# Limitations on the topological $BF$ scheme in Riemann-Cartan spacetime with torsion

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Cartan's structure equations in the Riemann-Cartan framework and some topological invariants of gravity are reanalyzed from the perspective of $BF$ theories. This is related to a variational approach to Chern-Simons terms and Bianchi identities employing Lagrange multipliers. Here, it is pointed out that the $BF$ scheme has some *limitations* to the effect that a coupling to matter would *leave the minimal coupling prescription* of gauge theories. In the case of gravity, the field equations would, generically, become *higher order* with a coupling to the *relocalized* Belinfante-Rosenfeld energy-momentum current.

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## I. INTRODUCTION

A rather new development in topological field theory is the $BF$ formalism, which provides potentially interesting relations to higher-dimensional knots, cf. Refs. [1,2]. $BF$ theory is a framework where the connection one-form $A = A_i dx^i$ and an auxiliary[1] two-form $B = B_{ij} dx^i \wedge dx^j/2$ are varied independently. In the Abelian case, the one-form $A$ can be interpreted as electromagnetic potential. In its primordial form, it starts from the metric-independent Lagrangian four-form

$$L_{BF} = -B \wedge F = -B \wedge dA. \tag{1.1}$$

Independent variations with respect to $A$ and $B$ lead to $dB = 0$ and the *constraint* of vanishing field strength $F := dA = 0$. This topological model has no local degrees of freedom.

This pure $BF$ system can be modified [3] via a boundary term such that the Lagrangian

$$\tilde{L}_{BF} = -B \wedge dA + \tfrac{1}{2} B \wedge B \cong -B \wedge dA + \tfrac{1}{2} dC \tag{1.2}$$

becomes, "on shell," quadratic[2] in $B$. Now, independent variations provide the definition of the field strength together with the corresponding *Bianchi identity*

$$B \cong dA := F, \qquad dB = dF \equiv 0, \tag{1.3}$$

respectively, in compliancy with the Poincaré lemma $dd \equiv 0$. It still defines a topological theory since, "on shell," Eq. (1.2) is equivalent to (1.1) amended by a boundary term $dC$ derived from a *Chern-Simons three-form* $C$. For an Abelian connection, this is simply given by $C = A \wedge F$. In general, it is well known [7] that Bianchi-type identities can be recovered via the variation of the associate Pontrjagin term $dC = F \wedge F$, e.g. $\delta dC/\delta A = 2dF \equiv 0$ in the Abelian case.

Bianchi *identities* do not allow for couplings to source terms thus seriously limiting applications of such a topological scheme. In order to proceed to more *realistic physical models* admitting matter couplings as in Maxwell's theory, the corresponding $BF$ Lagrangian

$$L_{\max} = -B \wedge {}^*dA + \tfrac{1}{2} B \wedge {}^*B + L_{\text{matter}} \tag{1.4}$$

necessarily involves the *Hodge dual* $^*$ depending on the metric, cf. Ref. [8]. Then, independent variations of (1.4) provide again the definition of the field strength $B = dA := F$ but, as a bonus, the nontrivial physical field equation $d^*B = d^*F \cong j$.

However, it should not be overlooked that in a coupling to matter such a $BF$ scheme would leave the minimal coupling prescription since it generates a current three-form

$$\begin{aligned} j := \frac{\delta L_{\text{matter}}}{\delta A} &= \frac{\partial L_{\text{matter}}}{\partial A} + d\frac{\partial L_{\text{matter}}}{\partial dA} \\ &= \Psi \wedge \frac{\partial L_{\text{matter}}}{\partial D\Psi} + D\frac{\partial L_{\text{matter}}}{\partial B}, \end{aligned} \tag{1.5}$$

which "on shell," is conserved classically, i.e. $dj \cong 0$. In general, this includes *Pauli-type terms* generated by the variation of the Lagrangian with respect to $dA$, cf. (5.2.18) of [9]. Because of Eq. (1.3), this additional term is equivalent to the one generated by the variation with respect to $B$, as indicated in (1.5).

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[1]In four dimensions, $B$ resembles the two-form potential for the gauge-invariant field strength or excitation $H = dB$, the Kalb-Ramond *axion* three-form.

[2]In three dimensions, non-Abelian $BF$ systems with a cubic term $-B \wedge F + B \wedge B \wedge B/3$ are directly related to Chern-Simons theories departing from the three-form $C := A \wedge F - A \wedge A \wedge A/3$, cf. Refs. [4,5]. The resulting field equations $F = -B \wedge B$ together with the Bianchi identity $DB = DF \equiv 0$ correspond, in 3D gravity [6], to those with a cosmological term.

 

## II. GRAVITY IN RIEMANN-CARTAN SPACETIME

In the case of topological gravity, the structure equations[3] for the curvature $R_{\alpha\beta}$ in Riemann-Cartan (RC) spacetime and its Bianchi identity can be recovered similarly as in the $BF$ scheme.

The Lagrangian corresponding to the *second Bianchi identity*

$$DR_\alpha{}^\beta \equiv 0 \tag{2.1}$$

is the boundary term

$$dC_{\mathrm{RR}} = -\frac{1}{2} R_\alpha{}^\beta \wedge R_\beta{}^\alpha. \tag{2.2}$$

As is well known, its integration yields a number proportional to the topological invariant of Pontrjagin. Because of the Poincaré lemma $dd \equiv 0$, there do not arise higher-order terms[4] in this chain, cf. our scrupulous related method [7] using *Lagrange multipliers* as well as Ref. [12]. Such a topological theory can, similarly to (1.2), be given the generalized $BF$ structure

$$B_\alpha{}^\beta \wedge R_\beta{}^\alpha - \frac{1}{2} B_\alpha{}^\beta \wedge B_\beta{}^\alpha = B_\alpha{}^\beta \wedge R_\beta{}^\alpha + dC_{\mathrm{RR}}. \tag{2.3}$$

However, for torsion the situation is more subtle: The linear or Lorentz connection $\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha} = \Gamma^{\{\}}_{\alpha\beta} - K_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} + e_\alpha \rfloor T_\beta + (e_\alpha \rfloor e_\beta \rfloor T_\gamma) \wedge \vartheta^\gamma$ can be regarded as a "deformation" [4] of the unique Levi-Civita connection $\Gamma^{\{\}\alpha\beta}$ of Riemannian geometry via the *contortion* $K_{\alpha\beta} = -K_{\beta\alpha}$ implicitly related to torsion via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. In order to account for the torsion content, one can consider a *change of variables* in the gravitational Lagrangian with optional matter couplings, i.e.

$$L(\vartheta^\alpha, \Gamma_\beta{}^\gamma, \Psi, D\Psi) \rightarrow \hat{L}(\vartheta^\alpha, T^\beta, \Psi, D\Psi) + \mu_\alpha \wedge (B^\alpha - D\vartheta^\alpha) + \frac{\theta_T}{2\ell^2} B_\alpha \wedge B^\alpha, \tag{2.4}$$

thereby leaving, as in the Maxwell case, the minimal coupling scheme. As is explained in more detail[5] in Section 5.6 of Ref. [9], when torsion is regarded as an *independent* $B$-type two-form in the variational procedure, the first Cartan structure equation

$$B^\alpha = D\vartheta^\alpha := d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = T^\alpha \tag{2.5}$$

needs to be enforced by a term involving the *Lagrangian multiplier* two-form $\mu_\alpha$. Then torsion couples via $\delta B^\alpha \wedge [\mu_\alpha + (\theta_T/\ell^2) B_\alpha]$ to the former Lagrange multiplier and, finally, the variation of the translational gauge field, the soldered coframe $\vartheta^\alpha$ together with (2.5), induces a *relocalization* of the canonical energy-momentum current $\Sigma_\alpha := \delta L/\delta\vartheta^\alpha$ into

$$\begin{aligned} \sigma_\alpha &= \Sigma_\alpha - D\mu_\alpha + e_\beta \rfloor (T^\beta \wedge \mu_\alpha) \\ &\cong \Sigma_\alpha - D\mu_\alpha - \frac{\ell^2}{\theta_T} e_\beta \rfloor (\mu^\beta \wedge \mu_\alpha), \end{aligned} \tag{2.6}$$

generalizing the familiar Belinfante-Rosenfeld symmetrization of general relativity (GR). Since $\tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge \mu_{\beta]}$ is the spin current, Eq. (2.6) reveals the physical interpretation of $\mu_\alpha$ as *spin energy potential* of matter. This relocalization [13,14] can be traced back to the *translational* nature of energy momentum and torsion ("translational" curvature) in the Lagrange-Noether machinery for the affine group and, for $\theta_T \neq 0$, a quadratic *contact-type* interaction to the spin energy potential emerges "on shell."

Since torsion involves the exterior derivative of the coframe $\vartheta^\alpha$, the above change of variables *cannot be regarded as a point-transformation* in the canonical formalism, and therefore its complete elimination would, in general, generate a change of the gravitational gauge energy-momentum current $E_\alpha$ according to

$$2D\left(e^\beta \rfloor DH_{\alpha\beta} - \frac{1}{4} \vartheta_\alpha e^\gamma \rfloor e^\delta \rfloor DH_{\gamma\delta}\right) - E_\alpha = \Sigma_\alpha - D\mu_\alpha, \tag{2.7}$$

where $H_{\alpha\beta} := -\partial L/\partial R^{\alpha\beta}$ are curvature excitations subjected to the constraint $T^\alpha = 0$. If the two higher derivative Cotton-type terms in the left-hand side of (2.7), cf. Eq. (5.8.25) of Ref. [9], are nonvanishing, the gravitational field equation becomes third order in the Levi-Civita connection $\Gamma^{\{\}\alpha\beta}$, i.e. fourth order in the holonomic metric with an induced coupling to the symmetric Belinfante-Rosenfeld energy-momentum current $\Sigma_\alpha - D^{\{\}}\mu_\alpha$, cf. Ref. [14].

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[3]Incidentally, the 3 + 1 decomposition of Cartan's structure equations and Bianchi identities in Ref. [10] are well known and can also be performed in a rather general slicing of spacetime, cf. Appendices B, C, and D of Ref. [11]. The tangential parts of both Bianchi identities are "constraints," which are preserved during time evolution, as proven in Eqs. (D.5) and (D.6) of Ref. [11]. The so-called "reducibility" equations (10) of Ref. [10] are contained in Eqs. (2.15) and (2.16) of Ref. [7] as tangential pieces. There, also the reducibility of the de Rham chain for curvature and torsion is explicitly derived, cf. the "Identities of the identities" in Sec. II of Ref. [7].

[4]This can also be seen by applying the Ricci formula to the first Bianchi identity (2.1), i.e. $DDR_\alpha{}^\beta = -R_\mu{}^\beta \wedge R_\alpha{}^\mu + R_\alpha{}^\mu \wedge R_\mu{}^\beta \equiv 0$.

[5]The nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$ is always put to zero here following the constraint formalism of Sec. 5.8.1 of Ref. [9], although a metric-affine framework would easily allow to liberate this constraint.

## III. NIEH-YAN TOPOLOGICAL TERM

As an example of a partial $BF$ structure in RC spacetime with torsion, let us consider the so-called Nieh-Yan (NY) term [15,16]

$$dC_{\rm TT} = \frac{1}{2\ell^2}(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta), \qquad (3.1)$$

which violates parity [11,17]. It simply can be obtained by multiplying the *first Bianchi identity*

$$DT^\alpha \equiv R_\beta{}^\alpha \wedge \vartheta^\beta \qquad (3.2)$$

of RC spacetime with $\vartheta_\alpha$ from the left. *Vice versa*, this identity can be recovered via the variation of (3.1) with respect to the coframe, as carefully demonstrated in Ref. [7]. Since (3.2) constitute a purely geometric identity, it does *not* admit couplings to extended matter. Even for models with "distributional" matter, like cosmic strings [12], nonvanishing distributional torsion and curvature compensate each other for (3.2) to hold exactly [18]. The four-form on the right-hand side of the NY identity (3.1) even vanishes, as explicitly demonstrated in Eqs. (2.21) and (2.22) of Ref. [19].

Again, in order to obtain more realistic models, one of the field strengths in (3.1) needs to be converted via a duality rotation into its Hodge dual. Then the NY term suggest two options [20] for a viable gravitational Lagrangian: Hilbert's original choice

$$L_{\rm HE} = -\frac{1}{2\ell^2} R^{\{\}}_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) = -\frac{1}{2\ell^2} R^{\{\}} \eta, \qquad (3.3)$$

where $R^{\{\}}_{\alpha\beta}$ denotes the Riemannian curvature for vanishing torsion and $R^{\{\}} := {}^*(R^{\{\}\alpha\beta} \wedge \eta_{\beta\alpha})$ the Riemannian curvature scalar, as in GR. Formally, it can be put into the $BF$ scheme [12] when choosing $B_{\alpha\beta} = \eta_{\alpha\beta} = {}^*(\vartheta^\alpha \wedge \vartheta^\beta)$ albeit the no-avoidance of the Hodge dual[6]. The purely torsion-square Lagrangian

$$L_\| := \frac{1}{2\ell^2} T^\alpha \wedge {}^*\left(-{}^{(1)}T_\alpha + 2{}^{(2)}T_\alpha + \frac{1}{2}{}^{(3)}T_\alpha\right) \qquad (3.4)$$

where the torsion excitation $H^\|_\alpha := -\partial L_\| / \partial T^\alpha = (1/\ell^2)\eta_{\alpha\beta\gamma}K^{\beta\gamma}$ is dual to the contortion one-form $K_{\alpha\beta}$. It leads to proper *teleparallelism* ($\rm GR_\|$) when constrained by vanishing RC curvature, i.e. $R_{\alpha\beta} = 0$ via a Lagrangian multiplier term $\lambda_{\alpha\beta} \wedge R^{\alpha\beta}$, as was suggested already by Einstein.

Because of the geometric identity

$$L_\| \equiv L_{\rm HE} + \frac{1}{2\ell^2} R_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) + \frac{1}{2\ell^2} d(\vartheta^\alpha \wedge {}^*T_\alpha), \qquad (3.5)$$

$\rm GR_\|$ with the teleparallel constraint $R_{\alpha\beta} = 0$ is classically *equivalent* to GR up to a boundary term $d(\vartheta^\alpha \wedge {}^*T_\alpha)$ constructed from the Hodge dual ${}^*T_\alpha$ of the torsion, cf. Ref. [11].

Both pieces on the right-hand side of the dimensionless boundary type four-form (3.2) have found tentative applications before: The pseudoscalar curvature $R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta = \eta^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}\eta$ has been used for generating a *self-dual* or chiral reformulation of GR with its Hilbert-Einstein Lagrangian[7] proportional to curvature scalar $R_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta)$. This was anticipated already by Plebanski [22], Hojman *et al.* [23] and Dolan [24], but only later "rediscovered" by Holst [25] without references to earlier work. On the other hand, the torsion-squared term $T^\alpha \wedge T_\alpha$ in Eq. (3.1) has been employed to induce a *chiral* reformulation of the *teleparallelism equivalent* of GR, cf. Ref. [11] where the limiting case of vanishing RC curvature is consistently enforced[8] again via Lagrange multipliers.

In the supersymmetric extension of EC theory, the NY boundary term (3.1) induces a chiral formulation [27] of simple ($\mathcal{N} = 1$) supergravity, whereas the *translational* Chern-Simons three-form $C_{\rm TT} := \vartheta^\alpha \wedge T_\alpha / 2\ell^2$ is instrumental for first order models of topological gravity [6] in 3D.

Interesting enough, the generalized $BF$ scheme (1.2) can also be employed to induce a "breaking" [28] of the de Sitter gauge symmetry down to Einstein's GR with cosmological constant. This and its relation to BRST quantization of gravity [29,30] needs to be seen.

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[6]In fact, there have been attempts, to restrict oneself to the Lie dual $^{(\star)}$ only, as is the case of the Lie dual $R^{(\star)}_{\alpha\beta} := 1/2\eta_{\alpha\beta\gamma\delta}R^{\gamma\delta}$ of the curvature featuring, e.g. in the Euler invariant. However, it is not always realized, cf. for example Eq. (33) of Ref. [10], that the Lie dual $\eta_{\alpha\beta} := \frac{1}{2}\eta_{\alpha\beta\gamma\delta}\vartheta^\gamma \wedge \vartheta^\delta = {}^*(\vartheta^\alpha \wedge \vartheta^\beta)$ of the "unit" two-form $\vartheta^\alpha \wedge \vartheta^\beta$ is *equivalent* to its Hodge dual $^*$ as a consequence of the soldering of the coframe $\vartheta^\alpha$, cf. Eq. (3.7.8) of Ref. [9].

[7]The currently widespread usage of the label "Palatini action" or "method" has been questioned in Ref. [21] from the historical point of view and, therefore, appears to be a misnomer.

[8]For instance, it is claimed that the pseudoscalar action $S_5$ and the torsion-squared action $S_6$ of Ref. [10] " ...define the same dynamical system." However, according to (3.1) they *differ* by the boundary term $2\ell^2 dC_{\rm TT}$, which invariantly characterizes nontrivial topologies [26]. In fact, integration over the boundary three-sphere at infinity yields the invariant $n_{\rm NY} := \int_{R^4 \cup \infty} dC_{\rm TT} = \int_{S^3_\infty} \underline{C}_{\rm TT} = 6\pi^2 k$, , where $k$ is the winding or instanton number of Pontrjagin. The Lagrangian $S_7$ is of the $BF$ type (1.2), but merely leads to the *truncated* Bianchi identity $DT^\alpha =^* dd\vartheta^\alpha \equiv 0$ of a teleparallel spacetime, where $\Gamma^{\alpha\beta} =^* 0$ locally.

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