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Lorentz-breaking massive gravity in curved space

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A systematic study of the different phases of Lorentz-breaking massive gravity in a curved background is performed. For tensor and vector modes, the analysis is very close to that of Minkowski space. The most interesting results are in the scalar sector where, generically, there are two propagating degrees of freedom (DOF). While in maximally symmetric spaces ghostlike instabilities are inevitable, they can be avoided in a FRW background. The phases with less than two DOF in the scalar sector are also studied. Curvature allows an interesting interplay with the mass parameters; in particular, we have extended the Higuchi bound of de Sitter to Friedman-Robertson-Walker and Lorentz-breaking masses. As in dS, when the bound is saturated there is no propagating DOF in the scalar sector. In a number of phases the smallness of the kinetic terms gives rise to strongly coupled scalar modes at low energies. Finally, we have computed the gravitational potentials for pointlike sources. In the general case we recover the general relativity predictions at small distances, whereas the modifications appear at distances of the order of the characteristic mass scale. In contrast with Minkowski space, these corrections may not spoil the linear approximation at large distances.

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I. INTRODUCTION

Massive gravity has recently received a lot of attention mainly due to its relation to large distance modifications of the gravitational force (for recent reviews, see e.g. [1,2]). Even if the addition of a Lorentz-invariant mass term to the standard action for the graviton in a flat background accomplishes the desired modification, it also implies the appearance of new problems, such as the van Dam-Veltman-Zakharov (vDVZ) discontinuity and the strong coupling of the scalar mode of the massive graviton [3–6]. It was realized in [7] (see also [8]) that some of these problems may be softened if the mass term breaks the Lorentz invariance to rotational invariance.

It is also known that some of the features of Lorentz-breaking (LB) massive gravity are peculiar to Minkowski space and do not hold in other backgrounds [9] (see also [10,11]). In this note, we will study the behavior of the gravitational perturbations in a curved background with a mass term breaking linearized general covariance.

A caveat to this restriction has to do with the choice of action for the graviton fluctuations in curved spacetime. Normally, one considers the perturbations of the general relativity (GR) action to second order around a background which solves the equations of motion (EOM). This ensures gauge invariance under linearized diffeomorphisms (diff) at the quadratic level. In this work, we modify GR by adding a mass term which breaks explicitly gauge invariance, but other diff-breaking corrections are possible. In particular, also the kinetic term may be modified once one relaxes the constraint of general covariance.[1] The motivation for considering only mass terms is that we want to focus on large distance (infrared) modifications of gravity. Besides, there are known physical examples generating this kind of mass terms for gravitational perturbations.

A first example is a model where the matter sector includes four scalar fields that condense, breaking spontaneously the symmetry of the background metric [13] (see also [14] for related previous work and [15] for some cosmological implications). In the gauge where those scalar modes are frozen (unitary gauge), the spectrum of the perturbations reduces to the gravitational modes with a mass term that violates the symmetry of the background. In this sense, the scalars are the Goldstones modes of the broken diff invariance. Another interesting example is bigravity, where a second rank-2 tensor interacts with the metric $g_{\mu\nu}$ [16]. In this case, there are exact flat backgrounds where the metrics do not share the whole group of invariance, but preserve a common SO(3). The spectrum of fluctuations around these backgrounds includes a Lorentz-breaking massive graviton which is a combination of both

[1]Recently, there has been some interest in modifying the kinetic structure of GR as a way to improve its UV behavior [12].

metrics [17,18] (see also [19] for some phenomenology and [20,21] for spherically symmetric solutions).

Inspired by the previous models[2] we will consider the presence in the action of a generic mass term (function of the metric) which breaks general covariance. This term both allows for FRW backgrounds (see e.g. [24,25]) and generates the LB mass terms for the gravitational perturbations. Thus, the analysis of just the gravitational degrees of freedom is consistent in this setup where diff invariance is broken, while in a diff-invariant context this is possible only in a de Sitter (dS) space.[3]

The paper is organized as follows: In Sec. II we introduce our notations and the setup of our investigations. Then we analyze the perturbations with LB terms in curved backgrounds for the tensor (Sec. IV), vector (Sec. V) and scalar modes (Sec. VI and appendix A). In Sec. VII we study the generalized Newton-like potentials and their deviations from GR. We present the conclusions in Sec. VIII.

II. ACTION, BACKGROUND, AND PERTURBATIONS

Our starting point is the Einstein-Hilbert (EH) Lagrangian with the addition of mass terms for the gravitational perturbations, breaking general covariance. In the flat limit these terms also break Lorentz invariance, and we will refer to them as LB terms.

This setup describes, at quadratic level, infrared modifications of gravity where only gravitational degrees of freedom are present. At full nonlinear level these deformations of the EH theory may be parametrized by adding to the Lagrangian a *nonderivative* function of the metric components, breaking general covariance:

$$S = \int d^4x\sqrt{-g}M_P^2[R - 2F(g_{\mu\nu})]. \tag{2.1}$$

General covariance can be restored [4,8] by introducing extra (Stückelberg) fields,[4] which in the equivalent of the unitary gauge yield the form (2.1).

It is clear that the term F will contribute to the background EOM, and exact solutions are known for certain F functions. For example, when $F = \lambda = \text{const}$, the background will be maximally symmetric and the theory will be gauge (diff) invariant. For $F \neq \text{const}$, FRW solutions can be found, which can modify the standard cosmological solutions of GR. For certain classes of F, solutions that exhibit late-time cosmic acceleration were studied in [13] (see also [25]).

Accordingly, we assume that the dynamics of modified gravity admits a spatially flat isotropic and homogeneous background (FRW henceforth)

$$\bar{g}_{\mu\nu} = a(\eta)^2\eta_{\mu\nu} \quad \text{with} \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \tag{2.2}$$

where η is the conformal time. We will use $\mathcal{H}(\eta) = a'/a$ and $H(\eta) = a'/a^2$, where $0000'$ is the derivative with respect to η (therefore $aH' = \mathcal{H}' - \mathcal{H}^2$).

We define the metric perturbations as

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}). \tag{2.3}$$

The second-order expansion of S can then be written as

$$S = S^{(2)}_{\text{GR}} + S^{(2)}_{\text{LB}}, \tag{2.4}$$

where[5]

$$S_{\text{GR}} = \int d^4x\sqrt{-g}M_P^2(R - 6H^2) \tag{2.5}$$

and the term F gives rise also to LB masses for h. Assuming rotations are preserved, these can be parametrized as

$$\begin{aligned} S^{(2)}_{\text{LB}} = \frac{M_P^2}{4}\int d^4x\sqrt{-\bar{g}}[&m_0^2h_{00}^2 + 2m_1^2h_{0i}^2 \\ &- (m_2^2 - 4H'a^{-1})h_{ij}^2 + (m_3^2 - 2H'a^{-1})h_{ii}^2 \\ &- 2m_4^2h_{00}h_{ii}]. \end{aligned} \tag{2.6}$$

Here spatial indices are contracted with δ_{ij}, and $m_i \equiv m_i(\eta)$ represent effective time-dependent masses. The terms proportional to H' in (2.6) are conveniently chosen to cancel similar contributions coming from the expansion of (2.5) in backgrounds different from dS. Notice that the parametrization in (2.6) is completely general as the mass parameters are arbitrary functions of the conformal time.

Diff gauge invariance is restored taking the limit $m_i = 0$, $H' = 0$ and it corresponds to the case $F = \lambda = \text{const}$. On the other hand, for a FRW background and vanishing masses the action is invariant only under longitudinal spatial diffs. This is a consequence of the fact that a generic FRW background is never a consistent background for GR without matter. A non–maximally-symmetric background breaks the time diffs, and accordingly in the limit of vanishing masses one recovers the invariance under spatial diffs.

In the Lorentz-invariant case the masses can be expressed in terms of two parameters α, β

[2]Other related models include theories with extra dimensions [22] and theories with condensing vector fields [23].

[3]We leave the analysis of gravitational perturbations coupled to additional fields in a FRW background for a forthcoming publication [26].

[4]This implies the addition of (at most) four scalar fields, but invariant actions can also be found by adding vector or tensor fields.

[5]We stress that in this expression, H depends only on the background.

$$m_0^2 = \alpha + \beta, \qquad m_1^2 = -\alpha, \qquad m_2^2 - 4a^{-1}H' = -\alpha,$$
$$m_4^2 = \beta, \qquad m_3^2 - 2a^{-1}H' = \beta, \tag{2.7}$$

and the mass term (2.6) can be written in terms of contractions of $h_{\mu\nu}$ with $\bar{g}^{\mu\nu}$. The Fierz-Pauli (FP) choice, free of ghosts in flat space, corresponds to $\alpha + \beta = 0$. In curved space also the ''non-Fierz-Pauli'' case $\alpha + \beta \neq 0$ can be free of ghosts (see Sec. VI).

The setup introduced here is suitable to describe a rather general class of massive gravity theories, and exhibits a rich set of phases, depending on the masses m_i and $H(\eta)$. In general we can have the following scenarios:

(i) The F term in the action (2.1) does not affect neither the background (dS limit) nor the propagation of the perturbations. This is the case for $H' = 0$ and $m_i = 0$ and is realized, e.g., when $F = \lambda$.

(ii) Only the perturbations are modified. This corresponds to $H' = 0$ and $m_i \neq 0$. Writing $F(g_{\mu\nu}) = \lambda + f(g_{\mu\nu})$, this happens when the scale of λ is much larger than the scale related to f.

(iii) Only the background is modified. This happens for $H' \neq 0$ and $m_i = 0$. As we will see, it can also be realized in less trivial situations.

(iv) Finally, in general both the background and the perturbations are modified by F.

In order to study the dynamics of the perturbations, it is convenient to decompose the metric fluctuations as irreducible representations of the rotation group[6]

$$h_{00} = \psi, \qquad h_{0i} = u_i + \partial_i v, \qquad \partial_i u_i = 0,$$
$$h_{ij} = \chi_{ij} + \partial_i s_j + \partial_j s_i + \partial_i\partial_j\sigma + \delta_{ij}\tau, \tag{2.8}$$
$$\partial_i s_i = \partial_j \chi_{ij} = \delta_{ij}\chi_{ij} = 0.$$

From those fields one can define two scalar and one vector gauge invariant quantities

$$\Psi \equiv \tau + \mathcal{H}(2v - \sigma'),$$
$$\Phi \equiv \psi - 2v' + \sigma'' - \mathcal{H}(2v - \sigma'), \qquad W_i = u_i - s_i', \tag{2.9}$$

while the transverse-traceless spin two field χ_{ij} is already gauge invariant. It is also convenient to define the field $\Sigma = \sigma/\Delta$.

We will couple the gravitational fields to a conserved[7] energy-momentum tensor $T_{\mu\nu}$,

$$S_T = -\int d^4x a^2 h_{\mu\nu}\eta^{\mu\alpha}\eta^{\nu\beta}T_{\alpha\beta}$$
$$= -\int d^4x a^2(\chi_{ij}T_{ij} + \Phi T_{00} - 2T_{0i}W_i + \Psi T_{ii}), \tag{2.10}$$

with $\bar{g}^{\alpha\nu}\bar{\nabla}_\alpha T_{\mu\nu} = 0$, where $\bar{\nabla}$ is the covariant derivative associated to the background metric. The field Φ is the generalization of Newtonian potential around the source in the linearized approximation. For FRW, the EMT conservation is equivalent to

$$T_{00}' = \partial_i T_{0i} - \mathcal{H}(T_{00} + T_{ij}\delta_{ij}),$$
$$\partial_j T_{ij} = T_{i0}' + 2\mathcal{H}T_{i0} = a^{-2}(a^2 T_{i0})'. \tag{2.11}$$

III. STABLE PERTURBATIONS IN CURVED BACKGROUNDS?

Our goal is to study the dynamics of perturbations in curved backgrounds and determine when one can get a theory free of instabilities. These instabilities can be of *ghost* or *Jeans* type. The *ghostlike* instabilities are related to an infinite phase-space volume. If they are present, the decay rate of the perturbative vacuum will be infinite unless a cutoff is introduced in the theory [28] (see also [29]). In the Lorentz-breaking case, the different masses provide a natural energy scale to place the cutoff. If we admit a hierarchy inside the mass scales we may freeze the nonstable degrees of freedom, while still keeping some of the masses below the cutoff. In an expanding universe, there is also another important dimensional parameter, H. We will focus on modifications such that at least some of the mass scales are inside the horizon scale $m_i \gg H$. In this case, there is a natural hierarchy inside the set of dimensional parameters, which allows to define a large momentum cutoff Λ_c keeping the masses small,

$$|\Delta| \leq m^2\left(\frac{m}{H}\right)^\alpha \sim \Lambda_c^2, \tag{3.1}$$

with $\alpha > 0$. Even if the addition of a cutoff may unveil phenomenologically acceptable phases, to keep the discussion simple we will consider theories free from *ghostlike* instabilities in the quadratic Lagrangian (see however [2,17]).

Jeanslike instabilities can be also present. By this we mean instabilities that appear in a certain finite range of momenta. They are the signature of the growth with time of the perturbation at certain scales, and may even be interesting phenomenologically as a contribution to the clustering of matter at large distances (see, e.g. [30]). Furthermore, in an expanding universe, they may be settled beyond the horizon, where they are presumably frozen. Again, we leave the study of this possibility for future research [26], and concentrate on Lagrangians with a stable spectrum.

[6]Here, we follow the notation of [7].
[7]Energy-momentum tensor(EMT) conservation is not strictly required in massive gravity. A study of nonconserved EMT in FP can be found in [27].

It is also important to recall that in a FRW universe, energy is not a conserved quantity, and its positivity does not guarantee stability. Nevertheless, for scales smaller than the horizon, we can still use the positivity of the energy associated to the conformal time as a *necessary* requirement for stability (see also [31,32]). As we will focus on these length scales, we will not discuss any global issue.

Finally, there are two concerns for massive gravity beyond the linear theory [4]. The first one is the *strong coupling* that emerges when one or more propagating states have their kinetic terms suppressed by a small parameter. In this case, the range of validity of the linear theory is drastically reduced. Furthermore, if we consider the action (2.1) as an effective action with a cutoff, one expects the contributions of higher order operators to become important at much lower energies than the initial cutoff scale. To study this behavior one should analyze the scaling of relevant interaction terms [33] which is beyond the scope of this paper. Here we just point out that when Lorentz invariance is violated, the strong coupling cutoff can be present in energy and/or momentum independently. We will accordingly speak of *time* and *space* cutoff Λ_t, Λ_s, by making canonical the relative quadratic terms in the action.

Besides, when the functions m_i satisfy certain conditions, there is a *reduction of the phase-space*, i.e. not all the six degrees of freedom (DOF) of the gravitational perturbations propagate. It turns out that in Minkowski those are the only ghost-free possibilities [8]. In general, unless there exists a symmetry that enforces them (see, for example, the case of bigravity [18], or the case described in [13]) these conditions are only satisfied for the quadratic Lagrangian in very finely tuned backgrounds. This means that the analysis is very sensitive to small changes in the background and probably to the interaction terms and higher order operators [1,8,10]. In the following we show how the generalization of LB massive gravity to curved backgrounds is useful to circumvent both concerns.

In the next sections we analyze the spectrum of tensor, vector, and scalar perturbations, that at linearized level are not coupled, by SO(3) symmetry.

IV. TENSOR MODES

The action for the tensor perturbations is

$$S^{(T)} = \frac{M_P^2}{4}\int d^4x a^2[-\eta^{\mu\nu}\partial_\mu\chi_{ij}\partial_\nu\chi_{ij} - a^2 m_2^2\chi_{ij}\chi_{ij}], \tag{4.1}$$

from which the EOM read

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\chi_{ij} - 2\mathcal{H}\chi'_{ij} - a^2 m_2^2\chi_{ij} = 0. \tag{4.2}$$

The absence of tachyonic instabilities requires

$$m_2^2 \geq 0. \tag{4.3}$$

One can also readily check from (4.1) that there are no ghost or gradient instabilities.[8]

V. VECTOR MODES

Extracting the vector part from (2.4) we get

$$S^{(v)} = \frac{M_P^2}{2}\int d^4x a^2\{-(u_i - s'_i)\Delta(u_i - s'_i) + a^2[m_1^2 u_i u_i + m_2^2 s_j\Delta s_j]\}. \tag{5.1}$$

The field u_i is not dynamical and it can be integrated out through its equation of motion,

$$\Delta(u_i - s'_i) - a^2 m_1^2 u_i = 0, \tag{5.2}$$

to yield

$$S^{(v)} = \frac{M_P^2}{2}\int d^4x a^4\Big[m_1^2 s'_i\frac{\Delta}{\Delta - a^2 m_1^2}s'_i + m_2^2 s_i\Delta s_i\Big]. \tag{5.3}$$

Therefore, the dispersion relation of the vector field s_i breaks Lorentz invariance at any scale (provided $m_1 \neq m_2$). The action is free from instabilities for

$$m_1^2 \geq 0 \quad \text{and} \quad m_2^2 \geq 0, \tag{5.4}$$

in complete analogy to what happens in Minkowski space.

The case $m_1 = 0$ is particularly interesting, as it implies the cancellation of the time-derivative term in (5.3), so that there is no propagating vector mode. As we will see in Sec. VI C, this case is also important for the scalar sector.

The canonically normalized field s_i^c can be defined by a rescaling:

$$s_i^c \equiv \Lambda_v(\eta)^2 s_i, \qquad \Lambda_v(\eta)^2 = a^2 m_1 M_P\sqrt{\frac{\Delta}{\Delta - a^2 m_1^2}}, \tag{5.5}$$

with the action

$$S^{(v)} = \frac{1}{2}\int d^4x\{(s_i^c)'(s_i^c)' + \frac{m_2^2}{m_1^2}s_i^c\Delta s_i^c - [a^2 m_2^2 + \Lambda_v^2(\Lambda_v^{-2})'']s_i^c s_i^c\}. \tag{5.6}$$

Therefore, the canonical field has a LB dispersion relation and a time dependent mass.

From (5.5) we can also read the naive *temporal* strong coupling scale of the vectors, that is momentum (and time) dependent. At large momenta, $|\Delta| > m_1^2 a^2$, we expect the physical strong coupling scale to be given by $\Lambda_t = \Lambda_v/a \sim \sqrt{m_1 M_P}$, that is also the expected cutoff for a

[8]In an expanding universe, the friction term appearing in (4.2) implies that the perturbation is frozen at large distance. When imposing $m_2^2 \geq 0$, we are assuming that the mass scale is well inside the horizon and unless otherwise stated, we will assume this to be the case.

gauge theory explicitly broken by a mass term m. This implies that the theory can be trusted only if the horizon scale does not exceed the cutoff, $H < \sqrt{m_1 M_P}$. In Sec. VIII we will comment on some physical consequences of this bound. A similar *spatial* strong coupling scale can be defined by making canonical the spatial gradients.

VI. SCALAR MODES

The scalar sector of the theory is the most interesting one. In flat space with the FP Lorentz-invariant mass term it is a scalar mode which has the lowest strong coupling scale, and it is this sector that shows crucial differences when Lorentz symmetry is violated or spacetime is curved [7,9]. As we will see later, it is also here that the difference between the maximally symmetric spacetimes and generic FRW spaces arises. The analysis will show that generically there are two scalar degrees of freedom, while the only possibilities with less degrees of freedom are $m_1 = 0$ or $m_0 = 0$.

The scalar part of (2.4) can be written as (modulo total derivatives)

$$S^{(s)} = \frac{M_P^2}{4}\int d^4x a^2\{-6(\tau' + \mathcal{H}\psi)^2 + 2(2\psi - \tau)\Delta\tau + 4(\tau' + \mathcal{H}\psi)\Delta(2v - \sigma') + a^2[m_0^2\psi^2 - 2m_1^2 v\Delta v - m_2^2(\sigma\Delta^2\sigma + 2\tau\Delta\sigma + 3\tau^2) + m_3^2(\Delta\sigma + 3\tau)^2 - 2m_4^2\psi(\Delta\sigma + 3\tau)]\}, \tag{6.1}$$

where the H, H' terms have canceled as promised.

In the de Sitter background, when all masses are set to zero, the action reduces to the first line of (6.1) and it is gauge invariant.[9] As previously remarked, for a FRW background and vanishing masses the action is invariant only under longitudinal spatial diffs (only σ is undetermined).

From (6.1) it is clear that ψ and v are Lagrange multipliers enforcing the following constraints

$$\psi = \frac{m_1^2 m_4^2(\Sigma + 3\tau)a^3 + 2Hm_1^2(\Sigma' + 3\tau')a^2 - 2\Delta m_1^2\tau a - 8H\Delta\tau'}{8H^2 a\Delta + (m_0^2 - 6H^2)m_1^2 a^3}, \quad v = \frac{2Ha^2 m_4^2(\Sigma + 3\tau) + 4H^2 a\Sigma' + 2m_0^2 a\tau' - 4H\Delta\tau}{8H^2 a\Delta + (m_0^2 - 6H^2)m_1^2 a^3}. \tag{6.2}$$

Notice that the behavior of these fields in FRW is qualitatively different from Minkowski space. In particular, whereas in flat space, the cases $m_0 = 0$ and $m_1 = 0$ are singular and must be treated separately, in curved spacetime, ψ and v are always determined by Eqs. (6.2). After integrating out v and ψ we are left with a Lagrangian for $\varphi = (\Sigma, \tau)^t$:

$$\mathcal{L}_{\Sigma,\tau} = \tfrac{1}{2}\varphi'^t\mathcal{K}\varphi' + \varphi^t\mathcal{B}\varphi' - \tfrac{1}{2}\varphi^t\mathcal{A}\varphi, \tag{6.3}$$

where

$$\mathcal{K} = \frac{-M_P^2 a^2}{8H^2\Delta + (m_0^2 - 6H^2)m_1^2 a^2} \times \begin{pmatrix} 2H^2a^2m_1^2 & a^2m_0^2m_1^2 \\ a^2m_0^2m_1^2 & m_0^2(3a^2m_1^2 - 4\Delta)\end{pmatrix}, \tag{6.4}$$

$$\mathcal{B} = \frac{M_P^2\Delta H(m_1^2 - 2m_4^2)}{8H^2\Delta + (m_0^2 - 6H^2)m_1^2 a^2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}. \tag{6.5}$$

We will first study the dynamics of (6.3) through the Hamiltonian, for which the explicit expression of the matrix $\mathcal{A}$ is not needed.[10] The conjugate momenta π are

$$\pi_i = \frac{\partial\mathcal{L}}{\partial\varphi_i'} = \mathcal{K}_{ij}\varphi_j' - \mathcal{B}_{ij}\varphi_j. \tag{6.6}$$

Thus, two DOF will propagate when the matrix $\mathcal{K}$ is nondegenerate, i.e. when

$$\det\|\mathcal{K}\| \propto m_0 m_1 \neq 0. \tag{6.7}$$

In this case one can express the velocities in terms of momenta, and the resulting Hamiltonian is

$$\mathcal{H}_{\Sigma,\tau} = \tfrac{1}{2}\pi^t\mathcal{K}^{-1}\pi + \tfrac{1}{2}\varphi^t\mathcal{M}\varphi, \quad \mathcal{M} = (\mathcal{A} + \mathcal{B}\mathcal{K}^{-1}\mathcal{B}), \tag{6.8}$$

with a rather simple kinetic term

[9] The first line of (6.1) differs from the standard action of the graviton in a FRW background by a term proportional to $(\mathcal{H}' - \mathcal{H}^2)\psi^2$ which cancels in dS (cf. [34]).

[10] As the form of this matrix in the general case is quite cumbersome and not particularly illuminating, we will not write it explicitly in this work.

$$\mathcal{K}^{-1} = \frac{1}{M_P^2 a^2}\begin{pmatrix} 3 - \frac{4\Delta}{a^2 m_1^2} & -2 \\ -2 & \frac{2H^2}{m_0^2} \end{pmatrix}. \tag{6.9}$$

The theory is free of ghosts when the kinetic energy matrix $\mathcal{K}^{-1}$ is positive definite, that translates into the following conditions:

$$m_1^2 > 0, \qquad 0 < m_0^2 \leq 6H^2. \tag{6.10}$$

Therefore, contrary to the flat space case, we can still have a well-defined kinetic term with two propagating degrees of freedom. In fact a window for m_0^2 opens up, and this allows even for a "non-FP" Lorentz-invariant mass term free of ghosts (and vDVZ discontinuity, see Sec. VII).[11]

It is also instructive to look at the no-ghost conditions in the low and high momentum regimes. We find,

$$\text{no ghost}\begin{cases}\text{at large momenta:} & m_1^2 > 0, \qquad m_0^2 > 0 \\ \text{at small momenta:} & m_1^2 > 0, \qquad 0 < m_0^2 \leq 6H^2.\end{cases} \tag{6.11}$$

Therefore, a nonzero curvature allows the scenario where the theory is free of ghosts in the ultraviolet but there is one ghost mode at large wavelengths; this happens for $m_1^2 > 0$ and $m_0^2 > 6H^2$. Such a ghost mode at very large distances would not necessarily render the theory phenomenologically sick, but would indicate a large scale instability of backgrounds with curvature smaller than $m_0^2/6$, including the limiting case of Minkowski (*Jeanslike* instability, in the language of Sec. III).

From (6.9) one can find when the scalar sector suffers from strong coupling due to a small kinetic term. As happens for the vector modes, one of the strong coupling scales is related to the smallness of m_1, whereas the other one depends on the ratio m_0^2/H^2. When this ratio is not small [and compatible with the ghost-free condition (6.10)], both the scalar and vector sector become strongly coupled at the same *time* scale $\Lambda_t \sim \sqrt{m_1 M_P}$.

The analysis of the positivity of the "mass" term $\mathcal{M}$ is rather cumbersome and we will consider just the high momentum limit (larger than the rest of the scales: m_i, H, and H'). In this case, requiring that the mass matrix in (6.8) is positive definite gives

$$\begin{aligned} m_3^2 - m_2^2 &< \frac{(m_1^2 - 2m_4^2)^2}{16 m_0^2}, \\ H' a^{-1} &< -\left[\frac{m_1^2}{4} + \frac{(m_1^2 - 2m_4^2)^2}{16 m_1^2}\right], \end{aligned} \tag{6.12}$$

(where we have used $m_1^2 > 0$, $m_0^2 > 0$ and $H^2 > 0$.) When the previous conditions are satisfied there is no gradient instability at small distances. Notice that the right-hand side(r.h.s.) in the last condition is always negative, meaning that only a FRW background with an expanding horizon can be stable. Besides, one can easily check that the previous conditions are inconsistent in the Lorentz-invariant case.

To summarize, in the *nondegenerate case* of $m_{0,1} \neq 0$, we found two DOF where

(i) there is no ghost provided $m_1^2 > 0$, $6H^2 \geq m_0^2 > 0$

(ii) there is no gradient instability when (6.12) are satisfied (so H' is negative).

The difference that we found with the maximally symmetric case, where there is necessarily a gradient instability, implies the presence of a *spatial* strong coupling problem in this limit. In fact in approaching the dS background the spectrum of the Hamiltonian must pass through the case in which one of the modes is frozen, because the determinant of $\mathcal{M}$ vanishes and accordingly the "spatial" part of its dispersion relation will vanish.[12]

In the degenerate cases, when m_0 or m_1 vanish, there are less DOF and a separate analysis is given in the following sections. The $m_0 = 0$ case is related to the Fierz-Pauli case [36], whereas the case $m_1 = 0$ appears naturally in the ghost condensate and bigravity theories [13,17,18].

A. The phase $m_0 = 0$

For $m_0 = 0$, the field τ is an auxiliary field as one can check from the action (6.3). Even if there is only one remaining DOF, the general treatment is quite involved and it is presented in appendix A. In this section we will just state the results and study some particular cases.

The EOM for τ yield the constraint (A1), which once substituted in the action gives a (quite complicated) effective Lagrangian for Σ. Its kinetic part is

$$\frac{\mathcal{K}}{a^4 M_P^2} = \frac{3a^3 m_1^2[a(m_4^2)^2 + 2(a m_\mu^2 H^2 - H(m_4^2)' + m_4^2 H')] - 4[a m_4^2(m_1^2 - m_4^2) + m_1^2 H']\Delta}{a m_1^2(2\Delta - 3a^2 m_4^2)^2 - 2(4\Delta - 3a^2 m_1^2)[3a^2 H(aH m_\mu^2 - (m_4^2)') - (2\Delta - 3a^2 m_4^2)H']}, \tag{6.13}$$

where $m_\mu^2 = 3(m_3^2 - m_4^2) - m_2^2$. The positivity of the kinetic energy (no ghost) for large momenta gives

[11] Recently, non-Fierz-Pauli Lagrangians with scale-dependent masses were also considered in [35]. Notice, though, that in that case Lorentz invariance made the masses depend on both space and time, whereas in this work we are dealing only with time-dependent masses.

[12] See [8] for a discussion of the modifications to these dispersion relations coming from higher order operators.

$$\frac{am_4^2(m_1^2 - m_4^2) + m_1^2 H'}{am_1^2 + 4H'} > 0. \tag{6.14}$$

On the other hand at small momenta the kinetic term reduces to

$$\mathcal{K}|_{\Delta=0} = a^2 M_P^2/3,$$

which is, remarkably, always positive.

One can show (cf. appendix A) that $\mathcal{K}$ is positive also at any intermediate momenta in the variable Δ provided that, in addition to (6.14), one has

$$m_1^2 \geq 0, \qquad \left(\frac{a(m_4^4 + 2H^2 m_\mu^2) + 2m_4^2 H' - 2H(m_4^2)'}{am_4^2(m_1^2 - m_4^2) + m_1^2 H'}\right) > 0. \tag{6.15}$$

When these conditions are saturated we are led to a case with a vanishing kinetic term, as discussed below [see (6.24)]. Because of its analogy with the special case discussed in [32], we will refer to this case as *partially massless*.

The condition (6.15) refers to modes at large distances (eventually outside the horizon) and is not present in the Minkowski spacetime. In dS, taking the Lorentz-invariant FP limit (2.7) with $m^2 \equiv \beta = -\alpha$, the previous conditions reduce to the Higuchi bound [37],

$$2H^2 \leq m^2. \tag{6.16}$$

Contrary to this case, the LB mass terms allow for a unitary massless limit. In fact, if the mass is an appropriate function of the conformal time, this limit can be free from ghosts also in the Lorentz-invariant case (see Sec. VI B 3).

From the same kinetic term we can also estimate the strong coupling scale of the field Σ, since the canonical field Σ^c is defined at high momentum by the rescaling

$$\Sigma^c = \Lambda_\Sigma \Sigma, \qquad \Lambda_\Sigma = a\frac{M_P m_1}{\sqrt{-\Delta}}\left[\frac{am_1^2 x(1-x) + H'}{am_1^2 + 4H'}\right]^{1/2}, \tag{6.17}$$

where $x = m_4^2/m_1^2$.

Concerning the potential term $\mathcal{M}$, it can be written as

$$\mathcal{M} = \frac{m_2^2 b^2 + c\Delta + d\Delta^2 + e\Delta^3 + (m_2^2 - m_3^2)\Delta^4}{q^2} \tag{6.18}$$

where b, c, d, e are functions of m_i and $\mathcal{H}$, whereas q is a second-order polynomial in Δ.

The absence of gradient instabilities, equivalent to the positivity of $\mathcal{M}$, requires in the ultraviolet and infrared regimes the following simple conditions:

$$\begin{aligned} &\text{at large momenta } m_2^2 > m_3^2 \\ &\text{at small momenta } m_2^2 > 0. \end{aligned} \tag{6.19}$$

We see that for $m_2^2 \geq m_3^2$ the potential is free from gradient instabilities that would be as dangerous as ghost instabilities as they would imply an infinitely fast instability [29]. Notice also that at zero momentum, the condition $m_2^2 > 0$ required for the stability of tensors and vectors, enforces positivity of the potential. This implies, together with the stability of the kinetic term, that at small momentum the theory is always stable.

At intermediate scales, the analysis becomes very technical, and a method to check for the positive definiteness is presented in the appendix A.

B. Particular cases with $m_0 = 0$

Some interesting subcases of the $m_0 = 0$ dynamics can be found looking at the numerator and denominator of Eq. (6.13). When the denominator vanishes, the field τ disappear from the EOM and the constraint (A1) does not hold anymore. The analysis of this situation is presented in Sec. VI B 1.

When the mass parameters are fine-tuned in such a way that the numerator of (6.13) vanishes, the Σ field does not propagate and it becomes an auxiliary field. This possibility is examined in Sec. VI B 2, where we also show its relation to a gauge invariance related to conformal invariance. Finally, the Fierz-Pauli Lagrangian in dS is a subcase of the phase $m_0 = 0$ where this fine-tuning can occur. We study this possibility in Sec. VI B 3.

1. Time diffeomorphisms in FRW

From the constraint (A1) for τ as a function of Σ, we see that it is singular for specific values of the masses. This happens when

$$m_1^2 = 2m_4^2 = -4a^{-1}H', \qquad m_\mu^2 = \frac{a(H^2)' - 2H''}{a^2 H}. \tag{6.20}$$

[If only the first condition holds, the constraint for τ (A1) reduces to $\tau = -\Sigma/3$.]

In this case, and away from dS (we are assuming $m_1 \neq 0$), the field τ does not appear at all in the action. This corresponds to a restoration of the gauge symmetry corresponding to time diffeomorphisms, $\delta(2v - \sigma') = 2\xi_0$.

The final Lagrangian in terms of Σ is then given by

$$\mathcal{L} = \frac{M_P^2 a^2}{2}\left\{\frac{aH'}{\Delta + 3aH'}\Sigma'^2 - \frac{1}{3}\left[m_2^2 + \Delta\frac{3(aH')^2 + (2aH' + H''/H)\Delta}{(\Delta + 3aH')^2}\right]\Sigma^2\right\}. \tag{6.21}$$

Notice that in Minkowski this phase has $m_0 = m_1 = m_4 = 0$ and features an enhanced gauge invariance mentioned in [18]. In contrast to the case of Minkowski, and to the case of dS, in a FRW background the field Σ propagates. It is also clear from (6.21) that the kinetic term is positive definite provided that $H' < 0$. Concerning the potential term, let us consider scales well inside the horizon. The requirement of positive energy at these scales gives the condition

$$m_2^2 \geq -(2aH' + H''/H), \tag{6.22}$$

When inequality (6.22) is exactly saturated, the scalar degree of freedom has vanishing speed in the at high momenta. Its dispersion relation is then $\omega^2 \simeq \text{const} + 1/\Delta$.

Finally, let us note that if for $H' < 0$ the scalar mode is well behaved, the limit of vanishing H' leads to a vanishing kinetic term and thus to strong coupling once interactions are taken into account. The resulting *time* strong coupling scale can be estimated as

$$\Lambda_s = aM_P\sqrt{\frac{aH'}{\Delta + 3aH'}}. \tag{6.23}$$

that is clearly more dangerous at short distances $|\Delta| \gg |aH'|$, where it may become sensibly lower than M_P.

2. Partially massless

Another particular case appears when, after integrating out τ, the kinetic term of Σ cancels. This happens when the inequalities (6.14) and (6.15) are saturated,

$$m_1^2 = \frac{am_4^4}{am_4^2 + H'},$$
$$a(m_4^4 + 2H^2m_\mu^2) + 2m_4^2H' - 2H(m_4^2)' = 0. \tag{6.24}$$

In the Fierz-Pauli limit this expression reduces to the partially massless case of de Sitter space (cf. [32]) and corresponds to a situation without propagating scalar degrees of freedom. In the Lorentz-invariant case with constant masses in de Sitter space, this fact is related to a conformal invariance [38]. In the most general case, one can prove that the system is invariant under the transformation

$$\delta\psi = -2(\xi' + \mathcal{H}\xi) + \phi_t, \qquad \delta v = -\xi + \zeta',$$
$$\delta\tau = 2\mathcal{H}\xi + \phi_s, \qquad \delta\sigma = 2\zeta, \tag{6.25}$$

with

$$\xi = -\frac{a_4^2(am_4^2\zeta + H'\zeta + H\zeta')}{H(am_4^2 + 2H')}, \qquad \phi_s = m_4^2\zeta$$
$$\phi_t = \frac{\zeta m_2(am_4(m_4^2 - 4H^2) + 2m_4H' - 4Hm_4')}{2aH^2}, \tag{6.26}$$

only when the extra condition

$$a[m_4^4 + 2H^2(2m_2^2 - 3m_4^2)] + 2[m_4^2H' - H(m_4^2)'] = 0 \tag{6.27}$$

is satisfied. The previous condition implies the cancellation of the potential part once (6.24) is satisfied. In the Lorentz-invariant limit with constant masses and dS background (6.27) is always satisfied when (6.24) holds. Notice also that the existence of this sort of scale invariance is general even if the kinetic term is not invariant under diff away from de Sitter.

3. Lorentz-invariant FP limit with time dependent masses

In the Fierz-Pauli limit [(2.7) with $m^2 = \beta = -\alpha$] the mode Σ propagates. However, the conditions (6.24) can be still be satisfied in dS (and only for this background) provided that m satisfies the differential equation

$$4\mathcal{H}m' = (a^2m^2 - 2\mathcal{H}^2)m. \tag{6.28}$$

This equation can be integrated to yield

$$m^2(\eta) = \frac{2H^2m_I^2}{m_I^2 + (2H^2 - m_I^2)a(\eta)}, \tag{6.29}$$

where m_I is the value of the mass at the time corresponding to $a(\eta) = 1$. The resulting mass runs from m_I^2 to $2H^2$ when a runs from 0 to 1. Notice that choosing the initial conditions corresponding to a constant mass, $m_I^2 = 2H^2$, we recover the partially massless case discussed in [32]. A similar situation could be studied for the non-Fierz-Pauli (Lorentz-invariant) case ($m_1 = m_2 \neq m_3 = m_4$, $m_0 = m_3 - m_2$).

C. The phase $m_1 = 0$

The case $m_1 = 0$ is particularly interesting in the Minkowski background, as only the tensor modes propagate. As we will show, there is a corresponding effect in dS, while one scalar mode starts to propagate in a FRW background. When $m_1 = 0$ the fields Σ is not dynamical as one can check in action (6.3). Accordingly, its EOM is

$$\mathcal{H}(m_2^2 - m_3^2)\Sigma = m_4^2\tau' - \mathcal{H}(m_2^2 - 3m_3^2)\tau. \tag{6.30}$$

Notice that again the Minkowski space limit $\mathcal{H} = 0$ is peculiar and the degree of freedom associated to τ is not present.[13] In curved space, generically Σ is determined by (6.30) and when it is substituted back in the action, after integration by parts, yields the Lagrangian

[13]Also, the case $m_2 = m_3$ should be treated differently.

$$\mathcal{L} = \frac{M_P^2 a^2}{H^2}\left\{\frac{m_\eta^4}{2(m_2^2 - m_3^2)}\tau'^2 - \left[\frac{H'}{a}\Delta + \frac{m_2^2[\mathcal{H}^2(m_2^2 - 3m_3^2 + 3m_4^2) - m_4^2 H' a]}{m_2^2 - m_3^2} - \frac{\mathcal{H}[m_4^2(m_3^2(m_2^2)' - m_2^2(m_3^2)') + m_2^2(m_3^2 - m_2^2)(m_4^2)']}{(m_2^2 - m_3^2)^2}\right]\tau^2\right\}. \tag{6.31}$$

where $m_\eta^4 = m_0^2(m_2^2 - m_3^2) + m_4^4$. From the previous expression we discover that in dS, the phase $m_1 = 0$ has no propagating degrees of freedom (in the sense that the action is Δ independent so that there is no dynamics in $\vec{x}$ space), even if, in comparison to the Minkowski case, the scalar sector has a kinetic term from which we expect a ghost condensate like dispersion relation coming from higher derivatives [8]. Besides, the potential strong coupling scales Λ_s and Λ_t are easily read out from the previous expression.

Thus, in general the phase $m_1 = 0$ is quite rich, and particularly simple. Ghostlike instabilities are avoided imposing $m_\eta^4(m_2^2 - m_3^2) \geq 0$. To get rid of gradient instabilities in this case, it is enough to impose $H' < 0$, whereas the tachyon free condition can also be read from (6.31). For the case with constant masses, it reduces to $m_2^2[\mathcal{H}^2(m_2^2 - 3m_3^2 + 3m_4^2) - m_4^2 H' a] \geq 0$.

D. Particular cases with $m_1 = 0$

A direct inspection of (6.31) and (6.30) shows some interesting subcases for the mass parameters. First, when the r.h.s. of Eq. (6.30) cancels, this equation is no longer a constraint for Σ. Besides, for $m_\eta = 0$, the kinetic term for τ cancels in the action. We devote the rest of this section to the analysis of these possibilities.

1. The case $m_2^2 = m_3^2$

When $m_2^2 = m_3^2$ the kinetic term of τ is zero. In this case τ is nondynamical and can be eliminated from the action. The only degree of freedom now is Σ with a Lagrangian

$$\mathcal{L} = \frac{a^4 M_P^2}{2}\frac{(6m_2^2 m_4^4 - 9m_4^6 + 4m_0^2 m_2^2)m_4^4}{(2m_2^2 - 3m_4^2)(2m_0^2 m_2^2 - 3m_4^4)^2} \times\left[\frac{m_4^4}{2(2m_2^2 - 3m_4^2)\mathcal{H}^2}\Sigma'^2 + m_2^2\Sigma^2\right]. \tag{6.32}$$

Again, this mode has no dynamics in space. From direct inspection we can derive the strong coupling scale, and the region of parameters where this mode disappears.

2. The case $m_\eta = 0$

Finally, for $m_\eta = 0$ we are back to a situation without scalar propagating degrees of freedom but still with a potential part at the linear level. In Minkowski also this part vanishes and the field τ is not determined (indeed, there is an additional gauge invariance). In dS, this happens when

$$m_\eta = 0,$$
$$\frac{\mathcal{H}(m_2^2 - 3m_3^2 + 3m_4^2) - (m_4^2)'}{m_4^2} = \frac{m_3^2(m_2^2)' - m_2^2(m_3^2)'}{m_2^2(m_2^2 - m_3^2)}, \tag{6.33}$$

and outside this region of the parameter space, the EOM gives $\tau = 0$.

VII. COUPLING TO MATTER AND VDVZ DISCONTINUITY

Though the vDVZ discontinuity is one of the main phenomenological difficulties of FP massive gravity in flat space, it is known that it may be circumvented in curved backgrounds [9] or when one considers Lorentz violating mass terms [7]. For AdS or dS, the vDVZ discontinuity is avoided by hiding the effects of the mass at distances larger than the horizon, and as a consequence there is no modification of gravity at scales smaller than the Hubble radius. In this section we will see that some of the massive gravity phases we have studied allow for a modification of gravity at scales shorter than the horizon scale and still compatible with GR at linear order. We will focus on the gravitational potentials produced by a "pointlike" conserved source.

The tensor part is described by a massive graviton with mass given by m_2^2. Phenomenologically, this mass is constraint by cosmological and astrophysical observations (see e.g. [13,39]), and has no impact on the gravitational potentials for pointlike sources. Also vectors modes do not affect these potentials (for cosmological constraints see [40]). For our purposes only scalar perturbations are relevant.

Let us briefly review the situation of standard GR in presence of "pointlike" conserved sources, in Minkowski or dS background:

$$T_{00} = \frac{\rho(r)}{a}, \qquad T_{0i} = T_{ij} = 0. \tag{7.1}$$

In GR, there is no scalar propagating DOF and the gauge invariant potentials are determined from the sources as

$$\Phi_{\rm GR} = \Psi_{\rm GR} = \frac{1}{M_P^2\Delta}T_{00}. \tag{7.2}$$

Recall that the perturbations are defined with respect to a nonflat metric. Thus, both the background and the perturbations play a role in the gravitational dynamics around local sources.

As described in Sec. VI, the generic massive gravity case has two propagating DOF in the scalar sector. In this section we are interested in static solutions in the presence of static sources. More concretely, we will consider time scales short enough such that we can consider the background metric constant.[14] By inspecting the EOM's in this limit, time derivatives can be neglected provided that ω, $\omega H \ll k^2$; $\omega H \ll m_i^2$, and $\omega H \ll E^2$ where ω^{-1} is the typical time scale for the variation of the gravitational perturbations and E is the energy scale of the sources.

Once that time derivatives of the two dynamical fields Σ and τ are neglected, and in the regime $H^2, H', m_i' \ll m_i^2 \ll \Delta$, the EOM can be solved in a straightforward though lengthy way. The generalization of the Newtonian potential is the quantity Φ and we get

$$\Phi = \frac{n_2\Delta^2 + n_1\Delta + n_0}{d_3\Delta^3 + d_2\Delta^2 + d_1\Delta + d_0}, \tag{7.3}$$

where the n_i and d_i are polynomials in the masses. The physics relevant for the vDVZ discontinuity is captured by expanding Φ in powers of $1/\Delta$, e.g. $\Delta \gg m_i^2$.

$$\begin{aligned}\Phi &= \frac{T_{ii} + T_{00}}{M_P^2\Delta} - \frac{uT_{00} + vT_{ii}}{2M_P^2\Delta^2(m_2^2 - m_3^2)} + O\left(\frac{1}{\Delta^3}\right),\\ u &= a^2[m_\eta^4 + m_2^2(6m_3^2 - 4m_4^2 - 2m_2^2)],\\ v &= a^2[m_\eta^4 - 2m_2^2m_4^2].\end{aligned} \tag{7.4}$$

Thus, at small distances we get the GR result plus corrections.[15] Also Ψ, that is important for post-Newtonian tests, has the same structure:

$$\Psi = \frac{T_{00}}{M_P^2\Delta} - \frac{a^2}{2M_P^2\Delta^2}\left[T_{00}\frac{m_\eta^4 - 2m_2^2m_4^2}{m_2^2 - m_3^2} + T_{ii}\frac{m_\eta^4}{m_2^2 - m_3^2}\right] + O\left(\frac{1}{\Delta^3}\right). \tag{7.5}$$

Clearly, no discontinuity is present at small distances provided that $m_2^2 \neq m_3^2$ (notice also that m_1 has disappeared from the previous expression). When $m_2^2 = m_3^2$, the previous expressions are not valid and a discontinuity is present, as it can be established by noting that in the UV the EOM imply

$$2m_3^2\Psi = m_4^2\Phi, \tag{7.6}$$

which does not hold in GR.

A. Coupling to matter for $m_1 = 0$

The case $m_1 = 0$ is of particular interest, as in flat space there is no scalar DOF and the potential features a correction linear with r, invalidating the linearized approximation at large distances. In a curved space the scalar τ propagates and the gauge invariant potentials Φ and Ψ can be written as a combination of the source, τ and its time derivatives as

$$\begin{aligned}\Psi &= \Psi_{\rm GR} + a\left(\frac{2aHm_2^2m_4^2\tau + m_\eta^4\tau'}{2\Delta H(m_2^2 - m_3^2)}\right),\\ \Phi &= \Psi + am_2^2\left(\frac{2aH(m_2^2 - 3m_3^2)\tau - m_4^4\tau'}{\Delta H(m_2^2 - m_3^2)}\right).\end{aligned} \tag{7.7}$$

Here we have used the expression for τ'' obtained from the EOM, namely

$$\tau'' = \frac{2(m_2^2 - m_3^2)H'}{am_\eta^4M_P^2}(T_{00} - M_P^2\Delta\tau) + q_1(m_i, H)\tau + q_2(m_i, H)\tau', \tag{7.8}$$

where $q_{1,2}$ are functions of the background and the masses, finite in the limit $m_i \to 0$. From these expressions one can study the behavior of potentials in the limit $m_i \to 0$.

First, consider the dS background, $H' = 0$. In this case, the first term in the r.h.s.of (7.8) vanishes. As a result, the only particular solution (vanishing for zero sources) is $\tau = 0$, and the potentials (7.7) coincide with those of GR. Remarkably, also the linear term appearing in Minkowski [8] is absent in dS.

A similar situation happens in FRW background: when $H' \neq 0$, the first term in the r.h.s. dominates in the $m_i \to 0$ limit, and τ remains finite:

$$\tau \sim \frac{T_{00}}{\Delta M_P^2} + O(m^2). \tag{7.9}$$

This implies that the corrections to Φ and Ψ with respect to GR vanish in this limit, and there is no vDVZ discontinuity.

Further insight can be gained by looking at explicit solutions of (7.8). These can be found by assuming a special time dependence of the masses and the scale factor:

$$a = \left(\frac{\eta}{\eta_0}\right)^\ell, \qquad m_i^2(\eta) = a^s\lambda_i, \tag{7.10}$$

where λ_i are constants of dimension two. The EOM for τ (7.8) then reduces to

[14]In this limit the standard Fourier analysis is well suited to analyze the EOM and energy is a conserved quantity. It is also clear that if the limit is not singular the results are equivalent to those of Minkowski space considered in [7,8].

[15]The expression (7.4) is valid for distances smaller than the inverse of mass. For distances of the order of the inverse of the mass, the appearance of a pole in (7.3) makes the series ill defined. The exact solution can be easily found and one can see that the perturbations acquire a Yukawa tail. Thus, this modification of Newtonian potential has the desirable feature of keeping the perturbations small at large distances.

$$\tau'' + \frac{2+\ell(4+s)}{\eta}\tau' + \frac{2a^{-s-3}\ell(\ell+1)\rho(r)(\lambda_2-\lambda_3)}{M_P^2\lambda_\eta^2\eta^2} + \frac{2a^{-s-2}\ell}{\lambda_\eta^2}(-(\ell+1)(\lambda_2-\lambda_3)\Delta + a^{2+s}\lambda_2[\lambda_4+\ell(\lambda_2-3\lambda_3)+\lambda_4\ell(s+4)])\tau = 0, \tag{7.11}$$

where $\lambda_\eta^2 = \lambda_4^2 + \lambda_0(\lambda_2 - \lambda_3)$.

For a general background one can easily find an exact solution of (7.11) when $s = -2$. The solution that is relevant to us can be written as[16]

$$\tau_P = \frac{T_{00}}{M_P^2(\Delta-\mu^2)},\qquad \mu^2 = \frac{2\lambda_2\lambda_4(1+2\ell)-(\ell+1)\lambda_\eta^2+2\ell(\lambda_2^2-3\lambda_2\lambda_3)}{2(\ell+1)(\lambda_2-\lambda_3)}. \tag{7.12}$$

In this case, the potentials (7.7) are

$$\Psi = \Psi_{\rm GR} + \left(\frac{2\lambda_2\lambda_4-\lambda_\eta^2}{2\Delta(\lambda_2-\lambda_3)}\right)\tau_P,\qquad \Phi = \Psi + \lambda_2\left(\frac{\lambda_2-3\lambda_3+\lambda_4}{\Delta(\lambda_2-\lambda_3)}\right)\tau_P. \tag{7.13}$$

and one can check that there is no discontinuity in the massless limit. As recalled in appendix B one can work out the explicit expression for the potentials in position space to get (we assume $\mu^2 > 0$)

$$\Psi = \Psi_{\rm GR}\left[1 + (\frac{2\lambda_2\lambda_4-\lambda_\eta^2}{2\mu^2(\lambda_2-\lambda_3)})(e^{-\mu r}-1)\right],\qquad \Phi = \Phi_{\rm GR}\left[1 + (\frac{2\lambda_2^2-6\lambda_2\lambda_3+4\lambda_2\lambda_4-\lambda_\eta^2}{2\mu^2(\lambda_2-\lambda_3)})(e^{-\mu r}-1)\right], \tag{7.14}$$

where μ can be read from (7.12). This result differs from the one found in flat space (see e.g. [24]) in some essential facts: first, instead of the linear correction to Φ that appears in Minkowski, we found an exponential function that decays to a constant at large values of r ($r \gg \mu^{-1}$). This is an infrared modification of GR whose magnitude depends on a ratio of masses, i.e. it gives finite $O(1)$ value in the generic $m \to 0$ limit. Second, also Ψ is modified at large distances, and the modification decays to a different constant. This implies that the modification is not simply a redefinition of M_P. Finally, in FRW τ is an ordinary propagating DOF (see footnote 16) and, in contrast to the Minkowski case (see e.g. [24]), there is no free time-independent function in the solution. At short distance both potentials reduce to GR and there is no discontinuity.

With the above explicit solution one can check that in the dS limit ($\ell \to -1$) the potentials reduce to GR, because $\mu \to \infty$ and $\tau \to 0$, in agreement with the previous discussion. On the other hand, also the flat limit $\ell \to 0$ can be safely taken in the last expression, but the result is not the Minkowski one. We conclude that the presence of a curved background removes the linearly growing term at large distance, or in other words regulates the infrared modification of the gravitational force.

One can also find the exact expression for the potential in the phases where there is no scalar DOF, e.g. $m_\eta = 0$ (see section VI D 2). As we have seen, in this case the kinetic term of τ is zero and we can explicitly solve for its EOM for any source obtaining an expression similar to (7.7). Moreover, τ will be of the form

$$\tau = \frac{T_{00}}{M_P^2(\Delta - M^2)},\qquad M^2 = \frac{q[m_i, a]}{(m_2^2-m_3^3)^2H'}, \tag{7.15}$$

where $q[m_i, a]$ is an analytic function of the masses, a and their derivatives. The correction to Newtonian potential can then be written in the form

$$\Phi = \Phi_{\rm GR}\left[1 + \frac{a^2m_2^2k_1}{(m_2^2-m_3^2)M^2}\left(e^{-Mr}-1 + \left[\frac{(m_2^2-3m_3^2+2m_4^2)}{k_1}-1\right]Mre^{-Mr}\right)\right],\qquad \Psi = \Psi_{\rm GR}\left[1+\frac{a^2m_2^2m_4^2}{(m_2^2-m_3^2)M^2}(e^{-Mr}-1)\right], \tag{7.16}$$

where

$$k_1 = (m_2^2 - 3m_3^2 + 2m_4^2) + \frac{m_4^2(M^2)'}{aHM^2}.$$

Again we see that the presence of a non trivial background gives rise to a modification of GR at large distances $r \sim M^{-1}$ (pushed to infinity for vanishing masses).

At short distances, the potential reduces to GR plus corrections:

$$\Phi = \Phi_{\rm GR} + \frac{a^2m_2^2T_{00}(m_2^2-3m_3^2+2m_4^2)}{(m_2^2-m_3^2)\Delta^2M_P^2} + O\left(\frac{1}{H'\Delta^3}\right). \tag{7.17}$$

On the other hand taking the dS limit carefully we recover $\Phi = \Phi_{GR}$.

[16]The general solution is of the form

$$\tau = t^{-1/2-\ell}(C_1t^{\beta[\Delta]/2} + C_2t^{-\beta[\Delta]/2}) + \tau_P.$$

Stability requires $|\beta(\Delta)| - (2l+1) < 0$, which at high energies implies $l(l+1)\lambda_\eta^2(\lambda_2-\lambda_3) \propto -H'\lambda_\eta^2(\lambda_2-\lambda_3) > 0$. This condition was readily derived in Sec. VI C from direct inspection of the Lagrangian. One can also check that the solution is stable at any scale if in addition $l > -1/2$ and $\lambda_2\lambda_\eta^2\{\lambda_4 + l(\lambda_2 - 3\lambda_2 + 2\lambda_4)\} < 0$.

B. Coupling to matter for $m_0 = 0$

For the $m_0 = 0$ case, one can express the potentials in a way similar to (7.7) and (7.8), this time in terms of Σ. The resulting expressions turn out to be very complicated, and here we will consider explicitly only the "partially massless" case discussed in Sec. VI B 2, where no DOF is present. In this case, we can write the gravitational potentials as

$$\Phi = \Phi_{\rm GR} + a\left(\frac{(3T_{00} + M_P^2\Delta\Sigma)(am_4^4 + 2m_4^2H' - 2H(m_4^2)') - 2a(3T_{00} + 2M_P^2\Delta\Sigma)H^2m_4^2}{4\Delta^2H^2M_P^2}\right),$$
$$\Psi = \Psi_{\rm GR} + a^2m_4^2\left(\frac{3T_{00} + M_P^2\Delta\Sigma}{2\Delta^2M_P^2}\right), \quad (7.18)$$

where Σ is

$$M_P^2\Delta\Sigma = -\frac{3T_{00}(am_4^4 + 2m_4^2H' - 2H(m_4^2)' - 4aH^2m_4^2)}{a[m_4^4 + 2H^2(2m_2^2 - 3m_4^2)] + 2[m_4^2H' - H(m_4^2)']}. \quad (7.19)$$

The previous two equations indicate that there is no vDVZ discontinuity and we recover GR in the massless limit.

Some care is needed in the special cases when the numerator or the denominator of (7.19) vanishes. If the denominator vanishes the theory has an extra gauge invariance (cf. Sec. VI B 2). As a result the EMT is coupled consistently only if $T_{00} = 0$ unless also the numerator vanishes. In any case, Σ can be set to zero by a gauge transformation, and the potentials can be read from (7.18).

One can readily see from these expressions that the corrections to the Newtonian potential simply amount to a linear correction (see appendix B), that invalidates the linear approximation at large distance. This modification vanishes for $m_4 = 0$, which also gives $m_1 = 0$ and $H(3m_3^2 - m_2^2) = 0$ [cf. (6.24)].

VIII. DISCUSSION AND CONCLUSIONS

In this work we have performed a systematic study of Lorentz-breaking massive gravity in a FRW background. For the *tensor* and *vector* sectors, the analysis is very close to that of Minkowski space: both sets of modes acquire independent masses constrained by phenomenological bounds. For vector modes, the naive strong coupling scale is similar to that of flat space (see also below).

The most interesting results are in the *scalar* sector where generically there are two propagating degrees of freedom. For maximally symmetric spaces, the study of the dispersion relations at high energy reveals the appearance of *ghostlike* instabilities, i.e. instabilities associated to an infinite volume phase-space, that can be cured only by introducing a momentum space cutoff. Remarkably, this is not necessarily the case in arbitrary spacetimes: high energy instabilities can be absent in a FRW background with expanding horizon, i.e. $H' < 0$, see (6.10) and (6.12). Indeed, at high energies the sign of the determinant of the mass matrix $\mathcal{M}$ [see (6.8) for the definition] is fixed in Minkowski and dS, whereas in FRW H' enters in the game allowing a region in the parameter space where $\mathcal{M}$ is positive definite. As a drawback, in the limit $H' \to 0$, the theory is strongly coupled in the scalar sector.

The scalar sector also features a number of phases with less than two DOF. Generically, in a FRW background the phase $m_0 = 0$ (which includes the FP phase) has one scalar DOF. We found the conditions that make the kinetic term of this mode positive definite, generalizing the Higuchi bound to LB masses in FRW spaces [see (6.14) and (6.15)]. Moreover, provided that $m_2^2 \geq m_3^2$, high energy instabilities are absent. We also sketched the method to avoid instabilities at intermediate momenta (which may be even interesting for cosmological perturbations).

In the presence of curvature there exist situations where the Lagrangian for the scalar modes becomes particularly simple as discussed in Sec. VI B 1. In particular the invariance under time diff can be recovered even when $m_1 \neq 0$. More interesting is the case where the absence of scalar DOF is due to a residual gauge invariance which is absent in flat space (*partially massless case*). Taking the FP limit in dS, the condition for having residual gauge invariance can be solved, and as a result all masses are determined in terms of the curvature scale [cf. (6.29)].

Also interesting is the phase $m_1 = 0$ where in general there is again a single propagating scalar DOF. For maximally symmetric backgrounds, the EOM for this scalar do not contain any gradient term and this mode effectively has no dynamics (zero velocity): it behaves as a collective mode. For generic FRW with expanding horizon, the propagating scalar has a dispersion relation that can be made free of instabilities.

We have then analyzed how the Newtonian potentials generated by conserved pointlike sources are modified. In the general case they agree with GR modulo corrections at scales related to the massive gravity scale. A typical form of the gauge invariant gravitational potentials is, for example, [see Eqs. (7.3), (7.14), and (7.16)]

$$\Phi = \Phi_{\rm GR}[1 + c_1(e^{-\mu_1 r} - 1) + c_2\mu_2 re^{-\mu_2 r}], \quad (8.1)$$

where the mass scales μ_i are combinations of curvature and mass parameters while c_i are dimensionless combinations. This form is valid also for the Ψ potential. Therefore in the massless limit or for scales $r \ll (aH)^{-1}$, m^{-1} the potentials reduce to the GR result, which makes these phases potentially very interesting. Comparing for instance to the $m_1 = 0$ phase in flat space, where a linearly growing term invalidates perturbation theory at large distance [13], in curved space the potential (8.1) is well behaved at large distance without imposing any fine-tuning in the mass parameters. In this sense, the presence of a curved background regularizes many of the peculiarities of Minkowski (also the case $m_\eta = 0$, singular in Minkowski, is regular in FRW). At short distance, the corrections with respect to GR in (8.1) can be estimated by expanding the exponentials.

In our analysis we found that some of the propagating states can have small kinetic terms (typically proportional to mass or H') giving rise to strongly coupled sectors at very low energy. This fact can be relevant for its possible cosmological implications. For instance, already in the vector sector when $m_1 \neq 0$, perturbation theory is reliable only for $H < \sqrt{m_1 M_P}$. This casts serious doubts on the possibility to use massive gravity in standard inflation, while keeping small the LB masses. In fact, to trust the linear approximation at the standard inflation scale $H_{\text{inflation}} \simeq 10^{13}$ GeV one would need $m_1 \gtrsim 10^{16}$ eV, and this would require a severe fine-tuning with respect to the other masses that are constrained by various gravitational tests (pulsar, solar system tests) to be much smaller (typically 10^{-21} eV). On the other hand, the value of $H_{\text{inflation}}$ is very model dependent, the only real upper bound comes from BBN, $T_{RH} \gtrsim 10$ MeV, i.e. $H \gtrsim 10^{-16}$ GeV. This gives the limit $m_1 > 10^{-30}$ eV, well below any other gravitational constraints on the masses. The cosmological constraints coming from the analysis of the modified gravitational perturbation's dynamics are presently under study [26].

Let us close with a comment on exact solutions. Besides the large distance modifications to GR found in this work at linearized level, some modifications have also been found in exact (spherically symmetric) solutions of massive gravity [20,41]. These solutions exist in dS space and feature a nonanalytic r^γ term in the gravitational potential.[17] Thus, they differ also asymptotically from the linearized gravitational potentials found in this work, which may be understood from the presence of long-range instantaneous interactions at linearized level. Therefore also for many of the phases analyzed in the present work, one may expect important nonlinear effects even at large distances.

[17] As well as a $1/\sqrt{r}$ term in a would-be gauge direction (i.e. for a Goldstone field). This is verified explicitly in bigravity (where $m_1 = 0$) [20], and in a decoupling limit in the FP case (Lorentz-invariant $m_0 = 0$) [41]. In [42], the solutions of [20] have been translated in the Goldstone formalism and extended numerically to other nonlinear Lagrangians.

Finally, our study suggests that the analysis of perturbations around other nontrivial backgrounds may also unveil phases where the perturbations have a stable spectrum, and is thus of definite interest. Of main importance would be a dedicated study addressing perturbations and their stability in the (exact) gravitational background produced by a star.

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APPENDIX A: THE PHASE $m_0 = 0$ IN THE GENERIC CASE

In this appendix we present the explicit analysis of the dynamical degrees of freedom of the case $m_0 = 0$ discussed in Sec. VI A. The EOM for τ yield the constraint

$$\begin{aligned} \tau = \frac{a^2}{D}\{&\Sigma'[4\Delta(m_1^2 - 2m_4^2)H] + \Sigma[2(4\Delta - 3a^2m_1^2) \\ &\times (am_\mu^2 H^2 - (m_4^2)'H + m_4^2 H') \\ &+ am_4^2 m_1^2(2\Delta - 3a^2 m_4^2)]\}, \qquad \text{(A1)} \\ D = 4\Delta^2&[am_1^2 + 4H'] - 12\Delta[2am_\mu^2 H^2 - 2(m_4^2)'H \\ &+ am_1^2 m_4^2 + (m_1^2 + 2m_4^2)H']a^2 \\ &+ 9m_1^2[2am_\mu^2 H^2 + a(m_4^2)^2 + 2(m_4^2 H' - (m_4^2)'H)]a^4, \end{aligned}$$

where $m_\mu^2 = 3(m_3^2 - m_4^2) - m_2^2$. Once τ is substituted in the action, we get a (quite involved) effective action for Σ whose kinetic part is written in (6.13). Requiring the positivity of the kinetic energy (no ghost) for large momenta we find the condition (6.14). As we saw, the kinetic term is always positive at small momenta.

To understand when $\mathcal{K}$ is positive also at intermediate momenta, first notice that $\mathcal{K}$ is expressed as a fraction of two polynomials with different roots[18] in the variable Δ. For the fraction to keep its sign those roots must be either at $\Delta > 0$ or be absent. The numerator is a linear polynomial,

[18] The two roots coincide only when

$$m_1^2(m_1^2 - 2m_4^2)^2[a(m_4^4 + 2H^2 m_\mu^2) + 2m_4^2 H' - H(m_4^2)'] = 0. \qquad \text{(A2)}$$

If the second factor cancels, we find a very simple kinetic term which is always positive. When the last factor cancels, the kinetic term is positive at any scale provided that (6.14) holds. Finally, when both terms cancel, the constraint (A1) reduces to $\tau = 0$ and the whole action is much simpler.

and setting its root at positive Δ corresponds to the condition (6.15). The denominator is a second-order polynomial that, once the coefficient of Δ^2 is taken as a common factor, has a the zeroth order term

$$9a^4 m_1^2\left(\frac{a(m_4^4 + 2H^2 m_\mu^2) + 2m_4^2 H' - 2H(m_4^2)'}{am_1^2 + 4H'}\right), \quad \text{(A3)}$$

which, from (6.14) and (6.15) is positive. Thus, the product of the roots of the polynomial is positive. Besides, the term proportional to Δ in the denominator reads

$$-12\left(\frac{a(m_4^4 + 2H^2 m_\mu^2) + 2m_4^2 H' - 2H(m_4^2)'}{9a^2(am_1^2 + 4H')}\right) - 12a^2\left(\frac{am_4^2(m_1^2 - m_4^2) + m_1^2 H'}{am_1^2 + 4H'}\right), \quad \text{(A4)}$$

which from (6.14) and (6.15) is negative definite for $m_1^2 \geq 0$. This finally means that in the case $m_1^2 \geq 0$ (required for stability of the vector sector), both roots are positive and instabilities in the kinetic term are absent at any scale provided that the inequalities (6.14) and (6.15) are satisfied.

Concerning the potential term, the analysis is more involved. The expansion in Δ provides a useful tool to analyze the absence of instabilities at any momentum scale. The potential term $\mathcal{M}$ can be written as (6.18). The absence of gradient instabilities in the ultraviolet and infrared regimes, equivalent to the positivity of $\mathcal{M}$ in these regimes, requires

$$\begin{aligned} &\text{at large momenta } m_2^2 > m_3^2 \\ &\text{at small momenta } m_2^2 > 0. \end{aligned} \quad \text{(A5)}$$

At intermediate scales, the instabilities are *Jeanslike*. The potential $\mathcal{L}_V$ is quartic in Δ which does not allow to find its zero exactly. Nevertheless, imposing that it is free from instabilities at high energy scales and at zero momentum we know that it will be positive definite at any scale provided that its minima in the interval $\Delta \in (-\infty, 0]$ are below zero. These minima can be exactly localized as they corresponds to the solutions of

$$c + 2d\Delta + 3e\Delta^2 + 4(m_2^2 - m_3^2)\Delta^3 = 0. \quad \text{(A6)}$$

From the fact that we have at much two minima localized in the interval $\Delta \in (-\infty, 0]$, and yet some extra freedom in the choice of the mass functions, we expect to find a large class of Lagrangians with a well-defined potential (see [2]).

APPENDIX B: GRAVITATIONAL GREEN'S FUNCTIONS

Once the Newtonian potentials are worked out in momentum space, they can easily be Fourier-transformed to the physical position r-space ($r = |\vec{x}|$). The potentials Φ and Ψ found in this work are always of the kind

$$\Phi = \frac{\text{Polynomial}(\Delta^{n} + \cdots)}{\text{Polynomial}(\Delta^{n+2} + \cdots)} = \sum_i \frac{Z_i}{(\Delta - M_i^2)^i} \quad \text{(B1)}$$

where Z_i and M_i are functions of the background and mass parameters. Once the fraction has been decomposed in poles, we can use the following correspondence to directly read the r dependence:

$$\frac{1}{\Delta} \to \frac{1}{r}, \qquad \frac{1}{\Delta^2} \to r, \qquad \frac{1}{\Delta - m^2} \to \frac{e^{-mr}}{r}, \qquad \frac{1}{(\Delta - m^2)^2} \to \frac{e^{-mr}}{m}, \quad \text{etc}, \quad \text{(B2)}$$

for suitable choices of integration constants.

[1] V. A. Rubakov and P. G. Tinyakov, Phys. Usp. **51**, 759 (2008).
[2] D. Blas, arXiv:0809.3744.
[3] H. van Dam and M. J. Veltman, Nucl. Phys. **B22**, 397 (1970); V. I. Zakharov, JETP Lett. **12**, 312 (1970); Y. Iwasaki, Phys. Rev. D **2**, 2255 (1970).
[4] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (N.Y.) **305**, 96 (2003).
[5] C. Deffayet, G. R. Dvali, G. Gabadadze, and A. I. Vainshtein, Phys. Rev. D **65**, 044026 (2002).
[6] G. Dvali, New J. Phys. **8**, 326 (2006).
[7] V. A. Rubakov, arXiv:0407104.
[8] S. L. Dubovsky, J. High Energy Phys. 10 (2004) 076.
[9] M. Porrati, Phys. Lett. B **498**, 92 (2001); A. Karch, E. Katz, and L. Randall, J. High Energy Phys. 12 (2001) 016; I. I. Kogan, S. Mouslopoulos, and A. Papazoglou, Phys. Lett. B **503**, 173 (2001); G. Gabadadze, A. Iglesias, and Y. Shang, arXiv:0809.2996.
[10] P. Creminelli, A. Nicolis, M. Papucci, and E. Trincherini, J. High Energy Phys. 09 (2005) 003.
[11] V. P. Nair, S. Randjbar-Daemi, and V. Rubakov, arXiv:0811.3781.
[12] P. Horava, Phys. Rev. D **79**, 084008 (2009).
[13] S. L. Dubovsky, P. G. Tinyakov, and I. I. Tkachev, Phys. Rev. Lett. **94**, 181102 (2005).
[14] N. Arkani-Hamed, H. C. Cheng, M. A. Luty, and S. Mukohyama, J. High Energy Phys. 05 (2004) 074.
[15] M. V. Bebronne and P. G. Tinyakov, Phys. Rev. D **76**, 084011 (2007).
[16] T. Damour and I. I. Kogan, Phys. Rev. D **66**, 104024 (2002).
[17] D. Blas, C. Deffayet, and J. Garriga, Phys. Rev. D **76**, 104036 (2007).
[18] Z. Berezhiani, D. Comelli, F. Nesti, and L. Pilo, Phys. Rev.

Lett. **99**, 131101 (2007).
[19] N. Rossi, Eur. Phys. J. Special Topics **163**, 291 (2008); Z. Berezhiani, L. Pilo, and N. Rossi, J. High Energy Phys. 07 (2009) 083; Z. Berezhiani, F. Nesti, L. Pilo, and N. Rossi, arXiv:0902.0144; M. Banados, A. Gomberoff, D. C. Rodrigues, and C. Skordis, Phys. Rev. D **79**, 063515 (2009).
[20] Z. Berezhiani, D. Comelli, F. Nesti, and L. Pilo, J. High Energy Phys. 07 (2008) 130.
[21] C. Deffayet, Classical Quantum Gravity **25**, 154007 (2008).
[22] P. Koroteev and M. Libanov, Phys. Rev. D **79**, 045023 (2009).
[23] M. V. Libanov and V. A. Rubakov, J. High Energy Phys. 08 (2005) 001.
[24] S. L. Dubovsky, P. G. Tinyakov, and I. I. Tkachev, Phys. Rev. D **72**, 084011 (2005).
[25] L. Grisa, J. High Energy Phys. 11 (2008) 023; S. Mukohyama, J. Cosmol. Astropart. Phys. 10 (2006) 011; T. Damour, I. I. Kogan, and A. Papazoglou, Phys. Rev. D **66**, 104025 (2002).
[26] D. Blas, D. Comelli, F. Nesti, and L. Pilo (unpublished).
[27] L. H. Ford and H. Van Dam, Nucl. Phys. **B169**, 126 (1980).
[28] J. M. Cline, S. Jeon, and G. D. Moore, Phys. Rev. D **70**, 043543 (2004).
[29] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, J. High Energy Phys. 03 (2006) 025.
[30] M. V. Libanov, V. A. Rubakov, O. S. Sazhina, and M. V. Sazhin, J. Exp. Theor. Phys. **108**, 226 (2009).
[31] L. F. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).
[32] S. Deser and A. Waldron, Phys. Lett. B **508**, 347 (2001).
[33] J. Polchinski, arXiv:9210046.
[34] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[35] G. Dvali, O. Pujolas, and M. Redi, Phys. Rev. Lett. **101**, 171303 (2008).
[36] M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).
[37] A. Higuchi, Nucl. Phys. **B282**, 397 (1987).
[38] S. Deser and R. I. Nepomechie, Ann. Phys. (N.Y.) **154**, 396 (1984).
[39] K. G. Arun and C. M. Will, Classical Quantum Gravity **26**, 155002 (2009); M. Pshirkov, A. Tuntsov, and K. A. Postnov, Phys. Rev. Lett. **101**, 261101 (2008).
[40] D. Bessada and O. D. Miranda, Classical Quantum Gravity **26**, 045005 (2009).
[41] E. Babichev, C. Deffayet, and R. Ziour, J. High Energy Phys. 05 (2009) 098.
[42] M. V. Bebronne and P. G. Tinyakov, J. High Energy Phys. 04 (2009) 100.