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Couplings between Chern-Simons gravities and $2p$-branes

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The interaction between Chern-Simons (CS) theories and localized external sources ($2p$-branes) is analyzed. This interaction generalizes the minimal coupling between a point charge (0-brane) and a gauge connection. The external currents that define the $2p$ branes are covariantly constant $(D-2p-1)$-forms coupled to $(2p-1)$ CS forms. The general expression for the sources—charged with respect to the corresponding gauge algebra—is presented, focusing on two special cases: 0-branes and $(D-3)$-branes. In any dimension, 0-branes are constructed as topological defects produced by a surface deficit of $(D-2)$-sphere in anti-de Sitter space, and they are not constant curvature spaces for $D>3$. They correspond to dimensionally continued black holes with negative mass. On the other hand, in the case of CS (super) gravities, the $(D-3)$-branes are naked conical singularities (topological defects) obtained by identification of points with a Killing vector. In $2+1$ dimensions, extremal spinning branes of this type are Bogomol'nyi-Prasad-Sommerfield states. Stable $(D-3)$-branes are shown to exist also in higher dimensions, as well. Classical field equations are also discussed, and in the presence of sources there is a large number of inequivalent and disconnected sectors in solution space.

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I. INTRODUCTION

Chern-Simons (CS) theories are a remarkable family of metric-free, background-independent, generally covariant gauge theories that extend the usual concept of minimal coupling between a current and the electromagnetic potential. Even in the simplest three-dimensional case where CS theory do not possess local degrees of freedom and is purely topological, it leads to a classification of three-dimensional manifolds [1], and gives an exactly solvable quantum theory of gravity [2]. Quantum CS theory describes the quantum Hall effect [3], and a CS term provides an alternative gauge-invariant procedure of mass generation [4]. If coupled to dynamical matter fields, CS theories exhibit spontaneous symmetry breaking and a Higgs mechanism that differs significantly from theories coupled to Yang-Mills fields [5].

There exist Chern-Simons formulations for gravity and supergravity in all odd dimensions [6–8] that are truly gauge theories with fiber bundle structure, which make them good candidates to tackle the quantization of gravity problem. On the other hand, CS theories have no adjustable couplings: all the constants in the Lagrangian are either particular combinatorial coefficients, or fixed by quantization. These are not coupling constants in the standard sense that could be used to define a perturbative quantum expansion. Moreover, the action is completely scale invariant, devoid of dimensionful coefficients or coupling constants that could get renormalized, and power-counting renormalizable.

In order to define the path integral formulation, it is necessary to couple the CS connection to external sources, and two natural options present themselves: i) to embed the symmetry as a subalgebra in a larger gauge algebra, as in CS supergravities, and ii) to add a minimal coupling of the form $\langle jA\rangle$, where j is covariantly constant $Dj=0$. The first option does not solve the problem, it just changes the setting to a larger gauge group. The second alternative seems acceptable but does not allow for more general couplings, for instance, to branes, which would be necessary to make contact and compare with the results in string theory.

Among the different branes, Bogomol'nyi-Prasad-Sommerfield (BPS) ones are extended objects that define natural localized vacua because they couple to the fields in a way that partially respects supersymmetry. This ensures their stability and makes them acceptable candidate ground states for the perturbative expansion of the theory expected to describe low energy phenomenology. Since the low energy limit of the gravitational sector of string theory should contain higher powers of curvature [9], here we will consider a particular case of Lovelock theories known as CS (super-)gravities, which have the added advantage of being genuine gauge theories with fiber bundle structure [8].

CS supergravities are interesting systems that offer a natural way to combine the gravitational field with other forms of matter and interactions under a unified scheme. However, coupling CS theories to branes in the same way

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that one does for standard supergravity as the low energy description of string theory, proves inconsistent. The problem stems from the clash between the fields that are expected to be turned on by the tensorial structure of the branes and the CS dynamics that requires some specific components of the connection to be present in order to ensure supersymmetry [10].

In standard supergravity the gravitino transforms as a covariantlike derivative of a spinor, including some curvature components as pieces of the connection. These extra terms allow nontrivial solutions if a chirality condition is fulfilled. On the other hand, in the CS theory the gravitino transforms as a pure covariant derivative of a spinor, which does not allow a similar solution. The conclusion is then that the naive minimal coupling between CS supergravity and branes cannot be consistent and still respect supersymmetry [10].

This negative conclusion could be seen as a no-go theorem or simply as one more indication of the fact that CS theories are exceptions to most of the standard rules of quantum field theory. In fact, the scope of this article is to show that a nonstandard coupling does exist between CS theories and that branes could respect supersymmetry. This claim is based on the observation that CS theories themselves have a structure that generalizes the minimal coupling between a gauge connection and a (point) particle to the case where the particle is replaced by an extended object.

In three-dimensional gravity with negative cosmological constant, it was shown that topological defects with angular deficit in anti-de Sitter (AdS) space corresponding to static or spinning 0-branes, represent pointlike external currents that couple in a gauge-invariant way to gravity. These topological defects are with negative energy states in the Bañados-Teitelboim-Zanelli (BTZ) black hole-like spectrum. Their energy range is between the AdS space ($M = -1$) and the zero-mass black hole, and if spinning, they become stable BPS 0-branes in CS supergravity for $M = -|J|$ [11] (see also [12] for a discussion related to exact solutions and thermodynamics of this kind of sources).

The presence of nondynamical external sources leads to explicit breaking of spacetime symmetries, as well as to spontaneous symmetry breaking, when the dynamical matter that couples to the gauge connection has nonzero vacuum expectation value. Since branes generically violate global (super)AdS symmetry, and, in particular, Lorentz and translational invariance, both bosonic and fermionic Goldstone modes are present in the theory [13]. Furthermore, since in the absence of sources the action is gauge invariant, external matter couplings can lead to a Higgs mechanism, and that the unbroken symmetries in non-Abelian case cannot be extended globally due to winding around a solution. Vortices and symmetry breaking in CS theories with matter have been discussed in ‘.

In what follows we focus on possible generalizations of the results in $2+1$ interactions, that is, gauge-invariant couplings between external p-brane sources and CS (super-)gravities, assuming that the sources are static and nondynamical.

II. CS THEORY AS GENERALIZED MINIMAL COUPLING

Given a connection A, a CS action in $2n+1$ dimensions is defined as

$$I_{\mathrm{CS},2n+1}[A] = \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(A) \rangle, \tag{1}$$

where M^{2n+1} is a $(2n+1)$-dimensional manifold, not necessarily endowed with a metric structure, the level κ is a dimensionless constant and

$$\langle \mathcal{C}_{2n+1}(A) \rangle = \frac{1}{n+1} \langle A(dA)^n + c_1 A^3 (dA)^{n-1} + \cdots + c_n A^{2n+1} \rangle. \tag{2}$$

Here, A stands for a 1-form with values in a certain Lie algebra G, $\langle \cdots \rangle$ denotes an invariant symmetric trace in the Lie algebra, and $c_1, \ldots, c_n$ are dimensionless rational coefficients uniquely determined by the condition that defines a Chern-Simons form,

$$d\langle \mathcal{C}_{2n+1}(A) \rangle = \frac{1}{n+1} \langle F^{n+1} \rangle, \tag{3}$$

where $F = dA + A^2$ is the curvature [6,17]. Exterior product is understood throughout, and the indices will be made explicit when needed.

It is clear from the construction that a CS action has no arbitrary constants apart from κ [18]. This is at the same time a virtue and a curse: the absence of free parameters means that quantization cannot renormalize the action (the beta function must vanish), lest gauge invariance is broken. On the other hand, it becomes extremely difficult to implement a working perturbative approximation to couple CS actions to matter sources, and it can be seen that the standard strategy that allows to couple supergravity to branes of various dimensions does not yield the desired results [10].

A. Review: Abelian *p*-branes

The interaction between gauge fields and matter is provided by the standard minimal coupling recipe

$$I^{\mathrm{EM}} = \int_{M^D} d^D x \, j^\mu(x) A_\mu(x), \tag{4}$$

where the $j^\mu(x)$ is the current produced by a point particle, charged with respect to the gauge group $U(1)$. The essential feature that selects (4) among all possible interaction terms is, apart from simplicity, gauge invariance. If the field A transforms as a connection, $A \to A + d\Lambda$, I^{EM}

remains invariant provided j^μ is localized in space and conserved, $\partial_\mu j^\mu = 0$. The current density of point charge $j^\mu(x) = q\dot{z}^\mu \delta^{(2n)}(x - z(\tau))$—where q is the magnitude of the electric charge and $z^\mu(\tau)$ represents its position along the worldline, parameterized by the affine parameter τ—, satisfies both requirements.

A point particle can also be viewed as a 0-brane whose time evolution is a one-dimensional manifold that supports the current: (4) is the integral of the 1-form A over the particle's history. The current j^μ is also the dual of a $(D-1)$-form $j_{[0]}$ that projects onto the worldline of the 0-brane. For a point source at rest at the origin and splitting spacetime between the worldline Γ^1 and the transverse space T^{D-1}, the source in (4) can be replaced by the $(D-1)$-form Dirac delta,

$$j_{[0]} = q_0 \delta(T^{D-1}) d\Omega^{D-1}, \tag{5}$$

where $d\Omega^{D-1}$ is the volume form in T^{D-1} (the Dirac delta $\delta(x)d^n x$ in R^n is naturally defined as an n-form that is ready to be integrated). Then, (4) could also be written as

$$I_{0\text{-brane}}[A, j] = \int_{M^D} j_{[0]} \wedge A = q_0 \int_{\Gamma^1} A = q_0 \int_{\Gamma^1} A_\mu dz^\mu, \tag{6}$$

where the conservation law $\partial_\mu j^\mu = 0$, is replaced by the closure of its dual, $dj_{[0]} = 0$. Comparing the second and third expressions in (6), it is obvious that the current acts by projecting the integral onto the worldline.

Similar couplings between gauge fields and higher-dimensional branes can also be considered. Attempts to couple gauge fields to branes of different dimensions were pioneered by Teitelboim [19], where (4) was generalized assuming extended sources represented by p-dimensional currents $j^{\mu_1\cdots\mu_p}$, coupled to p-form fields $A = A_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1}\cdots dx^{\mu_p}$. The field strengths (curvature $(p+1)$-form), $F = dA$, are invariant under Abelian transformations $A \to A + d\Lambda$, where Λ is a $(p-1)$-form. The direct extension of this idea to non-Abelian gauge fields, however, was shown to lead to inconsistencies [19,20]. Our approach circumvents those difficulties generalizing (6) in a different way: the minimal coupling can be regarded as a CS action in $0+1$ dimensions and, analogously, the worldvolume generated by a $2p$-brane can be seen as the action for a $2p+1$ CS form, Abelian or not [21].

The path-dependent but coordinate-independent expression (6) is the simplest example of CS action $I_{\text{CS},0+1}[A]$, obtained by setting $n = 0$ in (2) in the Abelian case. Then, an expression analogous to the right-hand side of (4) for a higher dimension can be taken as a $2p+1$ CS form with support on the worldvolume generated by the evolution of a $2p$-brane. Thus, the coupling of an Abelian CS connection in $D = 2n+1$ dimensions to a $2p$-brane can be similarly defined, with the source represented by a $2(n-p)$-form $j_{[2p]}$,

$$I_{2p\text{-brane}}[A, j] = \frac{1}{p+1}\int_{M^D} j_{[2p]} \underbrace{A dA \cdots dA}_{(2p+1)\text{-CS form}}. \tag{7}$$

This is an electromagnetically charged $2p$-brane coupled to a $(2p+1)$ Abelian CS form, where the current $j_{[2p]}$ is the $(2n-2p)$-form,

$$\begin{aligned} j_{[2p]} &= q_{2p}\delta(T^{2n-2p}) d\Omega^{2(n-p)} \\ &= q_{2p}\delta(x - z) dx^1 \wedge \cdots \wedge dx^{2n-2p}, \end{aligned} \tag{8}$$

with z labeling the points on the worldvolume of the $2p$-brane, $z \in \Gamma^{2p+1}$. The fact that the form $j_{[2p]}$ is closed implies that its dual, the current density $j^{\mu_1\cdots\mu_{2p+1}}$, is conserved, $\partial_\nu j^{\nu\mu_2\cdots\mu_{2p+1}} = 0$.

These sources are easily understood in two extreme cases, namely, $p = 0$ and $2p = D - 3$ $(p = n - 1)$. As discussed above, the first case describes a point singularity in the spatial section, whose worldvolume is a one-dimensional line; the second case is a brane whose worldvolume is a manifold of codimension 2 (conical defect). A few explicit examples of classical solutions for some sources are presented in Appendix A.

B. Non-Abelian generalization

The coupling between a non-Abelian gauge field A and a non-Abelian $(2n-2p)$-form source $j_{[2p]}$ that generalizes (7) is

$$I_{2p\text{-brane}}[A, j] = \kappa \int_{M^D} \langle j_{[2p]} \mathcal{C}_{2p+1}(A)\rangle. \tag{9}$$

The $2p$-brane source is given by

$$j_{[2p]}(x) = q_{2p}\delta(T^{D-1}) d\Omega^{D-1} G^{K_1\cdots K_{n-p}}, \tag{10}$$

where the indices $K_1, K_2, \cdots$ label the generators of the Lie algebra $\mathcal{G}$, and the operator $G^{K_1\cdots K_{n-p}}$ is a tensor in the corresponding representation. It is not guaranteed that the trace $\langle\cdots\rangle$ in (9) yields a nontrivial result; a matching between the Lie algebra, the invariant trace used and the specific operator G is required to produce interesting couplings.

The natural recipe to couple a gauge connection to a $2p$-brane is, take the algebra's invariant trace $\langle\cdots\rangle$ and consider any current of the form (10). In particular, to couple a CS theory in $2n+1$ dimensions to a $2p$-brane, we take

$$I_{2n+1}[A, j] = \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(A) - j_{[2p]} \mathcal{C}_{2p+1}(A)\rangle. \tag{11}$$

Here, we ignore boundary terms, which may be important in order to have a well-defined finite action principle and conserved charges [22] and in the quantum theory. The field equations obtained from the action (11) are

$$F^n = j_{[2p]} F^p. \tag{12}$$

Thus, off the worldvolume of the $2p$-brane, the gauge field is a solution of the source-free field equations $F^n = 0$, but on the worldvolume other options exist (see Appendix A).

Clearly, any value of p in the range $0 \leq p \leq (D-1)/2$ is allowed, and therefore one is naturally led to consider the most general coupling between a CS connection and all possible $2p$-branes,

$$I_{2n+1}[A, j] = \kappa \int_{M^{2n+1}} \sum_{p=0}^{n} (-1)^{n-p} \langle j_{[2p]} \mathcal{C}_{2p+1}(A) \rangle = \kappa \sum_{p=0}^{n} (-1)^{n-p} q_p \int_{\Gamma^{2p+1}} \langle \mathcal{C}_{2p+1}(A) \rangle, \quad (13)$$

where the alternating sign $(-1)^{n-p}$ is introduced to simplify the form of the field equations. Here, we have taken the simplest case of a static flat brane. A more realistic picture would include dynamically evolving branes that could also intersect, overlap, and even become embedded into each other.

Interaction of branes would require terms in (13) that combine CS densities with different p, that in general is not straightforward to construct as it would involve some generalization of transgression forms [23]. Intersecting branes with independent CS actions living on the different component bubbles were studied in [24]. This sort of "foam" can be viewed as a formal sum of CS forms integrated over a chain complex, in a system where the distinction between free theory and interaction terms is rather conventional. The fact that the coupling to branes enters on equal footing with the bulk action suggests a sort of "democracy" between brane worldvolumes and target space in CS theories [25].

C. Symmetry breaking

As mentioned above, off the brane the current is covariantly constant, $Dj \equiv dj + [A, j] = 0$ (that is, the dual current density is covariantly conserved, $D_\nu j^{\nu\mu_2\cdots\mu_{2p+1}} = 0$), which ensures gauge invariance of (9) from the point of view of an external observer living in M^D. On the brane the gauge invariance would be reduced to the subalgebra $\tilde{\mathcal{G}} \subset \mathcal{G}$ spanned by those generators that commute with $G^{K_1\cdots K_{n-p}}$, but this is not an issue for the dynamics off the brane's worldvolume.

On the other hand, the presence of the nondynamical source $j_{[2p]}$ as a fixed feature in the ambient space M^D does reduce the spacetime symmetries to those transformations that leave the source unchanged. This symmetry can be restored if the current is produced by some other particles or fields whose dynamics is included in the same action principle.

It is worthwhile noticing that the presence of dynamic matter coupled to a CS connection leads to spontaneous symmetry breaking when its vacuum expectation value is nonvanishing. If the broken symmetry is global, as for the brane that is noninvariant under spatial translations and Lorentz rotations, then massless Goldstone modes are present in the theory. In a supersymmetric extension of the theory, fermions may also contribute to the zero modes (see Ref. [13]).

On the other hand, since CS is a gauge theory, non-Abelian coupling to external matter may result in symmetry breaking of Schwarz type where the unbroken symmetries cannot be extended globally [26], or in a Higgs mechanism. Vortices are a class of stringlike solutions carrying magnetic flux confined in their center that arise in couplings, e.g., with scalar fields [27], and whose existence may make unbroken symmetries multivalued around the string; these are Alice strings [26,28] possessing nonlocalized electric charge (Cheshire charge), thus, related to $p(>0)$-branes. Vortices in CS theory are discussed, for example, in [14,15] and further analysis of Higgs mechanism in CS theories is analyzed in Ref. [16]. The issue of symmetry breaking is an open problem to be discussed in depth elsewhere.

III. COUPLING BRANES TO CS GRAVITY

CS gravities are theories where the gauge symmetry is the invariance group of the local tangents to the spacetime manifold; their supersymmetric extensions are CS supergravities [8]. The simplest of such theories occurs for $D = 2+1$, where it was observed that 0-branes corresponding to topological defects are naked singularities in the negative energy spectrum of the BTZ black hole [29]. It turns out that the extremal spinning 0-branes of this sort are BPS states. Here, we analyze the generalization of this picture to higher dimensions.

A. CS gravity

The cases in which the algebra $\mathcal{G}$ is either $so(D,1)$, $so(D-1,2)$ or $iso(D-1,1)$ represent an important class of CS theories that describe the dynamics of spacetime (gravitation) with positive, negative, or vanishing cosmological constant, respectively [6]. If the trace $\langle\cdots\rangle$ is given by the Levi-Civita tensor, these gravitation theories are particular cases of Lovelock theories, which take the form [30]

$$L = \sum_{p=o}^{[D/2]} \alpha_p \epsilon_{a_1 a_2 \cdots a_D} R^{a_1 a_2} R^{a_3 a_4} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}. \quad (14)$$

This Lagrangian describes the most general D-dimensional gravitation theory for the metric, if the spacetime is assumed to be Riemannian (torsion-free). For spacetimes of $D = 2n+1$ dimensions, the Lagrangian takes the CS form if the coefficients α_p are chosen as

$$\alpha_p = \frac{(\mp\ell^{-2})^p}{D-2p}\binom{n}{p},$$

where ℓ is the (A)dS radius. The cosmological constant is $\Lambda = \pm\ell^{-2}$ for the de Sitter (+), the anti-de Sitter (−), and $\Lambda = 0$ for the Poincaré (if $\alpha_p = \delta_p^n$) algebras, respectively, (see, e.g., [6–8]).

The general Lovelock Lagrangians are gauge theories for the Lorentz group $SO(D-1,1)$ and, in contrast with CS theories, they are not gauge theories for $SO(D-1,2)$, $SO(D,1)$, or $ISO(D-1,1)$, since the fields in the action are not the components of the connection for the respective algebras. Besides the connection $\omega^a{}_b$, Lovelock actions contain the vielbein e^a, which transforms in the vector representation of the $SO(D-1,1)$. The vielbein cannot decouple from the gauge connection, so Lovelock theories in general—and general relativity in particular—, are non-Abelian systems like QCD with e^a playing the role of quarks, but whose matter-free limit does not exist. Moreover, in the generic Lovelock theories the dimensionful parameters α_p can take arbitrary values and can get renormalized in the quantum theory because they are not protected by gauge invariance.

The functions $R^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$, e^a, and $\epsilon_{a_1a_2\cdots a_D}$ are Lorentz tensors, which makes all Lovelock theories—including Einstein gravity—invariant under local Lorentz transformations. CS theories enjoy an enhanced gauge symmetry that results from a particular choice of α_p. By virtue of this choice, e^a and $\omega^a{}_b$ can be combined into a single connection for the corresponding gauge group, all dimensionful constants can be absorbed in the fields, and the result is a fully scale-invariant gravitational action, where all the coupling constants are fixed rational numbers.

We shall mostly focus on Chern-Simons-AdS gravity, described by the action (11) for the AdS algebra $so(D-1,2)$, where the torsion tensor may enter the Lagrangian explicitly. The AdS generators are antisymmetric matrices J_{AB} $(A, B = 0, \ldots, D = 2n+1)$ acting on vectors of the local tangent to the spacetime manifold, an abstract covering space with the metric $\eta_{AB} = (-, +, \ldots, +, -)$. In this representation, the Lie algebra $so(D-1,2)$ reads

$$[J_{AB}, J_{CD}] = \eta_{AD}J_{BC} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC} + \eta_{BC}J_{AD}. \tag{15}$$

The symmetrized trace of the product of $n+1$ of these generators is given by the Levi-Civita invariant tensor

$$\langle J_{A_1B_1}\cdots J_{A_{n+1}B_{n+1}}\rangle = \epsilon_{A_1B_1\cdots A_{n+1}B_{n+1}}. \tag{16}$$

Using the decomposition of AdS indices $A = (a, 2n+1)$, $a = 0, 1, \ldots, 2n$, and calling $J_a := J_{a2n+1}$, the gauge field reads

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{\ell}e^aJ_a, \tag{17}$$

and the corresponding AdS curvature is

$$F = \frac{1}{2}\left(R^{ab} + \frac{1}{\ell^2}e^ae^b\right)J_{ab} + \frac{1}{\ell}T^aJ_a, \tag{18}$$

where $T^a \equiv De^a = de^a + \omega^a{}_be^b$ is the torsion 2-form. In case of CS supergravity, the gauge connection (17) contains the additional components along the supersymmetric extension of the AdS algebra, that contain fermionic generators, and may include also bosonic ones as required by the closure the superalgebra [31].

B. Gravitational sources

In the Abelian case, there are very few requirements for j to define an acceptable current source: it must have support on a $2p$-brane that generates a $(2p+1)$-dimensional submanifold of spacetime, and to be conserved in order to respect gauge invariance. In the non-Abelian case, since the charge is not invariant, but transforms irreducibly under the gauge group, the gauge invariance of the interacting theory may have fewer local symmetries than the source-free CS action.

Let us consider a static 0-brane sitting at the origin in a global AdS_{2n+1} spacetime. The invariance group of the brane worldline Γ^1 includes time translations, $SO(1,1)$, and spatial rotations, $SO(2n)$. The only operator in the AdS algebra $so(2n,2)$ that commutes with time translations—or rather AdS boosts in the time direction—, and spatial rotations is

$$G = \frac{1}{(2n)!}\epsilon^{0A_1B_1\cdots A_nB_n2n+1}J_{A_1B_1}\cdots J_{A_nB_n}, \tag{19}$$

$$= \frac{1}{(2n)!}\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}J_{\alpha_1\beta_1}\cdots J_{\alpha_n\beta_n}, \tag{20}$$

where the tangent space indices α, β, correspond to the tangent space of the spatial section of the manifold, so that $A = \{0, \alpha, 2n+1\}$. For example, in three dimensions, the $SO(2)$ generator is just the rotation $G = J^{12}$ in the 1-2 plane, whereas in five dimensions this is a spherically symmetric combination $G = (J^{12}J^{34} - J^{13}J^{24} + J^{14}J^{23})/3 = \epsilon_{\alpha\beta\gamma\rho}J^{\alpha\beta}J^{\gamma\rho}/24$. The fact that G commutes with all generators of spatial rotations, $[G, J_{\alpha\beta}] = 0$, follows directly from the identity

$$J_\beta^{\alpha_1}\epsilon^{\beta\alpha_2\cdots\alpha_{2n}} + \cdots + J_\beta^{\alpha_p}\epsilon^{\alpha_1\cdots\alpha_{p-1}\beta\alpha_{p+1}\cdots\alpha_{2n}} + \cdots + J_\beta^{\alpha_{2n}}\epsilon^{\alpha_1\cdots\alpha_{2n-1}\beta} = 0. \tag{21}$$

From (19), the symmetric invariant trace defining the interaction of the AdS connection and the 0-brane is

$$\langle GJ_{AB}\rangle = \delta_{[AB]}^{[0\,2n+1]}. \tag{22}$$

In this way, the introduction of a 0-brane breaks the symmetry from $SO(2n,2)$ down to $SO(1,1)\times SO(2n)$, with a net loss of $\Delta = 4n$ symmetry generators.

In the case of a $2p$-brane in AdS_{2n+1}, a worldvolume is a $(2p+1)$-dimensional timelike manifold Γ^{2p+1} that could have at most $SO(2p,2)$ symmetry on its tangent space. The $(2n-2p)$-dimensional spacelike transverse section can have a tangent space with local $SO(2n-2p)$ invariance. In this case, the reduction is from $SO(2n,2)$ down to $SO(2p,2)\times SO(2n-2p)$, with a loss of $\Delta = 4(n-p)\times(p+1)$ symmetry generators. The largest proper $2p$-brane that can be coupled this way is one with $p=n-1$, in which case the $\Delta = 4n$, as for the case with $p=0$.

This analysis also applies to cases where the gauge group for the CS action without external sources is de Sitter (dS), Poincaré, or a supersymmetric extension of Poincaré, but this point will not be analyzed further here.

C. 0-branes in $2n+1$ dimensions

The $2+1$ CS theory of gravity for the AdS group $SO(2,2)$ is the simplest analog of general relativity that is realistic enough to capture some of its essential features. Although this gravitational toy model has no Newtonian attraction, it gives rise to nontrivial black hole solutions that in many ways resemble astronomic black holes at many galactic nuclei [32]. For a comprehensive review, see [33].

In three dimensions, the only $2p$-brane that can be properly embedded is a point source in the two-dimensional spatial section, a 0-brane. The resulting spacetime is a negative energy naked singularity, generated by a topological defect in spacetime, similar to a string of angular deficit in a three-dimensional crystal [11]. For $D = 2n+1 > 3$, gravity with negative cosmological constant is described by a CS action for the $so(D-1,2)$ algebra, with connection (17). In this case, a pointlike source is a 0-brane describing a spherically symmetric topological defect, produced by a surface deficit on a $(D-2)$-dimensional sphere. This geometry is given by the classical solution

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{\ell^2}} + (1-\alpha)^2 r^2 d\Sigma_{D-2}^2, \tag{23}$$

where $d\Sigma_{D-2}^2$ is a metric of the unit $(D-2)$-sphere. For $\alpha = 0$ the metric reduces to that of the global AdS geometry, whereas for $0<\alpha<1$ there is a defect of magnitude $\alpha\Sigma(S^{D-2})$, where $\Sigma(S^{D-2})$ is the surface area of the whole sphere, and the metric exhibits a conical singularity at $r=0$. This metric includes the $2+1$ case where the topological defect is generated by an identification along a Killing vector in an Euclidean x^1-x^2-plane with a fixed point at $r=0$, producing a conical singularity in this plane. For $D>3$, the topological defect is not produced by a Killing vector identification and it changes the local geometry of spacetime and the curvature is not constant for $\alpha\neq 0$. It is straightforward to show that, for $r\neq 0$, the geometry has nonvanishing AdS curvature,

$$R^{pq} + \frac{1}{\ell^2}e^p e^q = -\frac{\alpha(\alpha-2)}{r^2(1-\alpha)^2}e^p e^q \quad \text{for } r\neq 0, \tag{24}$$

where the tangent space indices $p, q, \ldots$ correspond to the angular directions, so that the right-hand side in (24) vanishes for $D\leq 3$. The curvature is negative and not constant for $D>3$ and, for $r>0$, the Ricci scalar is

$$R = -\frac{D(D-1)}{\ell^2} - \frac{(D-2)(D-3)\alpha(\alpha-2)}{r^2(1-\alpha)^2}. \tag{25}$$

Changing coordinates as $(r,t) = (\frac{\rho}{1-\alpha}, (1-\alpha)\tau)$, the metric (23) becomes

$$ds^2 = -\left((1-\alpha)^2 + \frac{\rho^2}{\ell^2}\right)d\tau^2 + \frac{d\rho^2}{(1-\alpha)^2+\frac{\rho^2}{\ell^2}} + \rho^2 d\Sigma_{D-2}^2. \tag{26}$$

For $D=3$, this metric has the form of the $2+1$ black hole, but with negative mass, a naked singularity produced by a static 0-brane [11]. For $D = 2n+1\geq 5$, this solution describes dimensionally continued "black holes with negative mass" $(-1<M<0)$ [34]. The mass is related to the magnitude of the defect as

$$(1-\alpha)^2 = 1-(1+M)^{1/n}. \tag{27}$$

From (24) and (26) it is clear that for $\alpha = 0$ $(M=-1)$, the AdS space geometry is recovered. (For $\alpha = 2$ the geometry again has constant curvature, since the metric (23) possesses a discrete symmetry $(1-\alpha)\to -(1-\alpha)$, so that both $\alpha=0$ and $\alpha=2$ have no defect.) The surface deficit corresponds to α in the range $\alpha\in(0,1)\cup(1,2)$.

For $\alpha\neq 0$, the naked singularity is a static spherically symmetric configuration with mass in the range $-1<M<0$. For $M<-1$, the geometric interpretation becomes obscure as it would represent an angular sector greater than that of a full solid angle. It is unclear whether naked singularities of this type could exist at all.

In order to find the source generating this geometry, the AdS curvature for the metric (23) must be computed, regularizing it at $r=0$. If the regulated curvature is F_ϵ, the source is given by $j = \lim_{\epsilon\to 0} F_\epsilon^n$. The result is (see Appendix B)

$$\langle j(x) J_{AB}\rangle = q_0^{(D)}\delta^{(D-1)}(\vec{x})dx^1\cdots dx^{D-1}\delta_{[AB]}^{[0D]}, \tag{28}$$

where the "charge" $q_0^{(D)}$ in D dimensions is a polynomial of order $D-2$ in the constant α, given by

$$q_0^{(3)} = 2\pi\alpha, \tag{29}$$

$$q_0^{(5)} = \frac{4\pi}{3}\alpha^2(3-\alpha), \text{ etc.} \tag{30}$$

For the general form of the charge see Appendix B. Similar spherically symmetric angular defects can also be introduced in dS and in flat space, leading to geometries similar to (26), but with the metric functions $f_{\rm dS}^2 = \rho^2/\ell^2 - (1-\alpha)^2$ and $f_{\rm flat}^2 = \rho^2/\ell^2$, respectively. The dS case in 2 + 1 dimensions was discussed in [35].

The static 0-branes described above admit no globally defined, covariantly constant spinors and are therefore not BPS states—in absence of other fields that could couple to spinors—, except in the trivial case $M = 0$. In 3D, however, the source (28) of the form $j = 2\pi\alpha\delta^{(2)}(T_{12})dx^1dx^2J_{12}$ (here x^A, $A = 0, \ldots, 3$, are the coordinates in the embedding flat space $\mathbb{R}^{2,2}$ and J_{12} is the AdS_3 generator), can be extended to the full Cartan subalgebra of AdS_3 generated by J_{03} and J_{12}, allowing for the existence of two conserved charges, related to the mass and angular momentum M, J of the 0-brane. In the extreme case, $|M|\ell = |J|$, the current

$$j_{\rm ext} = 2\pi\alpha\delta^{(2)}(T_{12})dx^1dx^2(J_{03} - J_{12}) \tag{31}$$

leads to an extreme 0-brane produced by an identification with a Killing vector with fixed points at $r = 0$. As shown in [11], the extreme solution for this brane is a BPS solution admitting one globally defined Killing spinor, preserving 1/4 of supersymmetries of AdS, and which behaves asymptotically as the Killing spinor for zero-mass BTZ black hole [36].

D. Codimension 2 branes

In dimensions $D > 3$, it is possible to construct higher-dimensional $2p$-branes, for example, introducing an angular deficit in S^1 only, that leads to a geometry describing a (spinning and nonspinning) codimension 2 brane. The question arises whether those solution would be stable or not. In two examples in five dimensions, we show that stable BPS 2-branes exist.

1. Super CS in AdS space

Stable BPS solutions in five-dimensional CS-AdS supergravity were found in Ref. [37] for a supersymmetric extension of AdS_5 algebra $su(2,2|4)$. We show now that a CS gauge connection for this solution couples to a current of the type (10) that corresponds to a 2-brane. In this case, the field equations have the form $FF = jF$.

As shown in [37], the purely gravitational part of the solution is locally AdS (except at singularities) whose spatial boundary has topology isomorphic to the torus T^3. The bosonic sector of CS matter required by supersymmetry is given by $u(1) \times su(4)$ connection. Explicitly, the solution has the form

$$A = A_{\rm AdS} + bG_1 + a^{12}G_{12} + a^{34}G_{34},$$

where $A_{\rm AdS}$ is the AdS connection, and b and a^{IJ}, are $u(1)$ and $su(4)$ gauge fields, respectively. The Abelian field b describes currents that generate constant electric and magnetic fields whose strength has nonvanishing determinant. The generators G_{12} and G_{34} of $SU(4)$ commute, therefore the non-Abelian gauge field a^{IJ} breaks the symmetry to $U(1) \times U(1)$ and describes a soliton that winds up around one handle of the torus S^1 at spatial infinity. Its topological charge is associated to the arbitrary phase $d\theta = a^{12} - a^{34}$, which carries a nontrivial instanton number. The wrapping up around S^1 is obtained by the Killing vector identification and, just as in 3D case, produces a δ function in the transverse plane that has support in the three-dimensional worldvolume of a 2-brane.

If $dd\theta \sim \delta(T^2)$ (nontrivial winding), then $F \sim \delta(T^2)$, and consequently $j \sim \delta(T^2)$; thus, the current contains a combination of the generators G_1, G_{12}, and G_{34}. It is less obvious whether there exist components along some other AdS generators J_{AB} as well, since the phase θ is a general function of the local coordinates.

The asymptotic symmetries of the super CS-AdS_5 theory are described by the supersymmetric extension of WZW_4 algebra, and this particular solution corresponds to the ground state saturating the Bogomol'nyi bound.

2. 2-brane in AdS

It is possible to construct a solution of the CS equations with an angular defect in higher dimensions, in analogy with the case in 2 + 1 dimensions by making an identification of one angular coordinate ϕ. In this way, a brane with a worldvolume of codimension 2 is obtained. For simplicity, we focus again on the five-dimensional case. The AdS_5 space is given by the constraint $x \cdot x = -\ell^2$ in the embedding six-dimensional flat space given by the metric $ds^2 = \eta_{AB}dx^Adx^B$ with Lorentzian signature $(-++++-)$. Parameterizing the Euclidean planes (x^0-x^5), (x^1-x^2), and (x^3-x^4) in a form that explicitly reproduces the AdS_5 constraint, similarly to the three-dimensional expressions in Ref. [11],

$$\begin{aligned} x^0 &= A\cos\phi_{05}, \\ x^5 &= A\sin\phi_{05}, \\ x^1 &= B\sin\theta\cos\phi_{12}, \\ x^2 &= B\sin\theta\sin\phi_{12}, \\ x^3 &= B\cos\theta\cos\lambda, \\ x^4 &= B\cos\theta\sin\lambda, \end{aligned} \tag{32}$$

where $A^2 - B^2 = -\ell^2$. Note that this last condition implies that the origin of the $0-5$ plane in the embedding space, that is, $A = 0$, is not part of the AdS spacetime. Here, A and B are chosen as the following real functions of the radial coordinate:

$$A = \sqrt{\frac{\rho^2 + \ell^2a^2}{a^2 - b^2}}, \qquad B = \sqrt{\frac{\rho^2 + \ell^2b^2}{a^2 - b^2}}, \tag{33}$$

where a and b are allowed to take complex values but $a^2 \neq b^2$ (nonextremal case). In this parametrization, the metric takes the form

$$ds^2 = -dA^2 + dB^2 - A^2 d\phi_{05}^2 + B^2(d\theta^2 + \sin^2\theta d\phi_{12}^2 + \cos^2\theta d\lambda^2). \tag{34}$$

The lapse function can be read off directly from the radial part, $f^2(\rho) = a^2 + b^2 + \frac{\rho^2}{\ell^2} + a^2b^2\frac{\ell^2}{\rho^2}$, where $a^2 + b^2$ can be recognized as the mass parameter

$$a^2 + b^2 = -M.$$

For real a and b, M is negative (naked singularity) or zero, and the spin is given by $J/\ell = 2ab$. These are the same formal relations between (a, b) and (M, J) as for the 0-brane in $2+1$ dimensions. In order to compare with more standard forms, one can redefine and ϕ_{05}, ϕ_{12} as helicoidal coordinates given by

$$\phi_{05} = b\phi + \frac{a\tau}{\ell}, \quad \text{and} \quad \phi_{12} = a\phi + \frac{b\tau}{\ell}. \tag{35}$$

The form of the metric in Schwarzschild-like coordinates $(\tau, \rho, \theta, \lambda, \phi)$ is complicated and not very enlightening to write it explicitly here. However, it can be shown that the curvature is constant for $\rho \neq 0$, and hence, the spacetime is locally AdS, reflecting the fact that the geometry is AdS_5 with an appropriate identification, to wit, $\phi \simeq \phi + 2\pi$.

The identification connects points of the embedding AdS spacetime separated by the Killing vector $\xi = -2\pi\alpha J_{12} + 2\pi\beta J_{05}$, where $J_{AB} = x_A\partial_B - x_B\partial_A$ are the AdS generators. The coefficients α and β correspond to angular deficits $\alpha = 1 - a$ and $\beta = b$ in the planes (1–2) and (0–5), respectively, related to the mass and spin of the solution.

Choosing the vielbein as $e^0 = Ad\phi_{05}$, $e^1 = Cd\rho$, $e^2 = B\sin\theta d\phi_{12}$, $e^3 = Bd\theta$ and $e^4 = B\cos\theta d\lambda$, and assuming the spin connection to be torsionless for $\rho \neq 0$, it is straightforward to calculate the AdS curvature

$$F = \left(\frac{1}{\ell}AJ_{05} + \frac{A'}{\sqrt{B'^2 - A'^2}}J_{01}\right)dd\phi_{05} + \left(-\frac{B'}{\sqrt{B'^2 - A'^2}}\sin\theta J_{12} + \cos\theta J_{23} + \frac{1}{\ell}B\sin\theta J_{25}\right) \times dd\phi_{12}, \tag{36}$$

where the prime denotes radial derivative. Using the identities [38] $dd\phi_{05} = 2\pi\beta\delta(T_{05})dx^0dx^5$, $dd\phi_{12} = -2\pi\alpha\delta(T_{12})dx^1dx^2$, and the field equations $F(F - j) = 0$, we find that there is a sector of CS gravity where the current is $j = F$. In this sector, the current is

$$j = 2\pi bG_{05}\delta(T_{05})dx^0dx^5 + 2\pi\alpha G_{12}\delta(T_{12})dx^1dx^2. \tag{37}$$

It can be checked that the generators G_{05} and G_{12} are mutually commuting,

$$G_{05} = \frac{1}{\sqrt{a^2 - b^2}}(aJ_{05} + bJ_{01}), \tag{38}$$

$$G_{12} = \frac{\sin\theta}{\sqrt{a^2 - b^2}}(aJ_{12} - bJ_{25}) - \cos\theta J_{23}, \tag{39}$$

which is similar to the case in $2+1$ dimensions. Note however that since $A \neq 0$, the first term in the current (37) vanishes identically. Thus, j is not composed by two four-dimensional planar sources (in the embedding space) intersecting on the $3-4$ plane, as one could expect from (37). For the static solution ($b = 0$), one obtains

$$j_{\text{static}} = 2\pi\alpha(\sin\theta J_{12} - \cos\theta J_{23})\delta(T_{12})d\Omega_{12}^2. \tag{40}$$

Clearly, both in the static and spinning cases there is a conical singularity at $\rho = 0$, and also like in the $2+1$ case, the extremal 2-brane can be constructed as the limit $a = b$ ($A/B = 1$ or $|M|\ell = |J|$).

In general, for codimension 2 branes, the field equations are $F^n = j_{[2n-2]}F^{n-1}$, where the current $j_{[2n-2]}$ is a 2-form, and there always exist a sector where $j_{[2n-2]} = F$, and F corresponds to a conical singularity in a two-dimensional plane. Ideas similar in spirit were recently discussed in [39].

The nonextremal massive spinning 2-branes in five dimensions need not be BPS states. However, one might expect that stable (BPS) configurations can be constructed as extremal spinning 2-branes in analogy with the three-dimensional case. For example, a restriction of the metric (34) to a submanifold $\theta = \pi/2$, $\lambda = 1$ is the three-dimensional metric $ds^2 = -dA^2 + dB^2 - A^2d\phi_{05}^2$ that describes a 0-brane naked singularity in Schwarzschild-like coordinates $(\tau, \rho, \theta = \frac{\pi}{2}, \lambda = 1, \phi)$ [11]. The extremal spinning naked singularity $a = b$ ($|M|\ell = |J|$) can be similarly embedded in this submanifold, and as shown in [43], this defines a BPS state as well. Moreover, based on the fact that there exist BPS states in five-dimensional CS supergravity [37] where the space is also locally AdS, and that the asymptotic isometries of those states are $S^1 \times S^1 \times S^1$, or S^3, which correspond to the isometries of static branes, one can conjecture that those BPS states and the naked spinning codimension 2 brane are in fact related. However, the presence of other gauge fields in that theory makes that identification difficult. Moreover, five-dimensional CS supergravity has a very rich dynamical structure with various disconnected sectors in phase space characterized by different local symmetries and degrees of freedom [40]. In some of those sectors the AdS space is not a stable configuration. It would be interesting to see how the extremal BPS 2-brane fits in this scenario, since it might not carry maximal number of local degrees of freedom.

IV. SUMMARY AND PROSPECTS

The minimal coupling between an electric point charge and an electromagnetic potential is the simplest CS system—the $0+1$ case—, and at the same time the prototype of how a brane couples to a connection [21]. Then, it seems natural to regard any CS system as a form of coupling between a brane and a (non-) Abelian connection. Here, we have explicitly shown how this idea can be exploited to couple 0- and $(2n-2)$-branes to gravity, when the spacetime geometry is described by a CS action in $D = 2n + 1$ dimensions. The cases of other $2p$-branes are technically more complicated, but in principle can be worked out in a manner similar to these two extreme examples. Moreover, CS theories can be viewed as describing the dynamics of a (non-Abelian) connection living on the worldvolume of a $2p$-brane, which is a topological defect in the embedding space.

These topological defects are natural sources for gravity, which in the particular case of a 0-brane, has been shown to correspond to the negative energy spectrum of black holes in gravitational CS theories. This part of the spectrum corresponds to the gap between the massless black hole ($M = 0$) and anti-de Sitter spacetime ($M = -1$).

The fact that these topological defects are naked singularities does not mean they are necessarily unphysical. Moreover, it has been shown that if endowed with the right amount of angular momentum, they can be stable BPS objects [11]. The fact that these are negative energy states is not contradictory with their supersymmetric nature, because these are supersymmetric extensions of the AdS—and not the Poincaré—group. Furthermore, it has been argued that negative energy degrees of freedom are necessary for consistent microscopic description of the entropy of the BTZ black hole [41].

The coupling between a connection and a $2p$-brane of the form (9) can exist in any gauge system, described by a CS, or a Yang-Mills action, or even for some more exotic form of gauge-invariant action, such as the Born-Infeld theory. An interesting system to study, for example, could be that of the Maxwell field in $3 + 1$ dimensions coupled to a 2-brane,

$$\begin{aligned} I &= \frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}d^4x + \int j \wedge A \wedge dA, \\ &= \int\Big(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + q\delta(\Sigma)n^{\alpha}\epsilon_{\alpha\beta\mu\nu}A^{\beta}\partial^{\mu}A^{\nu}\Big)d^4x, \qquad (41)\end{aligned}$$

where q is the charge and Σ is the worldsheet of the membrane. Obviously, since the worldsheet singles out a direction, n^{α}, the membrane breaks Lorentz invariance, but the $U(1)$ gauge invariance is unaffected. This model has been recently proposed in relation to a possible mechanism for breaking Lorentz invariance [42].

The interaction term (9) is interesting in CS gravity because it is a natural gauge-invariant coupling, which does not require to introduce a metric. Moreover, CS gravity is a gauge theory where gauge invariance reflects the symmetries of the local tangents to spacetime. Hence, the presence of the brane reduces the AdS symmetry of the D-dimensional spacetime, $SO(D-1, 2)$, to those of the worldsheet $SO(2p, 2)$. In a supersymmetric theory, this would mean a reduction from the supersymmetric extension of the first, to a supersymmetric extension of the latter. This analysis will be the subject of a forthcoming paper [43].

The branes considered here are δ-like objects in the spacetime manifold. The interpretation of distributions of this sort in general relativity is obscure if they appear in the metric, and great care must be taken to avoid inconsistencies arising from products and inverses of metric components that enter in Einstein's equations [44–47]. In Chern-Simons theories, this problem does not arise because the field equations contain only exterior products of forms, which always produce well-defined distributional products.

An additional open question is how to generalize the notion of transgression for CS forms either of different degree or for intersection of several p-branes of different dimensionality.

In general, BPS states will give rise to fermionic zero modes on the branes due to the partial breaking of supersymmetry. In contrast with the standard supergravity, the transformation law for the fields in CS supergravities is that of the connection, $\delta A = D\lambda$. Therefore, the fermionic zero modes can be written as $\delta\psi = D\epsilon$, where the covariant derivative is evaluated on the BPS background retaining only the generators of the unbroken symmetries. Consequently, as in the standard case, BPS states give fermionic zero modes for free.

The question remains about the dynamics of branes as effected by their interaction with the connection field. In order to address this issue, one should postulate an action principle for free branes, and this in turn requires to decide whether the branes are fundamental objects themselves or are given as functions of more fundamental matter fields, as for example $j \sim \bar{\psi}\Gamma\psi$. In that case, those dynamic source can lead to spontaneous symmetry breaking or Higgs mechanism, as already mentioned.

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APPENDIX A: COUPLING BRANES TO ABELIAN CS THEORIES

Here, we show a few explicit examples of classical solutions for the coupling between an Abelian connection and simple external sources.

1. Point charge in 2 + 1 dimensions

Consider the CS system for a $U(1)$ connection in $2+1$ dimensions. The only brane that can couple to this connection is a 0-brane (point charge). In the presence of a point particle sitting at rest, the action reads

$$\begin{aligned} I_{2+1}[A,j] &= \kappa \int_{M^{2+1}} \left(\frac{1}{2} A dA - j_{[0]} A\right) \\ &= \kappa \left(\frac{1}{2} \int_{M^{2+1}} A dA - q_0 \int_{\Gamma^1} A\right). \end{aligned} \tag{A1}$$

The resulting field equation is

$$F = j_{[0]} = q_0 \delta(\Sigma_{12}) d\Omega^2, \tag{A2}$$

where Σ_{12} is the spatial section in the rest frame of the particle. The solution for the connection A is found by direct integration of (A2) on a disc. It reads, *modulo* gauge transformations,

$$A = \frac{q_0}{2\pi} d\phi_{12} + a(\Gamma^1), \tag{A3}$$

where ϕ_{12} is the angle around the origin in the Σ_{12} plane, and $a(x^0)$ is an arbitrary 1-form on the worldline. This configuration corresponds to a point charge that produces a timelike magnetic flux $\Phi = \iint F = q_0$, concentrated on the worldline. The curvature F can be obtained directly by differentiating (A3) and using the identity [38],

$$d(d\phi) = 2\pi \delta(\Sigma_{12}) d\Omega^2. \tag{A4}$$

This source generates a static magnetic field, a monopole in $2+1$ dimensions. The quantization of the magnetic flux, $\kappa F = 2n\pi\hbar$, would be a consequence of requiring that the holonomies around the monopole be quantum mechanically unobservable. In that case, the allowed values for the "magnetic charge" must be quantized by Dirac's rule, $\kappa q_0 = nh$.

2. Point source in 4 + 1 dimensions

The action for a $U(1)$ CS connection in $4+1$ dimensions coupled to a point source is

$$\begin{aligned} I_{4+1}[A,j] &= \kappa \int_{M^{4+1}} \left(\frac{1}{3} A (dA)^2 - j_{[0]} A\right) \\ &= \kappa \left(\frac{1}{3} \int_{M^{4+1}} A (dA)^2 - q_0 \int_{\Gamma^1} A\right). \end{aligned} \tag{A5}$$

The corresponding field equations read

$$FF = j_{[0]}, \tag{A6}$$

$$= q_0 \delta(T^4) d\Omega^4, \tag{A7}$$

which have many solutions for F, in particular,

$$F = \frac{\sqrt{q_0}}{2} [\delta(\Sigma_{12}) d\Omega_{12}^2 + \delta(\Sigma_{34}) d\Omega_{34}^2].$$

Permutations of the labels 1, 2, 3, 4 clearly yield the same result (A6). Hence, the general solution is a linear combination,

$$F = \sqrt{q_0} \sum_{i>j=1}^{4} \xi_{ij} \delta(\Sigma_{ij}) d\Omega_{ij}^2, \quad \text{where } \frac{1}{8} \xi_{ij} \xi_{kl} \epsilon^{ijkl} = 1, \tag{A8}$$

and the connection reads

$$A = \frac{\sqrt{q_0}}{2\pi} \sum_{i>j=1}^{4} \xi_{ij} d\phi_{ij} + \mathcal{A}(x^0), \tag{A9}$$

where $\mathcal{A}$ is an arbitrary 1-form. The six arbitrary coefficients ξ_{ij} reflect the fact that higher-dimensional CS theories have larger degeneracies, which are not found in more standard gauge theories, like in the Maxwell or Yang-Mills cases.

3. Two-brane in 4 + 1 dimensions

A five-dimensional CS theory for a $U(1)$ connection, can couple to a two-dimensional membrane. The action now reads

$$\begin{aligned} I_{4+1}[A,j] &= \kappa \int_{M^{4+1}} \left(\frac{1}{3} A (dA)^2 - j_{[2]} A dA\right) \\ &= \kappa \left(\frac{1}{3} \int_{M^{4+1}} A (dA)^2 - q_2 \int_{\Gamma^3} A dA\right). \end{aligned} \tag{A10}$$

The corresponding field equation

$$F \wedge (F - j_{[2]}) = 0 \tag{A11}$$

is degenerate: a portion of solution space is not determined by this equation. There are four (almost) obvious solutions for any source $j_{[2]}$:

(i) The connection is flat everywhere,

$$F^{(I)} = dA^{(I)} = 0. \tag{A12}$$

(ii) The curvature is given by the source

$$F^{(II)} = j_{[2]}. \tag{A13}$$

Since the current is a 2-form source on the section transverse to the worldvolume of the brane,

$$j_{[2]} = q_2 \delta(\Sigma_{34}) d\Omega^2, \tag{A14}$$

in the second case, the connection reads

$$A^{(II)} = \frac{q_2}{2\pi} d\phi_{34} + \mathcal{A}(\Gamma^3), \quad \text{(A15)}$$

up to gauge transformations. Here, $\mathcal{A}$ is an arbitrary 1-form defined on Γ^3.

(iii) A third obvious solution to (A11) contains (A12) and (A13) as particular cases,

$$F^{(III)} = d\mathcal{A}^{(III)}(T^2), \quad \text{(A16)}$$

where $\mathcal{A}^{(III)}$ is any 1-form defined on the two-dimensional transverse space T^2.

(iv) The fourth solution, independent from those above, is

$$F^{(IV)} = \tfrac{1}{2} j_{[2]} + d\mathcal{A}^{(IV)}(\Gamma^3), \quad \text{(A17)}$$

where $\mathcal{A}^{(IV)}$ is any 1-form on the worldvolume of the brane Γ^3. One could also add a term $\alpha(\Gamma^3) \wedge \beta(T^2)$, where α and β are 1-forms on the worldvolume and in the transverse space, respectively. However, this would require additional algebraic constraints, which could not be easily implemented. Up to gauge transformations, the connection for (A17) reads

$$A^{(IV)} = \frac{q_2}{4\pi} d\phi_{34} + \mathcal{A}^{(IV)}(\Gamma^3), \quad \text{(18)}$$

where ϕ_{34} is the angle in the transverse space surrounding the brane.

The fact that the phase space contains various disconnected sectors is a general feature, related to the presence of degeneracies in the symplectic form and to the existence of irregular constraint structures in higher-dimensional CS theories. These issues have been analyzed in [29,40,48].

4. Generalization to higher dimensions

The general recipe for a codimension 2 brane—or $(2n-2)$-brane—in $D = 2n+1$ dimensions can be easily presented. The transverse space is two dimensional (T^2), as in the examples discussed above ($D = 5$, $n = 2$), and the field equations read

$$F^n = j_{[n-1]} F^{n-1}, \quad \text{(A19)}$$

which have at least four sectors in the space of solutions (up to gauge),

$$F^{(I)} = 0, \qquad A^{(I)} = 0, \qquad F^{(II)} = j_{[n-1]},$$
$$A^{(II)} = \frac{q_{n-1}}{2\pi} d\phi_{D-2D-1}, \qquad F^{(III)} = dA^{(III)}(T^2),$$
$$A^{(III)}(T^2)\ \text{arbitrary}, \qquad F^{(IV)} = \frac{1}{n} j_{[n-1]} + d\mathcal{A}^{(IV)}(\Gamma),$$
$$A^{(IV)} = \frac{q_{n-1}}{2n\pi} d\phi_{D-2D-1} + \mathcal{A}^{(IV)}(\Gamma).$$

These sectors have different residual symmetries and, correspondingly, different degrees of freedom for the propagating modes around the corresponding vacua. Those degrees of freedom are governed by the "homogeneous" ($q_{n-1} = 0$) parts of the solutions and, as mentioned above, this is a generic feature of higher-dimensional CS theories.

Similar codimension 2 defects were also found in even-dimensional topological field theories: a 6D topological density in a six-dimensional AdS spacetime with a four-dimensional topological defect, induces an effective theory on the defect whose dynamics is that of 4D Einstein gravity [49].

In the other extreme, the case of 0-branes in $2n+1$ dimensions, can be easily generalized as well. The field equations read

$$F^n = j_{[0]},$$

where $j_{[0]}$ is a $2n$ form on the $2n$-dimensional transverse space to the worldline of the 0-brane. The solution takes the form

$$F = \sqrt{q_0} \sum_{i>j=1}^{2n} \xi_{ij} \delta(\Sigma_{ij}) d\Omega_{ij}^2,$$
$$\text{where } \frac{1}{2^n n!} \xi_{i_1 j_1} \cdots \xi_{i_n j_n} \epsilon^{i_1 j_1 \cdots i_n j_n} = 1, \quad \text{(A20)}$$

and the connection is

$$A = \frac{\sqrt{q_0}}{2\pi} \sum_{i>j=1}^{2n} \xi_{ij} d\phi_{ij} + \mathcal{A}(x^0). \quad \text{(A21)}$$

The even more radical case of a (-1)-brane could be conceived as well. It would correspond to an object whose worldvolume is zero dimensional, namely, a charged instanton [50].

APPENDIX B: REGULARIZATION OF THE CURVATURE FOR THE HIGHER-DIMENSIONAL 0-BRANE

Consider the metric (23) of D-dimensional AdS space with surface deficit $\alpha\Omega^{D-2}$, where $\alpha \in [0, 1]$ is the fraction of the topological defect and Ω^{D-2} is the surface area of a unit sphere S^{D-2}. For $\alpha > 0$, this metric has a singularity at the origin $r = 0$, and its regularization consists in smoothing out the topological defect α by making the replacement $\alpha \to \alpha_\epsilon(r)$, where $\alpha_\epsilon(r)$ is chosen as

$$\alpha_\epsilon(r) = \frac{\alpha r^2}{r^2 + \epsilon^2}. \quad \text{(B1)}$$

Note that for fixed finite r, $\alpha_\epsilon(r) \to \alpha$ in the limit $\epsilon \to 0$, and for fixed finite ϵ, $\alpha_\epsilon(r) \to 0$ in the limit $r \to 0$. The regularized vielbein can be taken as

$$e_\epsilon^0 = f(r) dt, \quad \text{(B2)}$$

$$e_\epsilon^1 = \frac{dr}{f(r)}, \quad \text{(B3)}$$

$$e_\epsilon^p = [1-\alpha_\epsilon(r)]r\tilde{e}^p, \tag{B4}$$

where $f^2 = 1+\frac{r^2}{\ell^2}$ is the AdS metric function, and $\tilde{e}^p$ is the vielbein of the $(D-2)$ unit sphere, $d\Sigma_{D-2}^2 = \delta_{pq}\tilde{e}^p\tilde{e}^q$. Assuming that torsion vanishes for $r>0$, the components of the regularized spin connection are

$$\omega_\epsilon^{01} = ff'dt, \tag{B5}$$

$$\omega_\epsilon^{1p} = f(r\alpha_\epsilon)'\tilde{e}^p, \tag{B6}$$

$$\omega_\epsilon^{pq} = \tilde{\omega}^{pq}, \tag{B7}$$

which gives the regularized AdS curvature $F_\epsilon^{ab} = R_\epsilon^{ab} + \frac{1}{\ell^2}e_\epsilon^a e_\epsilon^b$ in the form

$$F_\epsilon^{0p} = X_\epsilon(r)dt\tilde{e}^p, \tag{B8}$$

$$F^{1p} = Y_\epsilon(r)dr\tilde{e}^p, \tag{B9}$$

$$F^{pq} = Z_\epsilon(r)\tilde{e}^p\tilde{e}^q. \tag{B10}$$

For later use, it is convenient to analyze the behavior of X_ϵ, Y_ϵ, and Z_ϵ for small radius $r=\epsilon\eta$. Expanding in powers of ϵ and finite η, one finds

$$X_\epsilon(\epsilon\eta) = -\frac{2\alpha k\eta^3}{\ell^2(\eta^2+1)^2}\epsilon + \mathcal{O}(\epsilon^3), \tag{B11}$$

$$Y_\epsilon(\epsilon\eta) = \frac{2\alpha\eta(3-\eta^2)}{(\eta^2+1)^3}\frac{1}{\epsilon} + \mathcal{O}(\epsilon), \tag{B12}$$

$$Z_\epsilon(\epsilon\eta) = 1 - \frac{[(1-\alpha)\eta^4 + (2-3\alpha)\eta^2+1]^2}{(\eta^2+1)^4} + \mathcal{O}(\epsilon^2). \tag{B13}$$

The remaining components of F_ϵ^{AB} vanish. Note that the torsion $F_\epsilon^{aD} = (De^a)_\epsilon$ identically vanishes in this regularization.

We know that $(F_\epsilon)^n$ is singular and, on account of the field Eqs. (12) for 0-branes, and using (B8)–(B10), the nonvanishing components of the regularized current are

$$(j_\epsilon)_a = \frac{1}{2^n}\varepsilon_{aa_1b_1\cdots a_nb_n}F_\epsilon^{a_1b_1}\cdots F_\epsilon^{a_nb_n}. \tag{B14}$$

Moreover, from the properties of the functions X_ϵ, Y_ϵ, and Z_ϵ, it is straightforward to show that the current is

$$(j_0)_\epsilon = \frac{n}{2^{n-1}}\varepsilon_{01p_1\cdots p_{2n-1}}Y_\epsilon(r)Z_\epsilon^{n-2}(r)dr\tilde{e}^{p_1}\tilde{e}^{p_2}\cdots\tilde{e}^{p_{2n-1}}. \tag{B15}$$

Treating this source as a distribution that is regular everywhere, we multiply it by a test function Ψ with support on the spatial section of the spacetime. Since the source is spherically symmetric, one can take the average over the angular part. Denoting the area of a S^{2n-1} sphere as $\Omega^{2n-1} = \int\varepsilon_{01p_1\cdots p_{2n-1}}\tilde{e}^{p_1}\cdots\tilde{e}^{p_{2n-2}}$, one obtains

$$\lim_{\epsilon\to 0}\int\Psi(\vec{r})(j_0)_\epsilon = \lim_{\epsilon\to 0}\frac{n}{2^{n-1}}\Omega^{2n-1}\int_0^\infty dr\bar{\Psi}(r)Y(r)Z^{n-1}(r), \tag{B16}$$

and, after changing the integration parameter $r=\epsilon\eta$ and using the properties (B11)–(B13), one obtains in the limit

$$\lim_{\epsilon\to 0}\int\Psi(\vec{r})(j_0)_\epsilon \equiv q_0^{(2n+1)}(\alpha)\bar{\Psi}(0).$$

The charge in $2n+1$ dimensions is given by the polynomial in the topological defect α,

$$q_0^{(2n+1)} = \Omega^{2n-1}\frac{n\alpha^n}{2^{n-1}}\int_0^\infty dz\frac{z^{n-1}(3-z)(3+z)^{n-1}}{(z+1)^{4n-1}} \times[(2-\alpha)z^2 + (4-3\alpha)z+2]^{n-1}. \tag{B17}$$

For example, we have

$$q_0^{(3)} = 2\pi\alpha, \tag{B18}$$

$$q_0^{(5)} = \frac{4\pi\alpha^2}{3}(3-\alpha), \tag{B19}$$

$$q_0^{(7)} = \frac{2\pi^2\alpha^3}{5}\left(\alpha^2 - 5\alpha + \frac{20}{3}\right), \tag{B20}$$

in dimensions three, five, and seven, respectively.

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