

High-precision thermodynamics and Hagedorn density of states

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We compute the entropy density of the confined phase of QCD without quarks on the lattice to very high accuracy. The results are compared to the entropy density of free glueballs, where we include all the known glueball states below the two-particle threshold. We find that an excellent, parameter-free description of the entropy density between $0.7T_c$ and T_c is obtained by extending the spectrum with the exponential spectrum of the closed bosonic string.

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I. INTRODUCTION

The phase diagram of QCD is being actively studied in heavy ion collision experiments as well as theoretically. A form of matter with remarkable properties [1] has been observed in the Relativistic Heavy Ion Collider (RHIC) experiments [2–5]. It appears to be a strongly coupled plasma of quarks and gluons (QGP), but no consensus on a physical picture that accounts for both equilibrium and nonequilibrium properties has been reached yet. On the other hand, below the short interval of temperatures where the transition from the confined phase to the QGP takes place [6–9], it is widely believed that the most prominent degrees of freedom are the ordinary hadrons. From this point of view, the zeroth order approximation to the properties of the system is to treat the hadrons as infinitely narrow and noninteracting. We will refer to this approximation as the hadron resonance gas model (HRG). The HRG predictions were compared with lattice QCD thermodynamics data in [6,10], and lately they have been used to extrapolate certain results to zero temperature [7]. The HRG is also the basis of the statistical model currently applied to the analysis of hadron yields in heavy ion collisions [11], and recently the transport properties of a relativistic hadron gas have been studied in detail [12].

Since any heavy ion reaction ends up in the low-temperature phase of QCD, it is important to understand its properties in detail in order to extract those of the high-temperature phase with minimal uncertainty. In this paper we study whether the HRG model works in the absence of quarks, in other words in the pure $SU(N=3)$ gauge theory, where the low-lying states are glueballs. There are reasons to believe that if the HRG model is to work at any quark content of QCD, it is in the zero-flavor case. First, the mass gap in $SU(3)$ gauge theory is very large, $M_0/T_c \approx 5.3$. As we shall see, the thermodynamic properties up to quite close to T_c are dominated by the states below the two-particle threshold, which are exactly stable. Furthermore, because of their large mass, neglecting their thermal width should be a good approximation. Second, the scattering amplitudes between glueballs are parametrically $1/N^2$ suppressed while those between mesons are only $1/N$

suppressed [13]. This means that the glueballs should be free to a better approximation than the hadrons of realistic QCD.

An additional motivation to study the thermodynamics of the confined phase of $SU(3)$ gauge theory is that it is a parameter-free theory, simplifying the interpretation of its properties. Its spectrum is known quite accurately up to the two-particle threshold [14,15]. By contrast, in full QCD calculations, lattice data calculated at pion masses larger than in Nature are often compared out of necessity to the HRG model based on the experimental spectrum [6,7]. Finally, calculations in the pure gauge theory are at least 2 orders of magnitude faster, which allows us to reach a high level of control of statistical and systematic errors; in particular, we are able to perform calculations in very large volumes.

II. LATTICE CALCULATION

We use Monte-Carlo simulations of the Wilson action $S_g = \frac{1}{g_0^2} \sum_{x,\mu,\nu} \text{Tr}\{1 - P_{\mu\nu}(x)\}$ for $SU(3)$ gauge theory [16], where $P_{\mu\nu}$ is the plaquette. The lattice spacing a is related to the bare coupling through $g_0^2 \sim 1/\log(1/a\Lambda)$. The temperature T is set by the extent of the (periodic) Euclidean time direction, $1/T = N_\tau a$, where N_τ is an integer. We calculate the thermal expectation value of $\theta \equiv T_{\mu\mu}$, the (anomalous) trace of the energy-momentum tensor $T_{\mu\nu}$, and of $\theta_{00} \equiv T_{00} - \frac{1}{4}\theta$. In the thermodynamic limit,

$$Ts = e + p = \frac{4}{3}\langle\theta_{00}\rangle_T, \quad e - 3p = \langle\theta\rangle_T - \langle\theta\rangle_0. \quad (1)$$

Here e , p , s are, respectively, the energy density, pressure and entropy density. The operator $\theta_{00} = \frac{1}{2}(-\mathbf{E}^a \cdot \mathbf{E}^a + \mathbf{B}^a \cdot \mathbf{B}^a)$ requires no subtraction, because its vacuum expectation value vanishes. The choice of θ_{00} and θ as independent linear combinations is convenient because they both renormalize multiplicatively. We use the ‘‘HYP-clover’’ discretization of the energy-momentum tensor introduced in [17,18]. The normalization of the θ_{00} operator differs from its naive value by a factor that we parametrize as $Z(g_0)\chi(g_0)$. The factor $Z(g_0)$ is taken from [19] and rests on the results of [20]; its accuracy is

about 1%. The factor $\chi(g_0)$ is obtained by calibrating our discretization to the “bare plaquette” discretization in the deconfined phase at $N_t = 6$ [17]. We find, for $6/g_0^2$ between 5.90 and 6.41, $\chi(g_0) = 0.1306 \cdot (6/g_0^2) - 0.1865$ with an accuracy of half a percent. For the lattice beta-function that renormalizes θ , we use the parametrization [21] of the data in [22] and the same calibration method.

Our results for the entropy density from $N_t = 8$ and $N_t = 12$ simulations are shown on Fig. 3. The displayed error bars do not contain the uncertainty on the normalization factor, which is much smaller and would introduce correlation between the points. This factor varies by only 7% over the displayed interval and so to a first approximation amounts to an overall normalization of the curve. Our data is about 5 times statistically more accurate than that of previous thermodynamic studies [23,24], which were primarily focused on the deconfined phase. Just as importantly, we kept the finite-spatial-volume effects under good control, in particular, very close to the deconfining temperature T_c .

We use periodic boundary conditions in all three spatial directions, the extent of which is L . Figure 1 shows the size of finite-volume effects as a function of L in units of $1/T$, for T fixed. For instance, at $0.985T_c$ the conventional choice $LT = 4$ leads to an overestimate of the entropy density by a factor three. The fact that the $N_t = 12$ data fall on the same smooth curve as the $N_t = 8$ is strong evidence that discretization errors are small. We parametrize the volume dependence empirically by a $A + Be^{-cLT}$ curve, and use it to convert the $N_t = 12$ data to $LT = 8$. At $0.929T_c$, there is no statistically significant difference between $LT = 6$ and 8 and we do not apply any correction. It is the corrected $N_t = 12$ data that is then displayed on Fig. 3.

In [25], formulas for the leading finite-volume effects on the thermodynamic potentials were derived in terms of the energy gap of the theory defined on a $(1/T) \times L \times L$

spatial hypertorus. Close to T_c , this gap corresponds to the mass of the ground state flux loop winding around the cycle of length $1/T$. If $\delta s(T, L) \equiv s(T, \infty) - s(T, L)$, the formula then reads

$$\delta s(T, L) = \frac{e^{-m(T)L}}{2\pi L} \left[m^2(T) + \frac{3}{2} T \partial_T m^2(T) \right]. \quad (2)$$

Using the calculation of $m(T)$ described below, the predicted asymptotic approach to the infinite-volume entropy density for $0.985T_c$ is displayed on Fig. 1. While the sign is correct, the magnitude of the finite-volume effects is not reproduced for $LT \leq 8$. We conclude that the asymptotic approach to infinite volume sets in for very large values of LT . Since $m(T)L$ is only about 4 when $LT = 6$, it is not implausible that flux-loop states with high multiplicity dominate the finite-volume effects at that box size.

Next we obtain the correlation length $\xi(T)$ of the order parameter for the deconfining phase transition, the Polyakov loop. The method consists in computing the two-point function of zero-momentum operators, designed to have large overlaps with the ground state flux loop, along a spatial direction. We fit the lattice data for $m(T) \equiv 1/\xi(T)$ displayed on Fig. 2 with the formula

$$\left(\frac{m(T)T}{T_c} \right)^2 = a_0 - a_1 \left(\frac{T}{T_c} \right)^2 - a_2 \left(\frac{T}{T_c} \right)^4 \quad (3)$$

and find, either fitting a_2 or setting it to zero,

$$a_0 = 5.76(15), \quad a_1 = 4.97(65), \quad a_2 = 0.55(54) \quad (4)$$

$$a_0 = 5.90(9), \quad a_1 = 5.62(10), \quad a_2 = 0 \quad (5)$$

with in both cases a χ^2/dof of about 0.3. We remark that the parameters a_i are not far from the Nambu-Goto string [26] values $a_1 = \frac{2\pi}{3} \frac{\sigma}{T_c^2} = 5.02(5)$ [27] and $a_2 = 0$ (σ is the tension of the confining string). By open-closed string

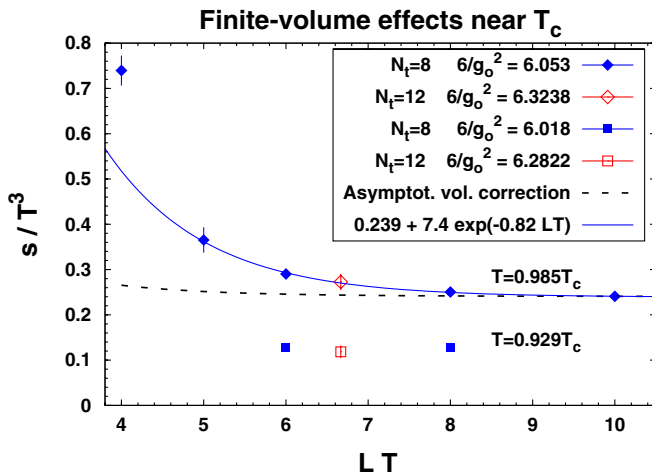


FIG. 1 (color online). Finite volume effects on the entropy density at two (fixed) temperatures close to T_c .

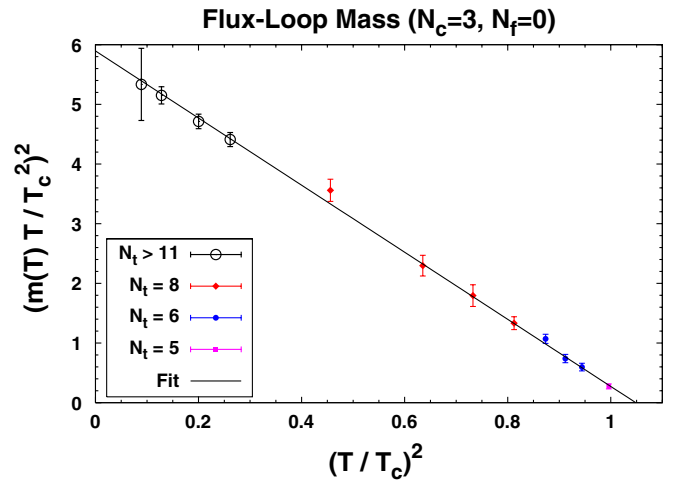


FIG. 2 (color online). The mass of the temporal flux loop as calculated from Polyakov loop correlators, and the fit (5). The $N_t > 11$ data are from [40], the $N_t = 5$ data from [27].

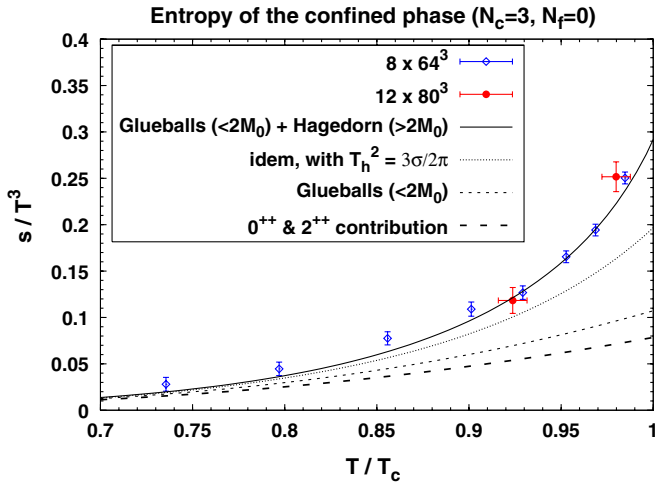


FIG. 3 (color online). The entropy density in units of T^3 for $LT = 8$. We applied a (modest) volume-correction to the $N_t = 12$ data.

duality, $m(T)T$ is also the effective string tension governing the linear rise with distance of the free energy of two static charges. As in [28], we define the ‘‘Hagedorn’’ temperature as the point where this effective string tension vanishes. We thus obtain from the second fit

$$T_h/T_c = 1.024(3). \quad (6)$$

This extraction amounts to assuming mean-field exponents near T_h (it is not clear which universality class should be used [29]). The result is stable if the fit interval is varied, and also if a_2 is fitted with a_0 and a_1 constrained to the known values of $(\sigma/T_c^2)^2$ and $\frac{2\pi}{3} \frac{\sigma}{T_c}$. We note that a direct extraction of the Hagedorn temperature [30] from the asymptotic density of glueball states is technically more difficult, and somewhat ambiguous at finite N . The authors of [30] find a much larger Hagedorn temperature, which we attribute to a preasymptotic, slower growth of the density of glueball states—an interesting fact by itself. In Sec. III we will instead assume that the two definitions coincide. (We expect them to coincide in the large- N limit; it would be very valuable to test this expectation by a direct calculation, although identifying highly excited states is numerically challenging.) Now that we have determined T_h for the gauge group $SU(3)$, we can ask whether treating $N = 3$ as large leads to a good description of the thermodynamic properties of the theory.

As a check on the normalization of the operators θ_{00} and θ , we calculate the latent heat L_h in two different ways. The latent heat is the jump in energy density at T_c . Since the pressure is continuous, we obtain it instead from the discontinuity in entropy density or the ‘‘conformality measure’’ $e - 3p$. We obtain s and $e - 3p$ on either side of T_c by extrapolating $LT = 10$ data from the confined (deconfined) phase towards T_c . The result is

$$\frac{\Delta s}{T_c^3} = 1.45(5)(5), \quad \frac{\Delta(e - 3p)}{T_c^4} = 1.39(4)(5), \quad (7)$$

where the first error is statistical and the second comes from the uncertainty in the extrapolation (taken to be the difference between a linear and quadratic fit). The compatibility between these two estimates of L_h/T_c^4 is strong evidence that we control the normalization of our operators. They are in good agreement with previous calculations of the latent heat performed on coarser lattices [27,31]. We have also verified more generally that the thermodynamic identity $T \partial_T (s/T^3) = (1/T^3) \partial_T (e - 3p)$ is satisfied within statistical errors.

III. INTERPRETATION

In infinite volume the pressure associated with a single noninteracting, relativistic particle species of mass M with n_σ polarization states reads

$$p = \frac{n_\sigma}{2\pi^2} M^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(nM/T) \quad (8)$$

where K_2 is a modified Bessel function. By linearity, the knowledge of the glueball spectrum leads to a simple prediction for the pressure and entropy density $s = \frac{\partial p}{\partial T}$, which is expected to become exact in the large- N limit. Since only the low-lying spectrum of glueballs is known, it is useful to consider how the density of states might be extended above the two-particle threshold $2M_0$, where M_0 is the mass of the lightest (scalar) glueball. The asymptotic closed bosonic string density of states with b degrees of freedom is given by [32]

$$\rho(M) = \frac{1}{T_h} \left(\frac{\pi b}{3}\right)^{b+1} \left(\frac{T_h}{M}\right)^{b+2} e^{M/T_h} \quad (9)$$

(for instance, $b = 24$ for a string living in 26 dimensions). In the string theory, the Hagedorn temperature T_h is related to the string tension, $T_h^2 = \frac{3\sigma}{2\pi}$, corresponding to $T_h/T_c = 1.069(5)$ [33]. Below we use this value as an alternative to the more direct determination (6).

On Fig. 3, we show the entropy contribution of the glueballs lying below the two-particle threshold $2M_0$. The curve is just about consistent with the smallest temperature lattice data point, but clearly fails to reproduce the strong increase in entropy density as $T \rightarrow T_c$. The figure also illustrates that the two lowest-lying states, the scalar and tensor glueballs, account for about three quarters of the stable glueballs’ contribution. We have used the continuum-extrapolated lattice spectrum [14,34].

Adding the Hagedorn spectrum contribution, Eq. (9) with T_h given by Eq. (6) and $b = 2$, leads to the solid curve on Fig. 3. It describes the direct calculation of the entropy density surprisingly well, particularly close to T_c . (We note that, based on Fig. 1, the data point closest to T_c may still be affected by an upward finite-volume effect at the one-sigma level.) By contrast, other integer values of b

lead to values of the entropy near T_c that are much further away from the lattice data: at $0.9855T_c$, the solid curve would go through $s/T^3 = 0.421, 0.239, 0.195, 0.191$ for $b = 1, 2, 3, 4$, respectively. This is a striking conclusion, since there is a large body of numerical evidence ([35] and Refs. therein) that on distances $r \gg T_c^{-1}$, the QCD string has $b = 2$ (transverse) degrees of freedom. On distance scales $r = O(T_c^{-1})$, it is thought, in part based on the AdS/CFT paradigm, that additional, massive degrees of freedom play a role (see for instance [36–38]). The integral over M that leads to the prediction for s/T^3 is dominated by very massive states, for which one might think that the massive string modes play an important role. For instance, at $0.9855T_c$, 17% of the total integral comes from the region $M > 5M_0$. A possible explanation is that the cost of exciting one such massive mode is numerically significantly greater than T_h .

The dominant uncertainty in comparing s/T^3 is firstly the matching prescription between the lattice spectrum and the Hagedorn spectrum (about 5%), and secondly the value of T_h . One should also bare in mind that Eq. (9) is an asymptotic expression based on the Hardy-Ramanujan formula, which overestimates the density of states.

At the lower temperatures, the displayed curve for $b = 2$ tends to underestimate somewhat the entropy density. This is likely to be a cutoff effect. Indeed, at fixed N_t lower temperatures correspond to a coarser lattice spacing, and the scalar glueball mass in physical units is known to be smaller on coarse lattices with the Wilson action [39]. If we use the stable glueball spectrum calculated at $g_0^2 = 1$ instead of the continuum spectrum, the agreement of the noninteracting glueball + Hagedorn spectrum with the lattice data at the lower four temperatures is again excellent. This difference provides an estimate for the size of lattice effects.

To summarize, we have computed to high accuracy the entropy of the confined phase of QCD without quarks. The low-lying states of the theory are therefore bound states called glueballs, and their spectrum is well determined [14,15]. If the size N of the gauge group is increased, the interactions of the glueballs are expected to be suppressed [13]. To what extent the glueballs really are weakly interacting at $N = 3$ is not known precisely. Some evidence for the smallness of their low-energy interactions was found some time ago by looking at the finite-volume effects on their masses [40]. But it seems unlikely that glueballs well above the two-particle threshold would have a small decay width. We have nevertheless compared the entropy density data to the entropy density of a gas of noninteracting glueballs. While restricting the spectral sum to the stable glueballs leads to an underestimate by at least a factor two of the entropy density near T_c , extending the spectral sum

with an exponential spectrum $\rho(M) \sim \exp(M/T_h)$, suggested long ago by Hagedorn [41], leads to a prediction in excellent agreement with the lattice data for the entropy density (Fig. 3). This is remarkable, since the analytic form of the asymptotic spectrum is completely predicted by free bosonic string theory, including its overall normalization [Eq. (9)]. Therefore, since we also separately computed the temperature (identified with T_h) where the flux loop mass vanishes, no parameter was fitted in the comparison with the thermodynamic data. By contrast, the entropy density is not nearly as well described if the Nambu-Goto value of T_h is used, see Fig. 3, or if the number of degrees of freedom of the string b is taken to be different from two. With detailed and precise data at larger N , the interpretation proposed here could potentially be put on a more solid footing, since glueball interactions are $1/N^2$ suppressed. And one could search more systematically for evidence of massive string modes in the glueball spectrum.

The success of the noninteracting string density of states in reproducing the entropy density suggests that once the Hagedorn temperature has been determined directly from the divergence of the flux-loop correlation length, the residual effects of interactions on the thermodynamic potentials are small. It may be that thermodynamic properties in general are not strongly influenced by interactions when a large number of states are contributing. A well-known example is provided by the $\mathcal{N} = 4$ super-Yang-Mills theory, whose entropy density at very strong coupling is only reduced by a factor 3/4 with respect to the free theory [42]. In this interpretation, the main effect of interactions among glueballs on thermodynamic properties is to slightly shift the value of the Hagedorn temperature T_h with respect to its free-string value. A possible mechanism is that the string tension that effectively determines T_h is an in-medium string tension that is $\sim 8\%$ lower than at $T = 0$.

Returning to full QCD, our results lend support to the idea that the hadron resonance gas model can largely account for the thermodynamic properties of the low-temperature phase. Whether the open string density of states reproduces the entropy calculated on the lattice can also be tested at quark masses not necessarily as light as in Nature using a simple open string model [43].

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