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Spontaneous Lorentz violation, negative energy, and the second law of thermodynamics

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We reconsider the possibility of violating the generalized second law of thermodynamics in theories with spontaneous Lorentz violation. It has been proposed that this may be accomplished in particular with a black hole immersed in a ghost condensate background, which may be taken to break Lorentz invariance without appreciably distorting the space-time geometry. In this paper we show that there in fact exist solutions explicitly describing the flow of negative energy into these black holes, allowing for violation of the second law in a very simple way. This second law violation is independent of any additional assumptions such as couplings of the ghost condensate to secondary fields, and suggests that violation of the null energy condition may be the true source of pathology in these theories.

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I. INTRODUCTION

In [1], Dubovsky and Sibiryakov discussed a relationship between theories with spontaneous violation of Lorentz invariance, and violation of the generalized second law of thermodynamics (GSL) [2] in the presence of black holes. That paper continues an intriguing line of inquiry in which one attempts to place bounds on possibilities for low energy physics, coming from what appear to be fundamental principles of quantum gravity. Other recent results in this direction have included an upper bound on particle masses in theories with large numbers of species [3,4], and a possible upper limit to the permitted size of the fine structure constant [5,6]. See also [7–12].

The specific setup discussed in [1] involved a "ghost condensate" background surrounding a Schwarzschild black hole. The ghost condensate is defined by a scalar field with an expectation value for its kinetic term $\langle\nabla_\mu\varphi\nabla^\mu\varphi\rangle > 0$, singling out a specific timelike direction for a breaking of Lorentz symmetry [13–17]. With some assumed couplings of the ghost condensate to secondary fields, it then becomes possible to construct a perpetual motion machine around the black hole, violating the second law. In this paper, we will reexamine this setup, the details of which will be reviewed in Sec. II. We will note that, in order to conclude that the GSL is actually violated, one must carefully examine the energy flowing into the black hole in the ghost condensate background under consideration. Although naively this energy flow vanishes, we will point out that even seemingly small corrections to this statement, from various possible sources, could ruin the purported GSL violation and must be checked carefully.

With this motivation, in Sec. III we will turn to an examination of possible ghost condensate flows into black holes. Using a perturbative approach, we will confirm the numerical result of [18] that, contrary to the expectation from fluid dynamics, there is a one parameter family of static spherically symmetric ghost condensate flows, rather than a unique one. We will discuss the range of parameters for which these flow solutions may be trusted, including possible effects from higher derivative operators.

In Sec. IV, we will calculate the energy-momentum tensor for these solutions, and show that they describe regions of either positive *or* negative energy flowing into the black holes. The negative energy solutions allow one to violate the GSL in a very simple way, independent of the couplings to secondary fields. We will also comment on the time scale for an instability in these flows, and argue that they should be sufficiently long-lived to easily allow violation of the GSL.

We will conclude in Sec. V, including a discussion of whether it is really violation of the null energy condition, rather than Lorentz invariance, which causes the difficulties with thermodynamics in these theories.

II. GENERALIZED SECOND LAW VIOLATION

The ghost condensate may be defined by a Lagrangian taking a form such as

$$\mathcal{L} = \frac{1}{2\Lambda^4}(X - \Lambda^4)^2, \qquad (1)$$

where $X \equiv \nabla_\mu\varphi\nabla^\mu\varphi$. The energy-momentum tensor is

$$T_{\mu\nu} = \frac{2}{\Lambda^4}(X - \Lambda^4)\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2\Lambda^4}(X - \Lambda^4)^2 g_{\mu\nu}, \qquad (2)$$

and the equation of motion is

$$\nabla_\mu\left(\frac{2}{\Lambda^4}(X - \Lambda^4)\nabla^\mu\varphi\right) = 0. \qquad (3)$$

It follows that there are zero-energy solutions for which $X = \Lambda^4$, and which therefore violate Lorentz invariance spontaneously. In flat space these solutions may be taken to have the form $\varphi = \Lambda^2 t$, with t identifying the timelike direction singled out for a breaking of Lorentz symmetry.

It is important to note that these $X = \Lambda^4$ zero-energy solutions exist not only in flat space-time, but in general curved backgrounds as well. In particular, we may set up such a configuration in a black hole background without affecting the underlying geometry [19]. The net effect is that we may construct an ordinary looking black hole, except that at each point there is a preferred direction as chosen by the ghost condensate field.

We write the Schwarzschild black hole geometry in the form

$$ds^2 = \left(1 - \frac{2M}{r}\right)d\tau^2 - 2\sqrt{\frac{2M}{r}}d\tau dr - dr^2 - r^2 d\Omega^2. \quad (4)$$

Here M is the mass of the black hole, and we have chosen natural units such that the Planck scale is $M_{\rm pl} = 1$. r is the usual radial coordinate of Schwarzschild space-time, such that spheres at radius r have area $4\pi r^2$, and $d\Omega^2$ is the standard metric on the unit sphere. The coordinate τ measures the proper time as seen by observers who freely fall into the black hole from rest at infinity, with surfaces of constant τ being orthogonal to the world lines of such observers. τ is related to the usual killing time t of Schwarzschild space-time via the relation

$$\tau = t + 2M\left(2\sqrt{\frac{r}{2M}} + \log\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right)\right), \quad (5)$$

in terms of which the metric takes the standard form

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{1}{1 - \frac{2M}{r}}dr^2 - r^2 d\Omega^2. \quad (6)$$

Recall that t labels the time-symmetry direction of the space-time, and that it becomes singular at the horizon ($r_h = 2M$), unlike τ.

In terms of τ, the zero-energy ghost condensate solution in Schwarzschild space-time with $X = \Lambda^4$ takes an extremely simple form:

$$\varphi_0 = \Lambda^2 \tau. \quad (7)$$

Indeed, the direction chosen by φ in this solution is simply that of freely falling observers, with $\nabla^\mu \varphi_0$ following their geodesics.

The argument in [1] now proceeds as follows: Since the ghost condensate picks out a special time-direction, it is simple to couple it to a secondary field, ψ, in such a way that the particles of ψ obtain a maximum speed which is different from the speed of light. We might imagine, for example, a coupling of the form

$$\mathcal{L} \supset \frac{1}{2}\nabla_\mu \psi \nabla^\mu \psi + \frac{\varepsilon}{2}\left(\frac{\nabla^\mu \varphi}{\Lambda^2}\nabla_\mu \psi\right)^2, \quad (8)$$

which gives the ψ particles a propagation speed of $v = \frac{1}{\sqrt{1+\varepsilon}}$. The key point is that this non-Lorentz invariant speed implies that the ψ particles have a horizon at a different radius from the usual horizon, and in turn a temperature different from the usual black hole temperature. The horizon radius and temperature for ψ are given by

$$r_\psi = r_h(1 + \varepsilon) \quad (9)$$

and

$$T_\psi = v^3 T, \quad (10)$$

where T is the standard black hole temperature $T = 1/8\pi M$. As might have been expected, particles with a slower speed have a larger horizon, and smaller temperature.

Particles without such a coupling to the ghost condensate of course still have the usual horizon radius and temperature, and in this way we obtain a black hole with rather peculiar thermodynamic properties; in particular, it is immediately unclear which temperature one should think of as the "real" temperature (if any), or which horizon area one should associate with the black hole entropy (if any). Indeed, the authors of [1] went on to show that one may violate the second law of thermodynamics in this setup with an appropriate set of hot shells surrounding the black hole. In particular, suppose we have two particle species 1 and 2 with speeds $v_1 < v_2$ and temperatures $T_1 < T_2$. Now imagine placing two shells around the black hole with temperatures T_A and T_B satisfying

$$T_1 < T_A < T_B < T_2. \quad (11)$$

Moreover, let us assume that shell A only interacts with particles of type 1, and shell B only interacts with particles of type 2. If one chooses the temperatures appropriately then we may satisfy the inequalities (11), while simultaneously satisfying the condition that the total energy flowing into the black hole is equal to zero. The net result of this setup for the type 1 particles is then a flow of energy *into* the black hole *from* the low temperature shell, while for the type 2 particles it is a flow of energy *out* of the black hole *onto* the high temperature shell. Since no total energy is being lost by the shells, this "machine" then has the net effect of transferring energy from the cold shell to the hot shell, lowering the entropy of matter outside the black hole without increasing the black hole area. It is in this way that this setup manages to violate the second law of thermodynamics.[1]

Now, a question one might immediately ask is: How can it be that backgrounds which violate Lorentz invariance allow for the existence of perpetual motion machines? In particular, why is it not possible to violate the second law of thermodynamics by using a regular fluid such as water flowing into a black hole? After all, water picks out a specific frame, and the phonons in water have a maximum speed which differs from the speed of light. One could imagine setting up the same sort of device as outlined

[1] In [20] the authors construct a process to violate the second law in this theory via a purely classical method, not involving Hawking radiation.

above, but with the slow particles replaced by the phonons in water. The reason such a device would fail to violate the GSL of course is simple: While the machine runs, there is a net flux of energy in water molecules flowing into the black hole, and this energy causes the horizon area to grow at a rate which is sufficient to overcome any purported GSL violation.

The point is that it is the special property of the ghost condensate of carrying essentially no energy-momentum which results in the difficulties with the laws of thermodynamics. For this reason, it is important to check carefully the claim that one can maintain the condition $T_{\mu\nu} = 0$ in the ghost condensate background. In particular, any perturbation to the equation of motion (3) has the potential to result in some small but non-zero-energy flow during the running of the perpetual motion machine, and could negate any GSL violation.

The first such perturbation one might worry about is that of higher derivative operators. In fact if one expands the ghost condensate in small perturbations π about the $X = \Lambda^4$ background, the equation of motion (3) becomes at linear order

$$\nabla_\nu \xi^\nu \xi^\mu \nabla_\mu \pi + \xi^\mu \xi^\nu \nabla_\mu \nabla_\nu \pi = 0, \tag{12}$$

with $\xi^\mu \equiv \nabla^\mu \varphi_0 / \Lambda^2$ being the normalized gradient of the background. In flat space-time this wave equation takes the form

$$\ddot{\pi} = 0. \tag{13}$$

It is therefore imperative for stability of these fluctuations that one adds to the original ghost condensate Lagrangian some higher derivative operators [13] such as for example[2]

$$\mathcal{L} \supset -\frac{2\alpha}{\Lambda^2} \Box\varphi \Box\varphi. \tag{14}$$

At energies much less than Λ, such a term modifies the π equation of motion to[3]

$$\ddot{\pi} + \frac{\alpha}{\Lambda^2} \vec{\nabla}^4 \pi = 0. \tag{15}$$

It is therefore important to analyze the effect of higher derivative operators on the energy-momentum carried by the ghost condensate into black holes.

A second potentially problematic perturbation to the ghost condensate flow comes from the couplings (8) being used to alter the speeds of the type 1 and type 2 particles. The nonzero fluxes of these particles, a key component in the construction of [1]'s perpetual motion machine, will lead to a perturbation to the ghost condensate solution, and any resulting energy flux could be important.[4]

It should be clear, therefore, that a more thorough examination of the energy flow in the ghost condensate during the running of the perpetual motion machine of [1] is crucial to determine whether or not the GSL is actually violated. For this reason, we will turn to consider the nature of such flows in more detail in the following section. There we will find that the perpetual motion machines can in fact be made to work, but that the reasons for this are closely related to the existence of a set of negative energy ghost condensate flows. These may then be used to violate the generalized second law in a very simple way.

III. GHOST CONDENSATE FLOWS

Note that we have been making an important implicit assumption in our discussion of ghost condensate flows thus far; that the stationary rate of energy flow in the ghost condensate into a black hole is actually a uniquely defined quantity. This assumption stems from an important result in fluid dynamics: Given a perfect fluid surrounding a black hole, and a particular asymptotic density, there exists a unique stationary spherically symmetric flow of the fluid into the hole (see [21] and references therein). Let us briefly review the reason for this:

Consider a perfect fluid with a density ρ, pressure P, four velocity u^μ, and "baryon" number density n. The speed of sound, which we will take to be subluminal for simplicity, is given by $a = \sqrt{\frac{dP}{d\rho}}$, and the energy-momentum tensor is

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu - g_{\mu\nu} P. \tag{16}$$

Assuming the black hole is much heavier than the energy of fluid flowing into it (so that we may ignore backreaction effects), the key equations describing the stationary, spherically symmetric flow of the fluid are

$$\text{baryon conservation:}\ \nabla_\mu (n u^\mu) = 0 \Rightarrow \partial_r (n u^r r^2) = 0. \tag{17}$$

$$\text{adiabaticity:}\ d\left(\frac{\rho}{n}\right) + P d\left(\frac{1}{n}\right) = 0 \Rightarrow \frac{d\rho}{dn} = \frac{\rho + P}{n}. \tag{18}$$

[2]Of course, such terms are automatically expected to be present in our effective field theory since no symmetry forbids them. We will assume α to be roughly of order 1.

[3]This form for the low energy π equation of motion is independent of the details of the structure of the higher derivative terms. This is because it follows from (15) that time derivatives of π are generally suppressed compared to its spatial derivatives, and so (15) automatically includes the dominant modification to (13).

[4]The setup of Ref. [20] presumably does not avoid this concern; the same issue could arise in their scenario if one imagines trying to violate the GSL with a continuous flux of particles undergoing the classical process they describe. Even if one takes their background to be given by, for example, Einstein-Aether theory, the presence of a flux of particles coupled to this background could result in it being slightly shifted. Such a shift might lead to a sufficient flow of positive energy into the black hole to save the GSL.

energy conservation: $\nabla_\mu T^\mu_t = 0 \Rightarrow \partial_r((\rho + P)u^r u_t r^2) = 0.$ (19)

Using these relations, as well as the fact that $u_\mu u^\mu = 1$, we may solve for the radial derivative of $u \equiv |u^r|$, obtaining

$$\frac{du}{dr} = \frac{N}{D}, \tag{20}$$

where

$$N = (1 - 2M/r + u^2)\frac{2a^2}{r} - \frac{M}{r^2}, \tag{21}$$

and

$$D = \frac{u^2 - (1 - 2M/r + u^2)a^2}{u}. \tag{22}$$

Since we assume the fluid to be at rest far from the black hole, we know that at large r, u approaches zero. In particular, the flow at large r becomes subsonic with $u^2 \ll a^2$, which in turn implies that $D < 0$. Close to the horizon on the other hand, inspection of (22) reveals that $D > 0$. We may therefore conclude that at *some* point outside the horizon, D passes identically through zero. In fact, it turns out that at this location the flow velocity as measured by a stationary observer is exactly passing through the speed of sound—it is the location of the sound horizon. In order for the flow to successfully pass through the sound horizon without becoming singular, it is clear that Eq. (20) requires that, at this radius, $N = 0$. This extra condition, required to ensure smoothness of the flow at the sound horizon, now provides us with enough information to solve for the flow exactly.

Naively, we have a very good reason to believe that this argument should apply equally well to the case of the ghost condensate. In fact, the ghost condensate may in a precise sense be thought of as a degenerate case of a perfect fluid [14]: Suppose we consider any Lagrangian of the form

$$\mathcal{L} = P(X), \tag{23}$$

with $X = \nabla_\mu\varphi\nabla^\mu\varphi$ as before. The ghost condensate is of course described by a special case of such a Lagrangian.[5] The energy-momentum tensor associated with (23) is then

$$T_{\mu\nu} = 2P'(X)\nabla_\mu\varphi\nabla_\nu\varphi - g_{\mu\nu}P(X). \tag{24}$$

The key point is then that this system describes precisely a perfect fluid, so long as we make a set of identifications:

$$P(X) \to P \qquad \nabla^\mu\varphi/\sqrt{X} \to u^\mu$$
$$2P'(X)X - P(X) \to \rho \qquad \frac{P'(X)}{2P''(X)X + P'(X)} \to a^2. \tag{25}$$

Additionally, n is taken proportional to $P'(X)\sqrt{X}$ so that the equation of motion for φ becomes the statement of baryon number conservation.

We might thus expect that ghost condensate flows into black holes should be unique, exactly as in the perfect fluid case. On the other hand, a closer inspection of the argument reveals at least a conceivable loophole: The speed of sound in the ghost condensate, in the absence of higher derivative terms, is equal to zero (since $P'(X) = 0$). Thus a ghost condensate flow, at the level of the perfect fluid picture, never actually passes through a sound horizon. In fact, as we will now show, thisloophole does indeed allow for the existence of a family of stationary, spherically symmetric ghost condensate flows.

Let us look for well-behaved solutions to the ghost condensate equation of motion in the Schwarzschild background which are small perturbations to the zero-energy solution (7). Again we will look for solutions which are stationary and spherically symmetric, and we will ignore backreaction of the flow on the background metric. Stationarity requires that the components of $\nabla^\mu\varphi$, in Schwarzschild coordinates (6), should be independent of the time t. This in turn requires that φ must take the form $\varphi = At + f(r)$, for some constant A and function f. Since we want the physics of the solution to approach that of the usual flat space configuration $\varphi = \Lambda^2 t$ at large distances from the black hole, we must continue to take $A = \Lambda^2$. The general perturbation of interest to the zero-energy solution (7) thus takes the form

$$\varphi = \Lambda^2\tau + \pi(r). \tag{26}$$

Plugging this into the equation of motion (12), we obtain an extremely simple differential equation

$$\frac{2M}{r}\pi''(r) + \frac{2M}{r^2}\pi'(r) = 0, \tag{27}$$

with a simple set of solutions

$$\pi(r) = C_1\log(r) + C_2. \tag{28}$$

The constant solution C_2 was to be expected, since our theory has a shift symmetry $\pi \to \pi + c$, but the existence of the log solutions is surprising; they are perfectly well-behaved through the event horizon out to infinity, and as such represent a one parameter family of stationary, spherically symmetric ghost condensate flows. Note that the fact that $\log(r)$ blows up as $r \to \infty$ is irrelevant, as the shift

[5] In fact, the ghost condensate is generally defined by a point X_0 such that $X_0 > 0$, $P'(X_0) = 0$, and $P''(X_0) > 0$. The specific form of the Lagrangian in Eq. (1) was taken only for simplicity.

symmetry ensures that only derivatives of π are physical. Indeed, let us check for what values of C_1 (28) actually represents a small perturbation about the zero-energy solution (7). Expanding X to linear order in the perturbation, we obtain

$$X \simeq \Lambda^4 - 2C_1 \frac{\Lambda^2}{r}\sqrt{\frac{2M}{r}}. \tag{29}$$

As long as we are only interested in values of r of order the horizon size or larger, we conclude that our perturbative solutions are valid if we take

$$|C_1| \ll \frac{\Lambda^2}{T}. \tag{30}$$

Another thing we should check is that these flow solutions are not significantly affected by the presence of higher derivative terms in the Lagrangian such as (14). In fact, as discussed in Sec. II, it is known that higher derivative terms have a small effect on the original zero-energy solution (7), suppressed by the small quantity T^2/Λ^2. More quantitatively, the leading order perturbation λ induced by these terms is a solution to the linear Eq. (27), but with a source given by the appropriate higher derivative term evaluated on the zeroth order background. With the term from (14) we then have

$$\frac{2M}{r}\lambda''(r) + \frac{2M}{r^2}\lambda'(r) = -\frac{\alpha}{\Lambda^2}\Box^2\varphi_0$$
$$= -\alpha\frac{9(r-6M)}{4\sqrt{2}r^4}\sqrt{\frac{M}{r}}. \tag{31}$$

Setting a boundary condition at infinity so that λ does not affect the radial energy flux, we obtain the solution

$$\lambda = -\alpha\frac{3(3r-2M)}{2r^2\sqrt{2M/r}}. \tag{32}$$

Comparing with the size of the perturbation defined by our flow solution (28), we may conclude that the perturbation coming from the higher derivative term is a subdominant effect on scales of horizon size or larger so long as we take

$$|C_1| \gg T. \tag{33}$$

Note that since the flows (28) with nonzero C_1 are perfectly well-behaved perturbations to the zeroth order background (7), they themselves change the effect of the higher derivative terms by an amount which is small so long as (30) is satisfied. In combination, the inequalities (30) and (33) thus leave us with a large range of sizes for these flows for which we may trust the form of our perturbative solutions, while ignoring the effects of higher derivative operators.

Before we move on to study the ghost condensate black hole flows in more detail, we would like to point out an interesting observation concerning the general perfect fluid flows discussed earlier. Specifically, suppose we have a Lagrangian of the type (23), with a nonzero speed of sound at large distances, and suppose we add to it some small perturbation. The perturbation could be, for example, a higher derivative operator, or a coupling to some secondary field, etc. The uniquely defined flow into a black hole will of course be changed by some small amount. In particular, it turns out that in the expression (20) for $\frac{du}{dr}$, the denominator "D" is unaffected, while the numerator "N" is shifted by a term small in the perturbation added to the action. It thus follows that the argument for uniqueness of the flow goes through unaffected- $D = 0$ still implies $N = 0$ in order to avoid singularities. On the other hand, the parameters determining the flow, as set by the $N = 0$ requirement, are changed by an amount given by the size of the perturbation as evaluated at the sound horizon. If the ghost condensate black hole flows were unique, this argument would tell us how they would become affected by small perturbations to the action such as those considered in Sec. II. Of course, since the uniqueness argument actually breaks down for the case of the ghost condensate, this point is moot.

In the next section, we will discuss the form of the energy-momentum tensor for the family (28) of ghost condensate black hole flows. We will show that for appropriate C_1 they carry negative energy, and we will also make some comments concerning their stability.

IV. ENERGY AND STABILITY

Expanding the energy-momentum tensor (2) to linear order about the $X = \Lambda^4$ background, we obtain

$$T^{\mu\nu} \simeq 4\Lambda^2\xi^\mu\xi^\nu\xi^\alpha\nabla_\alpha\pi. \tag{34}$$

With the specific form of the solutions (28) this yields

$$T^r_t \simeq \frac{8C_1 M\Lambda^2}{r^2} \tag{35}$$

and[6]

$$T^t_t \simeq -\frac{4C_1}{r}\sqrt{\frac{2M}{r}}\frac{\Lambda^2}{1-2M/r}. \tag{36}$$

We thus find that, depending on the sign of C_1, these flow solutions describe either regions of positive energy with positive energy flux into the black hole, or regions of *negative energy* with a corresponding *negative energy flux*. Such flows therefore allow one to trivially violate the generalized second law of thermodynamics in this

[6]Note that the singularity in T^t_t at the horizon is not physical, but due to the Schwarzschild coordinate singularity. Freely falling observers measure a perfectly finite energy at the horizon.

theory; they cause the horizon area to decrease without any corresponding increase in the total entropy of the exterior.[7] This demonstrates, in particular, that the ghost condensate is incompatible with the GSL independent of the size or nature of couplings to secondary fields. It had been argued, for example, that GSL violation might be avoided if direct couplings between the ghost condensate and other fields were induced only gravitationally [23]. Moreover, this shows that perpetual motion machines such as those of [1,20] may indeed be constructed; there will certainly be flows for which the ghost condensate energy accretion rate is sufficiently small.

Note that, if we take these ghost condensate solutions literally at extremely large distances from the black hole, then our approximation of ignoring the backreaction on the metric will become a bad one: T_t^t from (36) falls off at infinity like $1/r^{3/2}$, and thus the total integrated energy is not finite.[8] Of course, any stationary fluid flow solution into a black hole with a nonzero asymptotic density will have this same problem. The point is that one should think of these flows as being approximately stationary for a very long period of time, before the (appropriately finite) total energy available has been exhausted. This time period can be made arbitrarily longer than the Schwarzschild time scale by raising $M_{\rm pl}$ (for fixed r_h), thereby increasing the mass of the black hole and reducing the backreaction.

Another important issue for these negative energy flows is that of stability. Recall that in flat space, ghost condensate configurations with $X = \Lambda^4$ have perturbations which lack a $\vec{\nabla}^2\pi$ term in their wave equations. An important point though, is that if we consider configurations with $X < \Lambda^4$ instead, say $X = \Lambda^4 - \Delta$, a $\vec{\nabla}^2\pi$ term does appear, but with the wrong sign:

$$\left(1 - \frac{3\Delta}{2\Lambda^4}\right)\ddot{\pi} = -\frac{\Delta}{2\Lambda^4}\vec{\nabla}^2\pi - \frac{\alpha}{\Lambda^2}\vec{\nabla}^4\pi. \qquad (37)$$

Such background configurations are of course unstable, and in fact correspond to the negative energy region of the theory. In addition, it follows that there exists a *nonlinear* instability in the original $X = \Lambda^4$ background [14]; a short wavelength mode living on top of a longer wavelength mode with X values temporarily in the negative energy region will indeed display the instability.

It thus seems clear that the negative energy flows we have constructed will be unstable. Although analytic solutions for the growing modes have not been forthcoming, we can still attempt to make a rough estimate for the time scale of their growth. Note that for a given value of Δ, higher derivative operators put an upper limit on the wave number for an instability. In particular, in order for the $\vec{\nabla}^2\pi$ term in (37) to dominate over the stabilizing higher derivative term, we require the wave number k to be smaller than roughly $\sqrt{\Delta}/\Lambda$. The corresponding time scale for the growth of the mode, from (37), is then of order Λ^3/Δ.

In the ghost condensate flow solutions we have been considering, ΔX outside the horizon has a maximum magnitude of roughly $C_1\Lambda^2 T$, corresponding to a decay time scale of order $\Lambda/C_1 T$. Given the energy flux from Eq. (35), the total energy which may be transferred in this time is of order Λ^3/T^2. The corresponding change in the black hole area in Planck units is then of order Λ^3/T^3, and thus one may easily violate the generalized second law before the instability of these flows manifests itself.[9]

In fact, the nature of the instability of negative energy regions in the ghost condensate was studied numerically in [14]. Those authors found that negative energy lumps in the ghost condensate tend to shrink in size (rather than grow catastrophically), while maintaining a fixed total amount of negative energy. The fact that the tendency of these negative energy lumps is to coalesce suggests the nature of the instability in the black hole flows will not be a particularly remarkable one; presumably, the flows will very gradually accelerate, until eventually all of the negative energy around the black hole has been absorbed.

V. DISCUSSION

In this paper we have demonstrated the existence of a one parameter family of stationary, spherically symmetric solutions describing flow of ghost condensate fluid into black holes. These flows may carry either positive or

[7] We are assuming that the ghost condensate flows are accompanied by positive fluxes of entropy into the black hole. If the UV completion of the theory were such that large outward entropy fluxes were associated with these flows, then saving the generalized second law might become possible. This could conceivably occur, for example, if the UV completion of the theory contained particles with faster than light propagation. Note that Hawking radiation generates entropy in the exterior at a rate which is far too small to be relevant given the size of the negative energy flows under consideration (c.f. (30) and (35)). Hawking radiation of ghost condensate quanta themselves has been shown to be especially suppressed [22].

[8] This explains why these solutions were not identified in [19], since in that treatment the metric was assumed to be asymptotically flat.

[9] To be more precise, we could take $C_1 \gg \Lambda$ for simplicity, so that the evolution of the unstable modes takes place on scales much smaller than r_h. We could then roughly trust the form of the flat space wave-equation, and $\Lambda/C_1 T$ would be the approximate instability time scale as seen by a freely falling observer at rest relative to the ghost condensate. If we set up the initial conditions for the flow on a well-behaved spacelike hypersurface such as $\tau = 0$, for example, then the instability will become important at τ's of order $\Lambda/C_1 T$. At a fixed radius r, it then follows from (5) that the killing time t available is also given roughly by $\Lambda/C_1 T$, so that the total energy transferred before the flow becomes affected by the instability will be approximately Λ^3/T^2 as claimed. Flows with smaller values of C_1 may be used to violate the GSL on longer time scales, although precise general relativistic values for the lifetimes then become more difficult to estimate.

negative energy and may thus be used to violate the generalized second law of thermodynamics.

One point that is slightly puzzling about the existence of these flows is related to what happens if one perturbs slightly away from a pure ghost condensate solution at large distances from the black hole. In particular, suppose that as $r \to \infty$, X approaches $\Lambda^4(1 + \delta)$, so that the field approaches a configuration slightly different than the ghost condensate one at $X = \Lambda^4$. In this case the standard argument of Sec. III for uniqueness of perfect fluid flows into black holes applies, and thus the one parameter family of solutions must disappear. Presumably what is happening is that for nonzero δ, there is indeed a unique stationary flow, but also there is a family of flows with some small time dependence set by the size of δ. As $\delta \to 0$, the time dependence goes away, and we recover the complete family of stationary ghost condensate flows. It would certainly be interesting to see this explicitly, though this is beyond the scope of the present paper.

The fact that it is possible to violate the GSL through negative energy flow in the ghost condensate suggests that it is really violation of the null energy condition (NEC) which is the source of the problems this theory causes for thermodynamics. In particular, as we have discussed, the perpetual motion machines of [1,20] hinge fundamentally on the requirement that the background violating Lorentz invariance also carries no energy-momentum. Generically, the Lorentz violating background field can then be expected to have perturbations which contribute to $T_{\mu\nu}$ at linear order as in (34), leading directly to violations of the NEC.

It thus seems likely that it is not so much Lorentz violation which leads to potential problems with thermodynamics, as implied by the perpetual motion machines of [1,20], but simply the existence of negative energy configurations. Of course, the test of this hypothesis comes in trying to find a Lorentz violating theory for which the NEC is satisfied, but for which one can successfully set up a perpetual motion machine.

The first thing one might try in this direction would be to continue to consider a general Lagrangian of the form $\mathcal{L} = P(X)$ as in Sec. III. One could then demand the requirements that not only is there a point X_0 with $P'(X_0) = 0$ as in the ghost condensate (to allow a background with zero-energy), but that also $P''(X_0) = 0$ in order to remove the linear term (34) in $T_{\mu\nu}$ coming from fluctuations about X_0. The problem here, though, is that the action for these perturbations then begins at cubic order, and the theory then becomes completely nonperturbative. As a result, conclusions drawn in such a scenario could not be trusted.

Another interesting example to consider is that of Einstein-Aether theory [24–28]. This theory possesses a vector field with a timelike expectation value, and is known to have choices of parameters for which energies are apparently non-negative [28]. There are also parameter values for which black hole solutions have been studied [27], and it would be very interesting to investigate whether one can construct a perpetual motion machine in this theory without sacrificing the positive energy requirements. A natural conjecture would be that it will not in fact be possible. To check this one would have to study, in particular, whether one could maintain a sufficiently small energy flux in the Einstein-Aether fields into the black hole in the presence of the "slow" and "fast" particles being used to run the perpetual motion machine.

The fact that the pure ghost condensate theory is incompatible with the generalized second law of thermodynamics fits in nicely with the result of [22], that the Hawking spectrum of ghost condensate perturbations is highly suppressed and nonthermal. The obvious suggestion would be that the ghost condensate cannot emerge in a low energy effective theory coming from a quantum theory of gravity.[10]

The ghost condensate has been proposed as a concrete model for both dark energy and dark matter, as well as inflation. It is quite interesting that a seemingly consistent model describing low energy phenomena far below the Planck scale could turn out to be unacceptable due to the behavior of black holes as required by fundamental principles of gravity.

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[10] Of course it is always possible that quantum gravity will not ultimately behave as we expect.

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