Killing vectors and anisotropy

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We consider an action that can generate fluids with three unequal stresses for metrics with a spacelike Killing vector. The parameters in the action are directly related to the stress anisotropies. The field equations following from the action are applied to an anisotropic cosmological expansion and an extension of the Gott-Hiscock cosmic string.

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I. INTRODUCTION

General relativistic models with anisotropic stress have become increasing useful as the applications become more complex and precise [1–6]. There has been considerable interest in anisotropic spheres [7–11] because of applications to stellar models [12–14], and temperature anomalies in the cosmic microwave background (CMB) [15,16] have generated an increased interest in anisotropic cosmological models [3,17–21]. Anisotropy is usually discussed in the context of fluid stress-energy relations, and examining the Lagrangian actions that generate fluid equations of state provides a simple way to view their geometric and physical origins. This paper discusses a simple action describing a fluid with three unequal stresses. The fluid is supported by a metric with a spacelike Killing symmetry.

The vacuum Einstein field equations, for spacetimes with Killing vector ξ^a, can be generated from an action written in terms of the Killing vector norm $\lambda = \xi^a \xi_a$, and twist $\omega_i = \varepsilon_{iabc}\xi^a \nabla^b \xi^c$. The original vacuum action was developed by Ehlers [22], Harrison [23], and Geroch [24], who wrote the 3 + 1 Einstein equations on the 2 + 1 space of Killing vector orbits. For spacetimes with metric g_{ab}, $(-+++)$, the induced metric on the 3-manifold of Killing orbits is $h_{ab} = g_{ab} - \xi_a \xi_b/\lambda$. The action generating the vacuum field equations is, with h_{ab} replaced by $\gamma_{ab} := \lambda h_{ab}$,

$$S_{\rm vac} = \int d^3x\sqrt{\gamma}\left(\mathcal{R} - \frac{D_a\lambda D^a\lambda + D_a\omega D^a\omega}{2\lambda^2}\right), \tag{1}$$

where $\mathcal{R}$ and D_a are the Ricci scalar and covariant derivative on γ, respectively, and $\omega_a = D_a\omega$. This action was generalized to perfect fluids [25,26] by including a scalar function $\mathcal{K}$, and a vector function, s^a with $\xi^a s_a = 0$:

$$S_{\rm fluid} = \int d^3x\sqrt{\gamma}\left(\mathcal{R} - \frac{D_a\lambda D^a\lambda + D_a\omega D^a\omega}{2\lambda^2} - \mathcal{K} - s^a s_a\right). \tag{2}$$

Using this action, a stress energy for two isotropic equations of state can be generated. Krisch and Glass [27] showed that the same action may also be applied to fluids with two unequal stresses.

In this paper, which considers metrics with a spacelike Killing vector, the action $S_{\rm fluid}$ is extended to completely anisotropic fluids. Previous work using this action treated $\mathcal{K}$ and $s^a s_a$ separately. Considering both terms simultaneously allows extensions to fluids with three unequal stresses. In the next section we develop the anisotropic fluid content implicit in $S_{\rm fluid}$, and discuss some of the equations of state and their effect on the rate-of-expansion of the fluid 4-velocity. Some particular metric examples are given in Sec.III.

II. FLUID CONTENT

A. Field equations and geometry

The 2 + 1 field equations that follow from the action $S_{\rm fluid}$ are

$$\mathcal{R}_{ab} = \frac{D_a\lambda D_b\lambda + D_a\omega D_b\omega}{2\lambda^2} + s_a s_b + \mathcal{K}\gamma_{ab}, \tag{3}$$

$$\lambda^2 D^a(\lambda^{-1}D_a\lambda) + D_a\omega D^a\omega = 0, \tag{4}$$

$$D_a\left[\frac{D^a\omega}{\lambda^2}\right] = 0, \tag{5}$$

where D_a is the covariant derivative on γ. $\mathcal{R}_{ab} =$ Ricci(γ_{ab}) is related to $R_{ab} =$ Ricci(g_{ab}) [24,26] by

$$\begin{aligned} h_a{}^c h_b{}^d R_{cd} &= \mathcal{R}_{ab} - \frac{D_a\lambda D_b\lambda + D_a\omega D_b\omega}{2\lambda^2} \\ &= s_a s_b + \mathcal{K}\lambda h_{ab}, \end{aligned} \tag{6}$$

$$\xi^a\xi^b R_{ab} = 0, \tag{7}$$

$$h_a{}^c\xi^b R_{cb} = 0. \tag{8}$$

In the next section, we discuss the stress energy related to the matter parameters s_a and $\mathcal{K}$.

B. Stress energy

A simple anisotropic fluid description contains a density and three stresses $(\varepsilon, P_1, P_2, P_3)$, which depend on s_a and $\mathcal{K}$. In this section, we find the dependence of each of the stress-energy components on the action parameters and

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show that the spatial part of s_a is a measure of the stress anisotropy $P_1 - P_2$, while $\mathcal{K}$ enters into the anisotropy between the (1, 3) and (2, 3) planes. Consider a 3 + 1 metric, g_{ab}, described with unit tetrad vectors $[U^a, e^a_{(1)}, e^a_{(2)}, e^a_{(3)}]$, and with spacelike Killing vector ξ^a aligned with $e^a_{(3)}$: $\xi^a = \sqrt{\lambda} e^a_{(3)}$. The general stress-energy tensor is assumed to be

$$T_{ab} = \varepsilon U_a U_b + P_1 e_a^{(1)} e_b^{(1)} + P_2 e_a^{(2)} e_b^{(2)} + P_3 e_a^{(3)} e_b^{(3)}, \quad (9)$$

with the stress components carrying a tetrad index. The Ricci tensor for this stress energy is

$$R_{ab} = 8\pi\Big[\frac{\varepsilon + P_1 + P_2 + P_3}{2} U_a U_b + \frac{\varepsilon + P_1 - P_2 - P_3}{2} e_a^{(1)} e_b^{(1)} + \frac{\varepsilon - P_1 + P_2 - P_3}{2} e_a^{(2)} e_b^{(2)} + \frac{\varepsilon - P_1 - P_2 + P_3}{2} e_a^{(3)} e_b^{(3)}\Big].$$

Using the field equation $\xi^b \xi^a R_{ab} = 0$, P_3 is determined by the other stress-energy components

$$P_3 = -\varepsilon + P_1 + P_2, \quad (10)$$

and the Ricci tensor becomes

$$R_{ab} = 8\pi[(P_1 + P_2) U_a U_b + (\varepsilon - P_2) e_a^{(1)} e_b^{(1)} + (\varepsilon - P_1) e_a^{(2)} e_b^{(2)}]. \quad (11)$$

Expanding s^a, using $R_{ab} = s_a s_b + \lambda \mathcal{K} h_{ab}$ and taking scalar products, the complete stress-energy description is

$$s_a = s_0 U_a + s_1 e_a^{(1)} + s_2 e_a^{(2)}, \quad (12a)$$
$$16\pi\varepsilon = (s_0)^2 + (s_1)^2 + (s_2)^2 + \lambda\mathcal{K}, \quad (12b)$$
$$16\pi P_1 = (s_0)^2 + (s_1)^2 - (s_2)^2 - \lambda\mathcal{K}, \quad (12c)$$
$$16\pi P_2 = (s_0)^2 - (s_1)^2 + (s_2)^2 - \lambda\mathcal{K}, \quad (12d)$$
$$16\pi P_3 = (s_0)^2 - (s_1)^2 - (s_2)^2 - 3\lambda\mathcal{K}, \quad (12e)$$
$$s_1 s_2 = s_0 s_1 = s_0 s_2 = 0, \quad (12f)$$

where s_0, s_1, and s_2 are tetrad indexed. These relations between the stress energy and the action parameters show how s_a and $\mathcal{K}$ enter the fluid anisotropy. Using Eq. (12c) and (12d), the anisotropy in the 2 + 1 stress is described by the spatial components of s_a:

$$8\pi(P_1 - P_2) = (s_1)^2 - (s_2)^2. \quad (13)$$

The anisotropies involving the stress along the Killing direction require both s_a and $\mathcal{K}$:

$$8\pi(P_1 - P_3) = s_1^2 + \lambda\mathcal{K}, \qquad 8\pi(P_2 - P_3) = s_2^2 + \lambda\mathcal{K}. \quad (14)$$

TABLE I. Functions and their stress-energy relations.

Nonzero functions	Stress-energy relations
$\mathcal{K}$	$\varepsilon = -P_1 = -P_2 = -P_3/3$
s_1	$\varepsilon = P_1 = -P_2 = -P_3$
s_0	$\varepsilon = P_1 = P_2 = P_3$
$\mathcal{K}$, s_1	$\varepsilon = -P_2$, $P_3 = 2P_2 + P_1$
$\mathcal{K}$, s_0	$P_1 = P_2$, $P_3 = -\varepsilon + 2P_1$

A particularly useful relation emerging from this formalism is the equality of the stress along the Killing direction with the negative Ricci scalar: $16\pi P_3 = -R$.

C. Equations of state

Some equations of state for the 3 + 1 fluid can be written down by considering values for s_a and $\mathcal{K}$. The conditions in Eq. (12f) mean that any one of s_0, s_1, or s_2 can be nonzero. In the formalism so far, the choice s_1 or s_2 equal to zero only fixes the (1, 2) index. The stress-energy relations for each of the choices are described in Table I (with $s_2 = 0$) and s_1 identified as the index function for stress anisotropy in the (1, 2) plane. The $s_1 = 0$ choice simply replaces 2 by 1 in Table I.

The fluid parameter conditions can be generally related to the value of $s^a s_a$. For $s_2 = 0$, by using Eq. (12) we find

$$8\pi(\varepsilon + 2P_2 - P_1) = -s^a s_a. \quad (15)$$

If s_a is timelike, $s_1 = 0$, $\varepsilon + 2P_2 > P_1$, and $P_1 = P_2$. The timelike condition is $\varepsilon + P_1 > 0$, $P_1 = P_2$. If s_a is spacelike with $s_2 = 0$, we have $\varepsilon + 2P_2 < P_1$. Since only one component of s_a is nonzero, s_a cannot be null, and we do not have $\varepsilon + 2P_2 = P_1$. Completely anisotropic fluids will correspond to $s_1 (\text{or} s_2) \neq 0$ $\mathcal{K} \neq 0$. The fluid conditions for a completely anisotropic fluid are

$$s_2 = 0: \ \varepsilon = -P_2, \qquad P_3 = 2P_2 + P_1, \quad (16a)$$
$$s_1 = 0: \ \varepsilon = -P_1, \qquad P_3 = 2P_1 + P_2. \quad (16b)$$

D. Energy conditions

There are a number of energy conditions based on the structure of the stress energy and Ricci tensor [28]. In matter, the weak and dominant conditions require positive density and $-T_{ab} U^b = \varepsilon U_a$. With either s_1 or s_2 nonzero and a comoving observer, the strong and null conditions are

$$\text{Strong:} \ R_{ab} U^a U^b \geq 0, \quad (17)$$

$$-\lambda\mathcal{K} = 8\pi(P_1 + P_2) \geq 0, \quad (18)$$

$$\begin{aligned}
&\text{Null}: T_{ab}N^aN^b \geq 0\\
&1.\ N^a = U^a + e^a_{(1)}\colon \varepsilon + P_1 \geq 0 \Rightarrow s_1^2 \geq 0\\
&2.\ N^a = U^a + e^a_{(2)}\colon \varepsilon + P_2 \geq 0 \Rightarrow s_2^2 \geq 0\\
&3.\ N^a = U^a + e^a_{(3)}\colon \varepsilon + P_3 \geq 0 \Rightarrow (P_1 + P_2) \geq 0.
\end{aligned} \tag{19}$$

The weak and dominant conditions can be enforced, but the strong and some of the null conditions can be violated. For example, the $(s_0,\ s_2 = 0)$ condition, $\varepsilon = -P_2$, implies a violation if $|P_2| > P_1$. The violation of the strong energy condition is related to the development of the 4-velocity rate of expansion $\Theta = \nabla_a U^a$ in the Raychaudhuri equation

$$\frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - R_{ab}U^aU^b. \tag{20}$$

The action considered in this paper determines the form of the Ricci tensor. For the completely anisotropic fluid, the Ricci tensor is

$$\begin{aligned}
s_2 = 0\colon R_{ab} &= 8\pi[-(P_1 + P_2)(g_{ab} - e^{(3)}_a e^{(3)}_b)\\
&\quad + (P_1 - P_2)e^{(1)}_a e^{(1)}_b]\\
&= \lambda\mathcal{K}(g_{ab} - e^{(3)}_a e^{(3)}_b) + s_1^2 e^{(1)}_a e^{(1)}_b,
\end{aligned} \tag{21}$$

$$\begin{aligned}
s_1 = 0\colon R_{ab} &= 8\pi[-(P_1 + P_2)(g_{ab} - e^{(3)}_a e^{(3)}_b)\\
&\quad + (P_2 - P_1)e^{(2)}_a e^{(2)}_b]\\
&= \lambda\mathcal{K}(g_{ab} - e^{(3)}_a e^{(3)}_b) + s_2^2 e^{(2)}_a e^{(2)}_b,
\end{aligned} \tag{22}$$

The Raychaudhuri equation becomes

$$\begin{aligned}
\frac{d\Theta}{d\tau} &= -\frac{1}{3}\Theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} + \lambda\mathcal{K},\\
\lambda\mathcal{K} &= -8\pi(P_1 + P_2).
\end{aligned} \tag{23}$$

With zero vorticity and $\lambda\mathcal{K} = 0$, $\dot{\Theta} < 0$, these conditions describe a decreasing rate of expansion with an initially converging or diverging timelike congruence becoming more focused. However, even for zero vorticity, if $\lambda\mathcal{K} > 0$, the rate of change of the expansion can be positive, $\dot{\Theta} > 0$, and is unfocusing.

E. Fluid shear and anisotropy

The form of the stress-energy tensor, Eq. (9), identifies all of the spatial components as stress. However, equivalences [29,30] can be used to relate the completely anisotropic form to fluids with shear, with the differences in the shear tensor identified as the physical cause of the anisotropy. The general equivalence relation for $s_1 \neq 0$ is

$$\begin{aligned}
&\varepsilon U_aU_b + P_1 e^{(1)}_a e^{(1)}_b + P_2 e^{(2)}_a e^{(2)}_b + P_3 e^{(3)}_a e^{(3)}_b\\
&\quad = (\varepsilon + P)U_aU_b + Pg_{ab} - 2\eta\sigma_{ab}\\
&\varepsilon = -P_2\\
&P_1 = P - 2\eta\sigma_{(11)}\\
&P_2 = P - 2\eta\sigma_{(22)}\\
&P_3 = P - 2\eta\sigma_{(33)}.
\end{aligned} \tag{24}$$

The trace-free condition for σ_{ab} provides $P = (P_1 + P_2 + P_3)/3$. Using Eq. (12) the action parameters s^a and $\mathcal{K}$ describe the differences in the shear tensor components:

$$\begin{aligned}
P_i = P - 2\eta\sigma_{(ii)},\qquad s_1^2 &= 16\pi\eta[\sigma_{(22)} - \sigma_{(11)}],\\
\lambda\mathcal{K} &= 16\pi\eta[\sigma_{(33)} - \sigma_{(22)}].
\end{aligned} \tag{25}$$

Imposing an extra condition identifies an equation of state and the tetrad indexed shear tensor components. The evolution of spherically symmetric fluids of this type has been considered by Herrera *et al.* [2], who gave a metric for a shearing expansion-free evolving fluid with two unequal stresses.

III. EXAMPLES

The stress energy of a completely anisotropic fluid is described by

$$s_2 = 0\colon \varepsilon = P_2,\qquad P_3 = 2P_2 + P_1, \tag{26a}$$

$$s_1 = 0\colon \varepsilon = P_1,\qquad P_3 = 2P_2 + P_1. \tag{26b}$$

P_2 (P_1) must be negative for positive density. This suggests two possible application areas: Bianchi cosmological models, and cosmic strings with an axial tension.

A. Anisotropic cosmology

Bianchi metrics have been used to model anomalies in the CMB radiation [3,6,17–19], with anisotropies developing due to different expansion rates along coordinate axes. A general metric to consider is

$$ds^2 = -dt^2 + b^2(t)dx_1^2 + c^2(t)dx_2^2 + f^2(t)dx_3^2, \tag{27}$$

with three Hubble rates, $H_1 = \dot{b}/b$, $H_2 = \dot{c}/c$, and $H_3 = \dot{f}/f$. For a power-law expansion, one has $b = b_0t^\beta$, $c = c_0t^\gamma$, $f = f_0t^\delta$. The stress energy is

$$\begin{aligned}
8\pi\varepsilon(t^2) &= \beta\gamma + \beta\delta + \gamma\delta\\
8\pi P_1(t^2) &= -\gamma(\gamma - 1) - \delta(\delta - 1) - \delta\gamma\\
8\pi P_2(t^2) &= -\beta(\beta - 1) - \delta(\delta - 1) - \beta\delta\\
8\pi P_3(t^2) &= -\beta(\beta - 1) - \gamma(\gamma - 1) - \gamma\beta.
\end{aligned} \tag{28}$$

The stress-energy conditions $\varepsilon = -P_2$ and $P_3 = 2P_2 + P_1$ impose constraints

$$\gamma(\beta + \delta) = \beta(\beta - 1) + \delta(\delta - 1), \tag{29}$$

$$\gamma\beta = \beta(\beta - 1) + 3\delta(\delta - 1) + \delta(2\beta + \gamma). \qquad (30)$$

Combining the constraints yields

$$\delta(\delta - 1 + \beta + \gamma) = 0. \qquad (31)$$

Two cases emerge: $\delta = 0$ and $\delta = 1 - \beta - \gamma$. The $\delta \neq 0$ condition, when substituted back into Eq. (29), relates β and γ

$$\gamma + \beta = \beta^2 + \beta\gamma + \gamma^2 \qquad (32)$$

and results in zero energy density. Only the $\delta = 0$ case, with no expansion along the Killing direction, provides a fluid with nonzero density. With $\delta = 0$, Eq. (29) is

$$\beta(\beta - 1 - \gamma) = 0. \qquad (33)$$

When $\beta = 0$ the density is zero. In order to have an anisotropic fluid with positive density, expansions along at least two of the axes are necessary, with $b(t) = b_0 t^\beta$ and $c(t) = c_0 t^{\beta-1}$. For $\beta \neq 0$, the stress energy is

$$\begin{aligned} 8\pi\varepsilon &= \beta(\beta - 1)/t^2 \\ 8\pi P_1 &= -(\beta - 1)(\beta - 2)/t^2 \\ 8\pi P_2 &= -\beta(\beta - 1)/t^2 \\ 8\pi P_3 &= -(\beta - 1)(3\beta - 2)/t^2 \\ \mathcal{K} &= 2(\beta - 1)^2/t^2. \end{aligned} \qquad (34)$$

This example has positive density for $\beta < 0$ and $\beta > 1$, with $\beta > 1$ describing expansion. The rate of expansion for this case is

$$\Theta = \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = \frac{(2\beta - 1)}{t}, \qquad \dot{\Theta} = \frac{(1 - 2\beta)}{t^2}. \qquad (35)$$

The congruence will unfocus for $\beta < 1/2$. For the case of an expanding space with positive density the congruence is focusing. The family of radial stresses parameterized by β is especially interesting. For $1 < \beta < 2$, the stress is a pressure, for $\beta = 2$ it is dust, and for $\beta > 2$ it is a tension. P_3 also shows this range of behavior with the $P_3 = 0$ dust crossover at $\beta = 2/3$, out of the physical density region. Since P_3 is proportional to the negative Ricci scalar, the behavior of P_3 also reflects a change from positive scalar curvature, through zero, to a negative curvature manifold. In this model, only manifolds with nonzero Ricci scalar will have positive density. The timelike congruence will unfocus for $\beta < 0$. This is a positive density region of the parameter space with all three stresses as tensions.

In summary $\beta < 0$ unfocuses, $\beta > 1$ focuses.

B. Cosmic string

1. Metric and stress energy

Cosmic strings are of current interest both experimentally [31–34] and theoretically [35–37]. The usual cosmic string models have only a single axial tension. The simplest family of static 3 + 1 strings is described by the metric with $(0, 1, 2, 3) = (t, r, \varphi, z)$

$$ds^2 = -dt^2 + dr^2 + E^2(r)d\varphi^2 + dz^2. \qquad (36)$$

The density is $8\pi\varepsilon = -E''/E$, and the stresses are

$$P_r = P_\varphi = 0, \qquad P_z = E''/E. \qquad (37)$$

The Gott-Hiscock (GH) static string [38,39] is the constant density example with $a = \sqrt{8\pi\varepsilon}$, $E(r) = c_1 \sin(ar)$. The matter metric matches directly to vacuum Levi-Civita with an angular deficit and is Minkowski near the axis. There are two spacelike Killing vectors, $\xi_{(z)}$ and $\xi_{(\phi)}$.

As an anisotropic extension to the GH string interior, consider the metric

$$\begin{aligned} ds^2 = {}& -A_0^2\cos^2[a(R_0 - r)]dt^2 + dr^2 \\ & + a^{-2}\sin^2(ar + \gamma)d\phi^2 + dz^2. \end{aligned} \qquad (38)$$

The γ contribution to the sine argument is included for finite stress at $r = 0$. The stress-energy for this metric is

$$\begin{aligned} 8\pi\varepsilon &= a^2 = -8\pi P_\phi \\ 8\pi P_r &= a^2 \tan[a(R_0 - r)]\cot(ar + \gamma) \\ 8\pi P_z &= -a^2\{2 - \tan[a(R_0 - r)]\cot(ar + \gamma)\}, \end{aligned} \qquad (39)$$

with $\sin(\gamma) \neq 0$. The axial behavior of this stress energy is especially interesting. In the cosmological example, several of the stress components ranged through pressure, dust, and tension as the metric parameter varied. In this example, the same behavior is observed but is linked to the radial position or, along the axis, to the size of the string. To interpret this example as an anisotropic cosmic string with physical tension along the axis requires $1 < \tan[aR_0] \times \cot(\gamma) < 2$. The actual axial structure is related to the vacuum match and associated angular deficit. These are discussed in the next sections.

2. Vacuum matching

The metric can be matched to a vacuum metric across an Israel layer at $r = R_0$. The exterior (+) metric is a vacuum Levi-Civita metric with angular deficit δ

$$ds^2_{\text{LeviCivita}} = -dt^2 + dr^2 + r^2\delta^2 d\phi^2 + dz^2. \qquad (40)$$

The metric of the Israel layer is

$$ds^2_{\text{layer}} = -A_0^2 dt^2 + a^{-2}\sin^2(aR_0 + \gamma)d\phi^2 + dz^2. \qquad (41)$$

The Levi-Civita metric is matched to the layer. The matching conditions are

$$A_0 = 1, \qquad a^{-1}\sin(aR_0 + \gamma) = \pm R_0\delta. \qquad (42)$$

The stress energy of the Israel boundary layer is calculated from jumps in the extrinsic curvature K_{ab} going from Levi-Civita (+) across the Israel layer to the string interior (−), with the jumps calculated from $\langle K_{ab}\rangle = K^+_{ab} - K^-_{ab}$, $K = K^a_a$. The layer stress energy is

$$-8\pi S_{ab} = \langle K_{ab}\rangle - \langle K\rangle g_{ab}^{\text{layer}}. \tag{43}$$

The extrinsic curvatures needed to calculate the jumps are

$$\begin{aligned} K^+_{\phi\phi} &= -\delta^2 R_0, \\ K^+_{tt} &= 0, \\ K^+_{zz} &= 0, \\ K^-_{\phi\phi} &= a^{-1}\sin(aR_0+\gamma)\cos(aR_0+\gamma), \\ K^-_{tt} &= 0, \\ K^-_{zz} &= 0. \end{aligned} \tag{44}$$

The surface of the anisotropic string has an Israel stress-energy content

$$\begin{aligned} 8\pi S_{tt} &= -\langle K\rangle, \\ 8\pi S_{\phi\phi} &= 0, \\ 8\pi S_{zz} &= \langle K\rangle \\ \langle K\rangle &= -1/R_0 - a\cot(aR_0+\gamma). \end{aligned} \tag{45}$$

The equation of state of the GH string solution is found in the boundary layer of the anisotropic string.

3. Angular deficit

In the GH static string, the angular deficit is related to the mass/length, μ, calculated from a $t = \text{const}$, $z = \text{const}$ integral of the density. For the anisotropic string, that mass is composed of two parts. The contribution from the string interior is

$$\begin{aligned} \mu_1 &= \int_0^{2\pi}\int_0^{R_0} \varepsilon \frac{\sin(ar+\gamma)}{a} dr d\phi \\ &= 2\pi\int_0^{R_0} \frac{a^2}{8\pi}\frac{\sin(ar+\gamma)}{a} dr \\ &= \frac{1}{4}[\cos(\gamma) - \cos(aR_0+\gamma)], \end{aligned} \tag{46}$$

and the additional contribution from the boundary layer is

$$\begin{aligned} \mu_2 &= -2\pi \frac{\sin(aR_0+\gamma)}{8\pi a}\langle K\rangle \\ &= \frac{1}{4}\left[\frac{\sin(aR_0+\gamma)}{aR_0} + \cos(aR_0+\gamma)\right]. \end{aligned} \tag{47}$$

Combining the mass densities $\mu = \mu_1 + \mu_2$ yields

$$4\mu = \cos(\gamma) + \frac{\sin(aR_0+\gamma)}{aR_0}. \tag{48}$$

Substituting from the matching condition, Eq. (42), the relation between the linear mass density and the angular deficit is

$$4\mu - \cos(\gamma) = \pm\delta. \tag{49}$$

This is not the conventional "thin string" result $\delta = 1 - 4\mu$. A possible explanation is that there is a missing potential energy associated with the shell assembly, as is found in a matter shell bounding Schwarzschild and vacuum [28]. However, Futamase and Garfinkle [40] have pointed out that the relation between angular deficit and mass density depends on the matter in the string, and one could take Eq. (49) as that relation for this anisotropic string. A third possibility is that there is additional, unconsidered structure at the axis. Noting that the metric along $r = 0$ describes a $2 + 1$ hypersurface, the $r = 0$ axis could be an Israel layer boundary. This provides additional interior structure whose mass needs to be considered in the complete calculation. With this possibility, the actual string would be the interior axial structure with the anisotropic stress energy described in this example serving as an atmosphere around the string.

IV. CONCLUSIONS

The $3 + 1$ field equations can be written as a set of $2 + 1$ equations on the space orthogonal to the Killing trajectories. In this paper we have presented a formalism that generalizes a simple $2 + 1$ action for a spacelike Killing vector to describe a set of fluids with anisotropic stress. The Lagrangian extension uses both a function $\mathcal{K}$, and a oneform $s_a dx^a$. The spatial part of s_a describes the stress anisotropy in the plane orthogonal to the Killing vector. With the Killing vector in the (3) direction, $\mathcal{K}$ relates the energy and stress and can describe the anisotropies between the (1,2) and (3) planes. $s_a dx^a$ is not varied in the action. One advantage of using a fixed form rather than a field is its use as a modeling tool, with the anisotropies related to complex motions rather than new physical fields. The anisotropy is described in terms of stress but can be due to a number of physical mechanisms, such as fluid shear.

Two applications were considered. An anisotropic generalization of the GH cosmic string interior described an interior solution with positive density and anisotropic stress bounded by an Israel layer with the GH equation of state. The Bianchi I example described a family of power-law solutions containing both focusing and unfocusing expansions. The family of stress energies was parameterized by a single constant, with the range of the constant describing the entire stress range: tension through dust to pressure. With the current strong interest in explaining anomalies in the CMB, one could consider models with a single anisotropic fluid whose stress shifts from pressure, through dust to tension and focusing to unfocusing during a series of expansion eras. Because the stress associated with the Killing symmetry is proportional to the Ricci scalar, the curvature will also evolve. In this model there is no expansion along the direction associated with the Killing coordinate. A completely anisotropic expansion could be generated by considering a higher dimensional action and associating the Killing symmetry with a higher dimensional manifold. Both applications considered here underline the potential value of formally considering anisotropic stresses in relativity; while they form a more complicated stress-energy description, anisotropic fluids can have dynamic features leading to simpler matter models than those with equal stress.

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