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Globally regular deformation of De Sitter space

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We present a 1-parameter deformation of four-dimensional De Sitter space within Einstein-Maxwell–De Sitter gravity and show that it is causal-geodesically complete.

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In Ref. [1], generalizations of Kastor and Traschen's cosmological multiblack hole [2] solution were constructed by considering classes of supergravitylike theories. One of the possible theories is nothing but Einstein-Maxwell with a De Sitter cosmological constant, which in our conventions reads

$$\mathcal{S} = \int_4 \sqrt{g}[R - F^2 - 6\xi^2], \tag{1}$$

where ξ is an arbitrary real constant and $F = dA$ if the field strength for the Maxwell field is A. The structure of the metric of the solutions is that of a conformastationary metric, and the metric on the three-dimensional base space is restricted to that induced by a specific subclass of three-dimensional Einstein-Weyl spaces called Gauduchon-Tod spaces [3]. The simplest nontrivial, compact example of a Gauduchon-Tod–space is the so-called Berger sphere, which is a 1-parameter family of squashed three-spheres, where the squashing takes place along the $U(1)$ fiber in the Hopf fibration of the three-sphere. It is this Berger sphere which allows us to construct a 1-parameter family of deformations of four-dimensional De Sitter space, which is free of singularities and is causally geodesically complete.

The metric of the solution is given by, after a small coordinate transformation with respect to the results given in Ref. [1],

$$ds^2 = \left(dt + \frac{\sin(2\mu)}{2\xi}\tanh(\xi t)\varrho\right)^2 - \frac{\cos^2(\mu)}{4\xi^2}\cosh^2(\xi t)[dS^2_{[\theta,\phi]} + \cos^2(\mu)\varrho^2], \tag{2}$$

where $dS^2_{[\theta,\phi]}$ stands for the ordinary round metric on S^2 with coordinates θ and ϕ and the $U(1)$ direction is given by the missing Euler angle $\chi \in [0, 4\pi)$ and enters the metric in the combination $\varrho = d\chi + \cos(\theta)d\phi$. Furthermore, in order for the metric on the Berger sphere to be regular, the squashing constant μ must be constrained to $\mu \in [0, \pi/2)$. The expression for the Maxwell field is given by

$$A = \frac{\sin(2\mu)}{2\xi}[1 - 2\tanh^2(\xi t)]\varrho. \tag{3}$$

It is clear that when $\mu = 0$ the solution corresponds to four-dimensional De Sitter space in global coordinates: as DS_4 is a symmetric space it is automatically free of curvature singularities and is causal-geodesically complete; following Bonnor [4], we call a spacetime, which is free of singularities and is causal-geodesically complete, a globally regular spacetime.

The question, then, is: What characteristics of De Sitter space are also present in the family of solutions given by Eq. (2)? The fact that the space is free of curvature singularities can be seen rather quickly by using a computer program [5], which also tells us that the solution is generic of Petrov type I and becomes of type II when $\mu = 0$. The topology of the space is $\mathbb{R} \times S^3$ and the isometry algebra of this family is generically $\mathfrak{u}(2)$, which gets enlarged to $\mathfrak{so}(1, 4)$ when $\mu = 0$. As was shown in Ref. [6], squashing of highly symmetric solutions can lead to solutions with regions in which there are *closed timelike loops*, and in order to avoid those in our family, we must and will restrict the squashing parameter to the interval $\mu \in [0, \pi/4)$. The remaining question then is whether the metric is causal-geodesically complete or not.

In Ref. [7], sufficiency conditions were derived for a metric like the one in Eq. (2) to be globally hyperbolic and to be future causal-geodesically complete: one can see fairly rapidly that the solution satisfies [7]'s conditions for global hyperbolicity but not the ones for future causal-geodesically completeness. Still, we can employ techniques similar to the ones of Ref. [7] to show that the causal geodesics are complete.

Using the conserved charges for the geodesic motion due to the $U(2)$ isometry group, we can write the mass-shell condition for an affinely parametrized geodesic as, taking $\xi = 1$ without loss of generality,

$$M^2 = N^2\dot{t}^2 - \frac{4}{\cos^2(\mu)}ch^{-2}(t)\left[\vec{J}^2 + \tan^2(\mu)N^2 \times \left(1 + 4\frac{sh^2(t)}{ch^4(t)}\right)\pi_\chi^2\right], \tag{4}$$

where $\vec{J}$ and π_χ are the conserved charges due, respectively, to the $SU(2)$ and the $U(1)$ isometries; the lapse

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function N^2 is defined by

$$N^{-2} = 1 - 4\tan^2(\mu)\frac{\sinh^2(\xi t)}{\cosh^4(\xi t)}, \quad (5)$$

which is a strictly positive and uniformly bounded function of t, i.e. $1 - \tan^2(\mu) \leq N^{-2} \leq 1$.

It is instructive to first look at a null geodesic, whence $M = 0$, with vanishing $SU(2)$ charges: introducing the new coordinate $x = \sinh(\xi t)$ and the abbreviation $\aleph$ by $\cos(\mu)\aleph = 2\xi^2 \tan(\mu)\pi_\chi$, we can deduce from Eq. (4) that the affine parameter, s say, along the curve can be expressed as

$$\aleph s(x) = \int H(x)dx \equiv \int \frac{1 + x^2}{\sqrt{1 + 6x^2 + x^4}} dx, \quad (6)$$

where we have chosen the possible integration constant to vanish as to have $s(-x) = -s(x)$. The above integral can be done analytically in terms of elliptic functions, but for the more general geodesics this is not the case, and we will resort to a reasoning inspired by Ref. [7]: a geodesic is said to be complete if the affine parameter can take on all values on $\mathbb{R}$, which using the above means that the range of the function $s(x)$ should be $\mathbb{R}$. However, as the function $H(x)$ is uniformly bounded between $2^{-1/2}$ and 1, the function $s(x)$ is a strictly monotonic function, i.e. $s'(x) > 0$, and behaves asymptotically as $s(x) \sim x$, which implies that s: $\mathbb{R} \to \mathbb{R}$ is an invertible function. Coupling this to the fact that $x = \sinh(\xi t)$, we see that we are dealing with a complete geodesic starting off at "minus t" infinity and ending at "plus t" infinity.

For a general geodesic, we can still integrate the equations as in Eq. (6) but with $\cos(\mu)\aleph = 2\xi^2$ and

$$H^{-2} = N(\vec{J}^2 + \tan^2(\mu)\pi_\chi^2 N^2(1 + 4x^2y^{-4}) + \xi^2\aleph^{-2}M^2y^2), \quad (7)$$

where for convenience we used the abbreviation $y^2 = 1 + x^2$. As the generic H is an even function of x, we still have that s is an odd function, but H is not uniformly bounded: this in itself is not too problematic as we can see that $H > 0$, which implies that s is again a strictly monotonic function. The difference lies in the large x behavior: for the massless geodesics we obtain, as above, the fact that $s(x) \sim x$, whereas for massive geodesics we find $s(x) \sim \log(x)$ [8]. There is, however, no obstruction in taking x as large as we want, so that we must conclude that s is an invertible function from $\mathbb{R}$ to $\mathbb{R}$, and therefore also that our metric is causal-geodesically complete.

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