Analysis of the Sultana-Dyer cosmological black hole solution of the Einstein equations

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The Sultana-Dyer solution of general relativity representing a black hole embedded in a special cosmological background is analyzed. We find an expanding (weak) spacetime singularity instead of the reported conformal Killing horizon, which is covered by an expanding black hole apparent horizon (internal to a cosmological apparent horizon) for most of the history of the Universe. This singularity was naked early on. The global structure of the solution is studied as well.

I. INTRODUCTION

Two main areas of gravitational physics, cosmology and the study of black holes, come together in the attempt to find exact solutions of the Einstein equations describing black holes, which are asymptotically Friedmann-Lemaitre-Robertson-Walker (FLRW) instead of asymptotically flat. With the exception of static solutions, such as the Schwarzschild-de Sitter black hole or its generalizations, these exact solutions describe dynamical black holes. What is meant exactly by ''black hole'' when the metric is non-stationary and the usual teleological event horizon is not present is a nontrivial question that has led to extensive research on dynamical horizons and their mechanics and thermodynamics (e.g., [1–4] and references therein). The well-known black hole thermodynamics [5] hinges on Hawking's discovery of thermal radiation from black holes [6], and this calculation relies on neglecting the backreaction of radiation on the spacetime metric. The full treatment of backreaction and a full understanding of time-dependent dynamical horizons are still in the future. It is interesting, therefore, to find and study exact solutions of the Einstein field equations describing black holes, by which we mean central singularities covered by an *apparent* horizon—these can be used as test beds for various theoretical characterizations of surface gravity, energy fluxes, etc. Only a few such solutions are known. A complication arising when a black hole is somehow embedded in a cosmological background other than the (anti-)de Sitter one sourced by a cosmological constant, is that the cosmic fluid tends to accrete onto the central black hole. Forbidding this accretion flow results in a rather artificial condition and in the old McVittie solution [7].

There are other motivations for pursuing exact cosmological black hole spacetimes: one is the problem of the effect of the cosmological expansion on local dynamics ([8] and references therein), which generated the McVittie solution [7]. It has been realized [9] that participation in the

expansion of the cosmic substratum may be more difficult to achieve for strongly than for weakly bound local systems; hence, one wants to look at the most strongly bound local system, the black hole. This approach has led to the solutions of Sultana and Dyer [10] and those of [11–13]. Recent interest in phantom dark energy and its properties (including thermodynamics) [14,15] motivated the study of the backreaction due to the accretion of cosmological phantom energy onto black holes and the possibility that phantom energy may violate cosmic censorship [16]. Exact solutions describing black holes in a cosmological fluid may also lead to toy models for evaporating black holes. Finally, alternative gravitational theories such as $f(R)$ gravity have received much attention recently as possible models of the present acceleration of the Universe without postulating exotic dark energy ([17], see [18] for a review and [19] for shorter introductions to this subject). $f(R)$ gravity and all theories of modified gravity introduced for this purpose produce an effective time-varying cosmological constant and spherically symmetric solutions in these theories are not likely to be asymptotically flat but rather asymptotically FLRW. Black holes will not be Schwarzschild-like but dynamical: in fact, the Jebsen-Birkhoff theorem is not valid in these theories and spherically symmetric solutions will, in general, be nonstationary (see, for example, the dynamical solution proposed in [20] in the context of metric $f(R) = R^{1+\delta}$ gravity). The interest in cosmological black holes is not confined to this class of alternative theories (which are, anyway, special cases of scalar-tensor gravity [18,19]): interest has come from the possibility that inhomogeneities lead to local variations of the effective gravitational constant in scalar-tensor cosmologies [21]. Exact cosmological and time-dependent black holes are of interest also in higher-dimensional Gauss-Bonnet gravity [22] and arise from intersecting branes in supergravity [23].

In spite of all these motivations, only a handful of exact solutions describing dynamical black holes embedded in cosmological backgrounds are known, and even fewer are properly understood. Here, we analyze the Sultana-Dyer

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solution [10], which reserves a few surprises, and discuss its singularities, apparent horizons, global structure, and physical interpretation. We adopt the notations of Ref. [5].

II. THE SULTANA-DYER SOLUTION

The Sultana-Dyer solution [10] is a metric of Petrov type D interpreted by its discoverers as a black hole embedded in a spatially flat FLRW universe with scale factor $a(t) \propto t^{2/3}$ (where t is the comoving time). It is generated by conformally transforming the Schwarzschild metric $g_{ab}^{(S)} \to g_{ab} = \Omega^2 g_{ab}^{(S)}$ with the intention of mapping the Schwarzschild timelike Killing field ξ^c into a conformal Killing field (for $\xi^c \nabla_c \Omega \neq 0$), and the Schwarzschild event horizon into a conformal Killing horizon. The choice $\Omega = a(t) = a_0 t^{2/3}$ (the scale factor of a dust-filled $k = 0$ FLRW universe) generates the metric in the form given by Sultana and Dyer [10]

$$ds^2 = a^2(\eta)\left[-\left(1 - \frac{2m}{\tilde{r}}\right)d\eta^2 + \frac{4m}{\tilde{r}}d\eta d\tilde{r} \right.$$
$$\left. + \left(1 + \frac{2m}{\tilde{r}}\right)d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right], \qquad (2.1)$$

where m is a constant, $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit 2-sphere, $a(\eta) = a_0\eta^2 = a_0 t^{2/3}$. We use the metric signature opposite to that of [10] and the symbol η and t for conformal and comoving time, respectively, with $dt = ad\eta$ (these notations are switched with respect to those of [10] but follow standard practice in cosmology).

The Sultana-Dyer metric (2.1) is obtained by conformally transforming the Schwarzschild metric written in Painlevé-Gullstrand coordinates. Of course, the line element (2.1) can be cast in a form explicitly conformal to Schwarzschild written in the more familiar Schwarzschild coordinates. By introducing the new time coordinate $\bar{t}$ defined by

$$dt = d\bar{t} + \frac{2mad\tilde{r}}{\tilde{r}(1 - \frac{2m}{\tilde{r}})}, \qquad (2.2)$$

the line element (2.1) transforms to

$$ds^2 = -\left(1 - \frac{2m}{\tilde{r}}\right)d\bar{t}^2 + \frac{a^2 d\tilde{r}^2}{1 - \frac{2m}{\tilde{r}}} + a^2\tilde{r}^2 d\Omega^2$$
$$= a^2\left[-\left(1 - \frac{2m}{\tilde{r}}\right)d\bar{\eta}^2 + \frac{d\tilde{r}^2}{1 - \frac{2m}{\tilde{r}}} + \tilde{r}^2 d\Omega^2\right], \quad (2.3)$$

which is manifestly conformal to Schwarzschild with conformal factor a and $d\bar{t} = ad\bar{\eta}$. It is obvious that it reduces to a spatially flat FLRW metric as $r \to +\infty$.

By using the isotropic radius r defined by

$$\tilde{r} = r\left(1 + \frac{m}{2r}\right)^2, \qquad (2.4)$$

the line element (2.3) becomes

$$ds^2 = -\frac{(1 - \frac{m}{2r})^2}{(1 + \frac{m}{2r})^2}d\bar{t}^2 + a^2\left(1 + \frac{m}{2r}\right)^4(dr^2 + r^2 d\Omega^2)$$
$$(2.5)$$

in isotropic coordinates $(\bar{t}, r, \theta, \varphi)$. The metric (2.5) is recognized to be formally the same as the McVittie solution [7] but with the important difference that the metric coefficient m is now a constant, contrary to the McVittie case in which $dm/md\bar{t} = -da/ad\bar{t}$. We will refer to this equation as the ''McVittie no accretion condition.'' The latter stems from the explicit requirement that there is no accretion of cosmic fluid onto the central object. In modern language, this has been recognized as the constancy of the Hawking-Hayward quasilocal mass $M \equiv m(\bar{t})a(\bar{t})$ [24]. Since the McVittie no accretion condition is violated by the Sultana-Dyer solution, the latter describes an accreting object, as Sultana and Dyer recognize from inspection of the field equations and the fact that $T_0^1 \neq 0$.

A second important difference between the Sultana-Dyer and the McVittie metric is that the latter is sourced by a single perfect fluid, while the material source of the former is a mixture of two noninteracting perfect fluids, a null dust, and a massive dust [10]. The stress-energy tensor is

$$T_{ab} = T_{ab}^{(I)} + T_{ab}^{(II)}, \qquad (2.6)$$

where $T_{ab}^{(I)} = \rho u_a u_b$ describes an ordinary massive dust, and $T_{ab}^{(II)} = \rho_n k_a k_b$ describes a null dust with density ρ_n and $k^c k_c = 0$ [10]. The use of two fluids instead of one follows from the fact that once accretion is allowed and the energy density and pressure depend on the radial, as well as the time, coordinate, solutions sourced by a single perfect fluid do not exist, except for the Schwarzschild-de Sitter metric (which is a special case of the McVittie solution) [12,16].

We will make use of the quantity

$$M(\bar{t}) \equiv ma(\bar{t}), \qquad (2.7)$$

which is not constant in the Sultana-Dyer solution. These authors identify a conformal Killing horizon, the locus $\tilde{r} = 2m$ (or $r = m/2$ in terms of the isotropic radius r) [10]. This is obtained by mapping the Schwarzschild event horizon of the seed metric (a Killing horizon) into the 2-surface $\tilde{r} = 2m$. Therefore, Sultana and Dyer interpret their metric as describing a black hole embedded in a special (spatially flat) FLRW background universe. The authors identify certain problems with this solution: the accretion flow onto the black hole becomes superluminal, and the energy density of the cosmic fluid becomes negative after a certain time near the conformal Killing horizon at $\tilde{r} = 2m$. They proceed to study radial null geodesics and surface gravity on this surface, and the behavior of timelike geodesics representing, e.g., planetary orbits, in this metric. This is inspired by the old problem of general relativity

of whether the cosmic expansion affects local systems (see [8] for a recent review).

Here, we take the study of the Sultana-Dyer solution a step further. We find that it is quite different from the picture provided by these authors: in particular, we show that the locus $\tilde{r} = 2m$ is not a conformal Killing horizon but a spacetime singularity. It is a weak singularity in the sense of Tipler's classification [25]. If the current interpretation of the McVittie metric [24] applies to the Sultana-Dyer solution, the two disconnected regions $\tilde{r} > 2m$ and $0 < \tilde{r} < 2m$ correspond to a black hole and white hole region, respectively. The spacetime singularity $\tilde{r} = 2m$ is covered by an apparent horizon, which we locate together with a cosmological horizon. This alleviates the problems of negative energy density and superluminal flux near $\tilde{r} = 2m$. The singularity expands with time, comoving with the cosmic substratum but the horizon expands at a slightly smaller rate and, in the infinite future, it approaches the singularity asymptotically. The $\tilde{r} > 2m$ Sultana-Dyer spacetime has also a big bang spacelike singularity. In the early Universe, near the big bang, the $\tilde{r} = 2m$ singularity was naked, and got covered by an apparent horizon only later. All these features resemble certain characteristics of the Fonarev solution [26,27] or of generalized McVittie solutions [12,13,16].

III. SINGULARITIES AND APPARENT HORIZONS OF THE SULTANA-DYER SOLUTION

Let us begin by examining the locus $\tilde{r} = 2m$: this is not a conformal Killing horizon as stated in [10], but a spacetime singularity. In fact, the Ricci curvature is

$$R^a{}_a = \frac{12}{a^2(\bar{\eta})(1 - \frac{2m}{\tilde{r}})} = \frac{6(\partial_{\bar{t}}\bar{H} + 2\bar{H}^2)}{1 - \frac{2m}{\tilde{r}}} \qquad (3.1)$$

(where $\bar{H} = d\ln a/d\bar{t}$), and it diverges as $r \to m/2$ (the discrepancy with the Sultana-Dyer paper is discussed in the Appendix A). This is a covariant statement and not an artifact of the coordinate system adopted because the Ricci scalar is an invariant of the Riemann tensor. $R^a{}_a$ also diverges as $a \to 0$, the usual big bang singularity of the background FLRW universe. The metric determinant is

$$g = -a^8(\bar{\eta})\tilde{r}^4 \sin^2\theta. \qquad (3.2)$$

g and the area radius $R \equiv a\tilde{r}$ do not tend to zero as $\tilde{r} \to 2m$; hence, an object does not get crushed to zero volume as it approaches this singularity, which is a weak one in Tipler's classification [25]. It will be shown below that this singularity is covered by an apparent horizon and therefore, as in the McVittie metric, the regions $\tilde{r} > 2m$ and $0 < \tilde{r} < 2m$ may be interpreted as describing a black hole and a white hole region [24]. Although this interpretation does not seem absolutely compelling to us in view of the fact that a physical object or particle could potentially cross a weak singularity, we do not address this issue here and

refer the reader to the comprehensive discussion of Nolan [24] and to the references therein.

The issue is now to determine whether the $\tilde{r} = 2m$ singularity is naked or covered by an apparent horizon and, therefore, whether it really describes a black hole embedded in a cosmological background. Since the metric is nonstationary, there is no event horizon and the appropriate horizon notion is that of apparent horizon [1]. We proceed to rewrite the Sultana-Dyer solution in the Nolan gauge, in which it will be straightforward to decide whether apparent horizons exist and, if so, to locate them.

Using the area radius $R \equiv a\tilde{r}$, which is a geometric quantity, Eq. (2.7), and the fact that

$$d\tilde{r} = \frac{dR}{a} - HRd\eta, \qquad (3.3)$$

where $H \equiv \dot{a}/a$ and an overdot denotes differentiation with respect to t, the line element (2.1) becomes

$$ds^2 = -\left(1 - \frac{2M}{R} - \frac{H^2R^2}{1 - \frac{2M}{R}}\right)dt^2 + \frac{dR^2}{1 - \frac{2M}{R}} - \frac{2HRdtdR}{1 - \frac{2M}{R}}$$
$$+ R^2 d\Omega^2 \qquad (3.4)$$

in coordinates (t, R, θ, φ), with the singularity located at $R = 2M$. Let us use $A \equiv 1 - 2M/R$ and the new time coordinate T defined by

$$dT = \frac{1}{F}\left(dt + \frac{HR}{A^2 - H^2R^2}dR\right), \qquad (3.5)$$

where $F(T(t, R), R)$ is an integrating factor satisfying

$$\frac{\partial}{\partial R}\left(\frac{1}{F}\right) = \frac{\partial}{\partial t}\left(\frac{HR}{F(A^2 - H^2R^2)}\right) \qquad (3.6)$$

to guarantee that dT is an exact differential. After straightforward manipulations the line element (3.4) is recast in the Nolan gauge as

$$ds^2 = -\left(1 - \frac{2M}{R} - \frac{H^2R^2}{1 - \frac{2M}{R}}\right)F^2dT^2 + \frac{dR^2}{1 - \frac{2M}{R} - \frac{H^2R^2}{1 - (2M/R)}}$$
$$+ R^2 d\Omega^2. \qquad (3.7)$$

The apparent horizons, if they exist, are the locus $g^{RR} = 0$, or

$$HR = \pm\left(1 - \frac{2M}{R}\right). \qquad (3.8)$$

We discard the lower sign in Eq. (3.8) which corresponds to a contracting Universe, and the apparent black hole and cosmological horizons are given by

$$R_{bh}(t) = \frac{1 - \sqrt{1 - 8MH}}{2H}, \qquad R_c(t) = \frac{1 + \sqrt{1 - 8MH}}{2H}, \qquad (3.9)$$

respectively. It must be $M \leq \frac{H^{-1}}{8}$, or $\dot{a} \leq \frac{1}{8m}$ for these apparent horizons to exist. In the limit $m \to 0$ the black

hole and its apparent horizon disappear and the cosmological apparent horizon reduces to the familiar surface $R = \frac{1}{H}$. The fact that the proper radius of the latter is larger than R_c can be interpreted as the pull of the central black hole on the cosmic fluid.

Since $R(1 - HR) = 2M > 0$ at the apparent horizons and $H > 0$ in an expanding Universe, it is

$$R_{bh} = \frac{2M}{1 - HR_{bh}} > 2M \qquad (3.10)$$

and the singularity $R = 2M(t) = 2ma(t)$, at which the flow is superluminal and the energy density becomes negative definite, is hidden by the apparent horizon. This alleviates somehow the problems of this solution reported in [10]. However, since $a(t) = a_0 t^{2/3}$ in comoving time, as $t \to +\infty$, $MH = m\dot{a} = \frac{2ma_0}{3t^{1/3}} \to 0$ and $R_{bh} \to 2M$ approaching the singularity asymptotically.

By using the scale factor $a(t) = a_0 t^{2/3}$ of the Sultana-Dyer metric it follows that the constant m must satisfy $m \leq \frac{3t^{1/3}}{16a_0}$ for the apparent horizons to exist. This condition is violated at early times, implying that the $R = 2M$ singularity was naked at early times and later on (at $t = (\frac{16a_0}{m})^3$) an apparent horizon appeared that immediately bifurcated into a cosmological horizon and a black hole apparent horizon covering the singularity.

The global structure of the Sultana-Dyer solution is analyzed in the next section. To conclude this section, we discuss the Misner-Sharp [28] and Hawking-Hayward [29,30] quasilocal energies.

The Misner-Sharp mass M_{MS} is defined in terms of the area radius R by $1 - \frac{2M_{MS}}{R} = -\nabla^c R \nabla_c R$ [28] which, at the black hole apparent horizon, yields

$$M_{MS} = \frac{R_{bh}}{2} = \frac{1 - \sqrt{1 - 8MH}}{4H}. \qquad (3.11)$$

In order to compute the Hawking-Hayward quasilocal mass we introduce the null coordinates (u, v) defined by

$$du = \frac{1}{\sqrt{2}}\left[\sqrt{1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}}} F dT \right.$$
$$\left. - \frac{dR}{\sqrt{1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - (2M/R)}}}\right], \qquad (3.12)$$

$$dv = \frac{1}{\sqrt{2}}\left[\sqrt{1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}}} F dT \right.$$
$$\left. + \frac{dR}{\sqrt{1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - (2M/R)}}}\right], \qquad (3.13)$$

in terms of which the metric is reduced to Hayward's standard form [30]

$$ds^2 = -2dudv + R^2 d\Omega^2. \qquad (3.14)$$

Since $dR = \sqrt{1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}}}\frac{(dv - du)}{\sqrt{2}}$, the Hawking-Hayward quasilocal mass is easily computed using the prescription for spherical symmetry [30]

$$M_{HH} = R(R_u R_v + \tfrac{1}{2}) \qquad (3.15)$$

which yields, at the black hole apparent horizon,

$$M_{HH} = M_{MS} = \frac{R_{bh}}{2} \qquad (3.16)$$

and coincides with the Misner-Sharp mass. Both masses diverge in the limit $R \to 2M$.

IV. GLOBAL STRUCTURE

The causal nature of the singularity and the black hole apparent horizon are determined as follows: The singularity is characterized by the equation $R - 2M = 0$, and the normal is obtained by taking the gradient of this equation (the limit $R \to 2M$ is implicit in the following). Unfortunately, the integrating factor F appearing in the line element (3.7) is not determined explicitly; hence, it is more convenient to use the coordinates (t, R, θ, φ) in which the metric is given by Eq. (3.4) and its inverse by

$$(g^{ab}) = \begin{pmatrix} \frac{-1}{1 - \frac{2M}{R}} & \frac{-HR}{1 - \frac{2M}{R}} & 0 & 0 \\ \frac{-HR}{1 - \frac{2M}{R}} & (1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}}) & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2 \sin^2\theta} \end{pmatrix}. \qquad (4.1)$$

One obtains $n_a = \nabla_a R - 2m\dot{a}\delta_{a0} = (-2MH, 1, 0, 0)$ and

$$n^a = g^{ab} n_b = \left(-HR, -H^2 R^2 + 1 - \frac{2M}{R}, 0, 0\right) \qquad (4.2)$$

and, in the limit $R - 2M \to 0$ $n_a = (-2MH, 1, 0, 0)$, $n^a = (-2MH, -4M^2H^2, 0, 0)$, while $n^c n_c = 0$. The $R = 2M$ singularity is null, which is not surprising if one remembers that it is obtained by conformal mapping of the null event horizon of the Schwarzschild black hole used as the seed metric in this solution-generating technique [10].

The black hole apparent horizon has equation $f(t, R) = 2RH - 1 + \sqrt{1 - 8MH} = 0$. The normal has the direction of the gradient $\nabla_a f$, or

$$n_a = (B(t, R), 2H, 0, 0), \qquad (4.3)$$

where

$$B(t, R) = 2R\dot{H} - \frac{4M(H^2 + \dot{H})}{\sqrt{1 - 8MH}}, \qquad (4.4)$$

while

$$n^a = g^{ab} n_b$$

$$= \left(-\frac{(B + 2H^2 R)}{1 - \frac{2M}{R}}, \frac{-HRB}{1 - \frac{2M}{R}} \right.$$

$$\left. + 2H\left(1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}}\right), 0, 0 \right). \qquad (4.5)$$

Therefore,

$$n^a n_a = -\frac{1}{1 - \frac{2M}{R_{bh}}} \left\{ \left[2\dot{H}R_{bh} - \frac{4M(H^2 + \dot{H})}{\sqrt{1 - 8MH}} \right]^2 \right.$$

$$\left. + 4H^2 R\left[2\dot{H}R_{bh} - \frac{4M(H^2 + \dot{H})}{\sqrt{1 - 8MH}} \right] \right\}$$

$$= -\frac{1}{1 - \frac{2M}{R_{bh}}} (C^2 + 4H^2 R)$$

$$= -\frac{1}{1 - \frac{2M}{R_{bh}}} [(C + 2H^2 R)^2 - 4H^4 R^2]$$

$$= \frac{|C|}{HR_{bh}} (C + 4H^2 R), \qquad (4.6)$$

where

$$C = \frac{\frac{-\dot{H}}{2}(1 - \sqrt{1 - 8MH})^2 - 4MH^3}{H\sqrt{1 - 8MH}}. \qquad (4.7)$$

Now,

$$C + 4H^2 R = \frac{1}{H\sqrt{1 - 8MH}} \left[\frac{-\dot{H}}{2}(1 - \sqrt{1 - 8MH})^2 \right.$$

$$\left. + 12MH^3 - 2H^2(1 - \sqrt{1 - 8MH}) \right], \qquad (4.8)$$

and at late times $H = \frac{2}{3t}$ and $\dot{H} = \frac{-2}{3t^2}$; hence, the numerator of Eq. (4.8) is dominated by the term $12MH^3$ and $n^a n_a > 0$: the black hole apparent horizon is timelike at late times. At early times, but after the horizons have appeared, it is

$$\frac{1}{H\sqrt{1 - 8MH}} \left[\frac{-\dot{H}}{2}(1 - \sqrt{1 - 8MH})^2 + 12MH^3 \right.$$

$$\left. - 2H^2(1 - \sqrt{1 - 8MH}) \right] \simeq \frac{12ma_0}{t^{7/3}} \frac{1}{H\sqrt{1 - 8MH}} > 0, \qquad (4.9)$$

and the horizon is timelike early on.

The global picture of the Sultana-Dyer spacetime that emerges from these considerations is that of a universe with a spacelike big bang singularity at $t = 0$, the null singularity $R = 2M$ (the conformal mapping of a Schwarzschild event horizon), which was naked at early times, until it got covered by a timelike apparent horizon appearing at a finite time. The conformal diagram is sketched in Fig. 1.
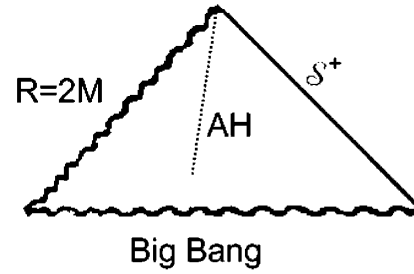

FIG. 1. The conformal diagram of the Sultana-Dyer spacetime. The horizontal wiggly line at the bottom describes the big bang singularity, the wiggly line at 45 degrees denotes the $R = 2M$ null singularity, and the solid line at 45 degrees describes future null infinity. Null geodesics end at future null infinity or at the black singularity (either when it is naked if started early on, or crossing the timelike black hole apparent horizon labeled AH).

V. DISCUSSION AND CONCLUSIONS

There are many motivations, discussed in the Introduction, to find and study exact solutions of the field equations of general relativity or alternative gravity theories describing dynamical black holes embedded in a cosmological background. Among these the Sultana-Dyer metric deserves some attention in view of its relative simplicity and of the technique used to generate it, which may lead to a wider class of solutions. There is evidence that among the class of generalized McVittie solutions [12,16] those conformal to the Schwarzschild spacetime ("comoving solutions") are generic [13], and this constitutes further motivation to understand solutions seeded by the Schwarzschild metric, such as the Sultana-Dyer one.

To summarize and discuss the salient features of the Sultana-Dyer spacetime, we found that the conformal image of the Schwarzschild event horizon is not a conformal Killing horizon but rather a spacetime singularity at $R = 2m$ (where R is the area radius). This singularity is weak in Tipler's sense and is null, as should be expected because the seed Schwarzschild event horizon is a null surface. This expanding singularity is interpreted as the effect of the pull of the cosmological matter stretching the $r = 0$ singularity of the Schwarzschild spacetime into one of finite radial extent. Early on, this singularity is naked and only later a timelike apparent horizon develops, which immediately bifurcates into a black hole apparent horizon covering the singularity and an apparent cosmological horizon. The radius of the cosmological horizon is smaller than the value H^{-1} of the Hubble radius in a FLRW spacetime, which may be interpreted as an effect due to the gravitational pull of the central object on the cosmic fluid.

The singularity expands comoving with the cosmic substratum, while the apparent horizon expands at a slightly smaller rate and eventually comes to coincide with the singularity in the infinite future. Sultana and Dyer report superluminal flow near the singularity, and one may question whether the notion of black hole makes sense at all in

the presence of superluminal flows which, in principle, allow particles to escape from the apparent horizon. However, it should be noted that this radial flow is always directed inward, and nothing actually escapes from the black hole apparent horizon. In our opinion, this unpleasant superluminal feature is due to the simplicity of the model under study and will not be present in more realistic and sophisticated models yet to come. For the moment we content ourselves with understanding the simple model (2.1). The Sultana-Dyer spacetime exhibits the spacelike big bang cosmological singularity, the null black hole singularity and, later on, the two apparent horizons. As is clear from the conformal diagram of Fig. 1, the timelike black hole apparent horizon meets the null singularity in the infinite future, with the dust-dominated universe expanding and diluting forever.

In addition to the superluminal flow, which we do not regard as a serious flaw, the Sultana-Dyer solution exhibits other less desirable features: the cosmological fluid has negative energy density at late times near $R = 2M$ [10]; it would be desirable if the cosmological matter were described by a single fluid composed of particles following timelike geodesics instead of an odd mixture of a null and a massive dust; and limiting the scale factor of the Universe to the special choice $a \propto t^{2/3}$ seems too restrictive. Work is in progress to find new exact solutions, both in general relativity and in alternative theories, which do not share these problems.

ACKNOWLEDGMENTS

APPENDIX

Equation (3.1) for the Ricci curvature shows a divergence at $\tilde{r} = 2m$, which is not noticed in the Sultana-Dyer paper. Here, we discuss the likely cause of this fact. The Ricci scalar coincides (up to the constant $\kappa = 8\pi G$) with the negative trace of the energy-momentum tensor T_{ab} and the matter energy density and (zero) pressure in the Sultana-Dyer paper are identified using their Einstein Eqs. (13) with the stress-energy tensor given by their Eq. (4). The latter is meant to provide the stress-energy tensor $\tilde{T}_{ab}$ obtained by conformally transforming the van-

ishing stress-energy tensor T_{ab} of the Schwarzschild solution, which we want to reconsider here. Under the conformal transformation of the metric $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$ the Ricci tensor and Ricci scalar transform according to

$$\tilde{R}_{ab} = R_{ab} - 2\nabla_a\nabla_b \ln\Omega - g_{ab}g^{ef}\nabla_e\nabla_f \ln\Omega$$
$$+ 2(\nabla_a \ln\Omega)(\nabla_b \ln\Omega) - 2g_{ab}g^{ef}(\nabla_e \ln\Omega)$$
$$\times (\nabla_f \ln\Omega), \tag{A1}$$

$$\tilde{R}^a_a = \Omega^{-2}\left(R^a_a - \frac{6\Box\Omega}{\Omega}\right), \tag{A2}$$

respectively [5]. Using the Einstein equations $\tilde{R}_{ab} - 1/2\tilde{g}_{ab}\tilde{R} = \kappa\tilde{T}_{ab}$ and $R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}$, one obtains

$$\kappa\tilde{T}_{ab} = \kappa T_{ab} - \frac{2\nabla_a\nabla_b\Omega}{\Omega^2} + \frac{4\nabla_a\Omega\nabla_b\Omega}{\Omega^2}$$
$$- g_{ab}\frac{g^{cd}\nabla_c\Omega\nabla_d\Omega}{\Omega^2} + 2g_{ab}\frac{g^{cd}\nabla_c\nabla_d\Omega}{\Omega}. \tag{A3}$$

Sultana and Dyer instead have, for vanishing T_{ab} (Eq. (4) of [10]),

$$\kappa\tilde{T}_{ab} = 2\tilde{g}_{ab}\frac{\tilde{\nabla}^2\Omega}{\Omega} - \frac{2\tilde{\nabla}_a\tilde{\nabla}_b\Omega}{\Omega} - \frac{3}{\Omega^2}\tilde{g}_{ab}\tilde{g}^{mn}\tilde{\nabla}_m\Omega\tilde{\nabla}_n\Omega. \tag{A4}$$

The discrepancy between these formulas, plus the fact that the tilded operator $\tilde{\nabla}$ is not the correct one to be used in the expression of $\tilde{T}_{ab}$, is likely to be the reason why the singularity at $\tilde{r} = 2m$ is missed in [10]. The correct Ricci scalar can be calculated using Eq. (A2) with $R^a_a = 0$ and $a = \bar{t}^{2/3} = \bar{\eta}^2$ yielding

$$\tilde{R}^a_a = -\frac{6\Box\Omega}{\Omega^3} = -\frac{6\Box(\bar{\eta}^2)}{\bar{\eta}^6} = 2g^{ab}\delta_{0a}\delta_{0b}$$
$$= \frac{12}{\bar{\eta}^6(1 - \frac{2m}{\tilde{r}})} = \frac{12}{a^3(1 - \frac{2m}{\tilde{r}})}. \tag{A5}$$

The singularity at $\tilde{r} = 2m$ was also noted recently in [31].

The Sultana-Dyer solution can be re-obtained using as material source a single imperfect fluid with a radial (spacelike) energy current instead of a two-fluid mixture, as shown in the first of Ref. [11]. Equation (90) of this work for $G^a_a = -R^a_a$ is clearly singular at $\tilde{r} = 2m$.

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