Scattering theory using smeared non-Hermitian potentials

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Local non-Hermitian potentials $V(x) \neq V^*(x)$ can, sometimes, generate stable bound states $\psi(x)$ at real energies. Unfortunately, the idea [based on the use of a non-Dirac *ad hoc* metric $\Theta(x, x') \neq \delta(x - x')$ in Hilbert space] cannot directly be transferred to scattering due to the related loss of the asymptotic observability of x [cf. H. F. Jones, Phys. Rev. D **78**, 065032 (2008)]. We argue that for smeared (typically, nonlocal or momentum-dependent) potentials $V \neq V^\dagger$ this difficulty may be circumvented. A return to the usual (i.e., causal and unitary) quantum scattering scenario is then illustrated via an exactly solvable multiple-scattering example. In it, the anomalous loss of observability of the coordinate remains restricted to a small vicinity of the scattering centers.

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I. INTRODUCTION

An intuitive understanding of various physical aspects of scattering can be facilitated when one turns attention to simplified, one-dimensional schematic models of experimental setup. An extremely exciting toy-model scenario has recently been proposed and analyzed in Ref. [1]. One of the most elementary and popular delta-function potentials $V_0(x) = -\alpha\delta(x)$ has tentatively been combined with a remote non-Hermitian interaction. The purpose of this *gedanken experiment* has been formulated as a study of an ‘‘interface’’ between Hermitian and non-Hermitian components of the potentials exemplified by the superposition

$$V(x) = V_0(x) + i\beta[\delta(x - L) - \delta(x + L)], \qquad L \gg 1. \tag{1}$$

Our present text offers an immediate continuation of this project. We feel motivated by the occurrence of many open questions in such a setting. In particular, the results of Ref. [1] indicated that it might be rather difficult to keep a non-Hermitian interaction model short ranged and compatible with the standard requirements of a local and causal physical interpretation of incoming and/or scattered waves.

Our present answer to these compatibility questions will be predominantly affirmative. More precisely, we shall emphasize that the number of problems which arose during the analysis of potential (1) may be attributed to its strict locality. We shall propose and advocate the replacement of the strictly local interaction operators $V \equiv V(x)$ by their slightly smeared descendants tractable as weakly momentum-dependent operators. For illustration purposes we shall use interactions $V \neq V^\dagger$ given by Eq. (6) in Sec. II and Eq. (14) in Sec. III.

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The latter choice of amended model will preserve its maximal similarity with the original potential of Ref. [1]. First, the role played by the real ‘‘measure of non-Hermiticity’’ β in Eq. (1) will be transferred to another real coupling constant g. Second, the distance L between the strictly localized interaction points of Eq. (1) will be replaced by a variable integer $\mathcal{N}$ representing a separation distance between two not entirely local, ‘‘smeared’’ domains of support of our interaction V.

We decided to parallel the majority of quantitative results of Ref. [1] by their close and explicit analogs. The major problems resulted from the manifest non-Hermiticity of the interaction which implies, for Eq. (1) at least, the necessity of a drastic change of the concept of the coordinate. In Ref. [2] this mathematical result has been identified as a source of deep conflict between the use of x in Eq. (1) (i.e., in the ‘‘input’’ definition of the interaction) and, simultaneously, in the asymptotic boundary conditions for the one-dimensional scattering,

$$\psi(x) = \begin{cases} e^{i\kappa x} + Re^{-i\kappa x}, & x \ll -1, \\ Te^{i\kappa x}, & x \gg 1. \end{cases} \tag{2}$$

It is necessary to keep in mind that the core of this conflict does not lie in the formalism of quantum mechanics itself. Formally, no problems occur since all the unitary transformations of a given model (cf., e.g., a nonlocal free-motion example given in Sec. 5 of Ref. [3]) *must* lead to equivalent physical predictions.

The differences in predictions can only occur when nonequivalent definitions of the dynamics are being compared. This is precisely in this sense that the difficulties emerged in Refs. [1,2] where a simultaneous validity of both the definition (1) of the local physical interaction $V \neq V^\dagger$ and an *a priori* assignment (2) of the usual physical meaning to the local free waves $\exp \pm i\kappa x$ in asymptotic domain has been required.

A key to the resolution of this misunderstanding has been described in our paper [4] where we showed that non-Hermitian models exist, where the simultaneous validity of the phenomenological postulates (1) and (2) *can* be achieved after a certain modification of their respective forms. In this sense, our present paper will just amend and strengthen the argumentation of Ref. [4]. Indeed, in a way emphasized by the note added in proof in [1], our old model "still involved a departure from standard quantum mechanics at large distances."

The present final resolution of the conflict between the locality of forces and waves will rely on a nontrivial extension of the class of interactions accompanied by an enhancement of efficiency of necessary mathematics. These technical details will be described in Secs. III and IV and in two Appendixes. *In nuce*, the Runge-Kutta coordinate-discretization method [5] will be shown superior, for the given purpose, to the usual perturbation expansions as employed, e.g., in Ref. [1]. In the latter study of Eq. (1), for this reason, *both* the inverse length $1/L$ *and* the variable coupling constant β had to be assumed small. In contrast, the variability of both our present parameters g and $\mathcal{N}$ will be, within their respective physical ranges, unrestricted.

The presentation of our explicit scattering solutions in Sec. IV will confirm the full consistency and unitarity of the scattering in our amended class of non-Hermitian models. In the Summary (Sec. V) several comments will finally be added clarifying the proposed changes of theoretical perspective in a broader, less model-dependent context.

II. TOWARD THE SHORT-RANGED NON-HERMITICITIES

A. Runge-Kutta discretization

We may treat any one-dimensional Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x), \qquad x \in (-\infty, \infty) \tag{3}$$

with a local and real or complex potential $V(x)$ as a continuous $h \to 0$ limit of its difference-equation approximation defined along the Runge-Kutta doubly infinite lattice of discrete coordinates $x = x_k = kh$, $k = 0, \pm 1, \ldots$,

$$-\frac{\psi(x_{k-1}) - 2\psi(x_k) + \psi(x_{k+1})}{h^2} + V(x_k)\psi(x_k) = E\psi(x_k). \tag{4}$$

Approximate wave functions may be then constructed via the methods of linear algebra reparametrizing, incidentally, the real energies $E = (2 - 2\cos\varphi)/h^2$ in terms of a real angle $\varphi = \varphi(E) \in (0, \pi)$. The scattering boundary conditions (2) may and should be rewritten in their discrete version,

$$\psi(x_m) = \begin{cases} e^{im\varphi} + Re^{-im\varphi}, & m \le -M \ll -1, \\ Te^{im\varphi}, & m \ge M - 1. \end{cases} \tag{5}$$

One could easily discretize the ultralocal non-Hermitian toy-model (1) and confirm the discouraging conclusions, formulated in Ref. [1], that one can "no longer talk in terms of reflection and transmission coefficients" so that "the only satisfactory resolution (of dilemmas) is to treat the non-Hermitian scattering potential as an effective one, and work in the standard framework of quantum mechanics, accepting that this effective potential may well involve the loss of unitarity" [1].

The loss of unitarity need not necessarily be perceived as a weakness of the theory, especially when one deals "with a subsystem of a larger system whose physics has not been taken fully into account" [1]. In this sense one may perceive Eq. (1) with a local non-Hermitian interaction $V(x) \neq V^*(x)$ as an "effective theory," i.e., as an *incomplete* picture of physical reality. This philosophy finds interesting phenomenological applications ranging from classical optics [6] or models with supersymmetry [7] up to the manifestly nonunitary scattering models in quantum phenomenology [8] and up to the descriptions of open systems in nuclear and solid state physics [9] and in quantum cosmology [10]. Nevertheless, in an alternative, theoretically much more ambitious approach to the localized non-Hermitian scattering potentials one should insist on the conservation of a suitable current and, hence, on the strict unitarity of the scattering realized by the asymptotically observable free plane waves.

B. Nonlocal updates of potentials

The first (and the least sophisticated) quantitative non-Hermitian model satisfying the above requirements has been constructed in our paper [4]. We replaced Eq. (4) by its generalization where the interaction operator V acquired the nearest-neighbor form,

$$-\frac{\psi(x_{k+1}) - 2\psi(x_k) + \psi(x_{k-1})}{h^2} + V_{k,k+1}\psi(x_{k+1}) + V_{k,k}\psi(x_k) + V_{k,k-1}\psi(x_{k-1}) = E\psi(x_k). \tag{6}$$

After a rescaling of the Hamiltonian $H = -d^2/dx^2 + V$ by an inessential numerical factor h^2 we obtained

$$H = -\triangle + V,$$

$$-\triangle = \begin{bmatrix} \ddots & \ddots & & & & \\ \ddots & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \ddots \\ & & & & \ddots & \ddots \end{bmatrix} \quad (7)$$

and chose the following, minimally nonlocal potential:

$$V = V^{(a,b,c,\ldots)} = \begin{bmatrix} & \ddots & & & & & & \\ \ddots & & -c & & & & & \\ & c & & -b & & & & \\ & & b & & -a & & & \\ & & & a & & -b & & \\ & & & & b & & -c & \\ & & & & & c & & \ddots \\ & & & & & & \ddots & \end{bmatrix}. \quad (8)$$

The resulting multiparametric Hamiltonian $H = -\triangle + V^{(a,b,c,\ldots)} \neq H^\dagger$ remains manifestly non-Hermitian in the "friendly" Hilbert space $\mathcal{H}^{(F)}$ endowed with the usual inner product

$$\langle\psi|\phi\rangle^{(F)} = \sum_k \psi^*(x_k)\phi(x_k) = \langle\psi|\phi\rangle \quad \text{in } \mathcal{H}^{(F)}. \quad (9)$$

Note that the summation would only be replaced by the integration in the continuous limit $h \to 0$. Now, the key point is that the *same* operator H may be found Hermitian after one moves into *another* Hilbert space $\mathcal{H}^{(S)}$. In the latter space the definition of the inner product must be different and more general,

$$\langle\psi|\phi\rangle^{(S)} = \sum_k \sum_n \psi^*(x_k)\Theta_{k,n}\phi(x_n) = \langle\psi|\Theta|\phi\rangle := \langle\langle\psi|\phi\rangle \quad \text{in } \mathcal{H}^{(S)}. \quad (10)$$

The "non-Dirac metric" matrix $\Theta = \Theta^\dagger$ must only remain positive definite and compatible with the Hamiltonian in question [11],

$$H^\dagger\Theta = \Theta H. \quad (11)$$

In the notation of Ref. [12] one writes $H = H^\ddagger$ and speaks about a "quasi-Hermiticity" [11] or "pseudo-Hermiticity" [13] or "crypto-Hermiticity" [14] of the Hamiltonian. In this context the core of the message delivered by our paper [4] was that there exists a metric-operator matrix $\Theta^{(a,b,c,\ldots)}$ which remains compatible with our interaction model (8) *as well as* with the asymptotic observability of Runge-Kutta coordinates x_k. This matrix has the following compact and fully diagonal form:

$$\Theta^{(a,b,c,\ldots)} = \begin{bmatrix} \ddots & & & & & & & \\ & \theta_{-5} & & & & & & \\ & & \theta_{-3} & & & & & \\ & & & \theta_{-1} & & & & \\ & & & & \theta_1 & & & \\ & & & & & \theta_3 & & \\ & & & & & & \theta_5 & \\ & & & & & & & \ddots \end{bmatrix}. \quad (12)$$

Its elements are given by closed formulas,

$$\theta_{\pm1} = (1 \pm a)(1 - b^2)(1 - c^2)(1 - d^2)\cdots,$$
$$\theta_{\pm3} = (1 \pm a)(1 \pm b)^2(1 - c^2)(1 - d^2)\cdots,$$
$$\theta_{\pm5} = (1 \pm a)(1 \pm b)^2(1 \pm c)^2(1 - d^2)\cdots.$$

One arrives at a causality-observing physical picture of scattering based on a clear separation of the "in" and "out" solutions not only in Hilbert space $\mathcal{H}^{(F)}$ but also in Hilbert space $\mathcal{H}^{(S)}$.

In our subsequent paper [15] the next step has been made. In the spirt of Eq. (1) we simulated the existence of several separate point interactions. Unfortunately, the construction of the metric only remained feasible under a very specific, left-right symmetric arrangement of the set of interaction centers. Sometimes, this type of symmetry is being called $\mathcal{PT}$ symmetry, for reasons and with motivations which are thoroughly explained elsewhere [16].

Our present continuation of development of the multiple-scattering idea will be based on a return to asymmetric models, allowing an independence of arrangement of *several* spatially separated scatterers. Paradoxically, the transition to asymmetric realizations of the set of interaction centers will be accompanied by a simplification of analysis of their mutual interference.

C. Limiting transition to continuous coordinates $h \to 0$

For a quantitative specification of the extent of nonlocality induced by multiparametric matrices $V^{(a,b,c,\ldots)}$ of Eq. (8) let us start from the simplest, coordinate-independent model where $a \approx b \approx c \approx \cdots$. Then, the limiting transition to $h = 0$ converts operator $V^{(a,a,a,\ldots)}$ into the first power of the momentum, $V^{(a,a,a,\ldots)} \sim d/dx$. In the subsequent step one may reintroduce a weak coordinate dependence (with $a \neq b \neq \cdots$) and evaluate the continuous limit perturbatively. Locally, the limit $h \to 0$ will preserve the same leading-order approximate proportionality of the coordinate-dependent potential to the momentum.

Admitting an unconstrained variability of the parameters in matrices $V^{(a,b,c,\ldots)}$ we obtain some less trivial coordinate- and momentum-dependent operators. For the sake of brevity let us restrict similar considerations solely to the models with just a few nonvanishing coupling parameters. Then, the limiting transition $h \to 0$ will certainly lead to point interactions. Their explicit definition will be given precisely by the matching of the wave functions. Just a slightly more complicated alternative to the delta-function point-interaction model (1) of Ref. [1] will be obtained. Our Appendixes A and B may be consulted for illustration of some technical aspects of such a type of matching recipe.

Our illustrative toy potential (8) has not been too well designed for phenomenological purposes since it did not allow us to remove the spatial asymmetry from the related metric matrix (12),

$$\frac{\theta_{-k}}{\theta_k} = \frac{(1-a)(1-b)^2(1-c)^2\cdots}{(1+a)(1+b)^2(1+c)^2\cdots}. \tag{13}$$

The effect of the localized non-Hermiticity in H remained long ranged.

III. TOY MODEL

Equation (13) indicates that the flow of the probability is different to the left and to the right of the scattering center. A weaker form of this shortcoming characterizes also the $\mathcal{PT}$-symmetric models of Ref. [15] where the metric remained rescaled (i.e., non-Dirac, $\Theta_{k,k} \neq 1$) along the spatial interval(s) separating the individual scatterers. This encouraged us to perform a series of computer-assisted trial-and-error experiments leading, at the end, to our present interaction-matrix candidate

$$V^{(g,\mathcal{N})} = \left[\begin{array}{cccccccccccc}
 & \ddots & & & & & & & & & & \\
\ddots & & & 0 & & & & & & & & \\
 & 0 & & -g & & & & & & & & \\
 & & g & & g & 0 & & & & & & \\
 & & & -g & & & & & & & & \\
 & & & & 0 & & \ddots & & & & & \\
 & & & & & \ddots & & & 0 & & & \\
 & & & & & & 0 & & -g & & & \\
 & & & & & \multicolumn{2}{c}{\underbrace{\qquad}_{\text{large gap, } 2\mathcal{N}+1 \text{ columns}}} & g & & g & 0 & \\
 & & & & & & & & -g & & & \\
 & & & & & & & & & 0 & & \ddots \\
 & & & & & & & & & & \ddots &
\end{array}\right], \tag{14}$$

where each scatterer is simulated by a three-dimensional submatrix. Although our particular model (14) comprises just two localized interaction centers at $x_{\pm(\mathcal{N}+2)}$, we shall not consider three [as in Eq. (1)] or more individual scatterers because such a generalization would remain routine, not necessitating any significant further technical improvements of our method.

A. Metric $\Theta^{(g,\mathcal{N})}$ with localized anomalies

Using heuristic arguments we arrived at *Ansatz* (14) and studied the scattering solutions. The Schrödinger equation with the smallest gaps $\mathcal{N}$ has been studied first of all. A sample of these calculations may be found collected in Appendixes A and B. They demonstrate that one of the specific merits of Eq. (14) lies in a maximal simplicity of necessary algebraic manipulations.

The second merit of the choice of Eq. (14) can be seen in its generic character. One can add several further interaction submatrices of the same form without worsening the feasibility of the calculations. On this background, without any real loss of generality we restricted our attention just to the first nontrivial example which is characterized by the occurrence of the mere two remote centers of interaction.

We decided to construct all the eligible metric matrices as linear-algebraic solutions of Eq. (11). After we imposed the condition of the compatibility of Θ with the asymptotic observability of the coordinate we revealed that our present models $V^{(g,\mathcal{N})}$ can be assigned the diagonal metric operators of the same doubly infinite diagonal matrix form

$$\Theta^{(g,\mathcal{N})} = \begin{bmatrix} \ddots &&&&&&&&&&&& \\ & 1 &&&&&&&&&&& \\ && 1 &&&&&&&&&& \\ &&& \frac{1+g}{1-g} &&&&&&&&& \\ &&&& 1 &&&&&&&& \\ &&&&& 1 &&&&&&& \\ &&&&&& \underbrace{\ddots}_{\substack{=2\mathcal{N}+1 \\ (\text{``distance''})}} &&&&&& \\ &&&&&&& 1 &&&&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& \frac{1+g}{1-g} &&& \\ &&&&&&&&&& 1 && \\ &&&&&&&&&&& 1 & \\ &&&&&&&&&&&& \ddots \end{bmatrix}. \tag{15}$$

This metric differs from the Dirac's $\Theta^{(\mathrm{Dirac})} = I$ *solely* at the centers of the nonvanishing three-by-three submatrices simulating the non-Hermitian pointlike scatterers.

B. Single-center limit $h \to 0$ at $\mathcal{N} = -1$

The picture of the scattering as offered by our toy potential (14) and by the related metric matrix (15) depends on the Runge-Kutta discretization length $h > 0$. Once we demand that a measured distance between two scattering centers is a macroscopic constant L, our parameter $\mathcal{N}$ must grow with the decrease of h as L/h. Vice versa, the use of an h-independent $\mathcal{N}$ will only lead to a single-centered scatterer. In the latter scenario the scattering is realized by a "quasilocal" potential. Its explicit specification will depend on the h dependence of $\mathcal{N} = \mathcal{N}(h)$. It remains compatible with $L = 0$ whenever $h\mathcal{N}(h) \to 0$ for $h \to 0$. Such a flexibility may make the interactions better suited for fine-tuning, say, of the strength of nonlocalities and/or of the extent of the violation of conservation laws at short distances, etc.

For illustration purposes let us pick up the elementary example of Appendix A. Relaxing the specification of some concrete asymptotic boundary conditions let us reinterpret its "distant" wave function components as an arbitrary free wave $\psi(x)$. At $x \le x_{-2}$ or $x \ge x_2$ this yields the coincidence of symbols

$$U_{-m} = \psi(x_{-m}) := \psi^{(\mathrm{free})}_{-m}, \qquad L_m = \psi(x_m) := \psi^{(\mathrm{free})}_{+m},$$
$$m \ge \mathcal{N} + 3 = 2,$$

respectively. Next, the first and last matching condition extend both the latter assignments by one more step,

$$U_{-1} = (1+g)\psi(x_{-1}), \qquad L_1 = (1+g)\psi(x_{+1}).$$

Finally, with $\psi_0 = \psi(x_0)$ we arrive at the three dynamically nontrivial requirements

$$\begin{bmatrix} 2\cos\varphi & -1 & 0 \\ -1+g^2 & 2\cos\varphi & -1+g^2 \\ 0 & -1 & 2\cos\varphi \end{bmatrix} \begin{bmatrix} U_{-1} \\ (1-g^2)\psi_0 \\ L_1 \end{bmatrix} = (1-g^2)\begin{bmatrix} \psi^{(\mathrm{free})}_{-2} \\ 0 \\ \psi^{(\mathrm{free})}_{2} \end{bmatrix}, \tag{16}$$

which define our wave function, implicitly, near the origin. Tentatively, we may Taylor expand

$$\psi^{(\mathrm{free})}_{-2} = \psi - 2h\psi' + 2h^2\psi'' + \cdots, \qquad (1+g)^{-1}U_{-1} = \psi - h\psi' + h^2\psi''/2 + \cdots, \qquad \psi_0 = \psi,$$
$$(1+g)^{-1}L_1 = \psi + h\psi' + h^2\psi''/2 + \cdots, \qquad \psi^{(\mathrm{free})}_{2} = \psi + 2h\psi' + 2h^2\psi'' + \cdots$$

and insert these approximants in Eq. (16), yielding

$$-(1-g)(\psi - 2h\psi' + 2h^2\psi'') + 2\cos\varphi(\psi - h\psi' + h^2\psi''/2) - (1-g)\psi = \mathcal{O}(h^3),$$
$$-(1+g)(\psi - h\psi' + h^2\psi''/2) + 2\cos\varphi\,\psi - (1+g)(\psi + h\psi' + h^2\psi''/2) = \mathcal{O}(h^3),$$
$$-(1-g)(\psi + 2h\psi' + 2h^2\psi'') + 2\cos\varphi(\psi + h\psi' + h^2\psi''/2) - (1-g)\psi = \mathcal{O}(h^3).$$

According to these relations, the Schrödinger equation $V - E = \psi''/\psi$ would make quantity V large and positive when extracted from the combination of the first and third equations, or large and negative when extracted from the middle equation. This means that our tentative assumption about the smoothness of wave functions near the origin leads to mathematical contradictions and must be abandoned.

Let us now modify our assumptions, distinguish between the left and right wave functions, and set $A(x) = \psi(x-h)$ and $B(x) = \psi(x+h)$, i.e.,

$$(1+g)^{-1}U_{-1} = A(0) := A,$$
$$(1+g)^{-1}L_1 = B(0) := B.$$

Naturally,

$$\psi_{-2}^{(\mathrm{free})} \approx A - hA' + \mathcal{O}(h^2),$$
$$\psi_{2}^{(\mathrm{free})} \approx B + hB' + \mathcal{O}(h^2),$$

while quantity ψ_0 acquires the two alternative first-order representations,

$$\psi_0 \approx A + hA' \approx B - hB'.$$

In the limit $h \to 0$ the latter relation yields

$$A = B, \qquad A' = -B'.$$

The insertion of our amended *Ansätze* in Eq. (16) leads just to the three alternative versions of the requirement of smallness of $A = \mathcal{O}(h^2)$, $B = \mathcal{O}(h^2)$ as well as of $\psi_0 = \mathcal{O}(h)$. Thus, in the continuous-coordinate extreme our simplest $\mathcal{N} = -1$ example degenerates to the opaque-wall-barrier dynamics generated by an additional Dirichlet boundary condition $\psi(0) = 0$.

We see that the role of non-Hermiticity is, in our model with $\mathcal{N} = -1$ at least, truly nonperturbative and dynamically highly influential. This conclusion may independently be confirmed by the inspection of the $\mathcal{N} = -1$ reflection and transmission coefficients given in Appendix A. We believe that also beyond this concrete example, at least some of its features will survive a transition to more-center models and/or to the two-center models at large separation distances $\mathcal{N}(h) = \mathcal{O}(1/h)$.

IV. THE UNITARITY OF THE SCATTERING AT ANY $\mathcal{N}$

The first encouraging surprise encountered during the inspection of the discretized metric (15) is that it remains asymptotically diagonal in the coordinate representation. This means that the asymptotic coordinate x remains observable. Moreover, the range of influence of individual non-Hermitian scatterers is shortened via Eqs. (14) and (15). Thus, the only missing component of the whole picture are formulas for the reflection and transmission coefficients, the determination of which may start from the linear Schrödinger equation for discretized wave functions $\psi = \psi(x_k) = \psi_k$,

$$H\psi = E\psi. \tag{17}$$

In its light the validity of boundary conditions (5) can be prolonged to all the subasymptotic free-motion domain,

$$\left.\begin{array}{r}\psi_{-m} = e^{-im\varphi} + Re^{im\varphi} \equiv U_{-m},\\ \psi_{+m} = Te^{im\varphi} \equiv L_m,\end{array}\right\} \qquad m \geq \mathcal{N} + 3. \tag{18}$$

In parallel, for larger integers $\mathcal{N} = \mathcal{N}(h)$ we may profit from adding another free-motion *Ansatz* at the smaller subscripts,

$$\psi_k = Ce^{ik\varphi} + De^{-ik\varphi}, \qquad |k| \leq \mathcal{N}. \tag{19}$$

One should add that the study of the large distances $\mathcal{N} \gg 1$ might be well motivated by its potential relevance in physics. In particular, its feasibility could offer a guide for simulation of macroscopic nonlocalities, the presence of which could, in its turn, lead to the violation of causality at small distances. In parallel it is important that the effect of our non-Hermitian V can be kept localized. This means that in contrast to virtually all of the published older models the simplicity of interaction (14) enables us to return to the "old-fashioned" definitions of the reflection coefficient R and transmission coefficient T.

A. The elimination of $\mathcal{N}$ from matching conditions

The second surprise offered by our example is that the matching remains easy even for remote interactions with $\mathcal{N} \gg 1$. In order to show this, let us now assume that the distance $2\mathcal{N} + 1$ between two three-dimensional interaction submatrices in (14) is arbitrary. We may abbreviate, in partitioned notation,

$$V^{(g,\mathcal{N})} = \begin{bmatrix} 0 & g & 0 & \vec{0}^T & 0 & 0 & 0\\ -g & 0 & -g & \vec{0}^T & 0 & 0 & 0\\ 0 & g & 0 & \vec{0}^T & 0 & 0 & 0\\ \vec{0} & \vec{0} & \vec{0} & \hat{0} & \vec{0} & \vec{0} & \vec{0}\\ 0 & 0 & 0 & \vec{0}^T & 0 & g & 0\\ 0 & 0 & 0 & \vec{0}^T & -g & 0 & -g\\ 0 & 0 & 0 & \vec{0}^T & 0 & g & 0 \end{bmatrix},$$

where $\hat{0}$ denotes a null matrix (of dimension $2\mathcal{N} + 1$) and where $\vec{0}$ are null column vectors. The superscripts T denote transpositions (i.e., row real vectors). In such a notation one has to consider the following $2\mathcal{N} + 7$ matching conditions:

$$M^{[\mathcal{N}]}(\varphi)\begin{bmatrix} U_{-\mathcal{N}-3}\\ U_{-\mathcal{N}-2} + \chi_{-2}\\ U_{-\mathcal{N}-1} + \chi_{-1}\\ \vec{\psi}_0\\ L_{\mathcal{N}+1} + \chi_1\\ L_{\mathcal{N}+2} + \chi_2\\ L_{\mathcal{N}+3}\end{bmatrix} = \begin{bmatrix} U_{-\mathcal{N}-4}\\ 0\\ 0\\ \vec{0}\\ 0\\ 0\\ L_{\mathcal{N}+4}\end{bmatrix},$$

where

$$M^{[\mathcal{N}]}(\varphi) = \begin{bmatrix} 2\cos\varphi & -1-g & 0 & \vec{0}^T & 0 & 0 & 0 \\ -1+g & 2\cos\varphi & -1+g & \vec{0}^T & 0 & 0 & 0 \\ 0 & -1-g & 2\cos\varphi & \vec{a}^T & 0 & 0 & 0 \\ \vec{0} & \vec{0} & \vec{a} & \hat{F}^{[\mathcal{N}]} & \vec{b} & \vec{0} & \vec{0} \\ 0 & 0 & 0 & \vec{b}^T & 2\cos\varphi & -1-g & 0 \\ 0 & 0 & 0 & \vec{0}^T & -1+g & 2\cos\varphi & -1+g \\ 0 & 0 & 0 & \vec{0}^T & 0 & -1-g & 2\cos\varphi \end{bmatrix},$$

and where $\vec{a}^T = (-1, 0, \ldots, 0)$ and $\vec{b}^T = (0, \ldots, 0, -1)$ are two $(2\mathcal{N}+1)$-dimensional auxiliary row vectors. The other auxiliary "free-motion" submatrix $\hat{F}^{[\mathcal{N}]}$ is tridiagonal and $(2\mathcal{N}+1)$ dimensional. Its elements $2\cos\varphi$ along the main diagonal are complemented by the elements -1 which lie along its two neighboring diagonals.

B. Exact solvability

What remains for us to demonstrate is that our model conserves the global or asymptotic flow of probability, i.e., that one obtains $|R|^2 + |T|^2 = 1$ in spite of the manifest non-Hermiticity of the Hamiltonian H. In this setting the final surprise comes with the observation that the reflection and transmission coefficients are obtainable in closed form. Even when the distance parameter $\mathcal{N}$ is arbitrarily large, the use of *Ansatz* (19) reduces the original set of $2\mathcal{N}+7$ matching conditions to the following two independent matching conditions consisting of four items each:

$$\begin{bmatrix} 2\cos\varphi & -1-g & 0 & 0 \\ -1+g & 2\cos\varphi & -1+g & 0 \\ 0 & -1-g & 2\cos\varphi & -1 \\ 0 & 0 & -1 & 2\cos\varphi \end{bmatrix} \begin{bmatrix} U_{-\mathcal{N}-3} \\ U_{-\mathcal{N}-2} + \chi_{-2} \\ U_{-\mathcal{N}-1} + \chi_{-1} \\ \psi_{-\mathcal{N}} \end{bmatrix} = \begin{bmatrix} U_{-\mathcal{N}-4} \\ 0 \\ 0 \\ \psi_{-\mathcal{N}+1} \end{bmatrix},$$

$$\begin{bmatrix} 2\cos\varphi & -1 & 0 & 0 \\ -1 & 2\cos\varphi & -1-g & 0 \\ 0 & -1+g & 2\cos\varphi & -1+g \\ 0 & 0 & -1-g & 2\cos\varphi \end{bmatrix} \begin{bmatrix} \psi_{\mathcal{N}} \\ L_{\mathcal{N}+1} + \chi_1 \\ L_{\mathcal{N}+2} + \chi_2 \\ L_{\mathcal{N}+3} \end{bmatrix} = \begin{bmatrix} \psi_{\mathcal{N}-1} \\ 0 \\ 0 \\ L_{\mathcal{N}+4} \end{bmatrix}.$$

Out of this octuplet of equations, the first and last lines can be solved,

$$(1+g)\chi_{-2} = -gU_{-\mathcal{N}-2}, \qquad (1+g)\chi_2 = -gL_{\mathcal{N}+2}.$$

This leads to the following two triplets of conditions:

$$\begin{bmatrix} 2\cos\varphi & -1+g^2 & 0 \\ -1 & 2\cos\varphi & -1 \\ 0 & -1 & 2\cos\varphi \end{bmatrix} \begin{bmatrix} U_{-\mathcal{N}-2} \\ U_{-\mathcal{N}-1} + \chi_{-1} \\ \psi_{-\mathcal{N}} \end{bmatrix} = \begin{bmatrix} (1-g^2)U_{-\mathcal{N}-3} \\ 0 \\ \psi_{-\mathcal{N}+1} \end{bmatrix},$$

$$\begin{bmatrix} 2\cos\varphi & -1 & 0 \\ -1 & 2\cos\varphi & -1 \\ 0 & -1+g^2 & 2\cos\varphi \end{bmatrix} \begin{bmatrix} \psi_{\mathcal{N}} \\ L_{\mathcal{N}+1} + \chi_1 \\ L_{\mathcal{N}+2} \end{bmatrix} = \begin{bmatrix} \psi_{\mathcal{N}-1} \\ 0 \\ (1-g^2)L_{\mathcal{N}+3} \end{bmatrix}.$$

Using the first and last equation we eliminate

$$(1-g^2)\chi_{-1} = g^2U_{-\mathcal{N}-1} + g^2U_{-\mathcal{N}-3},$$
$$(1-g^2)\chi_1 = g^2L_{\mathcal{N}+1} + g^2L_{\mathcal{N}+3}.$$

The net result of these manipulations are the four relations

$$\begin{bmatrix} 2\cos\varphi & -1 \\ -1 & 2\cos\varphi \end{bmatrix} \begin{bmatrix} U_{-\mathcal{N}-1} + g^2U_{-\mathcal{N}-3} \\ (1-g^2)\psi_{-\mathcal{N}} \end{bmatrix} = \begin{bmatrix} (1-g^2)U_{-\mathcal{N}-2} \\ (1-g^2)\psi_{-\mathcal{N}+1} \end{bmatrix},$$

$$\begin{bmatrix} 2\cos\varphi & -1 \\ -1 & 2\cos\varphi \end{bmatrix} \begin{bmatrix} (1-g^2)\psi_{\mathcal{N}} \\ L_{\mathcal{N}+1} + g^2L_{\mathcal{N}+3} \end{bmatrix} = \begin{bmatrix} (1-g^2)\psi_{\mathcal{N}-1} \\ (1-g^2)L_{\mathcal{N}+2} \end{bmatrix},$$

which can be simplified to read

$$(1-g^2)\psi_{-\mathcal{N}} = U_{-\mathcal{N}} + 2g^2U_{-\mathcal{N}-2} + g^2U_{-\mathcal{N}-4},$$
$$(1-g^2)\psi_{-\mathcal{N}-1} = U_{-\mathcal{N}-1} + g^2U_{-\mathcal{N}-3},$$
$$(1-g^2)\psi_{\mathcal{N}} = L_{\mathcal{N}} + 2g^2L_{\mathcal{N}+2} + g^2L_{\mathcal{N}+4},$$
$$(1-g^2)\psi_{\mathcal{N}+1} = L_{\mathcal{N}+1} + g^2L_{\mathcal{N}+3}.$$

These equations represent the two alternative definitions of the sum $C+D$ and of the difference $C-D$ of the two unknown coefficients in ψ_k,

$$2(1-g^2)(C+D)\cos\mathcal{N}\varphi$$
$$= A^*(\varphi) + A(\varphi)(R+T),$$
$$2(1-g^2)(C+D)\cos(\mathcal{N}+1)\varphi$$
$$= B^*(\varphi) + B(\varphi)(R+T) - 2i(1-g^2)(C-D)\sin\mathcal{N}\varphi$$
$$= A^*(\varphi) + A(\varphi)(R-T),$$
$$-2i(1-g^2)(C-D)\sin(\mathcal{N}+1)\varphi$$
$$= B^*(\varphi) + B(\varphi)(R-T),$$

where we abbreviated

$$A(\varphi) = e^{i\mathcal{N}\varphi} + g^2(2e^{i(\mathcal{N}+2)\varphi} + e^{i(\mathcal{N}+4)\varphi}),$$
$$B(\varphi) = e^{i(\mathcal{N}+1)\varphi} + g^2e^{i(\mathcal{N}+3)\varphi}.$$

In the next step we eliminate C and D and express

$$R - T = -\frac{u^*(\varphi)}{u(\varphi)},$$
$$u(\varphi) = \frac{B(\varphi)}{\sin(\mathcal{N}+1)\varphi} - \frac{A(\varphi)}{\sin\mathcal{N}\varphi},$$
$$R + T = -\frac{v^*(\varphi)}{v(\varphi)},$$
$$v(\varphi) = \frac{B(\varphi)}{\cos(\mathcal{N}+1)\varphi} - \frac{A(\varphi)}{\cos\mathcal{N}\varphi}.$$

The required amplitudes R and T are now found, in closed form, as the respective sum and difference of the latter two expressions. In the final step their probability conservation property

$$|R|^2 + |T|^2 = 1$$

is easily seen.

V. SUMMARY

Our main technical result is that via a discretization of the real axis of coordinates x (and using the matching method) an exact linear-algebraic solvability of our present model of scattering has been achieved. Constructively, the necessary unitarity requirement has been satisfied at the same time. Our model [containing several spatially separated and strictly localized interactions which *appear* non-Hermitian in $\mathcal{H}^{(F)} \equiv L_2(\mathbb{R})$] is being assigned the more or less unique Hilbert space of states $\mathcal{H}^{(S)} \equiv \mathcal{H}^{(\text{physical})}$ where the use of an anomalous inner product makes the Hamiltonians (crypto)Hermitian.

It is worth noticing that the metric operator which defines the inner product in $\mathcal{H}^{(\text{physical})}$ merely differs from the usual Dirac's delta function *locally*, viz., in a close vicinity of interaction points. This implies that the physical operator of the coordinate remains unmodified almost everywhere. An entirely consistent physical picture of scattering from multiple scatterers is obtained in this way. In contrast to the older models using non-Hermitian but strictly local potentials $V(x)$, the free motion between our present, slightly nonlocal individual non-Hermitian pointlike scatterers remains undistorted.

In conclusion let us reemphasize that the motivation and inspiration of our present study of a simplified model of multiple scattering resulted from several sources. One of the most important ones has to be seen in the recent enormous growth of interest in the models of quantum dynamics of bound states which look *manifestly non-Hermitian* in $L_2(\mathbb{R})$ and/or in similar mathematical representations of the Hilbert space of states [16].

The key to success can be seen in the discovery of feasibility of a strictly physics-motivated transition to correct Hilbert space $\mathcal{H}^{(\text{physical})}$ [12]. Our present paper can be read as an implementation and advertisement of such an approach where one chooses a slightly more complicated input physics (i.e., in our case, a slightly nonlocal Hamiltonian $H = T + V$) and where one is rewarded by a perceivable simplification of mathematics. In particular, we saw that the resulting metric Θ in $\mathcal{H}^{(\text{physical})}$ differed from the unit operator just in a finite number of matrix elements in our model.

We have only to repeat that our second, less abstract motivation grew from the emergence of several very recent studies of manifestly non-Hermitian models of quantum scattering [8]. Several issues may be addressed in this context. For example, in the less ambitious, effective-theory versions of these models (where one does not insist on the conserved probability) one can easily stay in the single, effective-theory Hilbert space $\mathcal{H}^{(F)}$. Moreover, various additional dynamical assumptions [like the strict locality of potentials $V(x)$] may easily be incorporated in the similar pragmatic applications of the theory.

In contrast, in the "fundamental" and unitary quantum theory a real challenge is to be seen in the existence of a correlation between non-Hermiticity of a local V and the long-range nonlocality emerging in Θ [17]. One can notice that this relationship seems highly model dependent. In this sense, our present message can be read as a methodical encouragement. Basically, we found that whenever one broadens the class of the eligible potentials the latter model dependence can be reinterpreted as an advantage.

It should be remembered that the increase of the non-Hermiticity of H need not necessarily be correlated with the growth of nonlocalities in Θ obscuring the clear physical picture of scattering. We succeeded here in showing that *both* the nonlocalities occurring in V and Θ can be kept under control *simultaneously*. After all, one may note that in our present one-dimensional model with $x \in \mathbb{R}$ the anomalies disappear ''almost everywhere'' in the continuous limit $h \to 0$.

In this manner our present text brought a rather surprising resolution of the puzzle formulated in Ref. [15] where we did not manage to get rid of the nonlocality in a non-Hermitian model comprising several spatially separate scattering centers. Here we revealed that sometimes it makes good sense to *sacrifice* some inessential symmetries of the model in order to preserve either its exact solvability or its phenomenological flexibility. We should note that the feasibility of our (computer-assisted) algebraic manipulations survived even the transition to unusually complicated point-interaction simulations by three-by-three matrices.

In the context of physics, good news concerns, first of all, the possibility of an explicit construction of an optimal metric Θ in the physical Hilbert space. Its ''optimality'' reflects the fact that with an obvious exception of the closest vicinities of the pointlike interaction centers of our model, the metric Θ itself has successfully been forced to commute with the operator of the coordinate almost everywhere. This means that in contrast to intuitive expectations (supported even by some solvable models), the concept of coordinate and of an asymptotically free (i.e., measurable) motion of a quantized object can survive the emergence of a finite number of pointlike non-Hermitian obstacles positioned arbitrarily along the real line.

The latter observation allows us to declare that our model represents an illustrative example of a standard quantum system where the non-Hermiticity as well as the resulting nonlocalities (in both the metric Θ and in wave functions) remains confined to a very small part of the domain of the coordinates. This means not only that up to the singular points the coordinates remain measurable but also that the clear physical picture and consistent probabilistic interpretation of the non-Hermitian systems are naturally being extended to the multiple-scattering scenario.

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APPENDIX A: CONSTRUCTION OF THE AMPLITUDES FOR MODEL (14) AT $\mathcal{N} = -1$

In the special case of our toy-model $H^{(g,\mathcal{N})}$ at $\mathcal{N} = -1$ the analysis of the respective transition and reflection amplitudes T and R can be based on the explicit solution of the Schrödinger equation which degenerates, in an obvious manner and under the notation conventions of Sec. IV, to the following set of the five linear relations representing matching conditions near the origin:

$$\begin{bmatrix} -1 & 2\cos\varphi & -1-g & 0 & 0 & 0 & 0 \\ 0 & -1+g & 2\cos\varphi & -1+g & 0 & 0 & 0 \\ 0 & 0 & -1-g & 2\cos\varphi & -1-g & 0 & 0 \\ 0 & 0 & 0 & -1+g & 2\cos\varphi & -1+g & 0 \\ 0 & 0 & 0 & 0 & -1-g & 2\cos\varphi & -1 \end{bmatrix} \begin{bmatrix} U_{-3} \\ U_{-2} \\ U_{-1}+\chi_{-1} \\ \psi_0 \\ L_1+\chi_1 \\ L_2 \\ L_3 \end{bmatrix} = 0.$$

Their solution may start from the first and last line giving

$$(1+g)\chi_{-1} = -gU_{-1} = -g(e^{-i\varphi} + Re^{i\varphi}), \qquad (1+g)\chi_1 = -gL_1 = -gTe^{i\varphi}.$$

This enables us to consider just the three modified matching conditions

$$\begin{bmatrix} -1+g^2 & 2\cos\varphi & -1 & 0 & 0 \\ 0 & -1+g^2 & 2\cos\varphi & -1+g^2 & 0 \\ 0 & 0 & -1 & 2\cos\varphi & -1+g^2 \end{bmatrix} \begin{bmatrix} U_{-2} \\ U_{-1} \\ (1-g^2)\psi_0 \\ L_1 \\ L_2 \end{bmatrix} = 0.$$

The first and last rows read

$$\begin{aligned}(1-g^2)\psi_0 &= U_0 + g^2U_{-2} \\ &= 1 + g^2e^{-2i\varphi} + (1+g^2e^{2i\varphi})R, \\ (1-g^2)\psi_0 &= L_0 + g^2L_2 = (1+g^2e^{2i\varphi})T,\end{aligned}$$

so that their combination

$$1 + g^2e^{-2i\varphi} = (1+g^2e^{2i\varphi})(T-R)$$

defines the difference between our two amplitudes as a complex number with unit norm,

$$T - R = \frac{1 - i\lambda}{1 + i\lambda} \equiv e^{i\alpha}, \qquad \lambda = \frac{g^2 \sin 2\varphi}{1 + g^2 \cos 2\varphi}.$$

The remaining central matching condition can be given the form of an equation for the sum $\Sigma = T + R$ of the amplitudes, with the solution equal to another complex number with unit norm,

$$T + R = -e^{-2i\varphi} \frac{1 - i\mu}{1 + i\mu} \equiv e^{i\beta},$$

$$\mu = \frac{(1 - g^2) \sin 2\varphi}{1 - 3g^2 - \cos 2\varphi - g^2 \cos 2\varphi}.$$

This gives the two final formulas

$$2T = e^{i\beta} + e^{i\alpha}, \qquad 2R = e^{i\beta} - e^{i\alpha}$$

with the two respective properties

$$4|T|^2 = (e^{i\beta} + e^{i\alpha})(e^{-i\beta} + e^{-i\alpha}) = 2 + e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)},$$

$$4|R|^2 = (e^{i\beta} - e^{i\alpha})(e^{-i\beta} - e^{-i\alpha}) = 2 - e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)},$$

which imply that

$$|R|^2 + |T|^2 = 1.$$

This means that in contrast to the observations made in some other non-Hermitian models [1,2,4], the flow of probability is conserved so that the standard physical picture of the scattering does not require any modifications.

APPENDIX B: CONSTRUCTION OF THE AMPLITUDES FOR MODEL (14) AT $\mathcal{N} = 0$

In place of the five-dimensional matching condition of our preceding Appendix let us now turn our attention to the family of nontrivial models where the two three-dimensional elementary-interaction submatrices are separated by a free-motion interval of the length $2\mathcal{N} + 1$. In the first nontrivial model with $\mathcal{N} = 0$ the nonvanishing submatrix of our interaction matrix is seven dimensional,

$$V^{(g,0)} = \begin{bmatrix} 0 & g & 0 & 0 & 0 & 0 & 0 \\ -g & 0 & -g & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & -g & 0 & -g \\ 0 & 0 & 0 & 0 & 0 & g & 0 \end{bmatrix}.$$

In such a case one has to consider seven matching conditions of the form

$$M^{[0]}(\varphi) \begin{bmatrix} U_{-3} \\ U_{-2} + \chi_{-2} \\ U_{-1} + \chi_{-1} \\ \psi_0 \\ L_1 + \chi_1 \\ L_2 + \chi_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} U_{-4} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ L_4 \end{bmatrix},$$

where

$$M^{[0]}(\varphi) = \begin{bmatrix} 2\cos\varphi & -1 - g & 0 & 0 & 0 & 0 & 0 \\ -1 + g & 2\cos\varphi & -1 + g & 0 & 0 & 0 & 0 \\ 0 & -1 - g & 2\cos\varphi & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2\cos\varphi & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2\cos\varphi & -1 - g & 0 \\ 0 & 0 & 0 & 0 & -1 + g & 2\cos\varphi & -1 + g \\ 0 & 0 & 0 & 0 & 0 & -1 - g & 2\cos\varphi \end{bmatrix}.$$

The separate subset of the first and last matching condition is solvable as follows:

$$(1 + g)\chi_{-2} = -gU_{-2}, \qquad (1 + g)\chi_2 = -gL_2.$$

The backward insertion of these formulas leads to the quintuplet of the reduced matching conditions

$$\begin{bmatrix} -1 + g^2 & 2\cos\varphi & -1 + g^2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2\cos\varphi & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2\cos\varphi & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2\cos\varphi & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 + g^2 & 2\cos\varphi & -1 + g^2 \end{bmatrix} \begin{bmatrix} U_{-3} \\ U_{-2} \\ U_{-1} + \chi_{-1} \\ \psi_0 \\ L_1 + \chi_1 \\ L_2 \\ L_3 \end{bmatrix} = 0.$$

Its first and last lines define the other two correction components,

$$(1 - g^2)\chi_{-1} = g^2(U_{-1} + U_{-3}), \qquad (1 - g^2)\chi_1 = g^2(L_1 + L_{-3}),$$

so that we are left with the three matching conditions

$$\begin{bmatrix} -1+g^2 & 2\cos\varphi & -1 & 0 & 0 \\ 0 & -1 & 2\cos\varphi & -1 & 0 \\ 0 & 0 & -1 & 2\cos\varphi & -1+g^2 \end{bmatrix} \begin{bmatrix} U_{-2} \\ U_{-1}+g^2U_{-3} \\ (1-g^2)\psi_0 \\ L_1+g^2L_3 \\ L_2 \end{bmatrix} = 0.$$

Their first and last items define the same quantity in two ways,

$$(1-g^2)\psi_0 = U_0 + g^2(U_{-2} + 2\cos\varphi U_{-3}) = U_0 + g^2(2U_{-2} + U_{-4}),$$

$$(1-g^2)\psi_0 = L_0 + g^2(L_2 + 2\cos\varphi L_3) = L_0 + g^2(2L_2 + L_4).$$

In effect, one can eliminate ψ_0,

$$(T-R)[1+g^2(2e^{2i\varphi}+e^{4i\varphi})] = [1+g^2(2e^{-2i\varphi}+e^{-4i\varphi})],$$

and specify the difference between T and R,

$$T - R = \frac{1-i\lambda'}{1+i\lambda'} \equiv e^{i\alpha'}, \qquad \lambda' = \frac{g^2(2\sin 2\varphi + \sin 4\varphi)}{1+g^2(2\cos 2\varphi + \cos 4\varphi)}.$$

Next, in a complete parallel to the previous construction, the sum Σ of T and R may and should be extracted again from the last and symmetrized middle items of our matching conditions,

$$2U_{-1} + 2L_1 + 2g^2(U_{-3} + L_3) = U_0 + L_0 + g^2(2U_{-2} + 2L_2 + U_{-4} + L_4).$$

After appropriate insertions this gives a similar formula as above,

$$T + R = -\frac{1-i\mu'}{1+i\mu'} \equiv e^{i\beta'},$$

$$\mu' = \frac{-2\sin\varphi + g^2(2\sin 2\varphi - 2\sin 3\varphi + \sin 4\varphi)}{[1 - 2\cos\varphi + g^2(2\cos 2\varphi - 2\cos 3\varphi + \cos 4\varphi)]}.$$

The same argumentation as above confirms the validity of the identity

$$|R|^2 + |T|^2 = 1$$

i.e., of the same probability conservation law as above.

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