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First law of *p*-brane thermodynamics

Marek Rogatko*

Institute of Physics, Maria Curie-Sklodowska University, 20-031 Lublin, pl. Marii Curie-Sklodowskiej 1, Poland

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We study the *physical process* version and the *equilibrium state* version of the first law of thermodynamics for a charged p-brane. The general setting for our investigations is $(n + p + 1)$-dimensional Einstein dilaton gravity with $(p + 2)$ strength form fields.

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I. INTRODUCTION

The unrelenting search for the unification scheme in contemporary high-energy physics leads to the idea that our Universe may have been more than four dimensional. One of the most promising approaches to the unification of fundamental forces of nature is the superstring/M theory, which is formulated on a higher dimensional manifold. On the other hand, black holes through their quantum behavior and thermodynamical properties have played an important role in our quest for understanding quantum theory of gravity. It triggers continuously growing interest in studying properties of black holes and other black objects appearing in higher dimensional theories of gravity.

The uniqueness theorem for higher dimensional static black holes is quite well justified [1]. On the contrary, the situation is far from obvious as far as stationary axisymmetric higher dimensional black holes is concerned. It was shown that even in five dimensions a kind of black object appeared. It was called a *black ring* and its topology of the event horizon is $S^2 \times S^1$ [2]. This object is equipped with the same mass and angular momentum as a five-dimensional spherically symmetric stationary axisymmetric black hole. For a black rings story, see Ref. [3] and references therein. However, the assumption about topology of the considered black object enables one to prove the uniqueness theorem for a five-dimensional vacuum stationary axisymmetric black hole [4] and for stationary axisymmetric self-gravitating σ models [5]. Taking into account the so-called *rod structure* [6] enables one to broaden these attempts to the case of asymptotically flat five-dimensional black hole solutions of vacuum Einstein equations [7] and charged five-dimensional stationary axisymmetric black holes [8] (where the gauge field appeared only in the fifth dimension). In Ref. [9], assuming the existence of two additional commutating axial Killing vector fields and the horizon topology of the black ring $S^1 \times S^2$, the only asymptotically flat black ring solution with a regular horizon is the Pomeransky-Sen'kov black ring [10]. In Ref. [11] it was show that in five dimensions admitting the self-gravitating σ model the only asymptotically flat black ring with a regular rotating event horizon is the black ring characterized by mass and two angular momenta with constant mapping.

However, in multidimensional theories the situation drastically changes. One can consider a product of $(d - m)$-dimensional Minkowski spacetime times a compact Ricci flat m-dimensional manifold Ricci^m. It happens that black objects with different horizon topology depending on the size of extra dimensions arise. Namely, when the size of a compact manifold is large compared to the event horizon of a black object, one obtains a black hole with topology of the event horizon S^{d-2}. On the contrary, when the size of the manifold is small one gets a black string with $S^{d-m-2} \times \text{Ricci}^m$. Such kinds of black objects were intensively studied in five dimensions ($\text{Mink}^4 \times S^1$) [12]. In the spacetime in question, the arising black holes were named *caged black holes*. Their numerical studies were presented in Ref. [13], while Ref. [14] was devoted to their analytical studies.

A string solution with topology $S^{d-3} \times S^1$ is z dependent and it is described by a $(d - 1)$-dimensional Schwarzschild solution with coordinate dz^2. It turns out that nonextremal stationary *translationally invariant* branes are unstable. In Ref. [15] the authors investigated the stability of a black p-brane and revealed that such a background was unstable as the compactification scale of extended directions became larger than the order of the horizon radius. Next, it was shown in [16] for a class of magnetically charged p-brane solutions of stringy action that the instability persisted to appear but decreased with the charge increase to the extended value. On the other hand, it happened that branes with extremal charge were stable [17]. It was conjectured [18] that for a black brane with translational symmetry, Gregory-Laflamme (GL) instabilities occurred when the brane in question was thermodynamically unstable. It was also demonstrated numerically that a certain class of black holes in anti–de Sitter spacetime were unstable against linear perturba-

*rogat@kft.umcs.lublin.p
marek.rogatko@poczta.umcs.lublin.p

tions [19]. Then, in Ref. [20] strong support for this conjecture was given. Among all, it was shown that it was possible to reveal that many black branes were classically unstable without implementing an arduous numerical analysis of the problem.

In Ref. [21] it was revealed that a nonextremal translationally invariant black brane decayed to some stable stationary configuration. It settled down to a stable but inhomogeneous p-brane. It was also found [22] that there existed a large class of new stable inhomogeneous black brane solutions which were unrelated to the GL instability. They exist even if the adequate homogeneous solution with the same mass and charge is stable.

But in principle it could be possible to derive the first law of thermodynamics for branes that are not translationally invariant. Some progress in this direction was made. In Ref. [23] a black brane spacetime that had at least one spatial translation Killing field tangent to the brane was considered. For a static charge black brane a law which related the tension perturbation, surface gravity, charge of the event horizon area, and variations of charges and their currents was derived. On the other hand, in Ref. [24] a generalization of the gravitational tension in a given asymptotically translationally invariant spacetime direction was presented. The tension was defined in analogy with the Hawking-Horowitz energy definition [25]. This definition was applied for finding a general tension formula in the case for near-extremal branes.

Higher dimensional gravity also possesses more complicated black objects such as p-branes, black strings, black Saturn, i.e., an n-dimensional spherically symmetric black hole surrounded by black rings. Various aspects of this blossomming subject of research were treated in Refs. [26,27] (see also references therein).

In our paper we shall examine the first law of thermodynamics for a charged black p-brane in $(n + p + 1)$-dimensional dilaton gravity with a $(p + 2)$-form strength field. In what follows one finds the *physical process* version of the first law of thermodynamics for a charged black p-brane and the so-called *equilibrium version* of this law.

The physical process version of the first law of black object thermodynamics can be established by changing a stationary black object by some infinitesimal physical process. For instance, it can be realized throwing matter into a black object. In addition, we assume that the final state of the black object settles down to a stationary one, and then it will be possible to extract the changes of the black object's parameters. This in turn enables one to gain information about the first law of its mechanics. The physical process version of the first law of black hole thermodynamics was widely studied in the context of Einstein and Einstein-Maxwell theory in Refs. [28,29] as well as in Einstein-Maxwell axion-dilaton gravity being the low-energy limit of the heterotic string theory in Ref. [30]. The case of Einstein gravity coupled to a $(n - 2)$-gauge form field strength was treated in Ref. [31]. On the other hand, the black ring case was examined in Ref. [32].

The equilibrium state version of the first law of black hole mechanics constitutes the other attitude to the problem in question. It was studied in the seminal paper of Bardeen, Carter, and Hawking [33]. These attempts are based on considering the linear perturbations of a stationary electrovac black hole to another one. Reference [34] was devoted to arbitrary asymptotically flat perturbations of a stationary black hole, while the first law of black hole thermodynamics valid for an arbitrary diffeomorphism invariant Lagrangian with metric and matter fields possessing stationary and axisymmetric black hole solutions was obtained in Refs. [35–38]. The cases of higher curvature terms and higher derivative terms in the metric were considered in [39], while the situation when the Lagrangian is an arbitrary function of metric, Ricci tensor, and a scalar field was elaborated on in Ref. [40]. In Ref. [41], a charged rotating black hole was treated, where fields were not smooth through the event horizon. The first law of black hole thermodynamics was also intensively studied in the case of n-dimensional black holes. The equilibrium state version was elaborated on in Ref. [42] under the assumption of spherical topology of black holes. Some of the works assume that the four-dimensional black hole uniqueness theorem extends to a higher dimensional case [43]. The Arnowitt-Deser-Misner (ADM) mass and Komar surface integrals for energy density, tension, and angular momentum density of a stationary p-brane were given in Ref. [44]. As far as the black ring first law of mechanics is concerned, the general form of this law was achieved in Ref. [45], using the notion of bifurcate Killing horizons and considering dipole charges. In n-dimensional gravity containing $(p + 1)$-form field strength and dilaton fields the first law of black ring mechanics choosing an arbitrary cross section of the event horizon to the future of the bifurcation surface was derived in Ref. [46]. In n-dimensional Einstein gravity with the Chern-Simons term the physical process version and the equilibrium state version of the first law of black ring thermodynamics were derived in Ref [47], while the case of black Saturn was presented in [48]. On the other hand, by means of the covariant cohomological methods to the conserved charges for p-form gauge fields coupled to gravity the first law of thermodynamics was found in Ref. [49].

The paper is organized as follows. We devoted Sec. II to the physical process version of the charged p-brane first law of thermodynamics. Section III studies the equilibrium state version of the first law of thermodynamics choosing an arbitrary cross section of the p-brane event horizon to the future of their bifurcation surfaces. Such an attitude enables one to take into account fields which are not necessarily smooth through the event horizon of a charged p-brane under consideration. Section IV concludes our investigations.

II. PHYSICAL PROCESS VERSION OF THE FIRST LAW OF BLACK HOLE MECHANICS

We begin with the Lagrangian describing $(n+p+1)$-dimensional dilaton gravity with $(p+2)$-form strength fields. It is subject to the relation as follows:

$$\mathbf{L} = \epsilon\Big({}^{(d)}R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2(p+2)!}e^{-\alpha\phi}F_{\mu_1\cdots\mu_{p+2}}F^{\mu_1\cdots\mu_{p+2}}\Big), \tag{1}$$

where ϵ is the volume element of dimension

$$d = n + p + 1,$$

ϕ is the dilaton field, while

$$F_{\mu_1\cdots\mu_{p+2}} = (p+2)!\nabla_{[\mu_1}A_{\mu_2\cdots\mu_{p+1}]}$$

is the $(p+2)$-form field strength, with a potential $A_{\mu_1\cdots\mu_{p+1}}$. By α we have denoted an arbitrary dilaton coupling parameter. One can check that equations of motion for the underlying theory are given by

$$G_{\mu\nu} - T_{\mu\nu}(F,\phi) = 0, \tag{2}$$

$$\nabla_{i_1}(e^{-\alpha\phi}F^{i_1\cdots i_{p+2}}) = 0, \tag{3}$$

$$\nabla_\mu\nabla^\mu\phi + \frac{\alpha}{2(p+2)!}e^{-\alpha\phi}F_{\mu_1\cdots\mu_{p+2}}F^{\mu_1\cdots\mu_{p+2}} = 0. \tag{4}$$

On the other hand, the energy momentum tensor for $(p+2)$-form field strength and dilatons has the form as

$$T_{\mu\nu}(F,\phi) = \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{4}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{2(p+2)!}e^{-\alpha\phi} \times\Big[(p+2)F_{\mu\nu_2\cdots\nu_{p+1}}F_\nu{}^{\nu_2\cdots\nu_{p+2}} - \frac{1}{2}g_{\mu\nu}F_{\mu_1\cdots\mu_{p+2}}F^{\mu_1\cdots\mu_{p+2}}\Big]. \tag{5}$$

In order to establish the physical version of the first law of p-brane thermodynamics we shall try to find the explicit expressions for the variation of mass and angular momentum and the tension of the brane. On evaluating the variations of the Lagrangian (1) with respect to the adequate fields, we find that one finally obtains

$$\delta\mathbf{L} = \epsilon(G_{\mu\nu} - T_{\mu\nu}(F,\phi))\delta g^{\mu\nu} - \epsilon\nabla_{j_1}(e^{-\alpha\phi}F^{j_1\cdots j_{p+2}})\delta A_{j_2\cdots j_{p+2}} + \epsilon\Big(\nabla_\mu\nabla^\mu\phi + \frac{\alpha}{2(p+2)!}e^{-\alpha\phi}F_{\mu_1\cdots\mu_{p+2}}F^{\mu_1\cdots\mu_{p+2}}\Big)\delta\phi + d\mathbf{\Theta}. \tag{6}$$

Using the above formula (6) we get the symplectic $(n+p)$-form $\Theta_{j_1\cdots j_{n+p}}[\psi_\alpha, \delta\psi_\alpha]$, which yields

$$\Theta_{j_1\cdots j_{n+p}}[\psi_\alpha,\delta\psi_\alpha] = \epsilon_{mj_1\cdots j_{n+p}}[\omega^m - e^{-\alpha\phi}F^{m\nu_1\cdots\nu_{p+1}}\delta A_{\nu_1\cdots\nu_{p+1}} - \nabla^m\phi\delta\phi]. \tag{7}$$

In relation (7) by ψ_α we denote fields in the considered theory while $\delta\psi_\alpha$ is equal to their variations. ω_μ stands for the expression

$$\omega_\mu = \nabla^\alpha\delta g_{\alpha\mu} - \nabla_\mu\delta g_\beta{}^\beta. \tag{8}$$

One can also remark that the adequate equations of motion can be read off relation (6). The standard procedure provided in Ref. [29] enables one to identify variations of the fields with a general coordinate transformation induced by an arbitrary Killing vector field ξ_α. Next, one can find that the Noether $(n+p)$-form with respect to this above mentioned Killing vector, i.e., $\mathcal{J}_{j_1\ldots j_{n+p}} = \epsilon_{mj_1\cdots j_{n+p}}\mathcal{J}^m[\psi_\alpha, \mathcal{L}_\xi\psi_\alpha]$. The result of doing that is of the following form:

$$\mathcal{J}_{j_1\cdots j_{n+p}} = d(Q^{\rm GR} + Q^B)_{j_1\cdots j_{n+p}} + 2\epsilon_{mj_1\cdots j_{n+p}}(G^m{}_\eta - T^m{}_\eta(F,\phi))\xi^\eta + (p+1)\epsilon_{mj_1\cdots j_{n+p}}\xi^d A_{d\alpha_3\cdots\alpha_{p+2}}\nabla_{\alpha_2}(e^{-\alpha\phi}F^{m\alpha_2\cdots\alpha_{p+2}}), \tag{9}$$

where $Q^{\rm GR}_{j_1\cdots j_{n+p-1}}$ yields

$$Q^{\rm GR}_{j_1\cdots j_{n+p-1}} = -\epsilon_{j_1\cdots j_{n+p-1}ab}\nabla^a\xi^b, \tag{10}$$

while $Q^A_{j_1\cdots j_{n+p-1}}$ has the following form:

$$Q^A_{j_1\cdots j_{n+p-1}} = \frac{p+1}{(p+2)!}\epsilon_{mkj_1\cdots j_{n+p-1}}\xi^d A_{d\alpha_3\cdots\alpha_{p+2}}e^{-\alpha\phi}F^{mk\alpha_3\cdots\alpha_{p+2}}. \tag{11}$$

Having in mind that $\mathcal{J}[\xi] = dQ[\xi] + \xi^\alpha\mathbf{C}_\alpha$, where $\mathbf{C}_\alpha$ is an $(n+p)$ form constructed from dynamical fields, i.e., from $g_{\mu\nu}$, $(p+2)$-form field $F^{j_1\cdots j_{p+2}}$, and dilaton fields, one may consequently identify the sum of relations (10) and (11) with the Noether charge for $(n+p+1)$-dimensional dilaton gravity theory with $(p+2)$-form strength field. $\mathbf{C}_\alpha$ implies

$$C_{kj_1\cdots j_{n+p}} = 2\epsilon_{gj_1\cdots j_{n+p}}[G_k{}^g - T_k{}^g(F,\phi)] + (p+1)\epsilon_{gj_1\cdots j_{n+p}}\nabla_{\alpha_2}(e^{-\alpha\phi}F^{g\alpha_2\cdots\alpha_{p+2}})A_{k\alpha_3\cdots\alpha_{p+2}}. \tag{12}$$

The source-free equations of motion are provided by the requirement that $\mathbf{C}_\alpha = 0$. On the contrary, when the source-free equations do not hold, they yield the following relations:

$$G_{\mu\nu} - T_{\mu\nu}(F,\phi) = T_{\mu\nu}(\text{matter}), \tag{13}$$

$$\nabla_{\xi_1}(e^{-\alpha\phi}F^{\xi_1\cdots\xi_{p+2}}) = j^{\xi_2\cdots\xi_{p+2}}(\text{matter}). \tag{14}$$

If one further assumes that $(g_{\mu\nu}, A_{\alpha_1\cdots\alpha_{p+1}}, \phi)$ are solutions of source-free equations of motion and $(\delta g_{\mu\nu}, \delta A^{\alpha_1\cdots\alpha_{p+1}}, \delta\phi)$ are the linearized perturbations satisfying equations of motion with sources $\delta T_{\mu\nu}(\text{matter})$ and $\delta j^{\mu_1\cdots\mu_{p+1}}(\text{matter})$, then one obtains the relation of the form as

$$\delta C_{km_1\cdots m_{n+p}} = 2\epsilon_{gm_1\cdots m_{n+p}}[\delta T_k{}^g(\text{matter}) + (p+1)A_{k\alpha_3\cdots\alpha_{p+2}}\delta j^{g\alpha_3\cdots\alpha_{p+2}}(\text{matter})]. \tag{15}$$

Because of the fact that the Killing vector field ξ_α describes also a symmetry of the background matter field, one gets the formula for a conserved quantity connected with ξ_α, namely,

$$\delta H_\xi = -2\int_\Sigma \epsilon_{mj_1\cdots j_{n+p}}[\delta T_a{}^m(\text{matter})\xi^a + (p+1)\xi^i A_{i\alpha_3\cdots\alpha_{p+2}}\delta j^{m\alpha_3\cdots\alpha_{p+2}}(\text{matter})] + \int_{\partial\Sigma}[\delta Q(\xi) - \xi\cdot\Theta]. \tag{16}$$

Let us choose ξ^α to be an asymptotic time translation t^α, then one can conclude that $M = H_t$ and finally obtain the variation of the ADM mass

$$\alpha\delta M = -2\int_\Sigma \epsilon_{mj_1\cdots j_{n+p}}[\delta T_k{}^m(\text{matter})t^k + (p+1)t^k A_{k\alpha_3\cdots\alpha_{p+2}}\delta j^{m\alpha_3\cdots\alpha_{p+2}}(\text{matter})] + \int_{\partial\Sigma}[\delta Q(t) - t\cdot\Theta], \tag{17}$$

where $\alpha = \frac{n-2}{n-1}$. Next, if we take the Killing vector fields $\varphi_{(i)}$, which are responsible for the rotation in the adequate directions, we arrive at the relations for angular momenta

$$\delta J_{(i)} = 2\int_\Sigma \epsilon_{mj_1\cdots j_{n+p}}[\delta T_a{}^m(\text{matter})\varphi^a_{(i)} + (p+1)\varphi^a_{(i)}A_{a\alpha_3\cdots\alpha_{p+2}}\delta j^{m\alpha_3\cdots\alpha_{p+2}}(\text{matter})] - \int_{\partial\Sigma}[\delta Q(\varphi_{(i)}) - \varphi_{(i)}\cdot\Theta]. \tag{18}$$

Moreover, p-brane spacetime may have more spatial Killing vector fields tangent to the brane. Consequently, with the above definition we introduce p translational Killing vectors $l^\alpha_{(j)}$ which are connected with the p-brane tension in the adequate direction. Thus, the change of the brane tension in the adequate direction yields

$$\delta\mathcal{T}_{(i)} = -2\int_\Sigma \epsilon_{mj_1\cdots j_{n+p}}[\delta T_k{}^m(\text{matter})l^k_{(i)} + (p+1)l^k_{(i)}A_{k\alpha_3\cdots\alpha_{p+2}}\delta j^{m\alpha_3\cdots\alpha_{p+2}}(\text{matter})] + \int_{\partial\Sigma}[\delta Q(l_{(i)}) - l_{(i)}\cdot\Theta]. \tag{19}$$

Let us suppose further that a stationary p-brane solution is regular on and outside the event horizon. Moreover, let us assume that the p-brane event horizon is a Killing horizon, which implies that there exists a Killing vector field χ^α normal to it. The Killing vector field χ^μ is of the form

$$\chi^\mu = t^\mu + \sum_i \Omega_{(i)}\varphi^{\mu(i)}. \tag{20}$$

The surface integrals in Eqs. (17)–(19) we understand as surface integrals over the p-brane event horizon and a bulk integral over the region bounded by the considered horizon and transverse spatial infinity on a n-surface having vector t^δ as one of its $(p+1)$ normals. One has in mind that we assume the existence of a Killing vector field responsible for stationarity t^δ, Killing vector fields $\varphi^{\mu(i)}$ which are connected with rotation in the adequate direction as well as Killing vector fields bounded to the translation in various directions. We consider both an *homogeneous* p-brane when $l_{(i)}$ are the same in every direction and an *inhomogeneous* brane when they differ in every direction. We call these states of p-brane *translationary invariant* (for brevity), keeping in mind what was written previously. All Killing vector fields in question are mutually commutating.

Let us perturb the black p-brane by dropping in some matter and assume that in the process of this action the p-brane will not be destroyed and will settle down to a stationary and translationary invariant final state. Then, the next task will be to find the changes of the black p-brane parameters. Changes of the event horizon area of a p-brane will be computed by means of the n-dimensional Raychaudhuri equation. In addition, we shall assume that Σ_0 is an asymptotically flat hypersurface which terminates on the p-brane event horizon. Then, one takes into account the initial data on Σ_0 for linearized perturbations of $(\delta g_{\mu\nu}, \delta A_{\alpha_1\cdots\alpha_{p+1}}, \delta\phi)$ with $\delta T_{\mu\nu}(\text{matter})$ and $\delta j^{\alpha_1\cdots\alpha_{p+1}}(\text{matter})$. We require that $\delta T_{\mu\nu}(\text{matter})$ and $\delta j^{\alpha_1\cdots\alpha_{p+1}}(\text{matter})$ disappear at infinity and the initial data for $(\delta g_{\mu\nu}, \delta A_{\alpha_1\cdots\alpha_{p+1}}, \delta\phi)$ vanish in the vicinity of the black p-brane event horizon $\mathcal{H}$ on the adequate hypersurface Σ_0. The above conditions provide that for the initial time Σ_0, the considered black p-brane is unperturbed. On its own, it causes the perturbations to vanish near the internal boundary $\partial\Sigma_0$. From relations (17) and (19) the following is fulfilled:

$$\alpha\delta M - \sum_i \Omega_{(i)}\delta J^{(i)} - \sum_i \delta\mathcal{T}_{(i)} = -2\int_{\Sigma_0} \epsilon_{mj_1\cdots j_{n+p}}[\delta T_f{}^m(\text{matter})\chi^f + (p+1)\chi^k A_{k\alpha_2\cdots\alpha_{p+1}}\delta j^{m\alpha_2\cdots\alpha_{p+1}}(\text{matter})]$$
$$- 2\sum_i\int_{\Sigma_0} \epsilon_{mj_1\cdots j_{n+p}}[\delta T_k{}^m(\text{matter})l^k_{(i)} + (p+1)l^k_{(i)}A_{k\alpha_3\cdots\alpha_{p+2}}\delta j^{m\alpha_3\cdots\alpha_{p+2}}(\text{matter})]$$
$$= \int_{\mathcal{H}} \gamma^\alpha k_\alpha \bar{\epsilon}_{j_1\cdots j_{n-1}} v(l), \tag{21}$$

where $\bar{\epsilon}_{j_1\cdots j_{n-1}} = n^\delta \epsilon_{\delta j_1\cdots j_{n-1}}$ and $v(l) = \int \epsilon_{j_1\cdots j_p}$. By n^δ we denoted the future directed unit normal to the hypersurface Σ_0. k_α is a tangent vector to the affinely parametrized null geodesics generators of the p-brane event horizon. Further, we assume that all of the matter falls into the considered black p-brane. We also keep in mind that the current γ^α is conserved. Because of the above facts we replace in relation (21) vector n^β by the vector k^β defined above.

We shall assume that the field strength $F_{\alpha_1\cdots\alpha_{p+2}}$ is invariant under symmetries generated by adequate Killing vector fields. Namely, the adequate Lie derivatives of gauge field $A_{\alpha_1\cdots\alpha_{p+1}}$ are equal to zero. One gets the following:

$$\mathcal{L}_\chi A_{\alpha_1\cdots\alpha_{p+1}} = 0, \qquad \mathcal{L}_l A_{\alpha_1\cdots\alpha_{p+1}} = 0. \tag{22}$$

The same relations are satisfied by the dilaton field

$$\mathcal{L}_\chi \phi = 0, \qquad \mathcal{L}_l \phi = 0. \tag{23}$$

It can be checked by the direct calculations that for ξ^α generating symmetries of the considered background the following relation takes place:

$$(p+1)!\mathcal{L}_\xi A_{\alpha_1\cdots\alpha_{p+1}}\delta j^{\alpha_1\cdots\alpha_{p+1}} - \xi^d F_{d\alpha_2\cdots\alpha_{p+2}}\delta j^{\alpha_2\cdots\alpha_{p+2}}$$
$$= (p+1)(p+1)!\nabla_{\alpha_2}(\xi^d A_{d\alpha_3\cdots\alpha_{p+2}})\delta j^{\alpha_2\cdots\alpha_{p+2}}. \tag{24}$$

Equation (24) will be useful in calculations of the integral over a black p-brane event horizon. In stationary background expansion θ and shear σ_{ij} will vanish. Using the higher dimensional Raychaudhuri equation of the form

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{(n+p-1)} - \sigma_{ij}\sigma^{ij} - R_{\mu\nu}\xi^\mu\xi^\nu, \tag{25}$$

where λ denotes the affine parameter corresponding to vector k_α, one concludes that $R_{\alpha\beta}k^\alpha k^\beta\,|_{\mathcal{H}} = 0$. Because of this fact we get a relation of the form

$$\frac{1}{2}k^\mu\nabla_\mu\phi k^\nu\nabla_\nu\phi + \frac{1}{2(p+1)!}e^{-\alpha\phi}F_{\mu\mu_2\cdots\mu_{p+2}}F_\nu{}^{\mu_2\cdots\mu_{p+2}}$$
$$\times k^\mu k^\nu|_{\mathcal{H}} = 0. \tag{26}$$

Using the fact that $\mathcal{L}_k\phi = 0$, it is easily seen that, $F_\alpha{}^{\mu_2\cdots\mu_{p+2}}k^\alpha = 0$. Because of the fact that $F_{\alpha\mu_2\cdots\mu_{p+2}}k^\alpha k^{\mu_2} = 0$, by asymmetry of $F_{\mu_1\cdots\mu_{p+2}}$ it turned out that $F_{\alpha\mu_2\cdots\mu_{p+1}}k^\alpha \sim k_{\mu_2}\cdots k_{\mu_{p+1}}$. It implies that the pullback of $F_\alpha^{\mu_2\cdots\mu_{p+2}}k^\alpha$ to the p-brane event horizon is equal to zero. In turn, it reveals the fact that $\chi^k F_{k\alpha_2\cdots\alpha_{p+2}}$ is a closed $(p+1)$ form on the p-brane event horizon.

The same considerations as above may be applied to the integrals concerning p-brane tension. Now, we proceed to the surface terms. It follows that the adequate surface terms will have a form of Φ, δQ, where Φ is the constant sum relating to the harmonic parts of $\xi^d F_{d\alpha_2\cdots\alpha_{p+2}}$ and δQ is the variation of local charges. These allow one to write the following:

$$\alpha\delta M - \sum_i \Omega_{(i)}\delta J^{(i)} - \sum_i \delta\mathcal{T}_{(i)} + \Phi\delta Q$$
$$= 4\int_{\mathcal{H}} \delta T_\mu{}^\nu \xi^\mu k_\nu v(l), \tag{27}$$

where $\Phi\delta Q = \Phi_{(\chi)}\delta Q_{(\chi)} + \sum_i \Phi_{l_{(i)}}\delta Q_{l_{(i)}}$ is the sum of the potentials and local charges connected with the adequate Killing fields. Our next task is to find the right-hand side of Eq. (27). It can be elaborated on by the same procedure as described in Refs. [29–31]. Namely, considering the $(n+p+1)$-dimensional Raychaudhuri equation and using the fact that the null generators of the event horizon of the perturbed black p-brane coincide with the null generators of the unperturbed black p-brane, leads to the relation of the form

$$\kappa\delta\mathcal{A}_{\text{eff}} = \int_{\mathcal{H}} \delta T^\mu{}_\nu(\text{matter})\xi^\nu k_\mu v(l), \tag{28}$$

where κ is the surface gravity of the black p-brane while $\delta\mathcal{A}_{\text{eff}} = v(l)\delta\mathcal{A}$ is just an $(n-1)$-dimensional effective area of the event horizon of the considered p-brane. The same reasoning enables us to find the same expression when on the right-hand side of Eq. (28) $l_{(i)}$ exists instead of ξ^ν.

In light of what has been shown above we arrive at the physical process version of the first law of black p-brane mechanics in Einstein $(n+p+1)$-dimensional gravity with additional $(p+2)$-form field strength and dilaton fields. It is provided by

$$\alpha\delta M - \sum_i \Omega_{(i)}\delta J^{(i)} - \sum_i \delta\mathcal{T}_{(i)} + \Phi\delta Q = 4\kappa\delta\mathcal{A}_{\text{eff}}. \tag{29}$$

We finally remark that in the sense of Ref. [29] the proof of the physical process version of the first law of thermodynamics for the $(n+p+1)$-dimensional black p-brane also provides support for cosmic censorship.

III. EQUILIBRIUM STATE VERSION OF THE FIRST LAW OF *P*-BRANE MECHANICS

In this section we shall look for the equilibrium state version of the first law of charged black p-brane thermodynamics. In Ref. [50] it was shown that in the spacetime with asymptotic conditions at infinity and possessing Killing vector field $\xi_{\mu(i)}$ which generates asymptotical symmetry it will be possible to define the *conserved* quantity $H_{\xi(i)}$, which is given by the relation

$$\delta H_{\xi(i)} = \int_\infty (\bar{\delta} Q(\xi_{(i)}) - \xi_{(i)}\Theta). \tag{30}$$

$\bar{\delta}$ denotes variation which has no effect on ξ_α since the Killing vector field in question is treated as a fixed background and it should not be varied in the above expression (30).

In our considerations we take into account $(n+p+1)$-dimensional spacetime with charge p-brane. As was mentioned in the preceding section, we have the Killing vector field χ^μ which is normal to the p-brane event horizon and translational Killing vectors $l^\mu_{(i)}$ in addition to the aforementioned ones. Moreover, we assume that all are mutually commutating. In what follows we choose an arbitrary cross section of the considered p-brane event horizon to the future of the bifurcation surface. In Ref. [41] it was revealed that such an attitude enabled one to treat fields which were not necessarily smooth through the event horizon of the black object. The only requirement is that the pullback of these fields in the future of the bifurcation surface be smooth.

To derive the equilibrium state version of the first law of charged p-brane mechanics let us consider asymptotically hypersurfaces Σ ending on the part of the p-brane event horizons $\mathcal{H}$ to the future of the bifurcation surfaces. The inner boundary $S_\mathcal{H}$ of the hypersurface Σ will be the cross sections of the black p-brane event horizon. Next, we shall compare variations between two neighboring states of the p-brane. One should recall [33] that there is a freedom in which the points can be chosen to correspond when one compares two slightly different solutions. In our consideration we choose the freedom of the generalized coordinate transformation and put $S_\mathcal{H}$ the same as the two solutions. Moreover, one takes into account the case when the null vector remains normal to $S_\mathcal{H}$. The stationarity, axisymmetricity, and translantionarity of the considered solutions will be conserved, which provides in turn that δt^μ, $\delta\varphi^{\mu(i)}$, and δl^μ will be equal to zero. On the other hand, the variation of the Killing vector field χ_μ normal to the charged p-brane event horizon will be given by the following:

$$\delta\xi^\mu = \sum_i \delta\Omega_{(i)}\varphi^{\mu(i)}. \tag{31}$$

Let us suppose that $(g_{\mu\nu}, A_{\alpha_1\cdots\alpha_{p+1}}, \phi)$ are solutions of the equations of motion and their variations $(\delta g_{\mu\nu}, \delta A^{\alpha_1\cdots\alpha_{p+1}}, \delta\phi)$ constitute their linearized perturbations to also fulfill equations of motion. One requires also that the pullback of the potential $A_{\alpha_1\cdots\alpha_{p+1}}$ to the future of the bifurcation surface be smooth, but not necessarily smooth on it [41]. We require further that $A_{\alpha_1\cdots\alpha_{p+1}}$ and its variation $\delta A_{\alpha_1\cdots\alpha_{p+1}}$ vanish sufficiently rapid at infinity. Consequently, for the charged black p-brane one obtains

$$\alpha\delta M - \sum_i \Omega_{(i)}\delta J^{(i)} - \sum_i \delta\mathcal{T}_{(i)} = \int(\bar{\delta}Q(\chi) - \chi\Theta) - \int(\bar{\delta}Q(l_{(i)}) - l_{(i)}\Theta). \tag{32}$$

To begin with we shall find the integral over the symplectic $(n+p)$ form connected to the dilaton field. In the case under consideration the volume element has the form

$$\epsilon_{\mu a j_1\cdots j_{n+p-1}} = \chi_\mu \wedge N_a \wedge \epsilon_{j_1\cdots j_{n-1}} \wedge \epsilon_{j_1\cdots j_p}, \tag{33}$$

where vector N_β is the *ingoing* future directed null normal to the p-brane event horizon $S_\mathcal{H}$. It is normalized as follows:

$$N^\mu \chi_\mu = -1. \tag{34}$$

We arrive at the relation of the form

$$\int \chi^{j_1}\Theta^\phi_{j_1\cdots j_{n+p}} = \int_{S_\mathcal{H}} v(l)\epsilon_{j_1\cdots j_{n-1}} N_\alpha \chi^\alpha \chi_\mu \nabla^\mu\phi\delta\phi = 0, \tag{35}$$

where we used the fact that $\mathcal{L}_\chi\phi = 0$. The arguments presented in the preceding section can be applied now. It leads to the following:

$$\int Q^A_{j_1\cdots j_{n+p-1}}(\chi) = \Phi_{(\chi)}Q_{(\chi)}. \tag{36}$$

Our next task will be to find the variation $\bar{\delta}$ of $Q^A_{j_1\cdots j_{n+p-1}}(\chi)$. Then, one obtains

$$\bar{\delta}\int Q^A_{j_1\cdots j_{n+p-1}}(\chi) = \delta(\Phi_{(\chi)}Q_{(\chi)}) - \frac{(p+1)v(l)}{(p+2)!}\int_{S_\mathcal{H}}\sum_i \delta\Omega_{(i)}\varphi^{\mu(i)}A_{\mu\alpha_3\cdots\alpha_{p+2}}\epsilon_{mkj_1\cdots j_{n-2}}e^{-\alpha\phi}F^{mk\alpha_3\cdots\alpha_{p+2}}. \tag{37}$$

As a direct consequence of relation (37) we arrive at the expression which can be written as

$$\delta\Phi_{(\chi)}Q_{(\chi)} = \frac{(p+1)v(l)}{(p+2)!}\int_{S_{\mathcal{H}}}\sum_i \delta\Omega_{(i)}\varphi^{\mu(i)}A_{\mu\alpha_3\cdots\alpha_{p+2}}\epsilon_{mjj_1\cdots j_{n-2}}e^{-\alpha\phi}F^{mj\alpha_3\cdots\alpha_{p+2}} + \frac{(p+1)v(l)}{(p+2)!}\int_{S_{\mathcal{H}}}\chi^d\delta A_{d\alpha_3\cdots\alpha_{p+2}}N_m\chi_j e^{-\alpha\phi}F^{mj\alpha_3\cdots\alpha_{p+2}}. \quad (39)$$

Using the fact that on the event horizon of black p-brane $F_{\mu\mu_2\cdots\mu_{p+2}}\chi^\mu \sim \chi_{\mu_2}\cdots\chi_{\mu_{p+1}}$ and expressing $\epsilon_{\mu a j_1\cdots j_{n-2}}$ in the same form as in the above case, one gets the following:

$$\int \chi^{j_1}\Theta^A_{j_1\cdots j_{n+p}} = \frac{(p+1)v(l)}{(p+2)!}\int_{S_{\mathcal{H}}}\epsilon_{j_1\cdots j_{n-2}}e^{-\alpha\phi}\chi_k F^{jk\nu_3\cdots\nu_{p+2}}N_j\chi^{\nu_2}\delta A_{\nu_2\cdots\nu_{p+2}}. \quad (40)$$

Having in mind Eqs. (39) and (40) one can conclude that

$$\bar\delta\int Q^A_{j_1\cdots j_{n+p-1}}(\chi) - \chi^{j_1}\Theta^A_{j_1\cdots j_{n+p}} = \Phi_{(\chi)}\delta Q_{(\chi)}. \quad (41)$$

Now, let us turn our attention to the contribution bounded with a gravitational field. Namely, for the p-brane one obtains

$$\int Q^{\rm GR}_{j_1\cdots j_{n+p-1}}(\chi) = 2\kappa\mathcal{A}_{\rm eff}, \quad (42)$$

where

$$\mathcal{A}_{\rm eff} = v(l)\int_{S_{\mathcal{H}}}\epsilon_{j_1\cdots j_{n-1}}$$

is the area of the p-brane event horizon. Then, it implies

$$\bar\delta\int Q^{\rm GR}_{j_1\cdots j_{n+p-1}}(\chi) = 2\delta(\kappa\mathcal{A}_{\rm eff}) + 2\sum_i \delta\Omega_{(i)}J^{(i)}, \quad (43)$$

which we have denoted by $J^{(i)} = \frac{1}{2}\times \int_{S_{\mathcal{H}}} v(l)\epsilon_{j_1\cdots j_{n-2}ab}\nabla^a\varphi^{(i)b}$ the angular momentum connected with the Killing vector fields responsible for the rotations in the adequate directions. Following the calculations presented in Ref. [33] it could be found that the following integral is satisfied:

$$\int \chi^{j_1}\Theta^{\rm GR}_{j_1\cdots j_{n+p-1}}(\chi) = 2\mathcal{A}_{\rm eff}\delta\kappa + 2\sum_i \delta\Omega_{(i)}J^{(i)}. \quad (44)$$

The above relation yields the conclusion that

$$\bar\delta\int Q^{\rm GR}_{j_1\cdots j_{n+p-1}}(\chi) - \chi^{j_1}\Theta^{\rm GR}_{j_1\cdots j_{n+p}} = 2\kappa\delta\mathcal{A}_{\rm eff} + 2\sum_a \kappa\delta\mathcal{A}_{\rm eff}. \quad (45)$$

The entirely analogous considerations can be applied to the second part of the right-hand side of Eq. (32) related to the brane tension and Killing vector fields $l_{(i)}$.

Thus, the direct consequence of relations (41) and (45) and the analogous for brane tension integrals provides the first law of charged black p-brane mechanics in Einstein $(n+p+1)$-dimensional gravity with additional $(p+2)$-form field strength and dilaton fields. The first law of mechanics for the considered black objects can be written in the form as

$$\alpha\delta M - \sum_i \Omega_{(i)}\delta J^{(i)} + \Phi\delta Q - \sum_i \delta\mathcal{T}_{(i)} = 4\kappa\delta\mathcal{A}_{\rm eff}, \quad (46)$$

where Φ and δQ are the adequate sums of constant potentials on $S_{\mathcal{H}}$ and the sum of local charges.

IV. CONCLUSIONS

In our paper we studied the first law of charged black p-brane thermodynamics in $(n+p+1)$-dimensional dilaton gravity with $(p+2)$-form field strength. We assumed stationarity and axisymmetricity of the considered p-brane. Moreover, we supposed that there were p translation Killing vectors in addition to the t^μ Killing vector field and $\varphi^\mu_{(i)}$ Killing vectors responsible for the rotations in adequate directions. All these Killing vectors commute mutually. We looked for both the *physical process* version and the *equilibrium state* version of the first law of charged p-brane thermodynamics.

Considering the physical process version of the first law of p-brane dynamics we change infinitesimally the p-brane under consideration by throwing matter into it. Assuming that this process will not destroy the black object in question we find changes of the ADM mass, angular momentum, tension, and effective area of the event horizon of the p-brane. As far as the equilibrium state version of the first law of p-brane thermodynamics is concerned we chose arbitrary cross sections of p-brane event horizons to the future of bifurcation surfaces, contrary to the previous derivations which are bounded to the considerations of bifurcation surfaces as the boundaries of hypersurfaces extending to spatial infinity. It turn, such attitude enables one to treat fields which are not necessary smooth through each event horizon of the adequate black object.

As was shown in Ref. [23] the modification of the derivation of the first law of black p-brane thermodynamics using the ADM formalism was fruitful. It will be not amiss to use this idea in higher dimensional p-brane spacetime. Perhaps the reasoning presented in [51] will be useful. We hope to return to the problem in question elsewhere.

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[1] G. W. Gibbons, D. Ida, and T. Shiromizu, Phys. Rev. Lett. **89**, 041101 (2002); M. Rogatko, Phys. Rev. D **67**, 084025 (2003); **73**, 124027 (2006); D.C. Robinson, in *Kerr Spacetime: Rotating Black Holes in General Relativity*, edited by D.L. Wiltshire, M. Visser, and S.M. Scott (Cambridge University Press, Cambridge, England, 2006); M. Rogatko, AIP Conf. Proc. **861**, 722 (2006).
[2] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002).
[3] R. Emparan and H. S. Real, Classical Quantum Gravity **23**, R169 (2006).
[4] Y. Morisawa and D. Ida, Phys. Rev. D **69**, 124005 (2004).
[5] M. Rogatko, Phys. Rev. D **70**, 084025 (2004).
[6] T. Harmark, Phys. Rev. D **70**, 124002 (2004); T. Harmark and P. Olesen, Phys. Rev. D **72**, 124017 (2005).
[7] S. Hollands and S. Yazadjiev, Commun. Math. Phys. **283**, 749 (2008).
[8] S. Hollands and S. Yazadjiev, Classical Quantum Gravity **25**, 095010 (2008).
[9] Y. Morisawa, S. Tomizawa, and Y. Yasui, Phys. Rev. D **77**, 064019 (2008).
[10] A. A. Pomeransky and R. A. Sen'kov, arXiv:hep-th/0612005.
[11] M. Rogatko, Phys. Rev. D **77**, 124037 (2008).
[12] B. Kol, E. Sorkin, and T. Piran, Phys. Rev. D **69**, 064031 (2004); T. Wiseman, Classical Quantum Gravity **20**, 1137 (2003); **20**, 1177 (2003); T. Harmark and N. A. Obers, Nucl. Phys. **B684**, 183 (2004); J. High Energy Phys. 05 (2002) 032.
[13] H. Kudoh and T. Wiseman, Prog. Theor. Phys. **111**, 475 (2004); Phys. Rev. Lett. **94**, 161102 (2005).
[14] T. Harmark, Phys. Rev. D **69**, 104015 (2004); D. Gorbonos and B. Kol, J. High Energy Phys. 06 (2004) 053; D. Karasik, C. Sahabandu, P. Suranyi, and L. C. R. Wijewardhana, Phys. Rev. D **71**, 024024 (2005).
[15] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70**, 2837 (1993).
[16] R. Gregory and R. Laflamme, Nucl. Phys. **B428**, 399 (1994).
[17] R. Gregory and R. Laflamme, Phys. Rev. D **51**, R305 (1995).
[18] S. S. Gubser and I. Mita, arXiv:hep-th/0009126.
[19] S. S. Gubser and I. Mita, J. High Energy Phys. 08 (2001) 018.
[20] H. Reall, Phys. Rev. D **64**, 044005 (2001).
[21] G. Horowitz and K. Maeda, Phys. Rev. Lett. **87**, 131301 (2001).
[22] G. Horowitz and K. Maeda, Phys. Rev. D **65**, 104028 (2002).
[23] J. Trashen and D. Fox, Classical Quantum Gravity **21**, 289 (2004).
[24] T. Harmark and N. A. Obers, J. High Energy Phys. 05 (2004) 043.
[25] S. W. Hawking and G. T. Horowitz, Classical Quantum Gravity **13**, 1487 (1996).
[26] N. A. Obers, Lect. Notes Phys. **769**, 211 (2009).
[27] R. Emparan and H. S. Real, Living Rev. Relativity **11**, 6 (2008).
[28] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).
[29] S. Gao and R. M. Wald, Phys. Rev. D **64**, 084020 (2001).
[30] M. Rogatko, Classical Quantum Gravity **19**, 3821 (2002).
[31] M. Rogatko, Phys. Rev. D **71**, 104004 (2005).
[32] M. Rogatko, Phys. Rev. D **72**, 074008 (2005); **72**, 089901 (E) (2005).
[33] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).
[34] D. Sudarsky and R. M. Wald, Phys. Rev. D **46**, 1453 (1992).
[35] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).
[36] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[37] V. Iyer and R. M. Wald, Phys. Rev. D **52**, 4430 (1995).
[38] V. Iyer, Phys. Rev. D **55**, 3411 (1997).
[39] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49**, 6587 (1994); **52**, 3518 (1995).
[40] J. Koga and K. Maeda, Phys. Rev. D **58**, 064020 (1998).
[41] S. Gao, Phys. Rev. D **68**, 044016 (2003).
[42] G. W. Gibbons, M. J. Perry, and C. N. Pope, Classical Quantum Gravity **22**, 1503 (2005); M. Korzynski, J. Lewandowski, and T. Pawlowski, *ibid.* **22**, 2001 (2005); M. Rogatko, Phys. Rev. D **71**, 024031 (2005).
[43] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986); J. P. Gauntlett, R. C. Myers, and P. K. Townsend, Classical Quantum Gravity **16**, 1 (1999).
[44] P. K. Townsend and M. Zamaklar, Classical Quantum Gravity **18**, 5269 (2001).
[45] K. Copsey and G. T. Horowitz, Phys. Rev. D **73**, 024015 (2006).
[46] M. Rogatko, Phys. Rev. D **73**, 024022 (2006).
[47] M. Rogatko, Phys. Rev. D **75**, 024008 (2007).
[48] M. Rogatko, Phys. Rev. D **75**, 124015 (2007).
[49] G. Compere, arXiv:hep-th/0703004.
[50] R. M. Wald and A. Zoupas, Phys. Rev. D **61**, 084027 (2000).
[51] M. Rogatko, Phys. Rev. D **71**, 024031 (2005).