Classical stability of sudden and big rip singularities

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We introduce a general characterization of sudden cosmological singularities and investigate the classical stability of homogeneous and isotropic cosmological solutions of all curvatures containing these singularities to small scalar, vector, and tensor perturbations using gauge-invariant perturbation theory. We establish that sudden singularities at which the scale factor, expansion rate, and density are finite are stable except for a set of special parameter values. We also apply our analysis to the stability of Big Rip singularities and find the conditions for their stability against small scalar, vector, and tensor perturbations.

I. INTRODUCTION

We have shown in [1] that general relativistic Friedmann-Robertson-Walker (FRW) universes allow finite-time singularities to occur in which the scale factor, $a(t)$, its time derivative, $\dot{a}$, and the density, ρ, remain finite while a singularity occurs in the fluid pressure, $p \rightarrow +\infty$, and the expansion acceleration, with $\ddot{a} \rightarrow -\infty$. Remarkably, the strong energy condition $\rho + 3p > 0$ continues to hold. Analogous solutions are possible in which the singularity can occur only in arbitrarily high derivatives of $a(t)$, [2]. This behavior occurs independently of the 3-curvature of the universe and can prevent closed FRW universes that obey the strong energy condition from recollapsing [3]. These singularities can be seen in a wider context by classifying the behaviors of FRW universes containing matter with a pressure-density relation defined by $\rho + p = \gamma \rho^{\lambda}$, as shown in [4], and reviewed further in [5]. The sudden singular behavior found for a range of values of (γ, λ) also encompasses the evolution found in a number of simple bulk viscous cosmologies studied in [6]. Subsequently, a number of studies have been carried out which generalize these results to different cosmologies, other density-pressure relationships, and theories of gravity [7,8].

Other studies [9] have also related the sudden singularity behavior to the formal classifications of 'weak' singularities according to the definitions introduced by Krolak [10] and Tipler [11], investigated the behavior of geodesics, classified the other types of future singularity that can arise during the expansion of the Universe [12], discussed the role of different energy conditions on different realizations of sudden singularities [7,13], and explored some observational constraints on their possible future occurrence in our visible Universe [14]. The Weyl invariant will not diverge on approach to a sudden singularity (and there is no geodesic incompleteness [15]), so it may represent part of a 'soft' future boundary of the universe with low gravitational entropy – which could be as close as 8.7 Myr in the future.

Most recently, the effects of quantum particle production have been studied and have been found to leave sudden singularities in place [16]. Specifically, it was shown that quantum particle production does not dominate over the classical background density on approach to a sudden singularity and does not stop it occurring or modify its properties, as can be the case for the Big Rip future singularities [17,18]. The effects of loop quantum gravity have been studied in cosmologies exhibiting classical sudden singularities and they may remove the sudden singularity under certain particular conditions [19]. Sudden singularities have also been studied due to their occurrence in various theories of modified gravity [20], and we would only expect these modifications to be significant in this respect if they also dominate over general relativity effects at late times.

In this paper, we will extend previous studies by investigating the classical stability of sudden singularities with respect to small inhomogeneous scalar, vector, and tensor perturbations using the gauge-invariant formalism introduced by Mukhanov [21]. We introduce a new characterization of sudden singularities in terms of the series expansion of the expansion scale factor on approach to the singularity. We show that, except for a subset of special parameter choices, they are stable to small perturbations if and only if the density does not diverge near the singularity. The latter is characteristic of sudden singularities. We also extend this analysis and apply it to "Big Rip" singularities and determine the conditions under which they are stable and unstable.

II. BACKGROUND THEORY

Following [22], we consider perturbations of a general FRW metric,

$$ds^2 = a^2(\tau)\left(d\tau^2 - \delta_{ij}\left(1 + \frac{K(x^2 + y^2 + z^2)}{4}\right)^{-2} dx^i dx^j\right),$$

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where $K = 0$, 1, -1 depending on whether the three-dimensional hypersurfaces of constant τ time are spatially flat, closed or open. Here, τ denotes conformal time, and is related to the comoving proper time, t, by $a \, d\tau = dt$. We shall also assume that the primary component of matter is a perfect fluid with energy-momentum tensor (index notations run $1 \le i, j \le 3, 0 \le \alpha, \beta \le 3$)

$$T_\beta^\alpha = (\rho + p)u^\alpha u_\beta - p\delta_\beta^\alpha, \qquad (1)$$

but we shall not necessarily be specifying a specific equation of state linking p and ρ.

For this background universe, the equations for the scale factor $a(t)$, density ρ and pressure p are as follows (in units with $8\pi G = 1$ and $c = 1$), where the overdot denotes d/dt:

$$\dot\rho = -3\frac{\dot a}{a}(\rho + p) \qquad (2)$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\rho}{3} - \frac{K}{a^2} \qquad (3)$$

$$\frac{\ddot a}{a} = -\left(\frac{\rho + 3p}{6}\right) \qquad (4)$$

and we shall assume an equation of state with a functional form $p = p(\rho)$. We can see by inspection that these equations permit finite-time singularities such that a, $\dot a$ and ρ are finite but p and $\ddot a$ diverge as $t \to t_s$.

We shall adopt the following definition [23]: a sudden singularity will be said to occur at time $t = t_s$ if the scale factor $a(t)$ can be written in the form

$$a(t) = c_0 + c_1(t_s - t)^{\lambda_1} + c_2(t_s - t)^{\lambda_2} + \cdots \qquad (5)$$

in a generalized power series about t_s, where the c_i, λ_i are real constants, with $c_i \ne 0$, $c_0 > 0$ and $0 < \lambda_1 < \lambda_2 < \ldots$. with at least one of the λ_i nonintegral (so that some derivative of $a(t)$ blows up near the singularity). Note that the series (5) need not be infinite. Also, our form of (5) assumes that the sudden singularity occurs in the future; however, the stability results we derive will also hold for past sudden singularities, defined in the analogous way using

$$a(t) = c_0 + c_1(t - t_s)^{\lambda_1} + c_2(t - t_s)^{\lambda_2} + \cdots. \qquad (6)$$

This characterization encompasses the particular finite series expressions for FRW models with sudden singularities introduced in Refs. [1,2], those arising in the solutions found in Ref. [4], and those studied in Refs. [5,7,8].

The energy-momentum tensor defined in (1) leads to the following gauge-invariant perturbations [21]

$$\delta T_0^0 = \delta\rho, \qquad \delta T_i^0 = \frac{1}{a}(\rho_0 + p_0)\delta u_i,$$
$$\delta T_j^i = -\delta p \delta_j^i = -p'(\rho)\delta\rho\delta_j^i. \qquad (7)$$

Here, $\delta\rho$, δp and δu_i are the gauge-invariant perturbations of the density, pressure and velocity. We shall now consider separately the behavior of the tensor, vector and scalar modes as $t \to t_s$.

III. TENSOR PERTURBATIONS

Under tensor perturbations, the most general form of the line element is

$$ds^2 = a^2(\tau)\Big(d\tau^2 - \Big[\Big(1 + \frac{K}{4}(x^2 + y^2 + z^2)\Big)\delta_{ij} - h_{ij}\Big]dx^i dx^j\Big),$$

where $h_i^i = 0$, $h_{ij}^{|j} = 0$, where a slash indicates a covariant derivative with respect to the spatial 3-metric. Note that the quantity h_{ij} is gauge-invariant.

In conformal time, the equation for tensor perturbations is [24]

$$h_{ij}'' + \frac{2a'}{a}h_{ij}' - \Delta h_{ij} + 2Kh_{ij} = 0, \qquad (8)$$

where a prime denotes a derivative with respect to conformal time. The cosmic time analogue of this equation is

$$a^2\ddot h_{ij} + 3a\dot a\dot h_{ij} + k^2h_{ij} + 2Kh_{ij} = 0 \qquad (9)$$

for a plane wave perturbation with wave number k. We now set $h_{ij} = ve_{ij}$, where e_{ij} is a time-independent polarization tensor. This leads to the differential equation

$$a^2\ddot v + 3a\dot a \, \dot v + (k^2 + 2K)v = 0. \qquad (10)$$

From (5), we see that in the limit $t \to t_s$, we have

$$a(t) = c_0 + c_1(t_s - t)^{\lambda_1} + \cdots \qquad (11)$$

$$\dot a(t) = -c_1\lambda_1(t_s - t)^{\lambda_1 - 1} + \cdots, \qquad (12)$$

where $\lambda_1 > 0$, and $c_0, c_1 \ne 0$. We can substitute these into (10) to obtain the asymptotic ordinary differential equation

$$c_0^2\ddot v - 3c_0c_1\lambda_1 T^{\lambda_1 - 1}\dot v + (k^2 + 2K)v = 0, \qquad (13)$$

where we have taken $T = t_s - t$ and dots now indicate differentiation with respect to T. We would like to investigate whether solutions to (13) exhibit blow-up near $T = 0$.

In the case $\lambda_1 < 1$, a very similar analysis to that described in the Appendix shows that v tends to a constant as $T \to 0$. Eqs. (A4) onwards still hold.

In the case $\lambda_1 \ge 1$, all coefficients of (13) are nonsingular, so the ordinary differential equation is regular and hence has no singularity at $T \to 0$. In fact, if $k^2 + 2K > 0$, we end up with a simple harmonic oscillator (which is damped if $\lambda_1 = 1$), whose solutions are bounded at $T = 0$.

Hence, both long- and short-wavelength tensor perturbations are bounded near the singularity. This clearly holds

for all values of the λ_i, and the sudden singularity is always stable against inhomogeneous gravitational-wave perturbations.

IV. VECTOR PERTURBATIONS

For vector perturbations, the most general form of the line element is

$$ds^2 = a^2(\tau)\Big(d\tau^2 + 2S_i dx^i d\tau$$
$$- \Big[\Big(1 + \frac{K}{4}(x^2 + y^2 + z^2)\Big)\delta_{ij} - F_{i;j} - F_{j;i}\Big]dx^i dx^j\Big),$$

where $S_i^{|i} = F_i^{|i} = 0$. The quantity $V_i \equiv S_i - F_i'$ is gauge-invariant.

The only part of the energy-momentum tensor which contributes to vector perturbations is [21]

$$\delta T_i^0 = \frac{1}{a}(\rho_0 + p_0)\delta u_{\perp i}, \tag{14}$$

where $\delta u_{\perp i}$ is the part of δu_i with zero divergence, and ρ_0 and p_0 are the background density and pressure, respectively. The equations for the vector perturbations are

$$(2K - k^2)V_i = 2a(\rho_0 + p_0)\delta u_{\perp i} \tag{15}$$

$$a(V_{i,j} + V_{j,i})^{\cdot} + 2\dot{a}(V_{i,j} + V_{j,i}) = 0. \tag{16}$$

For both long- and short-wavelength perturbations, we obtain as a consequence of the conservation of angular momentum:

$$V_i = \text{const} \times \frac{1}{a^2}, \tag{17}$$

$$\delta v^i = \text{const} \times \frac{1}{a^4(\rho_0 + p_0)}, \tag{18}$$

where the physical velocities δv^i are defined by $\delta v^i \equiv -a^{-1}\delta u_{\perp i}$. Hence, vector perturbations of the metric are bounded on approach to the sudden singularity since $a \to a_s < \infty$, $\rho_0 \to \rho_s < \infty$ and $p_0 \to \infty$.

V. SCALAR PERTURBATIONS

A. Overview

We have shown, above, that tensor and vector perturbations do not diverge near the sudden singularity, for all values of λ_i in the form (5). The analysis of scalar perturbations is slightly more involved, and we shall need to consider various cases according to the values taken by λ_1 and λ_2. This is not unexpected. The sudden singularity is primarily created by the behavior of the pressure and so we expect the scalar perturbation modes associated with pressure inhomogeneities to play a significant role in controlling the stability.

Under scalar perturbations, the most general form of the line element is

$$ds^2 = a^2(\tau)\Big((1 + 2\phi)d\tau^2 - 2B_{;i}dx^i d\tau$$
$$- \Big[(1 - 2\psi)\Big(1 + \frac{K}{4}(x^2 + y^2 + z^2)\Big)\delta_{ij} + 2E_{;ij}\Big]$$
$$\times dx^i dx^j\Big),$$

and we can define the usual gauge-invariant quantities

$$\Phi \equiv \phi - \frac{1}{a}[a(B - E')]' \quad \text{and} \quad \Psi \equiv \psi + \frac{a'}{a}(B - E').$$

The equations for scalar perturbations are, following [22]:

$$\Phi = \Psi \tag{19}$$

$$a^2\ddot{\Phi} + (4 + 3p'(\rho))a\dot{a}\,\dot{\Phi} + (2a\ddot{a} + (\dot{a}^2 - K)(1 + 3p'(\rho))$$
$$+ p'(\rho)k^2)\Phi = 0 \tag{20}$$

for plane wave perturbations with wave number k, where $p'(\rho) \equiv dp/d\rho$. We can then find the gauge-invariant perturbed quantities as follows:

$$\delta\rho = -\frac{6\dot{a}}{a}\dot{\Phi} - \Big(\frac{2k^2 + 6\dot{a}^2}{a^2}\Big)\Phi, \tag{21}$$

$$\delta p = p'(\rho)\delta\rho. \tag{22}$$

B. The general case

First, consider the case when $\lambda_1 \equiv \lambda$ is nonintegral. In this case we find, in the limit $t \to t_s$:

$$a(t) = c_0 + c_1(t_s - t)^\lambda + \cdots \tag{23}$$

$$\dot{a}(t) = -c_1\lambda(t_s - t)^{\lambda-1} + \cdots \tag{24}$$

$$\ddot{a}(t) = c_1\lambda(\lambda - 1)(t_s - t)^{\lambda-2} + \cdots \tag{25}$$

and from (3) and (4) we obtain the leading-order approximations

$$\rho = \frac{3K}{c_0^2} + \frac{3\lambda^2 c_1^2}{c_0^2}(t_s - t)^{2(\lambda-1)} + \cdots \tag{26}$$

$$p = -\frac{K}{c_0^2} - \frac{2c_1\lambda(\lambda - 1)}{c_0}(t_s - t)^{\lambda-2} + \cdots \tag{27}$$

for the density and pressure, respectively. (Note that, regardless of the value of K, the density diverges if and only if $\lambda < 1$, and the pressure diverges if and only if $\lambda < 2$.) We also find that

$$p'(\rho) = \frac{dp}{d\rho} = \frac{dp/dt}{d\rho/dt} = \frac{1}{3}\left(\frac{\dot{a}^3 - a^2\dddot{a} + K\dot{a}}{\dot{a}(a\ddot{a} - \dot{a}^2 - K)}\right)$$

$$= -\frac{c_0(\lambda - 2)}{3c_1\lambda}(t_s - t)^{-\lambda} \qquad (28)$$

to leading order, regardless of the value of K.

Substituting the above forms into (20), and neglecting higher-order terms, gives us the following equation (where $T = t_s - t$ and dots now indicate differentiation with respect to T):

$$\ddot{\Phi} - (\lambda - 2)T^{-1}\dot{\Phi} + \left(\frac{c_1}{c_0}\lambda^2 T^{\lambda-2}\right.$$

$$\left. - \frac{(k^2 - 3K)(\lambda - 2)}{3c_0c_1\lambda}T^{-\lambda} - \frac{K}{c_0^2}\right)\Phi = 0 \quad (29)$$

and we can neglect the last $\left(\frac{K\Phi}{c_0^2}\right)$ term since, for all values of λ, it will be dominated in magnitude by one of the other two terms making up the coefficient of Φ, as $T \to 0$. We will now consider the cases $\lambda < 1$ and $\lambda > 1$ separately.

If $\lambda < 1$, then (29) becomes

$$\ddot{\Phi} - (\lambda - 2)T^{-1}\dot{\Phi} + \frac{c_1}{c_0}\lambda^2 T^{\lambda-2}\Phi = 0 \qquad (30)$$

and applying the substitutions $P = T^{(1-\lambda)/2}\Phi$ and $x = CT^{\lambda/2}$ gives the equation

$$P''x^2 + P'x + \left(\frac{4c_1x^2}{c_0C^2} - \frac{(\lambda - 1)^2}{\lambda^2}\right)P = 30, \qquad (31)$$

where primes denote differentiation with respect to x. If $c_1 > 0$, we set $C = 2\sqrt{c_1/c_0}$, and obtain a Bessel equation with solution

$$P(x) = \tilde{A}J_\nu(x) + \tilde{B}Y_\nu(x)$$

$$\nu = \frac{\lambda - 1}{\lambda}$$

with $\tilde{A}$, $\tilde{B}$ arbitrary constants. Since we are considering the limit $x \to 0$, the leading-order solution is

$$P(x) = Ax^{(\lambda-1)/\lambda} + Bx^{(1-\lambda)/\lambda} \qquad (32)$$

or

$$\Phi(t) = A(t_s - t)^{\lambda-1} + B, \qquad (33)$$

where A, B are new constants. Therefore, since $\lambda < 1$, $\Phi(t)$ diverges in general as $t \to t_s$. The case $c_1 < 0$ can be treated similarly. Note that this result holds, independently of k, for both long- and short-wavelength perturbations and for all K.

Now suppose that $\lambda > 1$. Then (29) becomes

$$\ddot{\Phi} - (\lambda - 2)T^{-1}\dot{\Phi} - DT^{-\lambda}\Phi = 0 \qquad (34)$$

where $D = \frac{(k^2-3K)(\lambda-2)}{3c_0c_1\lambda}$. Analogously to the above, we

substitute $P = T^{(1-\lambda)/2}\Phi$, and then $x = CT^{1-\lambda/2}$, which leads to the Bessel equation

$$P''x^2 + P'x + \left(x^2 - \frac{(1-\lambda)^2}{(2-\lambda)^2}\right)P = 0 \qquad \nu = \frac{\lambda - 1}{2 - \lambda}$$

$$(35)$$

where we have set $C = 2\sqrt{|D|}/|\lambda - 2|$.

As $x \to 0$, this has the asymptotic solution

$$P(x) = Ax^{(\lambda-1)/(2-\lambda)} + Bx^{(1-\lambda)/(2-\lambda)} \qquad (36)$$

or

$$\Phi(t) = A(t_s - t)^{\lambda-1} + B. \qquad (37)$$

Thus, if $\lambda > 1$, the scalar metric perturbations do not diverge as $t \to t_s$. Again, note that this holds for all values of K, and for both long- and short-wavelength perturbations.

Hence, the scalar metric perturbations diverge for all $0 < \lambda_1 < 1$, and are bounded for all nonintegral $\lambda_1 > 1$. In fact, the analysis above also holds for all integral $\lambda_1 > 2$. In this case, the density and pressure tend to constants near the singularity, and the metric perturbations do not diverge. Thus, we just need to deal separately with the boundary cases of $\lambda_1 = 1$ and $\lambda_1 = 2$.

C. The case $\lambda_1 = 1$, $\lambda_2 \neq 2$

Consider the case $\lambda_1 = 1 < \lambda_2 < \ldots$. and assume first that $\lambda_2 \neq 2$. So, as $t \to t_s$:

$$a(t) = c_0 + c_1(t_s - t) + c_2(t_s - t)^{\lambda_2} + \cdots \qquad (38)$$

$$\dot{a}(t) = -c_1 - c_2\lambda_2(t_s - t)^{\lambda_2-1} + \cdots \qquad (39)$$

$$\ddot{a}(t) = c_2\lambda_2(\lambda_2 - 1)(t_s - t)^{\lambda_2-2} + \cdots \qquad (40)$$

$$\dddot{a}(t) = -c_2\lambda_2(\lambda_2 - 1)(\lambda_2 - 2)(t_s - t)^{\lambda_2-3} + \cdots \quad (41)$$

$$\rho = \frac{3(K + c_1^2)}{c_0^2} + \cdots \qquad (42)$$

$$p = -\frac{(K + c_1^2)}{c_0^2} - \frac{2c_2\lambda_2(\lambda_2 - 1)(t_s - t)^{\lambda_2-2}}{c_0} + \cdots.$$

$$(43)$$

We can see that ρ tends to a constant as we approach the singularity, but p tends to a constant if and only if $\lambda_2 \geq 2$, and diverges otherwise.

We can also obtain the following expressions:

$$p'(\rho) = \begin{cases} -\frac{c_0(\lambda_2-2)}{3c_1}(t_s-t)^{-1} & \text{if } 1<\lambda_2<2 \\ \frac{c_0^2 c_2 \lambda_2(\lambda_2-1)(\lambda_2-2)}{3c_1(K+c_1^2)}(t_s-t)^{\lambda_2-3} & \text{if } 2<\lambda_2<3 \\ \frac{6c_0^2 c_2 - c_1^3 - Kc_1}{3c_1(K+c_1^2)} & \text{if } \lambda_2=3 \\ -\frac{1}{3} & \text{if } \lambda_2>3 \end{cases}$$

if $K + c_1^2 \neq 0$, and

$$p'(\rho) = -\frac{c_0(\lambda_2-2)}{3c_1}(t_s-t)^{-1}$$

for all λ_2, if $K + c_1^2 = 0$ (i.e. $K=-1$, $c_1=\pm 1$).

First, we consider the case $K + c_1^2 \neq 0$. Substituting the forms of $p'(\rho)$ into (20) gives

$$\ddot{\Phi} + AT^{-1}\dot{\Phi} + BT^{-1}\Phi = 0 \quad \text{if } 1<\lambda_2<2$$
$$\ddot{\Phi} + AT^{\lambda_2-3}\dot{\Phi} + BT^{\lambda_2-3}\Phi = 0 \quad \text{if } 2<\lambda_2<3$$
$$\ddot{\Phi} + A\dot{\Phi} + B\Phi = 0 \quad \text{if } \lambda_2 \geq 3$$

where A and B are always constants. The first differential equation yields an asymptotic solution $\Phi(t) \approx A' + B'(t_s-t)^{\lambda_2-1}$, so the scalar perturbations are bounded in this case. Showing that solutions of the second differential equation are bounded as $T \to 0$ is more cumbersome: a derivation is given in the Appendix. For $\lambda_2 \geq 3$, the coefficients of the differential equation are constants, so the general solution has the form

$$\Phi(t) \approx A'\exp(\Lambda_1(t_s-t)) + B'\exp(\Lambda_2(t_s-t)), \quad (44)$$

which is bounded as $t \to t_s$.

The same analysis can be used to show that the solution in the case $K + c_1^2 = 0$ is also bounded as $T \to 0$. Notice that these results hold for both long- and short-wavelength perturbations.

D. The case $\lambda_1 = 1$, $\lambda_2 = 2$

Now let $\lambda_1 = 1$, $\lambda_2 = 2 < \lambda_3 < \cdots$. As $t \to t_s$:

$$a(t) = c_0 + c_1(t_s-t) + c_2(t_s-t)^2 + c_3(t_s-t)^{\lambda_3} + \cdots \quad (45)$$

$$\dot{a}(t) = -c_1 - 2c_2(t_s-t) - c_3\lambda_3(t_s-t)^{\lambda_3-1}\cdots \quad (46)$$

$$\ddot{a}(t) = 2c_2 + c_3\lambda_3(\lambda_3-1)(t_s-t)^{\lambda_3-2} + \cdots \quad (47)$$

$$\dddot{a}(t) = -c_3\lambda_3(\lambda_3-1)(\lambda_3-2)(t_s-t)^{\lambda_3-3} + \cdots \quad (48)$$

$$\rho = \frac{3(K+c_1^2)}{c_0^2} + \cdots \quad (49)$$

$$p = -\frac{(K+c_1^2)}{c_0^2} - \frac{4c_2}{c_0} + \cdots \quad (50)$$

so the density and pressure tend to constants near the

singularity. Also:

$$p'(\rho) = \begin{cases} \frac{c_0^2 c_3 \lambda_3(\lambda_3-1)(\lambda_3-2)}{3c_1(c_1^2+K-2c_0c_2)}(t_s-t)^{\lambda_3-3} & \text{if } 2<\lambda_3<3 \\ \frac{6c_0^2 c_3 - Kc_1 - c_1^3}{3c_1(c_1^2+K-2c_0c_2)} & \text{if } \lambda_3=3 \\ \frac{c_1^2+K}{3(2c_0c_2-c_1^2-K)} & \text{if } \lambda_3>3 \end{cases}$$

and we can now substitute these into (20) to get the following functional forms of the scalar perturbation equation (where $T = t_s - t$, dots indicate derivatives with respect to T, and A, B are constants):

$$\ddot{\Phi} + AT^{\lambda_3-3}\dot{\Phi} + BT^{\lambda_3-3}\Phi = 0 \quad \text{if } 2<\lambda_3<3$$
$$\ddot{\Phi} + A\dot{\Phi} + B\Phi = 0 \quad \text{if } \lambda_3 \geq 3$$

Equations of this form arose in the previous section, and we showed that their solutions are bounded as $T \to 0$.

The above expressions for $p'(\rho)$ are not well-defined if $K = 2c_0c_2 - c_1^2$. For this special case, we can calculate

$$p'(\rho) = \begin{cases} -\frac{c_0(\lambda_3-2)}{3c_1}(t_s-t)^{-1} & \text{if } 2<\lambda_3<3 \\ -\frac{c_0}{3c_1}(t_s-t)^{-1} & \text{if } \lambda_3 \geq 3 \end{cases}$$

and the corresponding scalar perturbation equation is

$$\ddot{\Phi} + A\dot{\Phi} + B\Phi/T = 0.$$

Solutions of this differential equation can be expressed in terms of Whittaker functions, and are bounded as $T \to 0$.

Hence, the scalar perturbations are bounded for $\lambda_1 = 1$, $\lambda_2 = 2$. This is quite a strong result, since most "nice" functions $a(t)$ can be expressed as power series. We have shown that for all such functions, as long as the coefficient of the t term is nonzero near the singularity (i.e. $\dot{a}(t_s) \neq 0$), the sudden singularity is a stable solution of the FRW equations.

E. The case $\lambda_1 = 2$

The final case to consider is:

$$a(t) = c_0 + c_1(t_s-t)^2 + c_2(t_s-t)^{\lambda_2} + \cdots, \quad (51)$$

where the $c_i \neq 0$, $c_0 > 0$ and $2 < \lambda_2 < \ldots$ with at least one of the λ_i nonintegral. As $t \to t_s$, we have:

$$\dot{a}(t) = -2c_1(t_s-t) - c_2\lambda_2(t_s-t)^{\lambda_2-1} + \cdots \quad (52)$$

$$\ddot{a}(t) = 2c_1 + c_2\lambda_2(\lambda_2-1)(t_s-t)^{\lambda_2-2} + \cdots \quad (53)$$

$$\dddot{a}(t) = -c_2\lambda_2(\lambda_2-1)(\lambda_2-2)(t_s-t)^{\lambda_2-3} + \cdots \quad (54)$$

and $\rho \to 3K/c_0^2$, $p \to -(4c_1c_0 + K)/c_0^2$.

First, let us assume $K = 0$. There are three cases to consider for $p'(\rho)$:

$$p'(\rho) = \begin{cases} -\frac{c_0 c_2 \lambda_2 (\lambda_2-1)(\lambda_2-2)}{12 c_1^2}(t_s - t)^{\lambda_2-4} & \text{if } 2 < \lambda_2 < 6 \\ \frac{2 c_1^3 - 30 c_0^2 c_2}{3 c_0 c_1^2}(t_s - t)^2 & \text{if } \lambda_2 = 6 \\ \frac{2 c_1}{3 c_0}(t_s - t)^2 & \text{if } \lambda_2 > 6 \end{cases}$$

We now substitute these expressions into the equation for scalar perturbations (20). Using $A, B, \ldots.$ to denote constants, the functional forms of the differential equations obtained are:

$$\ddot{\Phi} + AT^{\lambda_2-3}\dot{\Phi} + (BT^{\lambda_2-2} + Ck^2T^{\lambda_2-4})\Phi = 0 \quad \text{if } 2 < \lambda_2 < 4$$
$$\ddot{\Phi} + AT\dot{\Phi} + B\Phi = 0 \qquad\qquad\qquad \text{if } \lambda_2 \geq 4$$

The solutions of the second differential equation are bounded as $T \to 0$. If we assume that $k \neq 0$, the solution of the first differential equation can be expressed in terms of hypergeometric functions, and is bounded as $T \to 0$.

Now suppose that $K \neq 0$. There are two subcases: $K \neq 2c_1 c_0$ and $K = 2c_1 c_0$. For the first subcase, we have:

$$p'(\rho) = \begin{cases} \frac{c_0^2 c_2 \lambda_2 (\lambda_2-1)(\lambda_2-2)}{6 c_1 (K-2c_1 c_0)}(t_s - t)^{\lambda_2-4} & \text{if } 2 < \lambda_2 < 4 \\ \frac{24 c_0^2 c_2 - 2Kc_1}{6 c_1 (K-2c_1 c_0)} & \text{if } \lambda_2 = 4 \\ \frac{K}{3(2c_1 c_0 - K)} & \text{if } \lambda_2 > 4 \end{cases}$$

and the corresponding differential equations are

$$\ddot{\Phi} + AT^{\lambda_2-3}\dot{\Phi} + (BT^{\lambda_2-2} + Ck^2T^{\lambda_2-4})\Phi = 0 \quad \text{if } 2 < \lambda_2 < 4$$
$$\ddot{\Phi} + AT\dot{\Phi} + B\Phi = 0 \qquad\qquad\qquad \text{if } \lambda_2 \geq 4$$

with the same stability results as above.

Finally, we treat the special case $K = 2c_1 c_0$. We obtain:

$$p'(\rho) = \begin{cases} -\frac{c_0(\lambda_2-2)}{6c_1}(t_s - t)^{-2} & \text{if } 2 < \lambda_2 < 4 \\ -\frac{c_0}{3c_1}(t_s - t)^{-2} & \text{if } \lambda_2 \geq 4 \end{cases}$$

and this corresponds to:

$$\ddot{\Phi} - \frac{6c_1 C}{c_0}T^{-1}\dot{\Phi} + \frac{3(k^2 - K)C}{c_0^2}T^{-2}\Phi = 0$$

for all $\lambda_2 > 2$, where $p'(\rho) = CT^{-2}$. In general, the solutions of this equation are of the form $\Phi(T) = T^\gamma$ where

$$\gamma^2 + (-6c_1 C/c_0 - 1)\gamma + 3(k^2 - K)C/c_0^2 = 0.$$

For boundedness we need the two roots of this equation to be non-negative, i.e. we require $-6c_1 C \leq c_0$ and $(k^2 - K)C \geq 0$. It is easy to see that the first condition only holds for $\lambda_2 \leq 3$. So, in this subcase, we only have boundedness if $\lambda_2 \leq 3$ and $(k^2 - 2c_1 c_0)C \geq 0$, and divergence otherwise.

So all the perturbations are bounded except for the very special case $K \neq 0$, $K = 2c_1 c_0$, $\lambda_2 > 3$ (and perhaps $\lambda_2 \leq 3$).

F. Consideration of the sign of c_s^2

Note that our analysis here has not assumed a simple fluid equation of state of the form $p = w\rho$, or indeed any functional relation between the density and pressure of the matter source, as would characterize a perfect fluid of k-essence or its generalizations (see [25] for a discussion). Thus there is no general constraint arising from the positivity of the square of an effective speed of sound, c_s^2. In cases that reduce to fluids, or to k-essence and its relatives, it should be possible to introduce further constraints upon the series coefficients in order to preserve the positivity of c_s^2.

VI. EXTENSIONS TO BIG RIP MODELS

We can extend the above analysis easily to Big Rip models [26], where the scale factor behaves as:

$$a(t) = c_0(t_s - t)^{\eta_0} + c_1(t_s - t)^{\eta_1} + \cdots, \quad (55)$$

where $\eta_0 < 0$, $\eta_0 < \eta_1 < \ldots$, and also $c_0 > 0$, $c_i \neq 0$ for all i. Then, to leading order, we have

$$\dot{a}(t) = -\eta_0 c_0(t_s - t)^{\eta_0-1} + \cdots \quad (56)$$

$$\ddot{a}(t) = \eta_0(\eta_0 - 1)c_0(t_s - t)^{\eta_0-2} + \cdots \quad (57)$$

$$\dddot{a}(t) = -\eta_0(\eta_0 - 1)(\eta_0 - 2)c_0(t_s - t)^{\eta_0-3} + \cdots \quad (58)$$

$$\rho = 3\eta_0^2(t_s - t)^{-2} + \cdots \quad (59)$$

$$p = (2 - 3\eta_0)\eta_0(t_s - t)^{-2} + \cdots \quad (60)$$

$$p'(\rho) = 2/3\eta_0 - 1 + \cdots. \quad (61)$$

Note that $a(t)$, ρ, $p \to \infty$ as $t \to t_s$. It is easy to see that, near the singularity, the tensor perturbations h_{ij} and the vector perturbations V_i decay to zero. However, the physical velocities δv^i are proportional to $(t_s - t)^{-2-4\eta_0}$, so for them to be finite as $t \to t_s$ we need $\eta_0 \leq -1/2$.

Scalar perturbations obey the following equation:

$$\ddot{\Phi} + (2 + \eta_0)\frac{\dot{\Phi}}{T} + \left(\frac{2(K - k^2)}{c_0^2} + \frac{2(k^2 - 3K)}{3\eta_0 c_0^2}\right)T^{-2\eta_0}\Phi = 0 \quad (62)$$

where $T = t_s - t$ and dots denote differentiation with respect to T. A solution of (62) can be found in terms of Bessel functions, and we find that the solutions asymptotically tend to

$$\Phi(T) = C_1 + C_2 T^{-1-\eta_0}. \quad (63)$$

So there is divergence as $t \to t_s$ if $\eta_0 > -1$, otherwise Φ is bounded.

Note that these results match those of [27], which shows that, for equations of state of the form $p = \alpha\rho$ (where α is a constant), there is a discrepancy in behavior between the

cases $\alpha > -5/3$ and $\alpha < -5/3$. It can easily be checked that these correspond to the cases $\eta_0 < -1$ and $\eta_0 > -1$ respectively.

VII. CONCLUSION

We have produced a simple general characterization of sudden singularities in FRW universes. By the use of gauge-invariant perturbation theory we have investigated whether the existence of sudden singularities in a FRW cosmology is stable to small scalar, vector, and tensor inhomogeneities. We have shown that the existence of sudden singularities is stable when the density is bounded near the singularity except for some cases with special parameter choices. This result holds regardless of whether the background metric is spatially flat, closed or open. We also applied our analysis to a complementary characterization of Big Rip singularities and showed that they are stable if the leading term in the time-dependence of the FRW scale factor is proportional to $a(t) = (t_s - t)^{\eta_0}$ where $\eta_0 \leq -1$, and unstable otherwise. As discussed in the introduction, there have been a number of identifications of sudden singularity occurrence in theories of gravity other than general relativity. The approach described in this paper can also be straightforwardly applied in these theories to determine whether the sudden singularities that occur there are also stable.

ACKNOWLEDGMENTS

APPENDIX: EQUATIONS OF THE FORM $\ddot{\Phi} + AT^s\dot{\Phi} + BT^s\Phi = 0$

Consider the equation

$$\ddot{\Phi} + AT^s\dot{\Phi} + BT^s\Phi = 0 \qquad (A1)$$

where $\Phi = \Phi(T)$, $-1 < s < 0$ and A, B are arbitrary constants. We want to investigate the behavior of this equation as $T \to 0$.

We first reduce the equation to canonical form $\ddot{\phi} + f(T)\phi = 0$. The substitution $\Phi = \phi \exp(-AT^{s+1}/2(s+1))$ achieves this, and yields

$$\ddot{\phi} + \left(BT^s - \frac{1}{2}AsT^{s-1} - \frac{1}{4}A^2T^{2s}\right)\phi = 0. \qquad (A2)$$

We now let $Y(t) = t\phi(1/t)$ and this gives

$$\ddot{Y} + \left(Bt^{-s-4} - \frac{1}{2}Ast^{-s-3} - \frac{1}{4}A^2t^{-2s-4}\right)Y = 0. \qquad (A3)$$

where $t = 1/T$ and we now study the behavior as $t \to +\infty$. The substitutions $Y(t) = \exp(\phi(t))$ and $u(t) = \dot{\phi}(t)$ transform (83) into a Riccati equation, which (after neglecting subleading terms) becomes:

$$\dot{u} + u^2 - \frac{1}{2}Ast^{-s-3} = 0. \qquad (A4)$$

Finally we make the substitution $u = \dot{y}/y$ to obtain the Emden-Fowler equation

$$\ddot{y} = \frac{1}{2}Asyt^{-s-3}. \qquad (A5)$$

This has a solution in terms of Bessel functions:

$$y(t) = t^{1/2}\left(C_1 J_{1/(s+1)}\left(\frac{2\sqrt{-D}}{-s-1}t^{(-s-1)/2}\right) + C_2 Y_{1/(s+1)}\left(\frac{2\sqrt{-D}}{-s-1}t^{(-s-1)/2}\right)\right) \quad \text{if } D < 0 \quad (A6)$$

$$y(t) = t^{1/2}\left(C_1 I_{1/(s+1)}\left(\frac{\sqrt{D}}{-s-1}t^{(-s-1)/2}\right) + C_2 K_{1/(s+1)}\left(\frac{\sqrt{D}}{-s-1}t^{(-s-1)/2}\right)\right) \quad \text{if } D > 0 \quad (A7)$$

where $D = As/2$. We use the asymptotic behavior of the Bessel functions as $t \to +\infty$ to obtain

$$y(t) \sim C_3 + C_4 t \qquad (A8)$$

and $u(t) \sim 1/t$. So $Y(t) \sim t$ and $\Phi(T) \to \text{const}$ as $T \to 0$.

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