31), using formula (28) for $U(\Lambda)$ and (27) for the scalar product in $\mathcal{H}$, one readily obtains

$$\xi(w) = \int_{X_0} \sum_{k=1}^{n_\chi} |c_{k,\chi}|^2 \langle \phi_{0,\chi} | U_\chi(\Lambda) \phi_{0,\chi} \rangle d\mu(\chi) \tag{32}$$

It is then useful to introduce a notation

$$\xi_\chi(w) = \langle \phi_{0,\chi} | U_\chi(\Lambda) \phi_{0,\chi} \rangle \tag{33}$$

so that $\xi_\chi(w)$ may be called the *irreducible Isgur-Wise function* corresponding to χ. Introducing also the measure

$$d\nu(\chi) = \sum_{k=1}^{n_\chi} |c_{k,\chi}|^2 d\mu(\chi), \tag{34}$$

formula (32) writes

$$\xi(w) = \int_{X_0} \xi_\chi(w) d\nu(\chi) \tag{35}$$

and exhibits the IW function as a mean value of the irreducible IW functions, with respect to some *positive* normalized measure ν:

$$\int_{X_0} d\nu(\chi) = 1. \tag{36}$$

As we will see below, the irreducible IW function $\xi_\chi(w)$, which is the special case of (35) when ν is a δ function, could be interesting as the limiting case of a heavy hadron with light component in the irreducible representation $\chi \in X$. In fact, as will be seen, an example of such a limiting case is obtained when, for a given slope of $\xi(w)$, the lower bound of its curvature is saturated.

IV. IRREDUCIBLE UNITARY REPRESENTATIONS OF THE LORENTZ GROUP AND THEIR DECOMPOSITION UNDER ROTATIONS

A. An explicit form of the irreducible representations of the Lorentz group

Let us now describe an explicit form of the irreducible unitary representations of $SL(2, C)$. Their set X is divided into three sets, the set X_p of representations of the principal series, the set X_s of representations of the supplementary series, and the one-element set X_t made up of the trivial representation.

Principal series.—A representation $\chi = (p, n, \rho)$ in the principal series is labeled by an integer $n \in Z$ and a real number $\rho \in R$. Actually, the representations (p, n, ρ) and $(p, -n, -\rho)$ (as given below) turn out to be equivalent so that, in order to have each representation only once, n and ρ will be restricted as follows:

$$n = 0 \qquad \rho \geq 0 \qquad n > 0 \qquad \rho \in R \tag{37}$$

The Hilbert space $\mathcal{H}_{p,n,\rho}$ is made up of functions of a complex variable z with the standard scalar product

$$\langle \phi' | \phi \rangle = \int \overline{\phi'(z)} \phi(z) d^2 z, \tag{38}$$

with the measure $d^2 z$ in the complex plane being simply $d^2 z = d(\mathrm{Re} z) d(\mathrm{Im} z)$. So $\mathcal{H}_{p,n,\rho} = L^2(C, d^2 z)$.

The unitary operator $U_{p,n,\rho}(\Lambda)$ is given by

$$(U_{p,n,\rho}(\Lambda)\phi)(z) = \left(\frac{\alpha - \gamma z}{|\alpha - \gamma z|}\right)^n |\alpha - \gamma z|^{2i\rho - 2} \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right) \tag{39}$$

where α, β, γ, and δ are complex matrix elements of $\Lambda \in SL(2, C)$:

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1 \tag{40}$$

Supplementary series.—A representation $\chi = (s, \rho)$ in the supplementary series is labeled by a real number $\rho \in R$ satisfying

$$0 < \rho < 1. \tag{41}$$

The Hilbert space $\mathcal{H}_{s,\rho}$ is made up of functions of a complex variable z with the nonstandard scalar product

$$\langle \phi' | \phi \rangle = \int \overline{\phi'(z_1)} |z_1 - z_2|^{2\rho - 2} \phi(z_2) d^2 z_1 d^2 z_2. \tag{42}$$

The positivity of this scalar product (when $0 < \rho < 1$) can be seen by Fourier transforming. The Hilbert space can be obtained by completing the pre-Hilbert space of continuous functions vanishing outside a bounded region.

The unitary operator $U_{s,\rho}(\Lambda)$ is given by

$$(U_{s,\rho}(\Lambda)\phi)(z) = |\alpha - \gamma z|^{-2\rho - 2} \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right). \tag{43}$$

Trivial representation.—The trivial representation $\chi = t$ is of course the one-dimensional representation, with Hilbert space $\mathcal{H}_t = C$, scalar product $\langle \phi' | \phi \rangle = \overline{\phi'(z)} \phi(z)$ and unitary operator $U_t(\Lambda) = 1$.

The formulas above allow one, with some calculations, to see that they define unitary representations of $SL(2, C)$. For a proof that these representations are irreducible and that they exhaust the unitary irreducible representations, see Naïmark [12].

B. Decomposition under the rotation group

Next we need the decomposition of the restriction to the subgroup $SU(2)$ of each irreducible unitary representation of $SL(2, C)$.

The decomposition is by a direct sum (a direct integral is not needed) since $SU(2)$ is compact, so that, for each representation $\chi \in X$ we have an *orthonormal basis* $\phi^{\chi}_{j,M}$ of $\mathcal{H}_\chi$ adapted to $SU(2)$. Here we denote by j the spin of an irreducible representation of $SU(2)$ (having in mind the usual notation for the spin of the light component of a heavy hadron). It turns out [12] that each representation j of $SU(2)$ appears in χ with multiplicity 0 or 1, so that $\phi^{\chi}_{j,M}$ needs no more indices, and that the values taken by j are part of the integer and half-integer numbers. For j fixed, the functions $\phi^{\chi}_{j,M}$, $-j \leq M \leq j$ are chosen as a standard basis of the representation j of $SU(2)$. This leaves arbitrary the choice of a phase for each j. The choice in the formulas below is for simplicity. The normalization constants are computed in Appendix B.

It turns out [12] that the functions $\phi^{\chi}_{j,M}(z)$ are expressed in terms of the rotation matrix elements $D^j_{M',M}$ defined by (16). A matrix $R \in SU(2)$ being of the form

$$R = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1 \tag{44}$$

we shall also consider $D^j_{M',M}$ as a function of a and b (satisfying $|a|^2 + |b|^2 = 1$). Then one has the following simple generating function:

$$\sum_{M,M'} \frac{D^j_{M',M}(a, b)}{\sqrt{(j - M)!(j + M)!(j - M')!(j + M')!}} s^{j+M} s'^{j+M'} = \frac{1}{(2j)!} (bs' + \bar{a} + (as' - \bar{b})s)^{2j} \tag{45}$$

and the following explicit formula:

$$D^j_{M',M}(a, b) = \sqrt{\frac{(j - M')!(j + M')!}{(j - M)!(j + M)!}} \sum_k (-1)^k \binom{j + M}{k} \times \binom{j - M}{j - M' - k} a^{j+M-k} \bar{a}^{j-M'-k} b^{-M+M'+k} \bar{b}^k \tag{46}$$

The well-known scalar product in the space $L^2(SU(2), dR)$ of the rotation matrix elements will also be very useful:

$$\int D^{J_2}_{M'_2,M_2}(R)^* D^{J_1}_{M'_1,M_1}(R) dR = \frac{1}{2J_1 + 1} \delta_{M'_1,M'_2} \delta_{M_1,M_2} \delta_{J_1,J_2} \tag{47}$$

[dR is the normalized invariant measure on the group $SU(2)$].

We can now give explicit formulas for the orthonormal basis $\phi^{\chi}_{j,M}$ of $\mathcal{H}_{\chi}$.

Principal series.—The spins j which appear in a representation $\chi = (p, n, \rho)$ of the principal series are

(i) all the integers $j \geq \frac{n}{2}$ when n is even;

(ii) all the half-integers $j \geq \frac{n}{2}$ when n is odd.

Such a spin appears with multiplicity 1. The basis functions $\phi^{p,n,\rho}_{j,M}(z)$ are

$$\phi^{p,n,\rho}_{j,M}(z) = \frac{\sqrt{2j+1}}{\sqrt{\pi}}(1+|z|^2)^{i\rho-1} \times D^{j}_{n/2,M}\left(\frac{1}{\sqrt{1+|z|^2}}, -\frac{z}{\sqrt{1+|z|^2}}\right) \quad (48)$$

or, using the explicit formula for $D^{j}_{n/2,M}$,

$$\phi^{p,n,\rho}_{j,M}(z) = \frac{\sqrt{2j+1}}{\sqrt{\pi}}(-1)^{n/2-M} \times \sqrt{\frac{(j-n/2)!(j+n/2)!}{(j-M)!(j+M)!}}(1+|z|^2)^{i\rho-j-1} \times \sum_k (-1)^k \binom{j+M}{k}\binom{j-M}{j-n/2-k} z^{n/2-M+k}\bar{z}^k \quad (49)$$

where the range for k can be limited to $0 \leq k \leq j - n/2$ due to the binomial factors.

Supplementary series.—The spins j which appear in a representation $\chi = (s, \rho)$ of the supplementary series are all the integers $j \geq 0$.

Such a spin appears with multiplicity 1. The basis functions $\phi^{s,\rho}_{j,M}(z)$ are

$$\phi^{s,\rho}_{j,M}(z) = \frac{\sqrt{2j+1}}{\pi}\sqrt{\frac{\Gamma(j+\rho+1)\Gamma(1-\rho)}{\Gamma(j-\rho+1)\Gamma(\rho)}} \times (1+|z|^2)^{-\rho-1} D^{j}_{0,M}\left(\frac{1}{\sqrt{1+|z|^2}}, -\frac{z}{\sqrt{1+|z|^2}}\right) \quad (50)$$

or, using the explicit formula for $D^{j}_{0,M}$,

$$\phi^{s,\rho}_{j,M}(z) = \frac{\sqrt{2j+1}}{\pi}(-1)^M\sqrt{\frac{\Gamma(j+\rho+1)\Gamma(1-\rho)}{\Gamma(j-\rho+1)\Gamma(\rho)}} \times \frac{j!}{\sqrt{(j-M)!(j+M)!}}(1+|z|^2)^{-\rho-j-1} \times \sum_k (-1)^k \binom{j+M}{k}\binom{j-M}{j-k} z^{-M+k}\bar{z}^k \quad (51)$$

Trivial representation.—Of course the only spin j appearing in the trivial representation $\chi = t$ is $j = 0$, and $\phi^{t}_{0,0}$ is any normed element of the one-dimensional Hilbert space $\mathcal{H}_t$.

V. REPRESENTATIONS RELEVANT TO THE $j = 0$ CASE AND EXPLICIT INTEGRAL FORMULA FOR THE ISGUR-WISE FUNCTION

Let us return to the case $j = 0$ of a scalar light component. According to the description above, the subset X_0 of irreducible representations of $SL(2, C)$ containing a $SU(2)$ scalar is made of

(i) the subset $n = 0$ of the principal series;

(ii) all the supplementary series;

(iii) the trivial representation.

The $j = 0$ basis element [from (49) and (51))] is ($k = 0$ for $n = 0$, $j = 0$)

$$\phi^{p,0,\rho}_{0,0}(z) = \frac{1}{\sqrt{\pi}}(1+|z|^2)^{i\rho-1} \qquad \chi = (p, 0, \rho) \qquad \rho \geq 0 \quad (52)$$

$$\phi^{s,\rho}_{0,0}(z) = \frac{\sqrt{\rho}}{\pi}(1+|z|^2)^{-\rho-1} \qquad \chi = (s, \rho) \qquad 0 < \rho < 1 \quad (53)$$

$$\phi^{t}_{0,0}(z) = 1 \qquad \chi = t \quad (54)$$

The corresponding irreducible IW functions, according to (33), are

$$\xi_{\chi}(w) = \langle \phi^{\chi}_{0,0} | U_{\chi}(\Lambda_\tau)\phi^{\chi}_{0,0}\rangle \qquad w = \mathrm{ch}(\tau) \quad (55)$$

with

$$\Lambda_\tau = \begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{pmatrix}$$

The transformed elements $U_{\chi}(\Lambda_\tau)\phi^{\chi}_{0,0}$ are given by (39) and (43)

$$(U_{p,0,\rho}(\Lambda_\tau)\phi^{p,0,\rho}_{0,0})(z) = e^{(i\rho-1)\tau}\phi^{p,0,\rho}_{0,0}(e^{-\tau}z) = \frac{1}{\sqrt{\pi}}(e^{\tau} + e^{-\tau}|z|^2)^{i\rho-1} \quad (56)$$

$$(U_{s,\rho}(\Lambda_\tau)\phi^{s,\rho}_{0,0})(z) = e^{-(\rho+1)\tau}\phi^{s,\rho}_{0,0}(e^{-\tau}z) = \frac{\sqrt{\rho}}{\sqrt{\pi}}(e^{\tau} + e^{-\tau}|z|^2)^{-\rho-1} \quad (57)$$

$$U_t(\Lambda_\tau)\phi^{t}_{0,0} = 1 \quad (58)$$

and, the scalar products being given by (38) and (42), we have

$$\xi_{p,0,\rho}(w) = \frac{1}{\pi}\int (1+|z|^2)^{-i\rho-1}(e^{\tau} + e^{-\tau}|z|^2)^{i\rho-1} d^2z \quad (59)$$

$$\xi_{s,\rho}(w) = \frac{\rho}{\pi^2}\int(1+|z'|^2)^{-\rho-1}|z'-z|^{2\rho-2} \times (e^{\tau}+e^{-\tau}|z|^2)^{-\rho-1}d^2z'd^2z \quad (60)$$

$$\xi_t(w) = 1 \quad (61)$$

The integrals being computed in Appendix C, one obtains

$$\xi_{p,0,\rho}(w) = \frac{\sin(\rho\tau)}{\rho \mathrm{sh}(\tau)} \qquad (0 \le \rho) \quad (62)$$

$$\xi_{s,\rho}(w) = \frac{\mathrm{sh}(\rho\tau)}{\rho \mathrm{sh}(\tau)} \qquad (0 < \rho < 1) \quad (63)$$

$$\xi_t(w) = 1 \quad (64)$$

Then our fundamental integral formula (35) for the IW function writes:

$$\xi(w) = \int_{[0,\infty[}\frac{\sin(\rho\tau)}{\rho\mathrm{sh}(\tau)}d\nu_p(\rho) + \int_{]0,1[}\frac{\mathrm{sh}(\rho\tau)}{\rho\mathrm{sh}(\tau)}d\nu_s(\rho) + \nu_t \qquad w = \mathrm{ch}(\tau) \quad (65)$$

where ν_p and ν_s are positive measures on $[0,\infty[$ and $]0,1[$, and ν_t is a real number ≥ 0 (the same thing as a positive measure on the one-element set $\{t\}$), with the only condition that

$$\int_{[0,\infty[}d\nu_p(\rho) + \int_{]0,1[}d\nu_s(\rho) + \nu_t = 1 \quad (66)$$

(and the precise specification of the domain of integration is needed because ν_p and ν_s may include Dirac measures).

For the derivatives $\xi^{(k)}(1)$, formulas (62)–(65) give

$$\xi^{(k)}(1) = \int_{[0,\infty[}\xi^{(k)}_{p,0,\rho}(1)d\nu_p(\rho) + \int_{]0,1[}\xi^{(k)}_{s,\rho}(1)d\nu_s(\rho) + \nu_t\delta_{k,0} \quad (67)$$

with the surprisingly simple expressions for the lower derivatives obtained by direct calculation:

$$\xi_{p,0,\rho}(1) = 1 \qquad \xi_{s,\rho}(1) = 1 \qquad \xi'_{p,0,\rho}(1) = -\frac{1+\rho^2}{3} \qquad \xi'_{s,\rho}(1) = -\frac{1-\rho^2}{3} \qquad \xi''_{p,0,\rho}(1) = \frac{(1+\rho^2)(4+\rho^2)}{15}$$
$$\xi''_{s,\rho}(1) = \frac{(1-\rho^2)(4-\rho^2)}{15} \qquad \xi^{(3)}_{p,0,\rho}(1) = -\frac{(1+\rho^2)(4+\rho^2)(9+\rho^2)}{105} \qquad \xi^{(3)}_{s,\rho}(1) = -\frac{(1-\rho^2)(4-\rho^2)(9-\rho^2)}{105}\ldots \quad (68)$$

and we have

$$\xi(1) = 1 \qquad \xi'(1) = -\frac{1}{3}\left[\int_{[0,\infty[}(1+\rho^2)d\nu_p(\rho) + \int_{]0,1[}(1-\rho^2)d\nu_s(\rho)\right]$$
$$\xi''(1) = \frac{1}{15}\left[\int_{[0,\infty[}(1+\rho^2)(4+\rho^2)d\nu_p(\rho) + \int_{]0,1[}(1-\rho^2)(4-\rho^2)d\nu_s(\rho)\right] \quad (69)$$
$$\xi^{(3)}(1) = -\frac{1}{105}\left[\int_{[0,\infty[}(1+\rho^2)(4+\rho^2)(9+\rho^2)d\nu_p(\rho) + \int_{]0,1[}(1-\rho^2)(4-\rho^2)(4-\rho^2)d\nu_s(\rho)\right]\ldots$$

where ν_p and ν_s are arbitrary positive measures satisfying

$$\int_{[0,\infty[}d\nu_p(\rho) + \int_{]0,1[}d\nu_s(\rho) \le 1 \quad (70)$$

At first sight, the deduction of the constraints on the derivatives $\xi^{(k)}(1)$ from (69) under the condition (70) promises to be a tricky work. However, the problem will be reduced to an already solved one by rewriting (65) in a simpler form (namely, with just one integral). This is possible because the irreducible IW functions, principal, supplementary and trivial, can all be put into a *one-parameter family*:

$$\xi_x(w) = \frac{\mathrm{sh}(\tau\sqrt{1-x})}{\mathrm{sh}(\tau)\sqrt{1-x}} = \frac{\sin(\tau\sqrt{x-1})}{\mathrm{sh}(\tau)\sqrt{x-1}}. \quad (71)$$

Indeed we have

$$\begin{aligned} \xi_{p,0,\rho}(w) &= \xi_x(w) \qquad x = 1+\rho^2 \qquad \rho \in [0,\infty[\Leftrightarrow x \in [1,\infty[\\ \xi_{s,\rho}(w) &= \xi_x(w) \qquad x = 1-\rho^2 \qquad \rho \in]0,1[\Leftrightarrow x \in]0,1[\\ \xi_t(w) &= \xi_x(w) \qquad x = 0 \qquad x \in \{0\} \end{aligned} \tag{72}$$

so that $\xi_x(w)$ is $\xi_t(w)$ for $x=0$, is $\xi_{s,\rho}(w)$ for $0 < x < 1$, and is $\xi_{p,0,\rho}(w)$ for $1 \le x$. Then formula (66) writes

$$\xi(w) = \int_{[0,\infty[} \xi_x(w) d\nu(x) \tag{73}$$

with $\xi_x(w)$ given explicitly by (71) and ν is a positive measure on $[0,\infty[$, with the only condition that

$$\int_{[0,\infty[} d\nu(x) = 1 \tag{74}$$

According to (72), ν can be obtained in terms of ν_t, ν_s, and ν_p, through its restrictions to the subsets $\{0\}$, $]0,1[$, and $[1,\infty[$ of $[0,\infty[$, by changes of variable. In particular, the "trivial" ν_t term in (65) will be due to a Dirac δ contribution to ν at $x=0$.

For the derivatives $\xi^{(k)}(1)$, formula (73) gives

$$\xi^{(k)}(1) = \int_{[0,\infty[} \xi_x^{(k)}(1) d\nu(x) \tag{75}$$

As may be suspected from (68), $\xi_x^{(k)}(1)$ is a polynomial of degree k in x. As shown in Appendix D, it is given by

$$\xi_x^{(k)}(1) = (-1)^k 2^k \frac{k!}{(2k+1)!} \prod_{i=1}^{k} (x+i^2-1) \tag{76}$$

VI. THE CONSTRAINTS ON MOMENTS AND ON DERIVATIVES OF THE ISGUR-WISE FUNCTION

We may now deduce the constraints on the derivatives. From (75) and (76), the derivative $\xi^{(k)}(1)$ is given by the *expectation value* of a polynomial of degree k:

$$\xi^{(k)}(1) = (-1)^k 2^k \frac{k!}{(2k+1)!} \left\langle \prod_{i=1}^{k} (x+i^2-1) \right\rangle \tag{77}$$

with the expectation value defined by

$$\langle f(x) \rangle = \int_{[0,\infty[} f(x) d\nu(x) \tag{78}$$

for some normalized positive measure ν supported in $[0,\infty[$. One obtains

$$\begin{aligned} \xi(1) &= \langle 1 \rangle = 1 \\ \xi'(1) &= -\tfrac{1}{3}\langle x \rangle \\ \xi''(1) &= \tfrac{1}{15}\langle x(3+x) \rangle \\ \xi^{(3)}(1) &= -\tfrac{1}{105}\langle x(3+x)(8+x) \rangle \\ \xi^{(4)}(1) &= \tfrac{1}{945}\langle x(3+x)(8+x)(15+x) \rangle \end{aligned} \tag{79}$$

Expanding the polynomial, $\xi^{(k)}(1)$ is expressed as a combination of the moments $\mu_0, \mu_1, \ldots \mu_k$ of x, a moment μ_n being the expectation value of x^n:

$$\mu_n = \langle x^n \rangle \tag{80}$$

(notice that these moments could be infinite for $n \ge 1$, although, in the following, we will not consider this case).

So we have

$$\begin{aligned} \xi(1) &= \mu_0 = 1 \\ \xi'(1) &= -\tfrac{1}{3}\mu_1 \\ \xi''(1) &= \tfrac{1}{15}(3\mu_1 + \mu_2) \\ \xi^{(3)}(1) &= -\tfrac{1}{105}(24\mu_1 + 11\mu_2 + \mu_3) \\ \xi^{(4)}(1) &= \tfrac{1}{945}(360\mu_1 + 189\mu_2 + 26\mu_3 + \mu_4) \end{aligned} \tag{81}$$

These equations can be solved step by step, and the moment μ_k is expressed as a combination of the derivatives $\xi(1), \xi'(1), \ldots \xi^{(k)}(1)$:

$$\begin{aligned} \mu_0 &= \xi(1) = 1 \\ \mu_1 &= -3\xi'(1) \\ \mu_2 &= 3[3\xi'(1) + 5\xi''(1)] \\ \mu_3 &= -3[9\xi'(1) + 55\xi''(1) + 35\xi^{(3)}(1)] \\ \mu_4 &= 3[27\xi'(1) + 485\xi''(1) + 910\xi^{(3)}(1) + 315\xi^{(4)}(1)] \end{aligned} \tag{82}$$

Now, in [3] one has obtained a whole set of constraints on the moments of a variable with positive values, and also shown that this set of constraints is optimal, meaning that it cannot be improved (there are no other nor more strict constraints from the general definition of the moments).

In fact, [3] was concerned only with the particular case of a measure ν of the form

$$d\nu(x) = w(x)dx$$

with a weight function w (the measure ν is then said to be completely continuous with respect to the measure dx). In particular, this excludes Dirac δ contributions to ν and this, while perhaps physically reasonable, is not assumed in the present context.

However, the deduction of the constraints in [3] goes through in the present case of an arbitrary positive measure ν just by replacing strict inequalities $>$ by nonstrict ones $\ge$. Therefore, the set of constraints is as follows. For any $n \ge 0$, one has [3]

$$\det[(\mu_{i+j})_{0 \le i,j \le n}] \ge 0 \tag{83}$$

$$\det[(\mu_{i+j+1})_{0\leq i,j\leq n}] \geq 0 \tag{84}$$

(On the other hand, the optimality proof in [3] needs additional arguments in the present case.)

Since each moment μ_k is a combination of the derivatives $\xi(1), \xi'(1), \ldots \xi^{(k)}(1)$, the constraints on the moments translate into constraints on the derivatives.

We shall treat here in detail only the constraints on μ_1, μ_2, μ_3, and μ_4, which are given, respectively, by (84) ($n = 0$), (83) ($n = 1$), (84) ($n = 1$), (83) ($n = 2$).

$$\mu_1 \geq 0 \tag{85}$$

$$\det\begin{pmatrix} 1 & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} = \mu_2 - \mu_1^2 \geq 0 \tag{86}$$

$$\det\begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{pmatrix} = \mu_1\mu_3 - \mu_2^2 \geq 0 \tag{87}$$

$$\det\begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = (\mu_2 - \mu_1^2)\mu_4 - (\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_2^3) \geq 0 \tag{88}$$

Clearly, each moment μ_k is bounded from below, and the lower bound is given by (83) for k even and by (84) for k odd in terms of the lower moments. So (85)–(88) give

$$\mu_1 \geq 0 \tag{89}$$

$$\mu_2 \geq \mu_1^2 \tag{90}$$

$$\mu_3 \geq \frac{\mu_2^2}{\mu_1} \tag{91}$$

$$\mu_4 \geq \frac{\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_2^3}{\mu_2 - \mu_1^2} = \frac{(\mu_3 - \mu_1\mu_2)^2}{\mu_2 - \mu_1^2} + \mu_2^2 \tag{92}$$

If one of these inequalities is saturated, the inequalities following it may be meaningless. However, even in such a case, the inequalities (83) and (84) remain perfectly valid. Moreover we have in fact much more in this case, because the measure ν is then a *completely determined* combination of δ functions, and the IW function is completely fixed and explicitly given. In the following, we shall *explicitly* show this for the saturation of inequalities (89)–(91).

So, in the case $\mu_1 = 0$ of (89), Eq. (91) is meaningless. Equations (85)–(88) are nevertheless fully valid and, for instance, (87) gives $\mu_2 = 0$. In fact, when $\mu_1 = 0$ we have much more, since the condition

$$\mu_1 = \int_{[0,\infty[} x d\nu(x) = 0 \tag{93}$$

completely determines the measure ν:

$$\mu_1 = 0 \Leftrightarrow d\nu(x) = \delta(x)dx \tag{94}$$

(in particular then, $\mu_k = 0$ for all $k \geq 1$).

Also if the bound (90) is saturated, Eq. (92) becomes meaningless and, for instance, Eq. (88) gives $\mu_3 = \mu_1^3$. In fact, when $\mu_2 - \mu_1^2 = 0$ we have again much more, since the condition

$$\mu_2 - \mu_1^2 = \int_{[0,\infty[} (x - \mu_1)^2 d\nu(x) = 0 \tag{95}$$

also completely determines the measure ν. Since the integrand $(x - \mu_1)^2$ is >0 when $x \neq \mu_1$, the integral can vanish only if the measure is concentrated on the set $\{\mu_1\}$, and with the condition $\langle 1\rangle = 1$, we have

$$\mu_2 = \mu_1^2 \Leftrightarrow d\nu(x) = \delta(x - \mu_1)dx \tag{96}$$

(in particular then, $\mu_k = \mu_1^k$ for all $k \geq 1$).

In the same way, if (85) is strict ($\mu_1 > 0$) and (87) is saturated ($\mu_3 = \frac{\mu_2^2}{\mu_1}$), we have the condition

$$\mu_3 - \frac{\mu_2^2}{\mu_1} = \int_{[0,\infty[} x\left(x - \frac{\mu_2}{\mu_1}\right)^2 d\nu(x) = 0 \tag{97}$$

which completely determines the measure ν. The measure is concentrated on the set of two elements $\{0, \frac{\mu_2}{\mu_1}\}$, and with the conditions $\langle 1\rangle = 1$ and $\langle x\rangle = \mu_1$, we have

$$\begin{aligned} &\mu_1 > 0 \\ &\mu_3 = \frac{\mu_2^2}{\mu_1} \Leftrightarrow d\nu(x) \\ &\quad = \left[\left(1 - \frac{\mu_1^2}{\mu_2}\right)\delta(x) + \frac{\mu_1^2}{\mu_2}\delta\left(x - \frac{\mu_2}{\mu_1}\right)\right]dx \end{aligned} \tag{98}$$

[in particular then, $\mu_k = (\frac{\mu_2}{\mu_1})^{k-1}\mu_1$ for all $k \geq 1$].

Incidentally, this gives a hint on how to complete the optimality proof of [3]. If all the inequalities (83) and (84) are strict, then the proof in [3] is valid. If there is an equality, one can show that the values of the moments can be obtained by some finite combination of Dirac δ measures for ν.

Finally, using (82), the results on the moments μ_k are converted into results on the derivatives $\xi^{(k)}(1)$. With our usual notation for the slope and curvature

$$\rho_\Lambda^2 = -\xi'(1) \qquad \sigma_\Lambda^2 = \xi''(1) \tag{99}$$

the constraints (89)–(92) write

$$\rho_\Lambda^2 \geq 0 \tag{100}$$

$$\sigma_\Lambda^2 \geq \tfrac{3}{5}\rho_\Lambda^2(1 + \rho_\Lambda^2) \tag{101}$$

$$-\xi^{(3)}(1) \geq \frac{5}{7}\frac{\sigma_\Lambda^2}{\rho_\Lambda^2}(\rho_\Lambda^2 + \sigma_\Lambda^2) \tag{102}$$

$$\xi^{(4)}(1) \geq \frac{5}{63}\frac{1}{5\sigma_\Lambda^2 - 3\rho_\Lambda^2(1+\rho_\Lambda^2)}[3\sigma_\Lambda^2(\sigma_\Lambda^2+\rho_\Lambda^2)(5\sigma_\Lambda^2+8\rho_\Lambda^2+8) + 7(12\rho_\Lambda^2(1+\rho_\Lambda^2)+2(3\rho_\Lambda^2-2)\sigma_\Lambda^2+7\xi^{(3)}(1))\xi^{(3)}(1)]$$
$$= \frac{5}{63}\frac{1}{(\rho_\Lambda^2)^2}\sigma_\Lambda^2(\sigma_\Lambda^2+\rho_\Lambda^2)(5\sigma_\Lambda^2+12\rho_\Lambda^2) + \frac{5}{9}\left[\frac{2}{\rho_\Lambda^2}(\sigma_\Lambda^2+2\rho_\Lambda^2)+\frac{7}{5}\left(\frac{-\xi^{(3)}(1)-(-\xi^{(3)}(1))_{\min}}{\sigma_\Lambda^2-(\sigma_\Lambda^2)_{\min}}\right)\right][-\xi^{(3)}(1)-(-\xi^{(3)}(1))_{\min}] \quad (103)$$

Eliminating σ_Λ^2 from (102), we have a looser but simpler bound for $-\xi^{(3)}(1)$ depending only on the slope:

$$-\xi^{(3)}(1) \geq \tfrac{3}{35}\rho_\Lambda^2(1+\rho_\Lambda^2)(8+3\rho_\Lambda^2) \quad (104)$$

Eliminating $\xi^{(3)}(1)$ from (103), we have also a looser but simpler bound for $\xi^{(4)}(1)$, depending only on the slope and the curvature:

$$\xi^{(4)}(1) \geq \frac{5}{63}\frac{1}{(\rho_\Lambda^2)^2}\sigma_\Lambda^2(\sigma_\Lambda^2+\rho_\Lambda^2)(5\sigma_\Lambda^2+12\rho_\Lambda^2) \quad (105)$$

and eliminating σ_Λ^2 from (105), we have

$$\xi^{(4)}(1) \geq \tfrac{1}{35}\rho_\Lambda^2(1+\rho_\Lambda^2)(8+3\rho_\Lambda^2)(5+\rho_\Lambda^2) \quad (106)$$

A. Illustration of the inequalities for some models of the Isgur-Wise function

To have some feeling of what happens numerically, let us illustrate the preceding inequalities for two *Ansätze* of the IW function, the exponential and the "dipole" forms, that depend on a single parameter $c = \rho_\Lambda^2$.

In the case of the exponential form

$$\xi(w) = \exp[-c(w-1)] \quad (107)$$

From its derivatives

$$(-1)^k\xi^{(k)}(1) = c^k$$

the inequalities (100)–(103) yield, respectively, the following bounds on the parameter $c = \rho_\Lambda^2$:

$$c \geq 0 \qquad c \geq 1.5 \qquad c \geq 2.5 \qquad c \geq 4.28 \quad (108)$$

We observe that in this case the lower bound on the slope c increases strongly as we impose the constraint coming from higher order derivatives. We can suspect therefore that this function is not physically acceptable. Indeed, we will below rigorously demonstrate that this is the case.

Let us now consider another *Ansatz*, namely, the dipole form

$$\xi(w) = \left(\frac{2}{w+1}\right)^{2c} \quad (109)$$

From its derivatives

$$(-1)^k\xi^{(k)}(1) = \frac{2c(2c+1)\ldots(2c+k-1)}{2^k}$$

in this case we find, from (100), the trivial lower bound $c = \rho_\Lambda^2 \geq 0$, while from the three inequalities (101)–(103) we get the same lower bound, namely,

$$c = \rho_\Lambda^2 \geq \tfrac{1}{4} \quad (110)$$

Therefore it seems that, unlike the exponential form (107), the dipole *Ansatz* (109) satisfies in a regular way the inequalities that the derivatives of the IW function must fulfill. We will demonstrate in Sec. IX E that this function satisfies indeed the general constraints that the IW function must fulfill for $c = \rho_\Lambda^2 \geq \frac{1}{4}$.

As another example, let us consider the form

$$\xi(w) = \frac{1}{[1+\frac{c}{2}(w-1)]^2} \quad (111)$$

that, as discussed below, has been proposed in the literature. The first derivatives read

$$\rho_\Lambda^2 = c \qquad \sigma_\Lambda^2 = \tfrac{3}{2}(\rho_\Lambda^2)^2 \qquad -\xi^{(3)}(1) = 3(\rho_\Lambda^2)^3 \quad (112)$$

and the inequalities (101)–(103) yield, respectively, the following bounds on the parameter $c = \rho_\Lambda^2$:

$$\rho_\Lambda^2 \geq \tfrac{2}{3} \qquad \rho_\Lambda^2 \geq \tfrac{10}{13} \qquad \rho_\Lambda^2 \geq 0.86 \quad (113)$$

The successive lower bounds on the slope slowly grow and converge towards $\rho_\Lambda^2 = 1$. We will demonstrate in Sec. IX E that this function satisfies the general constraints that the IW function must fulfill for $c = \rho_\Lambda^2 \geq 1$.

B. Completely explicit form of the Isgur-Wise function when the inequality on one of the low order derivatives is saturated

We have seen above that if an equality is saturated, then the measure ν is completely fixed. According to (73), the IW function $\xi(w)$ is also *completely fixed*.

If (100) is saturated, namely, if $\rho_\Lambda^2 = 0$, then from (94) we have

$$\xi(w) = 1 \quad (114)$$

This being physically excluded, we may replace (100) by the strict inequality $\rho_\Lambda^2 > 0$.

If (101) is saturated, namely, if $\sigma_\Lambda^2 = \frac{3}{5}\rho_\Lambda^2(1+\rho_\Lambda^2)$, then from (96) we have

$$\xi(w) = \frac{\mathrm{sh}(\tau\sqrt{1-3\rho_\Lambda^2})}{\mathrm{sh}(\tau)\sqrt{1-3\rho_\Lambda^2}} = \frac{\sin(\tau\sqrt{3\rho_\Lambda^2-1})}{\mathrm{sh}(\tau)\sqrt{3\rho_\Lambda^2-1}}$$

$$w = \mathrm{ch}(\tau) \qquad (115)$$

We have here possible physically limiting cases (if $\rho_\Lambda^2 > 0$). In fact (115) means simply, as we have announced above, that the light component of the heavy hadron belongs to some *irreducible* representation of the Lorentz group.

In view of (72), when the slope ρ_Λ^2 goes from 0 to ∞, each irreducible representation occurs in turn: $\rho_\Lambda^2 = 0$ for the trivial representation, $0 < \rho_\Lambda^2 < \frac{1}{3}$ for the supplementary series, and $\frac{1}{3} \le \rho_\Lambda^2$ for the $n = 0$ principal series.

From (76), the derivatives $\xi^{(k)}(1)$ are then given by

$$\xi^{(k)}(1) = (-1)^k 2^k \frac{k!}{(2k+1)!}\prod_{i=1}^{k}(3\rho_\Lambda^2 + i^2 - 1) \qquad (116)$$

Finally, let us consider the case where (102) is saturated, $-\xi^{(3)} = \frac{5}{7}\frac{\sigma_\Lambda^2}{\rho_\Lambda^2}(\rho_\Lambda^2 + \sigma_\Lambda^2)$ (and $\rho_\Lambda^2 > 0$). Then, from (98) we have

$$\xi(w) = \frac{\mu_1^2}{\mu_2}\frac{\mathrm{sh}(\tau\sqrt{1-\frac{\mu_2}{\mu_1}})}{\mathrm{sh}(\tau)\sqrt{1-\frac{\mu_2}{\mu_1}}} + \left(1 - \frac{\mu_1^2}{\mu_2}\right) \qquad (117)$$

where

$$\mu_1 = 3\rho_\Lambda^2 \qquad \mu_2 = 3(5\sigma_\Lambda^2 - 3\rho_\Lambda^2) \qquad (118)$$

and $w = \mathrm{ch}(\tau)$. If we impose the physical condition that $\xi(w) \to 0$ when $w \to \infty$, then (117) is possible only if $\frac{\mu_1^2}{\mu_2} = 1$, that is, if (101) is saturated, a case fully discussed just above. So, we may conclude that if the inequality (101) is strict, then inequality (102) must also be strict [to avoid a nonvanishing limit of $\xi(w)$ when $w \to \infty$].

VII. DEMONSTRATION FROM SUM RULES THAT THE ISGUR-WISE FUNCTION IS OF POSITIVE TYPE: APPLICATION TO THE EXPONENTIAL FORM

A. Demonstration from sum rules that the Isgur-Wise function is of positive type

In this part we will demonstrate that the IW function $\xi(w)$ is of *positive type*, i.e. that, for any value of N and any complex numbers $a_1, \ldots, a_N$ and velocities $v_1, \ldots, v_N$ satisfies

$$\sum_{i,j=1}^{N} a_i^* a_j \xi(v_i.v_j) \ge 0 \qquad (119)$$

Notice that a positive-type function does not mean that this function is positive for all values of its argument.

As pointed out in the introduction, in the heavy quark limit of QCD from the OPE and the nonforward amplitude, we have demonstrated a sum rule for the $j = 0$ case [8], from which we have obtained the inequalities for the slope (100) [9] and for the curvature (101).

We have recently realized that the expression for this sum rule can be simplified enormously. Using the expression for the sum rule obtained in [8], this equivalent form is deduced in Appendix E (we replace the subindex f by j):

$$\xi(w_{ij}) = \sum_n \sum_L \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_j) \times \sum_{0\le k\le L/2} C_{L,k}(w_i^2-1)^k(w_j^2-1)^k(w_iw_j - w_{ij})^{L-2k} \qquad (120)$$

where $w_i = v_i.v'$, $w_j = v_j.v'$, $w_{ij} = v_i.v_j$ and v_i, v_j, and v' are the initial, final and intermediate state four-velocities in the sum rule, respectively, $\tau_L^{(n)}(w)$ are the IW functions for the transition $0^+ \to L^P$ with $P = (-1)^L$, and the coefficients $C_{L,k}$ are given by

$$C_{L,k} = (-1)^k \frac{(L!)^2}{(2L)!}\frac{(2L-2k)!}{k!(L-k)!(L-2k)!} \qquad (121)$$

The last sum in (120) can be expressed in terms of a Legendre polynomial, as demonstrated in Appendix A of the first reference of [7], and explicitly written in Appendix E of the present paper. We use now this derivation to express this Legendre polynomial in terms of spherical harmonics. Without loss of generality, let us use the rest frame for the intermediate states, i.e. $v' = (1, 0, 0, 0)$, that gives

$$w_i^2 - 1 = \vec{v}_i^2 \qquad w_j^2 - 1 = \vec{v}_j^2 \qquad w_iw_j - w_{ij} = \vec{v}_i.\vec{v}_j \qquad (122)$$

and using the results of Appendix A of the first reference of [7] we obtain

$$\sum_{0\le k\le L/2} C_{L,k}(\vec{v}_i^2)^k(\vec{v}_j^2)^k(\vec{v}_i.\vec{v}_j)^{L-2k} = 4\pi 2^L \frac{(L!)^2}{(2L+1)!}\sum_{m=-L}^{m=+L} \mathcal{Y}_L^m(\vec{v}_i)^* \mathcal{Y}_L^m(\vec{v}_j) \qquad (123)$$

Combining the previous equations, we find

$$\sum_{i,j=1}^{N} a_i^* a_j \xi(v_i.v_j) = 4\pi \sum_{i,j=1}^{N}\sum_n\sum_L \frac{2^L(L!)^2}{(2L+1)!} \times \sum_{m=-L}^{m=+L}[a_i\tau_L^{(n)}(\sqrt{1+\vec{v}_i^2})\mathcal{Y}_L^m(\vec{v}_i)]^* \times [a_j\tau_L^{(n)}(\sqrt{1+\vec{v}_j^2})\mathcal{Y}_L^m(\vec{v}_j)] \ge 0 \qquad (124)$$

and therefore the inequality (119) has been proved.

In conclusion, we have demonstrated that the IW function $\xi(w)$ is of positive type. The inequality (119), concerning a Riemann sum, would read, in a continuous (and covariant) form,

$$\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}'}{v'^0}\psi(v')^*\xi(v.v')\psi(v) \geq 0, \tag{125}$$

where $\psi(v)$ is an arbitrary function.

B. Inconsistency with the sum rules of an exponential form for the Isgur-Wise function

We have seen that if the sum rules are satisfied for a $j = 0$ IW function $\xi(w)$, then, for any function $\psi(v)$, we have the positivity condition (125).

We will now show that, for an exponential form, a function that one could guess from the harmonic oscillator potential

$$\xi(w) = \exp[-c(w-1)] \tag{126}$$

one can find a function $\psi(v)$ for which the integral in (125) is *strictly negative*, and this will prove that the exponential form of the IW function is incompatible with the sum rules.

For our purpose, it is enough to consider radial $\psi(v)$ functions

$$\psi(v) = \phi(|\vec{v}|) \tag{127}$$

First, let us integrate over the angles of $\vec{v}$ and $\vec{v}'$. Adopting just in a few lines below the notation $|\vec{v}| = v$ and $|\vec{v}'| = v'$, we obtain for the integral

$$\begin{aligned}\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}'}{v'^0}\phi(|\vec{v}'|)^*\exp[-c((v.v')-1)]\phi(|\vec{v}|) &= 8\pi^2 e^c\int_0^\infty \frac{dv}{v^0}\int_0^\infty\frac{dv'}{v'^0}\int_{-1}^1 ds\phi(v')^*\exp[-c(v^0v'^0-vv's)]v^2v'^2\phi(v)\\ &= 8\pi^2\frac{e^c}{c}\int_0^\infty \frac{dv}{v^0}\int_0^\infty\frac{dv'}{v'^0}\int_{-1}^1 ds\phi(v')^*\phi(v)\frac{d}{ds}(\exp[-c(v^0v'^0-vv's)])vv'\\ &= 8\pi^2\frac{e^c}{c}\int_0^\infty \frac{dv}{v^0}\int_0^\infty\frac{dv'}{v'^0}\phi(v')^*\phi(v)(\exp[-c(v^0v'^0-vv')]\\ &\quad -\exp[-c(v^0v'^0+vv')])vv'\end{aligned}$$

Next we change the variables of integration

$$v = \mathrm{sh}(\eta) \qquad v' = \mathrm{sh}(\eta') \tag{128}$$

and this gives

$$\begin{aligned}\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}'}{v'^0}\phi(|\vec{v}'|)^*\exp[-c((v.v')-1)]\phi(|\vec{v}|) &= 8\pi^2\frac{e^c}{c}\int_0^\infty\int_0^\infty d\eta d\eta'\mathrm{sh}(\eta')\phi(sh(\eta'))^*\mathrm{sh}(\eta)\phi(\mathrm{sh}(\eta))\\ &\quad\times(\exp[-c\mathrm{ch}(\eta'-\eta)]-\exp[-c\mathrm{ch}(\eta'+\eta)])\\ &= 4\pi^2\frac{e^c}{c}\int_{-\infty}^\infty\int_{-\infty}^\infty d\eta d\eta'\mathrm{sh}(\eta')\phi(\mathrm{sh}(\eta'))^*\\ &\quad\times\exp[-c\mathrm{ch}(\eta'-\eta)]\mathrm{sh}(\eta)\phi(\mathrm{sh}(\eta))\end{aligned}$$

In the last line, the function $\phi(v)$ is extended to negative arguments by

$$\phi(-v) = \phi(v) \tag{129}$$

Introducing the functions $f(\eta)$ and $K(\eta)$ by

$$f(\eta) = \mathrm{sh}(\eta)\phi(\mathrm{sh}(\eta)) \qquad K(\eta) = \exp[-c\mathrm{ch}(\eta)] \tag{130}$$

the last result writes

$$\begin{aligned}&\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}'}{v'^0}\phi(|\vec{v}'|)^*\exp[-c((v.v')-1)]\phi(|\vec{v}|)\\ &\quad = 4\pi^2\frac{e^c}{c}\int_{-\infty}^\infty\int_{-\infty}^\infty d\eta d\eta' f(\eta')^*K(\eta'-\eta)f(\eta)\end{aligned} \tag{131}$$

But we have here the matrix element of a convolution operator, and this is diagonalized by Fourier transforming. So, introducing

$$\tilde{f}(\rho) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\rho\eta}f(\eta)d\eta \tag{132}$$

$$\tilde{K}(\rho) = \int_{-\infty}^\infty e^{i\rho\eta}K(\eta)d\eta \tag{133}$$

we have

$$\begin{aligned}&\int_{-\infty}^\infty\int_{-\infty}^\infty d\eta d\eta' f(\eta')^*K(\eta'-\eta)f(\eta)\\ &\quad = 2\pi\int_{-\infty}^\infty \tilde{K}(\rho)|\tilde{f}(\rho)|^2 d\rho\end{aligned} \tag{134}$$

And finally we obtain

$$\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}'}{v'^0}\phi(|\vec{v}'|)^* \exp[-c((v.v')-1)]\phi(|\vec{v}|) = 8\pi^3 \frac{e^c}{c}\int_{-\infty}^{\infty}\tilde{K}(\rho)|\tilde{f}(\rho)|^2 d\rho \quad (135)$$

with $\tilde{f}(\rho)$ and $\tilde{K}(\rho)$ given by (129), (130), (132), and (133).

Now the Sommerfeld integral representation of the Macdonald function

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty}\exp[-z\mathrm{ch}(t)]e^{\nu t}dt \quad (136)$$

gives the following expression of $\tilde{K}(\rho)$:

$$\tilde{K}(\rho) = 2K_{i\rho}(c) \quad (137)$$

and we have

$$\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}'}{v'^0}\phi(|\vec{v}'|)^* \exp[-c((v.v')-1)]\phi(|\vec{v}|) = 16\pi^3 \frac{e^c}{c}\int_{-\infty}^{\infty}K_{i\rho}(c)|\tilde{f}(\rho)|^2 d\rho \quad (138)$$

Then, we observe that, whatever $c > 0$, the function $\rho \to K_{i\rho}(c)$ *takes negative values*, as shown by the asymptotic formula

$$K_{i\rho}(c) \sim \sqrt{\frac{2\pi}{\rho}}e^{-\rho\pi/2}\cos\left[\rho\left(\log\left(\frac{2\rho}{c}\right)-1\right)-\frac{\pi}{4}\right] \quad (\rho \gg c) \quad (139)$$

So, taking for $\tilde{f}(\rho)$ a function peaked at a point ρ_0 where $K_{i\rho_0}(c) < 0$, the right-hand side of (138) will be <0. Reversing the steps from $\psi(v)$ (where now v is the quadrivector velocity) to $\tilde{f}(\rho)$, we then have a function $\psi(v)$ for which the integral in (125) is <0.

VIII. EQUIVALENCE BETWEEN THE SUM RULE APPROACH AND THE LORENTZ GROUP APPROACH

We show here that the Lorentz group approach, introduced in the present work, is in fact equivalent to the (generalized Bjorken) sum rules. So, it must be considered just as a powerful way of exploring the consequences of these SR.

Here we must be more specific. When we say generalized Bjorken SR we do not mean the SR involving higher moments like Voloshin's [13] and its generalizations [14], but those that concern zero order moments. Of course, these sum rules are also not the ones of the QCD sum rule approach *à la* Shifman *et al.*

For baryons we mean the SR that we have formulated in all generality in [8] and, for mesons, Uraltsev SR [6] and the SR formulated in [7], that allow one to obtain bounds on all the derivatives of the IW function at zero recoil.

That the Lorentz group approach implies the sum rules is already stressed in [2]. In fact, the SR are just completeness relations in the Hilbert space of the light components:

$$\langle v_f, j', \epsilon', \alpha'|v_i, j, \epsilon, \alpha\rangle = \sum_{j'',M,\beta}\langle v_f, j', \epsilon', \alpha'|v, j'', \epsilon^M, \beta\rangle \times \langle v, j'', \epsilon^M, \beta|v_i, j, \epsilon, \alpha\rangle \quad (140)$$

when the overlaps are expressed in terms of the IW functions.

By the way, the fact that the $j = 0$ IW function $\xi(w)$ is of positive type, as expressed by (119), has also a very simple proof in the Lorentz group approach. Indeed, following (31), let us write

$$\xi(w) = \langle U(B_{v'})\psi_0|U(B_v)\psi_0\rangle \quad (141)$$

where B_v is the boost transforming the rest velocity v_0 into v. Then we have

$$\sum_{i,j=1}^{N}a_i^* a_j \xi(v_i.v_j) = \sum_{i,j=1}^{N}a_i^* a_j\langle U(B_{v_i})\psi_0|U(B_{v_j})\psi_0\rangle = \left\langle \sum_{i=1}^{N}a_i U(B_{v_i})\psi_0 \,\middle|\, \sum_{j=1}^{N}a_j U(B_{v_j})\psi_0\right\rangle = \left\|\sum_{j=1}^{N}a_j U(B_{v_j})\psi_0\right\|^2 \geq 0 \quad (142)$$

Let us now turn to the proof that the sum rule approach implies the Lorentz group approach, so that the results in this work are in fact consequences of the sum rules. The proof is based on a theorem about functions of positive type on a group (see Dixmier [11]).

A function $f(\Lambda)$ on the group $SL(2, C)$ is of positive type when

$$\sum_{i,j=1}^{N}a_i^* a_j f(\Lambda_i^{-1}\Lambda_j) \geq 0 \quad (143)$$

for any $N \geq 1$, any complex numbers $a_1, \ldots, a_N$, and any $\Lambda_1, \ldots, \Lambda_N \in SL(2, C)$. Then, according to Theorem 13.4.5 in [11] (*C*-algèbres*), for any such function $f(\Lambda)$ of positive type, there exists a unitary representation $U(\Lambda)$ of $SL(2, C)$ in a Hilbert space $\mathcal{H}$, and an element $\phi_0 \in \mathcal{H}$, such that

$$f(\Lambda) = \langle\phi_0|U(\Lambda)\phi_0\rangle. \quad (144)$$

Moreover, one may assume that $\mathcal{H}$ contains no invariant strict subspace containing ϕ_0, and then $\mathcal{H}$, $U(\Lambda)$ and ϕ_0 are unique up to isomorphisms. This will not be used here, but it will show that a $j = 0$ IW function completely determines the Lorentz representation and the scalar state (or in fact the subrepresentation generated by the scalar state).

Now, in Sec. VII we have proved, by using only the sum rules, the positivity-type property (119) of the $j = 0$ IW function $\xi(w)$. To apply the theorem quoted above, we must define, from $\xi(w)$, a function $f(\Lambda)$ of positive type on the group $SL(2, C)$. This is done as follows:

$$f(\Lambda) = \xi((\Lambda v_0)^0) \qquad v_0 = (1, 0, 0, 0) \tag{145}$$

Indeed we have

$$f(\Lambda_i^{-1}\Lambda_j) = \xi(v_0.\Lambda_i^{-1}\Lambda_j v_0) = \xi(\Lambda_i v_0.\Lambda_j v_0) \tag{146}$$

and, taking $v_i = \Lambda_i v_0$ and $v_j = \Lambda_j v_0$ in (119), one sees that (143) is satisfied.

We may conclude, from the sum rules, and from the positivity-type property of $\xi(w)$ which follows from them, that there exists a unitary representation $U(\Lambda)$ of $SL(2, C)$ in a Hilbert space $\mathcal{H}$, and an element $\phi_0 \in \mathcal{H}$, such that

$$\xi((\Lambda v_0)^0) = \langle \phi_0 | U(\Lambda)\phi_0 \rangle \tag{147}$$

But this is just the expression (31) of $\xi(w)$ which occurred in the Lorentz group approach.

There is a last point to be proved, namely, that the state $\phi_0 \in \mathcal{H}$ which occurs here is a scalar under rotations. This results from the following property of $f(\Lambda)$ as defined by (145):

$$f(R\Lambda) = f(\Lambda) \qquad R \in SU(2) \tag{148}$$

Looking at (144), this implies

$$\langle U(R)\phi_0 - \phi_0 | U(\Lambda)\phi_0 \rangle = 0 \tag{149}$$

for any $R \in SU(2)$ and any $\Lambda \in SL(2, C)$. Applying this with $\Lambda = 1$ and $\Lambda = R$, one obtains

$$\|U(R)\phi_0 - \phi_0\|^2 = \langle U(R)\phi_0 - \phi_0 | U(R)\phi_0 - \phi_0 \rangle = 0 \tag{150}$$

so that

$$U(R)\phi_0 = \phi_0 \tag{151}$$

as we wanted to demonstrate.

IX. ANOTHER APPLICATION OF THE LORENTZ GROUP APPROACH: CONSISTENCY TEST FOR ANY *ANSATZ* OF THE ISGUR-WISE FUNCTION

Section VII was based only on the traditional method of (generalized Bjorken) sum rules, leaving aside for a while the representation of the Lorentz group and its decomposition into irreducible representations. We have shown that the sum rules imply a positive-type property of the IW function, and that, in the case of an exponential form, there are functions on which the associated quadratic form is in fact negative.

Here we return to the main method of this work. As a result, we obtain a systematic way of testing the consistency of any *Ansatz* for the IW function. We prove again (in a more comprehensive way) the inconsistency of the exponential form, but, as a positive result, we propose other possible forms for the IW function that we demonstrate to be consistent.

A. Inversion of the integral representation of the Isgur-Wise function

Now, the fundamental question to be solved is the inversion of the integral representation (65), namely, given a function $\xi(w)$, to find the measures $d\nu_p(\rho)$ and $d\nu_s(\rho)$ and a number ν_t such that (65) holds.

The solution to that problem will be found by Fourier transforming the function $\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))$. After multiplying by $\mathrm{sh}(\tau)$, the dependence on τ of the right-hand side of (65) appears via the functions $\sin(\rho\tau)$ and $\mathrm{sh}(\rho\tau)$:

$$\mathrm{sh}\,(\tau)\xi(\mathrm{ch}(\tau)) = \int_{[0,\infty[} \frac{\sin(\rho\tau)}{\rho} d\nu_p(\rho) + \int_{]0,1[} \frac{\mathrm{sh}(\rho\tau)}{\rho} d\nu_s(\rho) + \nu_t \mathrm{sh}(\tau). \tag{152}$$

Defining

$$\hat{\xi}(\tau) = \mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau)) \tag{153}$$

a rather symbolic calculation of the Fourier transform gives

$$\begin{aligned}(\mathcal{F}\hat{\xi})(\sigma) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\sigma}\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))d\tau \\ &= \frac{i}{2}\int_{[0,\infty[} \frac{1}{\rho}[\delta(\sigma-\rho) - \delta(\sigma+\rho)]d\nu_p(\rho) \\ &\quad + \frac{1}{2}\int_{]0,1[} \frac{1}{\rho}[\delta(\sigma - i\rho) - \delta(\sigma + i\rho)]d\nu_s(\rho) \\ &\quad + \frac{1}{2}[\delta(\sigma - i) - \delta(\sigma + i)]\nu_t. \end{aligned} \tag{154}$$

Here, the mathematically oriented reader is probably horrified, because a Dirac function with a complex argument is not even a distribution. The usual (Schwartz) Fourier transformation of a distribution applies to a subclass, the tempered distributions, and gives tempered distributions. However, fulfilling our need, there is a theory of Fourier transformation of *all* distributions by Guelfand and Chilov [15]. The point is that, generally, a distribution is a linear functional on the space $\mathcal{D}(R)$ of the smooth functions with bounded support. The Fourier transform of a distribution will then be a linear functional on the space $\mathcal{Z}(R)$ made up of the Fourier transforms of the functions in $\mathcal{D}(R)$. This space is fully described in [15]. Because of the bounded support of the functions in $\mathcal{D}(R)$, the functions in $\mathcal{Z}(R)$ extend to entire functions (analytic in the whole complex plane), and, for example, the distribution $\delta(\sigma - i\rho)$ on $\mathcal{Z}(R)$ is defined by

$$\int_{-\infty}^{\infty} \delta(\sigma - i\rho)f(\sigma)d\sigma = f(i\rho) \qquad f \in \mathcal{Z}(R)$$

The precise definition [as a linear functional on $\mathcal{Z}(R)$] of the Fourier transform in (154) is then as follows:

$$(\mathcal{F}\hat{\xi},\tilde{u}) = \int_{-\infty}^{\infty} \mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))u(\tau)d\tau \qquad (155)$$

with

$$\tilde{u}(\rho) = \int_{-\infty}^{\infty} e^{-i\tau\rho}u(\tau)d\tau \qquad u \in \mathcal{D}(R)$$

The calculation of the right-hand side of (155) using (152) involves justified exchanges of integrals, and is almost trivial, giving

$$(\mathcal{F}\hat{\xi},\tilde{u}) = \frac{i}{2}\int_{[0,\infty[}\frac{1}{\rho}[\tilde{u}(\rho) - \tilde{u}(-\rho)]d\nu_p(\rho) + \frac{1}{2}\int_{]0,1[}\frac{1}{\rho}[\tilde{u}(i\rho) - \tilde{u}(-i\rho)]d\nu_s(\rho) + \frac{1}{2}[\tilde{u}(i) - \tilde{u}(-i)]\nu_t \qquad (156)$$

for any $\tilde{u} \in \mathcal{Z}(R)$. The above discussion shows that this condition (156) on $\xi(w)$ is fully equivalent to (65).

B. Illustration by the exponential form

Let us see what this gives for an exponential form $\xi(w) = e^{-c(w-1)}$ of the IW function. The function $\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))$ is integrable in this case, and its Fourier transform is an ordinary function, given by a convergent integral, which is easily calculated:

$$\begin{aligned}(\mathcal{F}\hat{\xi})(\rho) &= \frac{1}{2\pi}e^c\int_{-\infty}^{\infty} e^{i\tau\rho}\mathrm{sh}(\tau)\exp[-c\,\mathrm{ch}(\tau)]d\tau \\ &= -\frac{1}{2\pi}\frac{e^c}{c}\int_{-\infty}^{\infty} e^{i\tau\rho}\frac{d}{d\tau}\exp[-c\,\mathrm{ch}(\tau)]d\tau \\ &= \frac{i}{2\pi}\frac{e^c}{c}\rho\int_{-\infty}^{\infty} e^{i\tau\rho}\exp[-c\,\mathrm{ch}(\tau)]d\tau \\ &= \frac{i}{\pi}\frac{e^c}{c}\rho K_{i\rho}(c)\end{aligned} \qquad (157)$$

So, we have

$$(\mathcal{F}\hat{\xi},\tilde{u}) = \frac{i}{\pi}\frac{e^c}{c}\int_{-\infty}^{\infty}\rho K_{i\rho}(c)\tilde{u}(\rho)d\rho \qquad (158)$$

This is indeed of the form (156) [taking into account $K_{-i\rho}(c) = K_{i\rho}(c)$], with the following measures:

$$d\nu_p(\rho) = \frac{2}{\pi}\frac{e^c}{c}\rho^2K_{i\rho}(c)d\rho \qquad d\nu_s(\rho) = 0 \qquad \nu_t = 0 \qquad (159)$$

and for the integral representation (65), this gives

$$\exp[-c(w-1)] = \frac{2}{\pi}\frac{e^c}{c}\int_0^{\infty}\rho^2K_{i\rho}(c)\frac{\sin(\rho\tau)}{\rho\,\mathrm{sh}(\tau)}d\rho \qquad (w = \mathrm{ch}(\tau)) \qquad (160)$$

But (for any value $c > 0$), the function $K_{i\rho}(c)$ of ρ takes *negative values*, as we have seen from (139), so that the measure $d\nu_p(\rho)$ in (159) is not a positive one, contrarily to what is required in (65).

Nevertheless, we cannot yet conclude at the inconsistency of the exponential form. After all, as far as we know, a clever choice of the positive measures in (65) could perhaps give the same result as (160). What is still needed is a *unicity result*, namely, we must show that, if a function $\xi(w)$ can be put in the form (65), then the measures $d\nu_p(\rho)$ and $d\nu_s(\rho)$ and the number ν_t are unique. Also, we must show this without assuming these measures and this number positive [in order, for instance, to apply this result also to (160)].

C. Unicity of the representation of the Isgur-Wise function

Let us now demonstrate that the measures $d\nu_p(\rho)$ and $d\nu_s(\rho)$ and the number ν_t are unique in the representation (65) for the IW function. Moreover, here $d\nu_p(\rho)$, $d\nu_s(\rho)$ and ν_t are not assumed positive.

To be precise, let us specify that we consider here only bounded measures. A measure $d\nu$ is said to be bounded when $\int|d\nu| < \infty$. Then the integrals of bounded functions, as in (65), are convergent. The positive measures in (65) are bounded due to the condition (66).

Actually, an elementary unicity proof is obtained if one takes the *Laplace transform* of $\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))$. The integral giving the Laplace transform of the right-hand side of (152) is convergent for $\mathrm{Im}(z) > 1$. So the Laplace transform of $\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))$ is an analytic function in the half-plane $\mathrm{Im}(z) > 1$, and a simple calculation, valid for $\mathrm{Im}(z) > 1$, gives

$$\begin{aligned}\int_{[0,\infty[}\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))e^{iz\tau}d\tau &= \int_{[0,\infty[}\frac{d\nu_p(\rho)}{\rho^2 - z^2} + \int_{]0,1[}\frac{d\nu_s(\rho)}{-\rho^2 - z^2} + \frac{\nu_t}{-1 - z^2} \\ &= \int_{[-1,\infty[}\frac{d\nu(\sigma)}{\sigma - z^2}\end{aligned} \qquad (161)$$

In the last line we have introduced a measure $d\nu(\sigma)$ on the set $[-1,\infty[$ whose restrictions to the subsets $\{-1\}$, $]-1,0[$ and $[0,\infty[$ are related to ν_t, $d\nu_s(\rho)$ and $d\nu_p(\rho)$ by simple changes of variable, so that it is the same thing to know $d\nu(\sigma)$ or to know ν_t, $d\nu_s(\rho)$ and $d\nu_p(\rho)$.

Replacing z by $\sqrt{z}$ with $\mathrm{Im}(z) > 1$ in (161), we have a function $f_\xi(z)$, depending only on the function $\xi(w)$, given by

$$f_\xi(z) = \int_0^{\infty}\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))e^{i\tau\sqrt{z}}d\tau = \int_{[-1,\infty[}\frac{d\nu(\sigma)}{\sigma - z} \qquad (162)$$

Independently of ν_t, $d\nu_s(\rho)$ and $d\nu_p(\rho)$, this function

$f_\xi(z)$ is analytic when $\mathrm{Im}(z) > 1$ (for one of the two determinations of $\sqrt{z}$). This domain in z excludes a parabolic region containing $[-1, \infty[$ but, according to the last member of (162), $f_\xi(z)$ has an analytic continuation in the whole complex plane cut by $[-1, \infty[$. And according to the analytic functions theory, such an analytic continuation is unique.

Now we are done because, as is well known, the measure $d\nu(\sigma)$ can be recovered from the discontinuity across the cut. Precisely, one has

$$\int h(\sigma)d\nu(\sigma) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int [f_\xi(\sigma + i\epsilon) - f_\xi(\sigma - i\epsilon)]h(\sigma)d\sigma \qquad (163)$$

for any continuous function $h(\sigma)$ going to zero at ∞, and this defines $d\nu(\sigma)$.

This ends the present proof (in the Lorentz group approach) of the inconsistency of the exponential form for the IW function.

D. Summary of the general method to test the consistency of any *Ansatz* for the Isgur-Wise function

With Secs. IX A and IX C, we have now a general method to test the consistency of any form for the IW function in the case $j = 0$, that we summarize now.

Namely, given an *Ansatz* for the function $\xi(w)$:

(i) Compute the Fourier transform of $\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))$ (possibly in the Guelfand-Chilov generalized sense [15]), as defined by (155).

(ii) When this Fourier transform cannot be written in the form (156), the *Ansatz* is inconsistent.

(iii) When this Fourier transform is written in the form (156), the *Ansatz* is consistent if the measures $d\nu_p(\rho)$ and $d\nu_s(\rho)$ and the number ν_t are positive, and inconsistent if not.

E. *Ansätze* for the Isgur-Wise function compatible with the sum rules

Example 1.—We now use this method to establish the *consistency* of

$$\xi(w) = \left(\frac{2}{1+w}\right)^{2c} \qquad (164)$$

for any slope $c \geq \frac{1}{4}$ (and inconsistency for $0 < c < \frac{1}{4}$).

One has

$$\hat{\xi}(\tau) = \mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau)) = 2\mathrm{sh}(\tau/2)\mathrm{ch}(\tau/2)^{-4c+1}. \qquad (165)$$

When $c > \frac{1}{2}$, this function is integrable, and its Fourier transform is an ordinary function, given by a convergent integral, which can be calculated:

$$\begin{aligned}(\mathcal{F}\hat{\xi})(\rho) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{sh}(\tau/2)\mathrm{ch}(\tau/2)^{-4c+1}e^{i\tau\rho}d\tau \\ &= -\frac{1}{2c-1}\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{d}{d\tau}\mathrm{ch}(\tau/2)^{-4c+2}\right]e^{i\tau\rho}d\tau \\ &= \frac{1}{2c-1}i\rho\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{ch}(\tau/2)^{-4c+2}e^{i\tau\rho}d\tau\end{aligned} \qquad (166)$$

and by the change of variable $t = e^\tau$ we have

$$(\mathcal{F}\hat{\xi})(\rho) = 4^{2c-1}\frac{1}{2c-1}i\rho\frac{1}{2\pi} \times \int_0^\infty (1+t)^{-4c+2}t^{i\rho+2c-2}dt.$$

By the change of variable $t = \frac{s}{1-s}$ we obtain

$$\begin{aligned}(\mathcal{F}\hat{\xi})(\rho) &= 4^{2c-1}\frac{1}{2c-1}i\rho\frac{1}{2\pi} \times \int_0^1 s^{i\rho+2c-2}(1-s)^{-i\rho+2c-2}ds \\ &= 4^{2c}i\rho\frac{1}{2\pi}\frac{\Gamma(2c+i\rho-1)\Gamma(2c-i\rho-1)}{\Gamma(4c-1)}\end{aligned}$$

So, (155) writes, for the *Ansatz* (164):

$$(\mathcal{F}\hat{\xi}, \tilde{u}) = i\frac{4^{2c}}{2\pi}\int_{-\infty}^{\infty}\rho\frac{\Gamma(2c+i\rho-1)\Gamma(2c-i\rho-1)}{\Gamma(4c-1)} \times \tilde{u}(\rho)d\rho \qquad (167)$$

This is indeed of the form (156) with the following measures:

$$d\nu_p(\rho) = \frac{4^{2c}}{\pi}\rho^2\frac{|\Gamma(2c+i\rho-1)|^2}{\Gamma(4c-1)}d\rho \qquad d\nu_s(\rho) = 0 \qquad \nu_t = 0 \qquad (168)$$

Since these measures are positive, the consistency of (164) is established for the slopes $c > \frac{1}{2}$.

To treat the cases $c \leq \frac{1}{2}$, we have to find the generalized Fourier transform of $\mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau))$. To that end, we use analytic continuation in c. The above calculation of the Fourier transform (167) is valid for complex c provided $\mathrm{Re}(c) > \frac{1}{2}$. Considering (155), Eq. (167) writes

$$\int_{-\infty}^{\infty}\mathrm{sh}(\tau)\left(\frac{2}{1+w}\right)^{2c}u(\tau) = i\frac{4^{2c}}{2\pi}\int_{-\infty}^{\infty}\rho\frac{\Gamma(2c+i\rho-1)\Gamma(2c-i\rho-1)}{\Gamma(4c-1)} \times \tilde{u}(\rho)d\rho \qquad (169)$$

for any $u \in \mathcal{D}(R)$. This formula is proved for $\mathrm{Re}(c) > \frac{1}{2}$, but the left-hand side is an entire function of c (due to the bounded support of u). So, the right-hand side must have an analytic continuation, which will give the needed generalized Fourier transform of $\hat{\xi}(\tau)$.

First, we can directly take the limit $c \to \frac{1}{2}$ in (169), and also in (167) and (168), establishing the consistency of (164) also for the slope $c = \frac{1}{2}$.

It is more tricky to go to $\mathrm{Re}(c) < \frac{1}{2}$. Let us write

$$c = \tfrac{1}{2}(1 + \gamma e^{i\theta}) \tag{170}$$

with $\gamma > 0$ small, and see what happens when θ goes from 0 to π. Writing

$$\Gamma(2c + i\rho - 1)\Gamma(2c - i\rho - 1) = \frac{\Gamma(\gamma e^{i\theta} + i\rho + 1)\Gamma(\gamma e^{i\theta} - i\rho + 1)}{(\rho - i\gamma e^{i\theta})(\rho + i\gamma e^{i\theta})} \tag{171}$$

one sees that, in the right-hand side of (171), there are two poles $\rho = \pm i\gamma e^{i\theta}$, which, when θ goes to $\frac{\pi}{2}$ from below, approach the path of integration (which is the real axis). To have an analytic function, one must then deform the path of integration to avoid these poles. For $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ the new path can be decomposed into the old path (the real axis), a counterclockwise small circle around the pole $\rho = i\gamma e^{i\theta}$ in the lower complex half-plane, and a clockwise small circle around the pole $\rho = -i\gamma e^{i\theta}$ in the upper half-plane. The integrals along the small circles are given by the residue theorem, and the obtained analytic continuation of Eq. (169) to $0 < \mathrm{Re}(c) < \frac{1}{2}$ is

$$\int_{-\infty}^{\infty} sh(\tau)\left(\frac{2}{1+w}\right)^{2c} u(\tau)d\tau = i\frac{4^{2c}}{2\pi}\int_{-\infty}^{\infty}\rho\frac{\Gamma(2c + i\rho - 1)\Gamma(2c - i\rho - 1)}{\Gamma(4c-1)} \times \tilde{u}(\rho)d\rho + 2^{4c-1}[\tilde{u}(i(1-2c)) - \tilde{u}(-i(1-2c))] \tag{172}$$

When c is real with $0 < c < \frac{1}{2}$, this is indeed of the form (156) with the following measures:

$$d\nu_p(\rho) = \frac{4^{2c}}{\pi}\rho^2\frac{|\Gamma(2c + i\rho - 1)|^2}{\Gamma(4c-1)}d\rho$$
$$d\nu_s(\rho) = (1-2c)2^{4c}\delta(\rho - (1-2c)) \qquad \nu_t = 0 \tag{173}$$

Since these measures are positive when $\frac{1}{4} \le c$, the consistency of (164) is established for the slopes $\frac{1}{4} \le c < \frac{1}{2}$. If $0 < c < \frac{1}{4}$, the measure $d\nu_p(\rho)$ is *negative* and inconsistency follows.

The decomposition (65) of the *Ansatz* (164) into irreducible IW functions [given by (62)–(64)] is

$$\left(\frac{2}{1+w}\right)^{2c} = \frac{4^{2c}}{\pi}\int_0^{\infty}\rho^2\frac{|\Gamma(2c + i\rho - 1)|^2}{\Gamma(4c-1)}\xi_{p,0,\rho}(w)d\rho + \theta(1-2c)(1-2c)2^{4c}\xi_{s,1-2c}(w) \tag{174}$$

When $c \ge \frac{1}{2}$, the decomposition of the representation of the Lorentz group involves only the irreducible representations of the principal series, with a continuous combination of all ($n = 0$) of them. When $\frac{1}{4} < c < \frac{1}{2}$, we have a direct sum of such a continuous combination and of *one* of the irreducible representations of the supplementary series. When $c = \frac{1}{4}$, the representation is reduced to the $\rho = \frac{1}{2}$ irreducible representation of the supplementary series:

$$\left(\frac{2}{1+w}\right)^{1/2} = \xi_{s,1/2}(w) \tag{175}$$

When $0 < c < \frac{1}{4}$, the weight of $\xi_{p,0,\rho}$ is <0, and this of course cannot occur from decomposition of a representation.

Example 2.—As another application, we establish the consistency of the IW function

$$\xi(w) = \frac{1}{[1 + \frac{c}{2}(w-1)]^2} \tag{176}$$

for any slope $c \ge 1$ (and inconsistency for $0 < c < 1$).

The function

$$\hat{\xi}(\tau) = \mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau)) = \frac{\mathrm{sh}(\tau)}{[1 + \frac{c}{2}(\mathrm{ch}(\tau) - 1)]^2} \tag{177}$$

is integrable (for $c > 0$), and its Fourier transform is an ordinary function, given by a convergent integral:

$$(\mathcal{F}\hat{\xi})(\rho) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{sh}(\tau)}{[1 + \frac{c}{2}(\mathrm{ch}(\tau) - 1)]^2}e^{i\tau\rho}d\tau \tag{178}$$

Let us compute it. One has

$$(\mathcal{F}\hat{\xi})(\rho) = -\frac{2}{c}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{d}{d\tau}\frac{1}{1 + \frac{c}{2}(\mathrm{ch}(\tau) - 1)}\right]e^{i\tau\rho}d\tau$$
$$= \frac{2}{c}i\rho\frac{1}{2\pi}\int_0^{\infty}\frac{1}{1 + \frac{c}{2}(\mathrm{ch}(\tau) - 1)}e^{i\tau\rho}d\tau \; (t = e^{\tau})$$
$$= \frac{2}{c}i\rho\frac{1}{2\pi}\int_0^{\infty}\frac{t^{i\rho}}{t + \frac{c}{4}(1-t)^2}dt$$

The denominator is written as follows:

$$t + \frac{c}{4}(1-t)^2 = \frac{c}{4}(t - t_1)(t - t_2)$$
$$t_{1,2} = 1 - \frac{2}{c} \pm \sqrt{\left(1 - \frac{2}{c}\right)^2 - 1}$$

and the integral is obtained by a calculus of residues:

$$(\mathcal{F}\hat{\xi})(\rho) = \frac{8}{c^2} i\rho \frac{1}{2\pi} \int_0^\infty \frac{t^{i\rho}}{(t-t_1)(t-t_2)} dt$$
$$= \frac{4}{c^2} \frac{i\rho}{\mathrm{sh}(\pi\rho)} \frac{1}{2\pi} \times \int_0^\infty \frac{(-t-i0)^{i\rho} - (-t+i0)^{i\rho}}{(t-t_1)(t-t_2)} dt$$
$$= -\frac{4}{c^2} \frac{\rho}{\mathrm{sh}(\pi\rho)} (\mathrm{res}|_{t=t_1} + \mathrm{res}|_{t=t_2}) \times \frac{(-t)^{i\rho}}{(t-t_1)(t-t_2)}$$
$$= -\frac{4}{c^2} \frac{\rho}{\mathrm{sh}(\pi\rho)} \frac{(-t_1)^{i\rho} - (-t_2)^{i\rho}}{t_1 - t_2}$$

So, (155) writes

$$(\mathcal{F}\hat{\xi}, \tilde{u}) = -\frac{4}{c^2} \int_{-\infty}^\infty \frac{\rho}{\mathrm{sh}(\pi\rho)} \frac{(-t_1)^{i\rho} - (-t_2)^{i\rho}}{t_1 - t_2} \tilde{u}(\rho) d\rho \tag{179}$$

This is indeed of the form (156) with the following measures:

$$d\nu_p(\rho) = i\frac{8}{c^2} \frac{\rho^2}{\mathrm{sh}(\pi\rho)} \frac{(-t_1)^{i\rho} - (-t_2)^{i\rho}}{t_1 - t_2} d\rho \qquad d\nu_s(\rho) = 0 \qquad \nu_t = 0 \tag{180}$$

and the decomposition (65) of (176) into irreducible IW functions (62)–(64) is

$$\frac{1}{[1 + \frac{c}{2}(w-1)]^2} = i\frac{8}{c^2} \int_0^\infty \frac{\rho^2}{\mathrm{sh}(\pi\rho)} \times \frac{(-t_1)^{i\rho} - (-t_2)^{i\rho}}{t_1 - t_2} \xi_{p,0,\rho}(w) d\rho \tag{181}$$

Now, when $0 < c < 1$, one has $1 < \frac{2}{c} - 1$ and, replacing c by a parameter $\gamma > 0$ by $\mathrm{ch}(\gamma) = \frac{2}{c} - 1$, one has $t_1 = -e^{-\gamma}$ and $t_2 = -e^{\gamma}$. Then (181) writes

$$\frac{1}{[1 + \frac{c}{2}(w-1)]^2} = \frac{8}{c^2} \int_0^\infty \frac{\rho^2}{\mathrm{sh}(\pi\rho)} \frac{\sin(\gamma\rho)}{\mathrm{sh}(\gamma)} \xi_{p,0,\rho}(w) d\rho, \tag{182}$$

and when $1 < c$, one has $-1 < \frac{2}{c} - 1 < 1$ and, replacing c by a parameter $0 < \gamma < \pi$ by $\cos(\gamma) = \frac{2}{c} - 1$, one has $t_1 = -e^{i\gamma}$ and $t_2 = -e^{-i\gamma}$. Then (181) writes

$$\frac{1}{[1 + \frac{c}{2}(w-1)]^2} = \frac{8}{c^2} \int_0^\infty \frac{\rho^2}{\mathrm{sh}(\pi\rho)} \frac{\mathrm{sh}(\gamma\rho)}{\mathrm{sh}(\gamma)} \xi_{p,0,\rho}(w) d\rho. \tag{183}$$

For the case $c = 1$, we can take the limit $\gamma \to 0$ in (182) or (183):

$$\frac{1}{[1 + \frac{c}{2}(w-1)]^2} = \frac{8}{c^2} \int_0^\infty \frac{\rho^3}{sh(\pi\rho)} \xi_{p,0,\rho}(w) d\rho. \tag{184}$$

Since the weight of $\xi_{p,0,\rho}(w)$ in (182) takes negative values, the *Ansatz* (176) is incompatible with the sum rules when $0 < c < 1$.

Since the weight of $\xi_{p,0,\rho}(w)$ in (183) and (184) is positive, the *Ansatz* (176) is compatible with the sum rules when $1 \le c$.

Example 3.—As a last application, we establish the consistency with the sum rules of

$$\xi(w) = e^{-c\tau} = \frac{1}{(w + \sqrt{w^2 - 1})^c} \tag{185}$$

for any value $c \ge 0$ of the parameter. Here c is not the slope. In fact, for $c > 0$ all the derivatives at $w = 1$ are infinite, so that the bounds on these derivatives are helpless in this case.

We have to compute the Fourier transform of the function

$$\hat{\xi}(\tau) = \mathrm{sh}(\tau)\xi(\mathrm{ch}(\tau)) = \mathrm{sh}(\tau)e^{-c|\tau|} \tag{186}$$

When $\mathrm{Re}(c) > 1$, this functions is integrable, and its Fourier transform is an ordinary function given by a convergent integral

$$(\mathcal{F}\hat{\xi})(\rho) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{sh}(\tau)e^{-c|\tau|} e^{i\tau\rho} d\tau \tag{187}$$

which is easily calculated:

$$(\mathcal{F}\hat{\xi})(\rho) = \frac{1}{2\pi} i\rho \frac{4c}{[(c-1)^2 + \rho^2][(c+1)^2 + \rho^2]} \tag{188}$$

This is indeed of the form (156) with the following measures:

$$d\nu_p(\rho) = \frac{4c}{2\pi} \frac{\rho^2}{[(c-1)^2 + \rho^2][(c+1)^2 + \rho^2]} d\rho \qquad d\nu_s(\rho) = 0 \qquad \nu_t = 0 \tag{189}$$

Since these measures are positive for c real, the consistency of (185) is established for $c > 1$.

To treat the cases $c \le 1$, we have to find the generalized Fourier transform of $\hat{\xi}(\tau)$, and to that end, as in the case of the "dipolar" form (164), we use analytic continuation in c. Combining the general definition (155) of the Fourier transform with the Fourier transform (186) already calculated, we have

$$\int_{-\infty}^\infty \hat{\xi}(\tau)u(\tau)d\tau = i\frac{2c}{\pi} \int_{-\infty}^\infty \frac{\rho}{[(c-1)^2 + \rho^2][(c+1)^2 + \rho^2]} \times \tilde{u}(\rho) d\rho \tag{190}$$

for any $u \in \mathcal{D}(R)$. This formula is proved for $\mathrm{Re}(c) > 1$, but the left-hand side is an entire function of c (due to the bounded support of u), so the right-hand side must have an analytic continuation, which will give the needed generalized Fourier transform of $\hat{\xi}(\tau)$.

First, we can directly take the limit $c \to 1$ in (190), and also in (188) (in the sense of tempered distributions) and (189), establishing the consistency of (185) also for $c = 1$.

Going down to $\mathrm{Re}(c) < 1$ is quite similar to the case of the dipolar form (164). In the integrand of (190), there are two poles

$$\rho = \pm i(c-1) \tag{191}$$

which, when c goes around $c = 1$ from above, approach the path of integration (which is the real axis). To have an analytic function, one must then deform the path of integration to avoid these poles. The new path can be decomposed into the old path (the real axis), a counterclockwise small circle around the pole $\rho = i(c-1)$ in the lower complex half-plane, and a clockwise small circle around the $\rho = -i(c-1)$ in the upper complex half-plane. The integrals along the small circles are given by the residue theorem, and the obtained analytic continuation of (186) to $0 \le \mathrm{Re}(c) < 1$ is

$$\int_{-\infty}^{\infty} \hat{\xi}(\tau)u(\tau)d\tau = i\frac{2c}{\pi}\int_{-\infty}^{\infty}\frac{\rho}{[(c-1)^2+\rho^2][(c+1)^2+\rho^2]} \times \tilde{u}(\rho)d\rho + \frac{1}{2}[\tilde{u}(i(1-c)) - \tilde{u}(-i(1-c))] \tag{192}$$

When c is real with $0 < c < 1$, this is indeed of the form (156) with the following measures:

$$d\nu_p(\rho) = \frac{4c}{2\pi}\frac{\rho^2}{[(c-1)^2+\rho^2][(c+1)^2+\rho^2]}d\rho \qquad d\nu_s(\rho) = (1-c)\delta(\rho-(1-c))d\rho \qquad \nu_t = 0 \tag{193}$$

Since these measures are positive when $0 < c < 1$, the consistency of (185) is established for these values of c. For $c = 0$, $\xi(w) = 1$ is just the irreducible IW function given by the trivial representation. So, we have consistency for all $c \ge 0$.

The decomposition (65) of (185) into irreducible IW functions [given by (62)–(64)] is, for $c > 0$,

$$\frac{1}{(w+\sqrt{w^2-1})^c} = \frac{4c}{2\pi}\int_0^{\infty}\frac{\rho^2}{[(c-1)^2+\rho^2][(c+1)^2+\rho^2]} \times \xi_{p,0,\rho}(w)d\rho + \theta(1-c)(1-c)\xi_{s,1-c}(w) \tag{194}$$

The fact that all the derivatives at $w = 1$ are infinite is due to the slow decrease of the measure $d\nu_p(\rho)$, for which the moments $\mu_k = \langle x^k\rangle$ defined in (80) are divergent for $k \ge 1$.

X. PHENOMENOLOGICAL APPLICATIONS

Before concluding, let us summarize the main phenomenological consequences of the present paper. From a practical perspective, we have a number of interesting results for possible simple forms of the $j = 0$ IW function.

We have illustrated our different general results with some one-parameter *Ansätze* for the IW function (from now on we make the replacement $c = \rho_\Lambda^2$ for the slope, when it is finite), namely:

(i) the dipole form (109):

$$\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho_\Lambda^2} \qquad \rho_\Lambda^2 \ge \frac{1}{4} \tag{195}$$

(ii) the true dipole shape (111):

$$\xi(w) = \frac{1}{[1+\frac{\rho_\Lambda^2}{2}(w-1)]^2} \qquad \rho_\Lambda^2 \ge 1 \tag{196}$$

(iii) the form found in Sec. VI B (115):

$$\xi(w) = \frac{\mathrm{sh}(\tau\sqrt{1-3\rho_\Lambda^2})}{\mathrm{sh}(\tau)\sqrt{1-3\rho_\Lambda^2}} = \frac{\sin(\tau\sqrt{3\rho_\Lambda^2-1})}{\mathrm{sh}(\tau)\sqrt{3\rho_\Lambda^2-1}} \qquad w = \mathrm{ch}(\tau) \qquad \rho_\Lambda^2 \ge 0 \tag{197}$$

(iv) the form proposed in Sec. IX E:

$$\xi(w) = \frac{1}{(w+\sqrt{w^2-1})^c} \qquad c \ge 0 \tag{198}$$

for which at $w = 1$ *all derivatives are infinite* if $c > 1$.

Let us comment on these different possible one-parameter models for the IW function and briefly remind the results obtained above.

(i) The dipole form (195) was proposed in the case of the *meson* IW function [16]. For this function we have shown in Sec. VI A that all the bounds (101)–(103) imply $\rho_\Lambda^2 \ge \frac{1}{4}$, suggesting that this form is acceptable if this lower bound is fulfilled. Indeed, following the consistency test of Sec. IX, we have shown that this form is consistent for any slope $\rho_\Lambda^2 \ge \frac{1}{4}$ (and inconsistent for $0 < \rho_\Lambda^2 < \frac{1}{4}$).

(ii) The true dipole form is a model proposed in [17] for baryon decay (196). For it, we have shown in Sec. VI A that the bounds (101)–(103) imply $\rho_\Lambda^2 \ge \frac{2}{3}$, $\rho_\Lambda^2 \ge \frac{10}{13}$, and $\rho_\Lambda^2 \ge 0.86$, lower bounds that slowly converge towards 1 with the constraints on increasing order derivatives. Indeed, following the consistency test of Sec. IX, we have shown that this form is consistent for any slope $\rho_\Lambda^2 \ge 1$ (and inconsistent for $0 < \rho_\Lambda^2 < 1$).

(iii) The form (197) is a result of the present paper if the lower bound on the curvature (101) is saturated, i.e. $\sigma_\Lambda^2 = \frac{3}{5}\rho_\Lambda^2(1+\rho_\Lambda^2)$. It satisfies all the constraints for any value of the slope $\rho_\Lambda^2 \ge 0$.

(iv) The form (198) is interesting because it satisfies all the constraints for $c \geq 0$, with all its derivatives being infinite at zero recoil if $c > 0$.

These simple one-parameter forms will be useful in the future to fit the differential decay width for the process $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$ with quite different possibilities, and thus guess a possible variation of $|V_{cb}|$. In this sense, the form (198) constitutes an extreme case, since all the derivatives are infinite at $w = 1$.

Another extreme case is the one-parameter form

$$\xi(w) = 1 - e^{-c/(w-1)} \qquad c > 0 \tag{199}$$

for which $\xi(w) \to 0$ for $w \to \infty$, and all its derivatives vanish at $w = 1$. But according to Sec. VI B [see (114)], if the slope vanishes one gets $\xi(w) = 1$, and only the trivial representation contributes to the integral formula (65), with $\nu_t = 1$. Therefore, the *Ansatz* (199) is inconsistent with the sum rules.

Finally, let us comment on the exponential form (107)

$$\xi(w) = \exp[-\rho_\Lambda^2(w-1)] \tag{200}$$

that was proposed by Jenkins, Manohar, and Wise [18] and by Pervin, Roberts, and Capstick [17]. In the large N_c and heavy quark limit studied in [18] there is an important subtlety that we discuss at the end of this section.

For the exponential form we have shown in Sec. VI A that the bounds (101)–(103) imply, respectively, $\rho_\Lambda^2 \geq 1.5$, $\rho_\Lambda^2 \geq 2.5$, and $\rho_\Lambda^2 \geq 4.28$. The lower bound on the slope grows with the constraints on higher and higher derivatives, suggesting that the exponential form is not consistent. Indeed, we have demonstrated in Sec. VII from the sum rules that the IW function is of *positive type*, and that the exponential *Ansatz* is inconsistent with this property for any value of the slope $\rho_\Lambda^2 > 0$. We have exposed an alternative demonstration following the general consistency test formulated for any form of the IW function in Sec. IX B.

Let us make a final remark on the exponential form (200). This form was suggested in the paper by Jenkins, Manohar, and Wise [18] within a model based on QCD in the heavy quark and large N_c limits, with a slope of the order

$$\rho_\Lambda^2 = \lambda N_c^{3/2} \qquad \lambda = O(1) \tag{201}$$

However, one must keep in mind that formula (200) with the slope (201) is valid, according to [18], for

$$w - 1 = O(N_c^{-3/2}) \tag{202}$$

This means that in this scheme (200) would be valid in the heavy quark limit, but for *fixed*

$$x = \rho_\Lambda^2(w-1) \tag{203}$$

As we said in [8], the bound (101) obtained in the physical situation $N_c = 3$ is trivially satisfied in the large N_c limit, as it is obvious from (200) and (201). However, the phenomenological guess (3.8) from [18], $\rho_\Lambda^2 = 1.3$, slightly violates the bound, and we have more generally demonstrated that the exponential form is inconsistent.

But there is a subtle point concerning the exponential form (200) at *fixed* $x = \rho_\Lambda^2(w-1)$ (203). Indeed, the dipole form (195), that satisfies all the theoretical constraints, becomes, performing the change of variables (203) and taking the limit $\rho_\Lambda^2 = O(N_c^{3/2}) \to \infty$:

$$\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho_\Lambda^2} = \left(\frac{1}{1+\frac{x}{2\rho_\Lambda^2}}\right)^{2\rho_\Lambda^2} = e^{-2\rho_\Lambda^2 \log(1+(x/2\rho_\Lambda^2))} \sim e^{-x} = e^{-\rho_\Lambda^2(w-1)} \tag{204}$$

Therefore, within the conditions (201) and (202), i.e. for a very large slope and an infinitesimally small phase space, the exponential form (200) can be rigorously replaced by the dipole form (195), that satisfies all the theoretical constraints formulated in the present paper. Therefore, for finite slope and the whole phase space it would be convenient on theoretical grounds to replace the exponential form by the dipole form.

XI. CONCLUSION

The present paper explores new methods to study Isgur-Wise functions based on the Lorentz group. The IW function is expressed in terms of the scalar product of the initial and final light clouds of the heavy hadron, that involves a unitary representation of the Lorentz group. The method uses the decomposition of this unitary representation into irreducible representations and under the $SU(2)$ subgroup of rotations. The approach has practical consequences, namely, constraints on the possible IW functions, that can be applied to the different parametrizations proposed in the literature.

For the moment, we have applied this method to the case of a light cloud with $j^P = 0^+$, relevant to the decay $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$. This case is more involved from the point of view of group theory than the ground state meson case $j^P = \frac{1}{2}^-$. We leave the latter, being more complicated from the spin point of view, for future work.

We have shown, in the present baryon case, that the enumeration and explicit formulas for the relevant irreducible representations allows one to give an integral formula for the IW function $\xi(w)$ involving *positive measures*. Not only the *principal series* of the unitary representations of the Lorentz group appear, but also the so-called *supplementary series*. This powerful formula allows one in turn to express the derivatives of the IW function at zero recoil as moments of a variable with positive values.

The corresponding positivity constraints on determinants of these moments imply in turn bounds on the kth derivative of the IW function $\xi^{(k)}(1)$ in terms of the lower derivatives $\xi^{(n)}(1)$ $(n = 0, 1, \ldots k-1)$. We have illustrated these bounds for three one-parameter models of the IW function proposed in the literature, namely, the exponential form and two different kinds of the dipole forms. The exponential form, unlike the dipole forms, appears to be somewhat pathological in this respect.

We have also demonstrated that if one of the bounds is saturated (e.g. the bound on the curvature in terms of the slope), then one gets a completely explicit and simple one-parameter form of the IW function.

Then we have used the sum rule approach [8], and demonstrated that the IW function is *a function of positive type*. This allows one to show, as an example, that the exponential form for the IW function is not consistent with this property.

We demonstrate also, using the positive-type property, that the Lorentz group method developed in the present paper, and this is important, is equivalent to the sum rule approach. Moreover, the Lorentz group method sheds another light on the long distance physics, and summarizes all the possible constraints of the sum rule approach.

Finally, we have formulated a general consistency test for any given *Ansatz* of the IW function. We have applied this criterion to several phenomenological one-parameter forms proposed in the literature, like the exponential and the dipole forms, and shown that the former is inconsistent, while the two latter forms are consistent when the slope satisfies some lower bounds.

Hopefully, LHCb will provide new data on the decay $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$, that we know has a large branching ratio, of the order of 5.10^{-2}, measured at LEP, roughly half of the total semileptonic rate $\Lambda_b \to X_c \ell \bar{\nu}_\ell$. One expects at LHCb roughly $3.10^{10} \Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$ events/year [19] and possibly one could give a precise measurement of the differential rate. This measurement has a twofold interest. One concerns heavy quark hadronic physics, namely, in particular, the shape of the IW function, the object of the present paper. The other is an independent useful exclusive determination of $|V_{cb}|$, since there is still at present some tension between the exclusive and the inclusive determinations in B decays, the former giving a smaller value, although with a larger error.

For the decay $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$, theoretical work remains to be done. One should include radiative corrections within heavy quark effective theory and $1/m_Q$ corrections, as well as the Wilson coefficients that make the matching with the physical form factors, a program that was realized in the case of mesons by Dorsten [20]. This would allow one to compare with the future data and with other theoretical or phenomenological schemes of baryon form factors at finite mass. Also, once these necessary improvements are realized, any future fit to the differential distribution of $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$ should take into account the constraints formulated here for the IW function.

It is important to apply the method of the present paper to mesons. In this case one has the complication of spin, since the light cloud has $j^P = \frac{1}{2}^-$, but we have noticed that from the point of view of the Lorentz group the problem seems simpler because only the principal series of the representations of the Lorentz group appears. This program will be the object of a forthcoming work.

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APPENDIX A: SCALAR PRODUCTS IN HILBERT SPACES OF THE SUPPLEMENTARY SERIES

In this appendix, we describe a trick (which can be found in [12]) useful to compute scalar products (42) in the Hilbert space of a representation of the supplementary series.

The matrices $R \in SU(2)$ are of the form

$$R = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1. \tag{A1}$$

Parametrizing a and b by

$$a = \cos\left(\frac{\theta}{2}\right) e^{i\phi} \qquad b = \sin\left(\frac{\theta}{2}\right) e^{i\psi}$$
$$(0 \leq \phi, \psi \leq 2\pi, 0 \leq \theta \leq \pi) \tag{A2}$$

the normalized invariant measure on $SU(2)$ writes

$$dR = \frac{1}{8\pi^2} \sin\theta d\theta d\phi d\psi \tag{A3}$$

Defining $R(z, \alpha) \in SU(2)$ for $(z, \alpha) \in C \times [0, 2\pi[$ by

$$a = \frac{1}{\sqrt{1 + |z|^2}} e^{i\alpha} \qquad b = \frac{z}{\sqrt{1 + |z|^2}} e^{-i\alpha} \tag{A4}$$

the Jacobian is

$$dR(z, \alpha) = \frac{1}{2\pi^2} \frac{1}{(1 + |z|^2)^2} d^2 z d\alpha \tag{A5}$$

Noting that, abbreviating $R(z, \alpha)$ to R and $R(z', \alpha')$ to R', one has

$$z = \frac{R_{12}}{R_{22}} \tag{A6}$$

and (using $R'^{-1}_{12} = -R'_{12}$ and $R'^{-1}_{11} = R'_{22}$, which follow from $\det R = 1$)

$$z' - z = \frac{R'_{12}}{R'_{22}} - \frac{R_{12}}{R_{22}} = \frac{R'_{12} R_{22} - R'_{22} R_{12}}{R'_{22} R_{22}} = -\frac{(R'^{-1} R)_{12}}{R'_{22} R_{22}} \tag{A7}$$

the scalar product (42), rewritten here for convenience

$$\langle \phi' | \phi \rangle = \int \overline{\phi'(z')} |z' - z|^{2\rho - 2} \phi(z) d^2 z' d^2 z \tag{A8}$$

can be written as follows:

$$\langle\phi'|\phi\rangle = \pi^2\int |R'_{22}|^{-2\rho-2}\bar{\phi}'\left(\frac{R'_{12}}{R'_{22}}\right)|(R'^{-1}R)_{12}|^{2\rho-2}|R_{22}|^{-2\rho-2}\phi\left(\frac{R_{12}}{R_{22}}\right)dR'dR \tag{A9}$$

and by a change of varible $R \to R'R$, one obtains

$$\langle\phi'|\phi\rangle = \pi^2\int |R'_{22}|^{-2\rho-2}\bar{\phi}'\left(\frac{R'_{12}}{R'_{22}}\right)|R_{12}|^{2\rho-2}|(R'R)_{22}|^{-2\rho-2}\phi\left(\frac{(R'R)_{12}}{(R'R)_{22}}\right)dR'dR \tag{A10}$$

Next, applying the transformation law (39) with $\Lambda = R'^{-1}$, we have

$$\begin{aligned}
(U_{s,\rho}(R'^{-1})\phi)(z) &= |R'^{-1}_{11} - R'^{-1}_{21}z|^{-2\rho-2}\phi\left(\frac{R'^{-1}_{22}z - R'^{-1}_{12}}{R'^{-1}_{11} - R'^{-1}_{21}z}\right)\\
(U_{s,\rho}(R'^{-1})\phi)(z) &= |R'_{22} + R'_{21}z|^{-2\rho-2}\phi\left(\frac{R'_{11}z + R'_{12}}{R'_{22} + R'_{21}z}\right)\\
(U_{s,\rho}(R'^{-1})\phi)\left(\frac{R_{12}}{R_{22}}\right) &= \left(\frac{|R'_{22}R_{22} + R'_{21}R_{12}|}{|R_{22}|}\right)^{-2\rho-2}\phi\left(\frac{R'_{11}R_{12} + R'_{12}R_{22}}{R'_{22}R_{22} + R'_{21}R_{12}}\right)\\
(U_{s,\rho}(R'^{-1})\phi)\left(\frac{R_{12}}{R_{22}}\right) &= |R_{22}|^{2\rho+2}|(R'R)_{22}|^{-2\rho-2}\phi\left(\frac{(R'R)_{12}}{(R'R)_{22}}\right)
\end{aligned} \tag{A11}$$

This expresses a part of the integrand in (A10), which becomes

$$\langle\phi'|\phi\rangle = \pi^2\int |R'_{22}|^{-2\rho-2}\bar{\phi}'\left(\frac{R'_{12}}{R'_{22}}\right)|R_{12}|^{2\rho-2} |R_{22}|^{-2\rho-2}(U_{s,\rho}(R'^{-1})\phi)\left(\frac{R_{12}}{R_{22}}\right)dR'dR \tag{A12}$$

Returning to the variables z' and α' for R' and z and α for R, we obtain the following expression for the scalar product (42) in the Hilbert space $\mathcal{H}_{s,\rho}$ of the representation in the supplementary series labeled by ρ:

$$\langle\phi'|\phi\rangle = \frac{1}{2\pi}\int(1+|z'|^2)^{\rho-1}\overline{\phi'(z')}|z|^{2\rho-2} \times (U_{s,\rho}(R(z',\alpha')^{-1})\phi)(z)d^2zd^2z'd\alpha' \tag{A13}$$

This result will be used in Appendix B and in Appendix C.

APPENDIX B: CALCULATION OF NORMALIZATION CONSTANTS

In this appendix, we compute the normalization constants in (48) and (50).

Principal series.—Using the notation $R(z,\alpha)$ for the rotation defined by (A4), we write (48) as follows:

$$\phi^{p,n,\rho}_{j,M}(z) = N^{p,n,\rho}_{j,M}(1+|z|^2)^{i\rho-1}D^j_{n/2,M}(R(z,0)^{-1}) \tag{B1}$$

where $N^{p,n,\rho}_{j,M}$ is the normalization constant here to be found. We have to compute the following scalar product:

$$\langle\phi^{p,n,\rho}_{j',M'}|\phi^{p,n,\rho}_{j,M}\rangle = N^{p,n,\rho}_{j',M'}N^{p,n,\rho}_{j,M}\int(1+|z|^2)^{-2} \times D^{j'}_{n/2,M'}(R(z,0)^{-1})^* \times D^j_{n/2,M}(R(z,0)^{-1})d^2z \tag{B2}$$

From (A4), one sees that

$$R(z,\alpha) = R(z,0)\begin{pmatrix} e^{i\alpha} & 0\\ 0 & e^{-i\alpha}\end{pmatrix} \tag{B3}$$

and because $D^j_{n/2,M}(R)$, as defined by (16), is a matrix element with on the left an eigenstate of the Oz component of the angular momentum, with eigenvalue $n/2$, one has

$$D^j_{n/2,M}(R(z,\alpha)^{-1}) = e^{in\alpha}D^j_{n/2,M}(R(z,0)^{-1}) \tag{B4}$$

and we may rewrite (B2) as follows:

$$\langle\phi^{p,n,\rho}_{j',M'}|\phi^{p,n,\rho}_{j,M}\rangle = N^{p,n,\rho}_{j',M'}N^{p,n,\rho}_{j,M}\frac{1}{2\pi}\int(1+|z|^2)^{-2} \times D^{j'}_{n/2,M'}(R(z,\alpha)^{-1})^* \times D^j_{n/2,M}(R(z,\alpha)^{-1})d^2zd\alpha \tag{B5}$$

since the integrand does not in fact depend on α. Using the Jacobian (A5), this gives

$$\langle\phi^{p,n,\rho}_{j',M'}|\phi^{p,n,\rho}_{j,M}\rangle = N^{p,n,\rho}_{j',M'}N^{p,n,\rho}_{j,M}\pi\int D^{j'}_{n/2,M'}(R^{-1})^* \times D^j_{n/2,M}(R^{-1})dR \tag{B6}$$

By the change of variable of integration $R \to R^{-1}$, which leaves invariant the measure dR, this reduces to the scalar product (47) of the rotation matrix elements, and one obtains

$$\langle\phi^{p,n,\rho}_{j',M'}|\phi^{p,n,\rho}_{j,M}\rangle = N^{p,n,\rho}_{j',M'}N^{p,n,\rho}_{j,M}\frac{\pi}{2j+1}\delta_{j,j'}\delta_{M,M'} \tag{B7}$$

So, the normalization constant is

$$N^{p,n,\rho}_{j,M} = \sqrt{\frac{2j+1}{\pi}} \tag{B8}$$

Supplementary series.—Using the notation $R(z, \alpha)$ for the rotation defined by (A4), we write (50) as follows:

$$\phi^{s,\rho}_{j,M}(z) = N^{s,\rho}_{j,M}(1 + |z|^2)^{-\rho-1} D^j_{0,M}(R(z, 0)^{-1}) \tag{B9}$$

where $N^{s,\rho}_{j,M}$ is the normalization constant to be found.

Using (A13) for the scalar product in the supplementary series, we have

$$\langle \phi^{s,\rho}_{j',M'}|\phi^{s,\rho}_{j,M}\rangle = \frac{1}{2\pi}\int (1 + |z'|^2)^{\rho-1}\phi^{s,\rho}_{j',M'}(z')^*|z|^{2\rho-2} \times (U_{s,\rho}(R(z', \alpha')^{-1})\phi^{s,\rho}_{j,M})(z) d^2z d^2z' d\alpha' \tag{B10}$$

Now, because the $\phi^{s,\rho}_{j,M}$ (for $-j \leq M \leq j$) are the standard basis of the representation j of $SU(2)$, we have

$$(U_{s,\rho}(R(z', \alpha')^{-1})\phi^{s,\rho}_{j,M} = \sum_{M''} D^j_{M'',M}(R(z', \alpha')^{-1})\phi^{s,\rho}_{j,M''} \tag{B11}$$

Using (B9) and (B11), the scalar product (B10) writes

$$\begin{aligned}\langle \phi^{s,\rho}_{j',M'}|\phi^{s,\rho}_{j,M}\rangle &= \sum_{M''} N^{s,\rho}_{j',M'} N^{s,\rho}_{j,M''} \\ &\times \frac{1}{2\pi}\int |z|^{2\rho-2}(1 + |z|^2)^{-\rho-1} D^j_{0,M''}(R(z, 0)^{-1}) d^2z \\ &\times \int (1 + |z'|^2)^{-2} D^{j'}_{0,M'}(R(z', \alpha')^{-1})^* \\ &\times D^j_{M'',M}(R(z', \alpha')^{-1}) d^2z' d\alpha' \end{aligned} \tag{B12}$$

where we have also used the fact that $D^{j'}_{0,M'}(R(z', \alpha')^{-1})$ does not depend on α' [see (B4)]. Using the Jacobian (A5), the second integral is

$$2\pi^2 \int D^{j'}_{0,M'}(R^{-1})^* D^j_{M'',M}(R^{-1}) dR = \frac{2\pi^2}{2j + 1}\delta_{j,j'}\delta_{0,M''}\delta_{M,M'} \tag{B13}$$

so that (B12) reduces to

$$\begin{aligned}\langle \phi^{s,\rho}_{j',M'}|\phi^{s,\rho}_{j,M}\rangle &= N^{s,\rho}_{j',M'} N^{s,\rho}_{j,M}\frac{\pi}{2j + 1}\delta_{j,j'}\delta_{M,M'} \\ &\times \int |z|^{2\rho-2}(1 + |z|^2)^{-\rho-1} \\ &\times D^j_{0,0}(R(z, 0)^{-1}) d^2z \end{aligned} \tag{B14}$$

Now, from (46), with $a = 1/\sqrt{1 + |z|^2}$ and $b = -z/\sqrt{1 + |z|^2}$, we have

$$D^j_{0,0}(R(z, 0)^{-1}) = (1 + |z|^2)^{-j}\sum_k (-1)^k \binom{j}{k}\binom{j}{j-k}|z|^{2k} \tag{B15}$$

The calculation of the remaining integral then goes as follows:

$$\begin{aligned}\int |z|^{2\rho-2}(1 + |z|^2)^{-\rho-1} D^j_{0,0}(R(z, 0)^{-1}) d^2z &= \sum_{0\leq k\leq j} (-1)^k \binom{j}{k}\binom{j}{j-k}\int |z|^{2\rho+2k-2}(1 + |z|^2)^{-\rho-j-1} d^2z \\ &= \pi \sum_{0\leq k\leq j} (-1)^k \binom{j}{k}\binom{j}{j-k}\int_0^\infty x^{\rho+k-1}(1 + x)^{-\rho-j-1} dx \left(x \to \frac{y}{1-y}\right) \\ &= \pi \sum_{0\leq k\leq j} (-1)^k \binom{j}{k}\binom{j}{j-k}\int_0^1 y^{\rho+k-1}(1 - y)^{j-k} dy \\ &= \pi \sum_{0\leq k\leq j} (-1)^k \binom{j}{k}\binom{j}{j-k}\frac{\Gamma(\rho + k)\Gamma(j - k + 1)}{\Gamma(\rho + j + 1)} \\ &= \pi\frac{j!\Gamma(\rho)}{\Gamma(\rho + j + 1)}\sum_k (-1)^k \binom{j}{k}\frac{\Gamma(\rho + k)}{k!\Gamma(\rho)} \\ &= \pi\frac{j!\Gamma(\rho)}{\Gamma(\rho + j + 1)}\sum_k (-1)^k \binom{j}{k}\binom{\rho + k - 1}{k} \\ &= \pi\frac{j!\Gamma(\rho)}{\Gamma(\rho + j + 1)}\sum_k \binom{j}{j-k}\binom{-\rho}{k} = \pi\frac{j!\Gamma(\rho)}{\Gamma(\rho + j + 1)}\binom{j - \rho}{j} \end{aligned}$$

and gives

$$\int |z|^{2\rho-2}(1 + |z|^2)^{-\rho-1} D^j_{0,0}(R(z, 0)^{-1}) d^2z = \pi\frac{\Gamma(j - \rho + 1)\Gamma(\rho)}{\Gamma(j + \rho + 1)\Gamma(1 - \rho)}. \tag{B16}$$

With (B14), this gives the final result for the scalar product

$$\langle \phi^{s,\rho}_{j',M'} | \phi^{s,\rho}_{j,M} \rangle = N^{s,\rho}_{j',M'} N^{s,\rho}_{j,0} \frac{\pi^2}{2j+1} \times \frac{\Gamma(j-\rho+1)\Gamma(\rho)}{\Gamma(j+\rho+1)\Gamma(1-\rho)} \delta_{j,j'} \delta_{M,M'} \quad \text{(B17)}$$

So, the normalization constant is

$$N^{s,\rho}_{j,M} = \frac{\sqrt{2j+1}}{\pi} \sqrt{\frac{\Gamma(j+\rho+1)\Gamma(1-\rho)}{\Gamma(j-\rho+1)\Gamma(\rho)}} \quad \text{(B18)}$$

APPENDIX C: CALCULATION OF THE IRREDUCIBLE ISGUR-WISE FUNCTIONS FOR THE $j = 0$ CASE

In this appendix, we compute the IW functions (62) and (63) for the $j = 0$ state in the irreducible representations of $SL(2, C)$. These functions are in fact known, and can be found in [12] (with some changes of notation) under the name of "elementary spherical functions."

Principal series.—The integral (59) for $\xi_{p,0,\rho}(w)$ is directly computed. Integrating over the angle gives

$$\xi_{p,0,\rho}(w) = \int_0^\infty (1+x)^{-i\rho-1}(e^\tau + e^{-\tau}x)^{i\rho-1} dx \quad \text{(C1)}$$

and by the change of variable $x = \frac{y}{1-y}$, we obtain

$$\xi_{p,0,\rho}(w) = \int_0^1 [e^\tau(1-y) + e^{-\tau}y]^{i\rho-1} dy = \frac{\sin(\rho\tau)}{\rho \mathrm{sh}(\tau)} \quad \text{(C2)}$$

Supplementary series.—The integral for $\xi_{s,\rho}(w)$ is more involved. In order to use (A13), it is convenient to rewrite $\xi_{s,\rho}(w)$ in the form

$$\xi_{s,\rho}(w) = \langle U_{s,\rho}(\Lambda_{-\tau}) \phi^{s,\rho}_{0,0} | \phi^{s,\rho}_{0,0} \rangle \quad \text{(C3)}$$

where we have used the unitarity of $U_{s,\rho}(\Lambda)$,

$$U_{s,\rho}(\Lambda)^\dagger = U_{s,\rho}(\Lambda)^{-1} = U_{s,\rho}(\Lambda^{-1}) \qquad \Lambda_\tau^{-1} = \Lambda_{-\tau} \quad \text{(C4)}$$

In (A13), we have then the following simplification:

$$(U_{s,\rho}(R(z', \alpha')^{-1}) \phi^{s,\rho}_{0,0})(z) = \phi^{s,\rho}_{0,0}(z) \quad \text{(C5)}$$

due to the fact that $\phi^{s,\rho}_{0,0}$ is a scalar under the subgroup $SU(2)$ of $SL(2, C)$, and we obtain

$$\xi_{s,\rho}(w) = \left[\int (1+|z'|^2)^{\rho-1} (U_{s,\rho}(\Lambda_{-\tau}) \phi^{s,\rho}_{0,0})(z')^* d^2 z'\right] \times \left[\int |z|^{2\rho-2} \phi^{s,\rho}_{0,0}(z) d^2 z\right] \quad \text{(C6)}$$

The function $\phi^{s,\rho}_{0,0}$ is given by (53) and $U_{s,\rho}(\Lambda_{-\tau})\phi^{s,\rho}_{0,0}$ is then given by (43)

$$\phi^{s,\rho}_{0,0}(z) = \frac{\sqrt{\rho}}{\pi}(1+|z|^2)^{-\rho-1}$$
$$(U_{s,\rho}(\Lambda_{-\tau})\phi^{s,\rho}_{0,0})(z) = \frac{\sqrt{\rho}}{\pi}(e^{-\tau} + e^\tau |z|^2)^{-\rho-1} \quad \text{(C7)}$$

The integrals in (C6) are then directly computed:

$$\begin{aligned}\int (1+|z'|^2)^{\rho-1}(e^{-\tau} + e^\tau|z'|^2)^{-\rho-1} d^2 z' &= \pi \int_0^\infty (1+x)^{\rho-1}(e^{-\tau} + e^\tau x)^{-\rho-1} dx \\ &= \pi \int_0^1 [e^{-\tau}(1-y) + e^\tau y]^{-\rho-1} dy \\ &= \frac{\pi}{\rho} \frac{e^{\rho\tau} - e^{-\rho\tau}}{e^\tau - e^{-\tau}} = \pi \frac{\mathrm{sh}(\rho\tau)}{\rho \mathrm{sh}(\tau)} \end{aligned} \quad \text{(C8)}$$

$$\int |z|^{2\rho-2}(1+|z|^2)^{-\rho-1} d^2 z = \pi \int_0^\infty x^{\rho-1}(1+x)^{-\rho-1} dx = \pi \int_0^1 y^{\rho-1} dy = \frac{\pi}{\rho} \quad \text{(C9)}$$

and we obtain

$$\xi_{s,\rho}(w) = \frac{\mathrm{sh}(\rho\tau)}{\rho \mathrm{sh}(\tau)} \quad \text{(C10)}$$

APPENDIX D: EXPANSION IN POWERS OF $(w - 1)$ OF THE IRREDUCIBLE $j = 0$ ISGUR-WISE FUNCTION

In this appendix, we obtain the whole expansion in powers of $w - 1$ of the IW function for the $j = 0$ state in the irreducible representations of $SL(2, C)$.

We work with the function $\xi_x(w)$ defined by (71) which, when $x \geq 0$, covers all the cases (62)–(64). It is not easy to obtain the $w - 1$ expansion directly from (71) since in this formula the dependence in w occurs through $\tau = \mathrm{Arcch}(w)$.

We now obtain an integral representation (D3) for $\xi_x(w)$ in which the dependence on w is explicit and simple. To this end, let us compute the integral

$$I_a(w) = \int_0^\infty \frac{s^a}{(1+2ws+s^2)} ds = \int_0^\infty \frac{s^a}{(s+e^\tau)(s+e^{-\tau})} ds \quad \text{(D1)}$$

which is convergent for $-1 < \mathrm{Re}(a) < 1$. A standard calculus of residues gives

$$\begin{aligned} I_a(w) &= -\frac{\pi}{\sin(\pi a)} \frac{1}{2i\pi} \int_0^\infty \frac{(-z-i0)^a - (-z+i0)^a}{(z+e^\tau)(z+e^{-\tau})} dz \\ &= -\frac{\pi}{\sin(\pi a)} (\mathrm{res}_{z=-e^\tau} + \mathrm{res}_{z=-e^{-\tau}}) \times \frac{(-z)^a}{(z+e^\tau)(z+e^{-\tau})} \\ &= -\frac{\pi}{\sin(\pi a)} \left[\frac{e^{a\tau}}{(-e^\tau + e^{-\tau})} + \frac{e^{-a\tau}}{(-e^{-\tau} + e^\tau)}\right] \\ &= \frac{\pi}{\sin(\pi a)} \frac{\mathrm{sh}(a\tau)}{\mathrm{sh}(\tau)}. \end{aligned} \quad \text{(D2)}$$

Then for $\xi_x(w) = \mathrm{sh}(\tau\sqrt{1-x})/\mathrm{sh}(\tau)\sqrt{1-x}$ we have

$$\xi_x(w) = \frac{\sin(\pi\sqrt{1-x})}{\pi\sqrt{1-x}} \int_0^\infty \frac{s^{\sqrt{1-x}}}{1+2ws+s^2} ds \quad \text{(D3)}$$

valid for $x > 0$.

We can now expand at $w = 1$:

$$\xi_x(w) = \frac{\sin(\pi\sqrt{1-x})}{\pi\sqrt{1-x}} \times \sum_{k\geq 0} (-1)^k 2^k (w-1)^k \int_0^\infty \frac{s^{k+\sqrt{1-x}}}{(1+s)^{2k+2}} ds \quad \text{(D4)}$$

The integral is directly calculated:

$$\begin{aligned}\int_0^\infty \frac{s^{k+\sqrt{1-x}}}{(1+s)^{2k+2}} ds &= \int_0^1 t^{k+\sqrt{1-x}}(1-t)^{k-\sqrt{1-x}} dt \\ &= \frac{\Gamma(k+\sqrt{1-x}+1)\Gamma(k-\sqrt{1-x}+1)}{(2k+1)!} \\ &= \frac{\pi\sqrt{1-x}}{\sin(\pi\sqrt{1-x})} \frac{1}{(2k+1)!} \\ &\quad \times \prod_{i=1}^{k} [(i^2-1)+x] \end{aligned} \quad \text{(D5)}$$

where we have used

$$\Gamma(\sqrt{1-x}+1)\Gamma(-\sqrt{1-x}+1) = \frac{\pi\sqrt{1-x}}{\sin(\pi\sqrt{1-x})} \quad \text{(D6)}$$

and we obtain

$$\xi_x(w) = \sum_{k\geq 0} (-1)^k 2^k \frac{1}{(2k+1)!} \prod_{i=1}^{k} (x+i^2-1)(w-1)^k \quad \text{(D7)}$$

This deduction of (D7) is valid only when $x > 0$. Considering the case $x = 0$, we have

$$\prod_{i=1}^{k} (i^2-1) = \delta_{k,0}$$

so that the formula (D7) reduces to $\xi_0(w) = 1$, and is therefore true also in this case.

APPENDIX E: SUM RULE FOR THE ISGUR-WISE FUNCTION IN THE $j = 0$ CASE

In [8], from the OPE and the nonforward amplitude, we have demonstrated the following sum rule for the $j = 0$ case:

$$\begin{aligned}\xi(w_{if}) = \sum_n \sum_{L\geq 0} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f) \sum_{0\leq k\leq L/2} C_{L,k}(w_i^2-1)^k(w_f^2-1)^k \Big\{ (w_iw_f - w_{if})^{L-2k} - \frac{2}{2L+1}[(L-2k)(w_i+1) \\ \times (w_f+1)(w_iw_f-w_{if})^{L-2k-1} + 2k(w_iw_f-w_{if})^{L-2k}] + \frac{2}{(2L+1)^2}[(L-2k)(3+4k)(w_i+1)(w_f+1) \\ \times (w_iw_f-w_{if})^{L-2k-1} + (L-2k)(L-2k-1)(w_i+1)(w_f+1)(w_iw_f+w_i+w_f-1-2w_{if}) \\ \times (w_iw_f-w_{if})^{L-2k-2} + 4k^2(w_iw_f-w_{if})^{L-2k}] \Big\} \end{aligned} \quad \text{(E1)}$$

where $w_i = v_i.v'$, $w_f = v_f.v'$, $w_{if} = v_i.v_f$ and v_i, v_f, and v' are the initial, final and intermediate state four-velocities in the sum rule, respectively, $\tau_L^{(n)}(w)$ are the IW functions for the transition $0^+ \to L^P$ with $P = (-1)^L$, and the coefficients $C_{L,k}$ are given by

$$C_{L,k} = (-1)^k \frac{(L!)^2}{(2L)!} \frac{(2L-2k)!}{k!(L-k)!(L-2k)!} \quad \text{(E2)}$$

From this sum rule we have demonstrated the inequalities for the slope (100) [9] and for the curvature (101).

We have recently realized that the expression for this sum rule can enormously be simplified.

First, let us group the terms in (E1), that gives

$$\begin{aligned}\xi(w_{if}) = \sum_n \sum_{L\geq 0} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f) \frac{1}{(2L+1)^2} \sum_{0\leq k\leq L/2} C_{L,k}(w_i^2-1)^k(w_f^2-1)^k(w_iw_f-w_{if})^{L-2k-2}\{[(2L+1)^2 \\ - 4(2L+1)k + 8k^2](w_iw_f-w_{if})^2 - 2(L-2k)(L-2k-1)(w_i^2-1)^k(w_f^2-1)\} \end{aligned} \quad \text{(E3)}$$

Next, using the expression for the Legendre polynomials

$$P_L(x) = \frac{1}{2^L} \sum_k (-1)^k \frac{(2L-2k)!}{k!(L-k)!(L-2k)!} x^{L-2k} \quad \text{(E4)}$$

one gets

$$\sum_{0\leq k\leq L/2} C_{L,k}x^{L-2k} = 2^L \frac{(L!)^2}{(2L)!} P_L(x) \tag{E5}$$

Defining now the variable x_{if} as

$$x_{if} = \frac{w_i w_f - w_{if}}{\sqrt{(w_i^2 - 1)(w_f^2 - 1)}} \tag{E6}$$

one obtains, from (E3), the following expression for the sum rule:

$$\begin{aligned}\xi(w_{if}) = \sum_n \sum_{L\geq 0} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f) \frac{1}{(2L+1)^2}(w_i^2-1)^{L/2} \\ \times (w_f^2-1)^{L/2} \sum_{0\leq k\leq L/2} C_{L,k} x_{if}^{L-2k}[2L^2+6L-8k \\ +1-2(L-2k)(L-2k-1)(1-x_{if}^2)x_{if}^{-2}]\end{aligned} \tag{E7}$$

that can be written in terms of derivatives of Legendre polynomials:

$$\begin{aligned}\xi(w_{if}) = \sum_n \sum_{L\geq 0} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f) \frac{2^L}{(2L+1)^2}\frac{(L!)^2}{(2L)!} \\ \times (w_i^2-1)^{L/2}(w_f^2-1)^{L/2}[(2L^2+2L+1)P_L(x_{if}) \\ +4x_{if}P_L'(x_{if}) - 2(1-x_{if}^2)P_L''(x_{if})]\end{aligned} \tag{E8}$$

and from the differential equation satisfied by the Legendre polynomials

$$(1-x^2)P_L''(x) - 2xP_L'(x) + L(L+1)P_L(x) = 0 \tag{E9}$$

one gets

$$\begin{aligned}\xi(w_{if}) = \sum_n \sum_{L\geq 0} \frac{2^L(L!)^2}{(2L)!} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f) \\ \times (w_i^2-1)^{L/2}(w_f^2-1)^{L/2} P_L(x_{if})\end{aligned} \tag{E10}$$

that gives finally the simple expression for the sum rule, used in Sec. VII:

$$\begin{aligned}\xi(w_{if}) = \sum_n \sum_{L\geq 0} \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_f) \\ \times \sum_{0\leq k\leq L/2} C_{L,k}(w_i^2-1)^k(w_f^2-1)^k \\ \times (w_i w_f - w_{if})^{L-2k}\end{aligned} \tag{E11}$$

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