Cosmological fluctuations from infrared cascading during inflation

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We propose a qualitatively new mechanism for generating cosmological fluctuations from inflation. The nonequilibrium excitation of interacting scalar fields often evolves into infrared (IR) and ultraviolet cascading, resulting in an intermediate scaling regime. We observe elements of this phenomenon in a simple model with inflaton ϕ and isoinflaton χ fields interacting during inflation via the coupling $g^2(\phi - \phi_0)^2\chi^2$. Isoinflaton particles are created during inflation when they become instantaneously massless at $\phi = \phi_0$, with occupation numbers not exceeding unity. Previous studies have focused on the momentary slowing down of the condensate $\phi(t)$ by back-reaction effects. Here, we point out that very quickly the produced χ particles become heavy and their multiple rescatterings off the homogeneous condensate $\phi(t)$ generates Bremsstrahlung radiation of light inflaton IR fluctuations with high occupation numbers. The subsequent evolution of these IR fluctuations is qualitatively similar to that of the usual inflationary fluctuations, but their initial amplitude is different. The IR cascading generates a bump-shaped contribution to the cosmological curvature fluctuations, which can even dominate over the usual fluctuations for $g^2 > 0.06$. The IR cascading curvature fluctuations are significantly non-Gaussian, and the strength and location of the bump are model dependent, through g^2 and ϕ_0. The effect from IR cascading fluctuations is significantly larger than that from the momentary slowing down of $\phi(t)$. With a sequence of such bursts of particle production, the superposition of the bumps can lead to a new broadband non-Gaussian component of cosmological fluctuations added to the usual fluctuations. Such a sequence of particle creation events can, but need not, lead to trapped inflation.

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I. INTRODUCTION

In addition to the standard mechanism for generating cosmological perturbations during inflation from the vacuum fluctuations of the inflaton field [1], there are alternative mechanisms including modulated fluctuations (inhomogeneous preheating) [2,3] and the curvaton [4], both of which are based on the vacuum fluctuations of isoinflaton fields during inflation. In this paper, we propose a new and qualitatively different mechanism for generating cosmological fluctuations during inflation.

Physical processes during inflation may leave their imprint as features in the cosmological fluctuations. These can, in principle, be observed if they fall in the range of the wavelengths between 10^4 Mpc and 100 Kpc, which corresponds to about ten e-folds during inflation. There may also be signatures such as additional features at the horizon scale or potential observables on much smaller scales. Relevant dynamical models were studied in the early days of the inflationary theory, for example, the model with phase transitions during inflation yielding associated features in the cosmological fluctuations [5–7] (see also [8]). There the time-dependent dynamics of the inflaton field ϕ can trigger a phase transition in the isoinflaton χ field. The growth of χ inhomogeneities induces curvature fluctuations on scales leaving the horizon at the moment of the phase transition.

Recently, several studies [5,9–12] considered features in the cosmological fluctuations from the effect of particle creation during inflation, which can be modeled by the simple interaction

$$\mathcal{L}_{\rm int} = -\frac{g^2}{2}(\phi - \phi_0)^2\chi^2, \tag{1}$$

with some value of the scalar field ϕ_0, which the rolling $\phi(t)$ crosses during inflation that must be tuned to give a signal in the observable range of e-folds. There are different motivations for the model (1). The early study [13] introduced the possibility of slowing down the fast rolling inflaton using particle creation via the interaction (1). Imagine there are a number of field points ϕ_{0i}, $i = 1, 2, \cdots, n$, where the isoinflaton field becomes massless and χ particles are created. The produced χ particles are diluted by the expansion of the Universe, however, the back-reaction effect from multiple bursts of particle creation may slow down the motion of ϕ sufficiently to allow for slow-roll inflation. This is called trapped inflation. A more concrete string theory realization of trapped inflation, based on the sequence of $D3$ branes interactions, was discussed in [10]. The work [12], which is complementary to this study, provides a detailed realization of trapped inflation in the context of the string theory model [14].

The instant of χ-particle creation and the slow down of the rolling inflaton shall generate a feature in the power spectrum of scalar curvature fluctuations from inflation $P_\zeta(k)$. This was noticed in [9], where the features in the power spectrum were estimated from the simple-minded formula $P_\zeta(k) \sim (\frac{H^2}{\dot\phi})^2$. The inflaton slow down was described by the mean-field equation

$$\ddot\phi + 3H\dot\phi + V_{,\phi} + g^2(\phi - \phi_0)\langle\chi^2\rangle = 0. \qquad (2)$$

The vacuum expectation value $\langle\chi^2\rangle$ can be calculated with the analytic machinery of particle creation with the coupling (1), which was developed in the theory of preheating after inflation [15,16]. The quantum field theory (QFT) of χ particles interacting with the time-dependent condensate $\phi(t)$ deals with the eigenmodes $\chi_k(t)e^{i\vec{k}\cdot\vec{x}}$, where the time-dependent mode function obeys an oscillatorlike equation in an expanding Universe

$$\ddot\chi_k + 3H\dot\chi_k + \left[\frac{\vec{k}^2}{a^2} + g^2(\phi(t) - \phi_0)^2\right]\chi_k = 0, \qquad (3)$$

with time-dependent frequency $\omega_k(t)$. When $\phi(t)$ crosses the value ϕ_0, the χ_k mode becomes massless, and $\omega_k(t)$ varies nonadiabatically. Around this point $(\phi(t) - \phi_0) \approx \dot\phi_0(t - t_0)$, where $t = t_0$ is corresponding time instant. With this very accurate [16] approximation, one can solve Eq. (3) analytically to obtain the occupation number of created χ particles

$$n_k = \exp\left(-\frac{\pi k^2}{k_\star^2}\right), \qquad k_\star^2 = g|\dot\phi_0|, \qquad (4)$$

presuming that $k_\star > H$. The latter condition requires coupling constant $g > H^2/|\dot\phi_0| \sim 10^{-4}$. It is useful to note that, independent of the details of $V(\phi)$ and $\phi(t)$, the scale $k_\star$ can be related to the naively estimated amplitude of vacuum fluctuations as $k_\star/H = \sqrt{g/(2\pi\mathcal{P}_\zeta^{1/2})}$. Thus, $k_\star/H \sim 30$ if $\mathcal{P}_\zeta^{1/2} = 5\times 10^{-5}$ as suggested by the CMB, and the coupling is $g^2 \sim 0.1$.[1]

The vacuum expectation value $\langle\chi^2\rangle$, which controls the back reaction on the homogeneous field $\phi(t)$, can be calculated from (4) and estimated as $\langle\chi^2\rangle = \int\frac{d^3k}{(2\pi)^3}|\chi_k|^2 \approx \int\frac{d^3k}{(2\pi)^3}\frac{n_k}{\omega_k} \approx \frac{n_\chi a^{-3}}{g|\phi-\phi_0|}$ for $\phi > \phi_0$. Substitution of this results back into (2) gives expected velocity dip of $\phi(t)$ and, correspondingly, a bump in the power spectrum $P_\zeta(k)$. In Fig. 1 we illustrate this velocity dip for the model (1) with $g^2 = 0.1$.

The calculation of curvature fluctuations in the model (1) was reconsidered in [11], where the linearized equations of motion for the quantum fluctuations $\delta\phi$ coupled with the metric fluctuations were treated again in the mean-field approximation, using $\langle\chi^2\rangle$ to quantify the back reaction. This study shows that the bump in the curvature power spectrum is the most prominent part of an otherwise wiggling pattern. Similar to us, the work [12] further refined the calculation of the curvature perturbation in this model, going beyond the mean-field treatment of ϕ.[2]

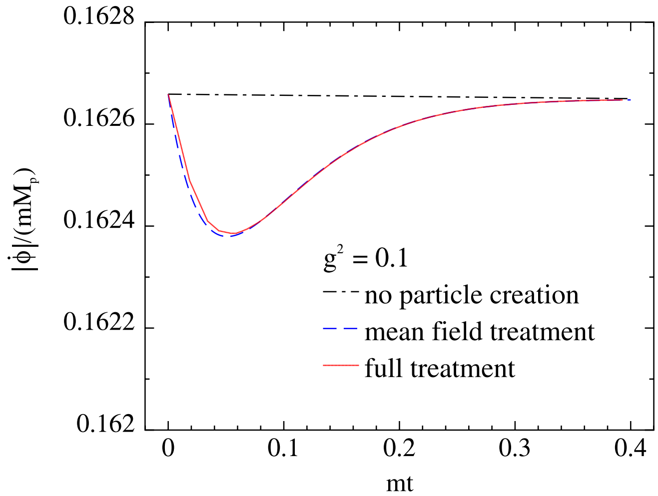

FIG. 1 (color online). $|\dot\phi|/(M_p m)$ plotted against mt for $g^2 = 0.1$ (where $m = V_{,\phi\phi}$ is the effective inflaton mass). Time $t = 0$ corresponds to the moment when $\phi = \phi_0$ and χ particles are produced copiously. The solid red line is the lattice field theory result taking into account the full dynamics of rescattering and IR cascading, while the dashed blue line is the result of a mean-field theory treatment, which ignores rescattering [11]. The dotted-dashed black line is the inflationary trajectory in the absence of particle creation.

In a parallel development, scalar fields interactions of the type (1) are the subject of studies in nonequilibrium QFT and its application to the theory of preheating after inflation, as we mentioned above. Although we study particle production *during* inflation (as opposed to during preheating, after inflation) there are many similarities. For example, in the case of parametric resonant preheating due to the oscillating inflaton background $\phi(t)$, χ particles are created in successive bursts whenever $\phi(t)$ crosses zero and the χ particles become instantaneously massless. This leads to huge occupation numbers of the created χ fluctuations. On the other hand, in the scenario described above there is only a single burst of particle production, and the resulting χ occupation number (4) is always less than unity.

The full dynamics of interacting scalars during preheating also includes not only bursts of particle production but also rescattering effects where $\delta\phi$ fluctuations (particles) are created very quickly due to the interaction of created χ

[1] We are assuming that supersymmetry protects the inflaton potential from radiative corrections at $t = t_0$. An explicit realization of the type of coupling we are interested in, based on global $\mathcal{N} = 1$ supersymmetry, has been provided in [17]. For string theory models the reader is referred to [10,12,14].

[2] Below we use QFT methods to study correlators of inhomogeneous fluctuations $\delta\phi$ induced by χ^2 inhomogeneities. Reference [12] considers the effect induced by quantum mechanical fluctuations of the total particle number n_χ. Owing to the relationship between χ^2 and n_χ, our calculations below capture this effect.

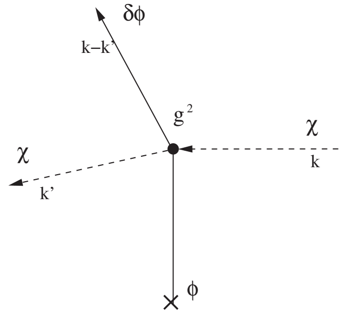

FIG. 2. Rescattering diagram.

particles with the condensate $\phi(t)$ [16,18,19]. The diagram for this process is illustrated in Fig. 2. The $\delta\phi$ particles produced by rescattering are far from equilibrium and evolve toward an intermediate regime, which is well described by the scaling "turbulent" solution. To understand the dynamics, one needs to use lattice numerical simulations of time evolution of the classical scalar fields based on the LATTICEASY [20] or DEFROST [21] codes, designed for this purpose. The turbulent regime of interacting scalars was investigated in several recent works. The papers [22,23] used numerical simulations to demonstrate the scaling regime in the model of self-interacting classical scalar $\lambda\phi^4$. The papers [24,25] show numerically the scaling solution for the fully QFT treatment of the same model, and advocate the new regime, the nonthermal fixed point, which may be asymptotically long (before the system evolves, if ever, to another fixed point: thermal equilibrium).

In this paper, we will study in detail the back reaction of χ particles produced during inflation on the inflaton field, resulting in Bremsstrahlung radiation and IR cascading of $\delta\phi$ fluctuations. Our results will also apply to the early stages of rescattering in preheating after inflation (this is so because the time scale for rescattering is short and hence the results are insensitive to the expansion of the Universe).

II. NUMERICAL STUDY OF RESCATTERING

To study the creation of $\delta\phi$ fluctuations by rescattering of produced χ particles off the condensate $\phi(t)$ in the model (1) we have adapted the numerical DEFROST code for the problem of a single burst of instantaneous particle creation during inflation.[3] To run the classical scalar field simulation, we must first choose the appropriate initial conditions. The field χ on the lattice is modeled by the random Gaussian field realized as the superposition of planar waves $\chi_k(t)e^{i\vec{k}\vec{x}}$ with random phases. The initial conditions for the models $\chi_k(t)$ are chosen to emulate the exact quantum mode functions corresponding to the physical occupation number (4) [see appendix A for more details] while ensuring that the source term for the $\delta\phi$ fluctuations turns on smoothly at $t = 0$. The box size of our 512^3 simulations corresponds to a comoving scale, which initially is $\frac{20}{2\pi} \sim 3$ times the horizon size $1/H$, while $k_\star \cong 60\sqrt{g}H$. We run our simulations for roughly three e-folding from the initial moment t_0 when χ particles are produced, although a single e-folding would have been sufficient to capture the effect. We are interested in the power spectrum of inflaton fluctuations $P_\phi = k^3|\delta\phi|^2/(2\pi^2)$, and also the number density of inflaton fluctuations $n_\phi(k) = \frac{\Omega_k}{2}(\frac{|\delta\dot{\phi}_k|^2}{\Omega_k^2} + |\delta\phi_k|^2)$ (where we introduce the notation $\Omega_k = \sqrt{V_{,\phi\phi} + k^2}$ for the inflaton frequency). For the sake of illustration we have chosen the standard chaotic inflationary potential $V = m^2\phi^2/2$ with $m^2 = 10^{-6}M_p$ and $\phi_0 = 3.2M_p$, however, our qualitative results will be independent of the choice of background

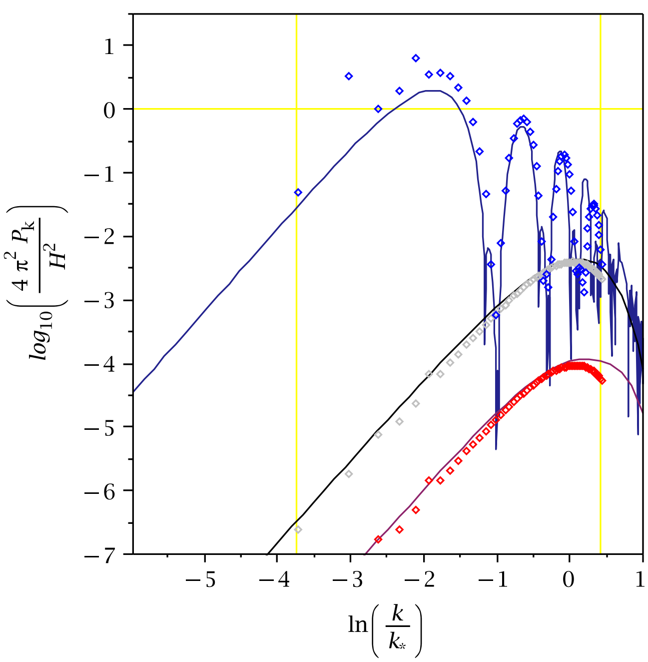

FIG. 3 (color online). The power spectrum of inflaton modes induced by rescattering (normalized to the usual vacuum fluctuations) as a function of $\ln(k/k_\star)$, plotted for three representative time steps in the evolution, showing the cascading of power into the IR. For each time step we plot the analytical result (the solid line) and the data points obtained using lattice field theory simulations (diamonds). The time steps correspond to the following values of the scale factor: $a = 1.03$, 1.04, 2.20 (where $a = 1$ at the moment when $\phi = \phi_0$). By this time the amplitude of fluctuations is saturated due to the expansion of the Universe. The vertical lines show the range of scales from our lattice simulation.

[3]Since the production of long wavelength $\delta\phi$ modes is so energetically inexpensive, a major requirement for successfully capturing this effect on the lattice is respecting energy conservation to very high accuracy. In our modified version of DEFROST energy conservation is respected with an accuracy of order 10^{-8}, compared to $10^{-3} - 10^{-5}$ obtained using previous codes. A minimum accuracy of roughly 10^{-4} is required for this problem.

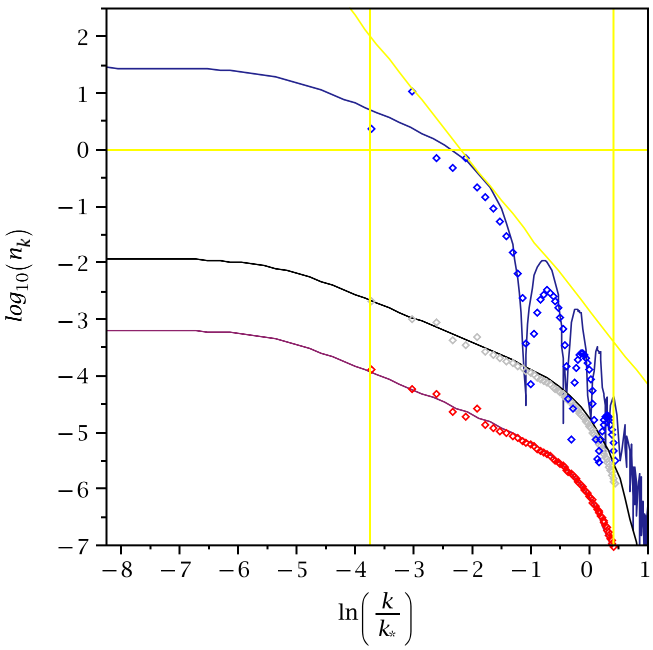

FIG. 4 (color online). Physical occupation number n_k as a function of $\ln(k/k_\star)$ for $g^2 = 0.1$. The three curves correspond to the same series of time steps used in Fig. 3, and demonstrate the growing number of long-wavelength inflaton modes, which are produced as a result of IR cascading. Because the same χ particle can undergo many rescatterings off the background condensate $\phi(t)$, the $\delta\phi$ occupation number is larger than the initial χ particle number (for $g^2 = 0.1$ one can achieve $n_\phi(k) \sim 30$ even though initially $n_\chi(k) \leq 1$). When $g^2 = 0.06$ the IR $\delta\phi$ occupation number exceeds unity within a single e-folding. The yellow envelope line shows the early onset of scaling behavior associated with the scaling turbulent regime.

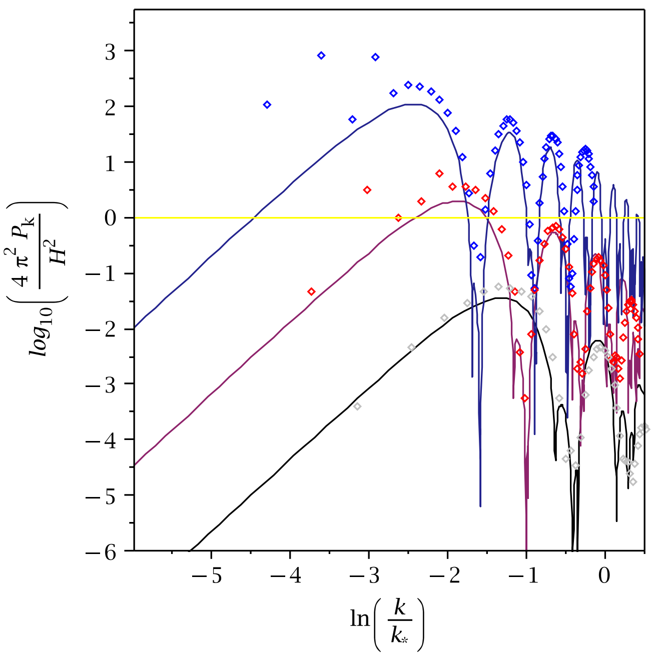

FIG. 5 (color online). The dependence of the power spectrum P_ϕ on the coupling g^2. The three curves correspond to P_ϕ for $g^2 = 0.01, 0.1, 1$, evaluated at a fixed value of the scale factor, $a = 2.20$. We see that even for small values of g^2 the inflaton modes induced by rescattering constitute a significant fraction of the usual vacuum fluctuations after only a single e-folding.

inflation model[4] and, in particular, are applicable to trapped inflation. We have considered three different values of the coupling constant, $g^2 = 0.01, 0.1, 1$, although we focus most of our attention on the case $g^2 = 0.1$. Figure 3 shows time evolution of the rescattered inflaton power spectrum $P_\phi(k)$ for three different time steps, while Fig. 4 shows the corresponding evolution of the particle number density $n_\phi(k)$. In Fig. 5 we illustrate the dependence of our results on the coupling constant g^2.

In Fig. 3 we see clearly how multiple rescatterings lead to a cascading of power into the IR. These rescattered inflaton perturbations are complementary to the usual long-wavelength inflaton modes produced by quantum fluctuations. As long as $g^2 > 0.06$ the rescattered power

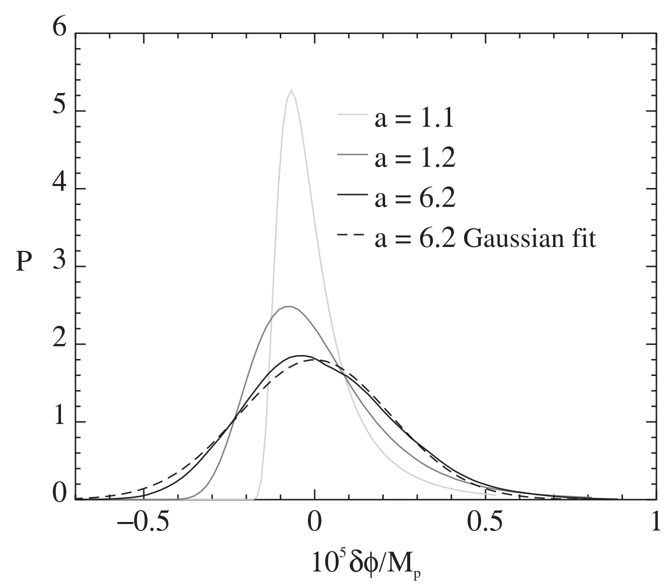

FIG. 6. Probability density function of $\delta\phi$ for $g^2 = 1$ at a series of different values of the scale factor a. The dashed curve shows a Gaussian fit at late time $a = 6.2$.

[4]The choice of background inflationary potential will alter the functional form of $\phi(t)$, however, all the dynamics of rescattering occurs within a single e-folding of the moment when $\phi = \phi_0$. Hence, for any inflation model it will be a good approximation to expand $\phi(t) = \phi_0 + vt$, and our results should depend only on the velocity $v \equiv \dot{\phi}(t = 0)$ [assuming this to be nonzero], which is related to the Hubble scale and the observed amplitude of the curvature perturbation. The claim of model independence is borne out by explicit analytical calculations in the next section. There we find that the dominant contribution to P_ζ from IR cascading depends sensitively only on the ratio $k_\star/H$ which, as we have shown, is determined entirely by the coupling g^2.

spectrum outside the horizon comes to dominate over the usual vacuum fluctuations within a single e-folding. At much later times the IR portion of the power spectrum remains frozen, while the UV portion is damped out by the Hubble expansion. The effect of IR cascading on the power spectrum is much more significant than features that are produced by the momentary slowing down of the background $\phi(t)$.[5]

Figure 1 illustrates the impact of rescattering on the dynamics of the velocity of the background field. The evolution of $\dot{\phi}(t)$ including rescattering is not changed significantly (as compared to the mean-field theory result), which show the energetic cheapness of IR cascading.

The long-wavelength inflaton fluctuations produced by IR cascading are non-Gaussian. This is illustrated in Fig. 6 where we study the probability density function and compare to a Gaussian fit.

III. ANALYTICAL THEORY OF RESCATTERING

We now develop an analytical theory of this effect. Here, we provide only a cursory discussion, the reader is referred to Appendices A and B for a detailed exposition and technical details of the calculation. At leading order the physics of rescattering (see Fig. 2) is described by the equation

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\vec{\nabla}^2\delta\phi + m^2\delta\phi \cong -g^2[\phi(t) - \phi_0]\chi^2, \tag{5}$$

where we introduce the notation $m^2 = V_{,\phi\phi}$ for the effective inflaton mass (hence, we are not assuming a background potential of the form $m^2\phi^2/2$ in this section, only that $V_{,\phi\phi} \neq 0$ in the vicinity of the point $\phi = \phi_0$). The solution of (5) consists of two components: the solution of the homogeneous equation, which simply corresponds to the usual vacuum fluctuations produced during inflation and the particular solution, which is due to the source term. We will focus our attention on this latter solution which, physically, corresponds to rescattered inflaton perturbations. Since the process of IR cascading takes less than a single e-folding, we can safely neglect the expansion of the Universe when studying analytically the particular solution of (5). [In all of our lattice simulations the inflationary expansion of the Universe is taken into account consistently.] Solving for the particular solution $\delta\phi_k$ of (5) and defining the rescattered power spectrum P_ϕ in terms of the QFT correlation function in the usual manner we arrive at an expression for P_ϕ in terms of the c-number mode functions χ_k, which obey Eq. (3)[6] The result is

$$P_\phi = \frac{g^4\dot{\phi}_0^2}{8\pi^5}\frac{k^3}{\Omega_k^2}\int dt'dt''t't''\sin[\Omega_k(t-t')] \times \sin[\Omega_k(t-t'')]\int d^3k'\chi_{k-k'}(t')\chi^\star_{k-k'}(t'')\chi_{k'}(t') \times \chi^\star_{k'}(t''), \tag{6}$$

where again we have $\Omega_k = \sqrt{k^2+m^2}$ for the $\delta\phi$-particle frequency.

To evaluate this power spectrum we need an expression for the solutions of (3) in the regime of interest. Let us choose the origin of time so that $t = 0$ corresponds to the moment when $\phi = \phi_0$. At the moment $t = 0$ the parameter $|\dot{\omega}_k|/\omega_k^2$ is order unity or larger and ω_k varies non-adiabatically. At this point χ_k modes are produced in the momentum band $k \lesssim k_\star$. However, within a time $\Delta t \sim k_\star^{-1}$ (which is tiny compared to the Hubble time H^{-1}) the χ particles become extremely heavy and their frequency again varies adiabatically. At times $t \gtrsim k_\star^{-1}$ we can safely approximate $\omega_k = \sqrt{k^2 + k_\star^4t^2} \cong k_\star^2 t$ for the modes of interest and χ_k takes the simple form

$$\chi_k(t) \cong \sqrt{1+n_k}\frac{e^{-i(k_\star t)^2/2}}{k_\star\sqrt{2t}} - i\sqrt{n_k}\frac{e^{+i(k_\star t)^2/2}}{k_\star\sqrt{2t}}, \tag{7}$$

where the occupation number was defined in (4). The factors $\sqrt{1+n_k}$, $-i\sqrt{n_k}$ are the Bogoliubov coefficients, while the factors proportional to $e^{\pm i(k_\star t)^2/2}$ come from the positive and negative frequency adiabatic mode solutions [16]. As we see, very quickly after $t = 0$ the χ particles become very massive and their multiple rescatterings off the condensate $\phi(t)$ generates Bremsstrahlung radiation of IR $\delta\phi$ particles.

We have computed the full renormalized power spectrum analytically in closed form, and the result is presented in Eq. (A17). This formula is used for all of our figures. Since the exact analytical result is quite cumbersome, it is useful to consider the following representative contribution to (6):

$$P_\phi \simeq \frac{g^2k^3k_\star^3}{32\sqrt{2}\pi^5}\left[\frac{1-\cos(\Omega_k t)}{\Omega_k^2}\right]^2 e^{-\pi k^2/(2k_\star^2)}, \tag{8}$$

which captures the properties of the full analytical solution. In particular, the simple expression (8) nicely describes the IR cascade. The spectrum has a peak, which initially (near $t \sim k_\star^{-1}$), is close to $k_\star$. As time evolves the peak moves to smaller-and-smaller k as power builds up in the IR. From

[5] To avoid confusion: here we use "cascading" to refer to the dynamical process of building up $\delta\phi$ fluctuations in the IR. If the Universe were not expanding, a scaling turbulent regime would be established. Here, we see this scaling regime only in an embryonic form, see the envelope in Fig. 4.

[6] We are only interested in connected contributions to the correlation functions, which is equivalent to subtracting the expectation value from the source term in (5): $\chi^2 \rightarrow \chi^2 - \langle\chi^2\rangle$. Thus, our rescattered inflaton modes are only sourced by the variation of χ^2 from the mean $\langle\chi^2\rangle$.

(8) we see that modes with $\Omega_k t < 1$ gain power as $P_\phi(k) \sim t^4$. For a given k mode the growth of the power spectrum saturates when $\Omega_k t \sim 1$, however, the cascade still continues at some lower k. If we had $m = 0$ then the cascade would continue forever, otherwise formula (8) predicts that the growth of the spectrum saturates at $t \sim m^{-1}$ when the peak has reached $k \sim m$. After this point the character of the IR cascade is expected to change, however, our analytic calculation is no longer reliable because $m \ll H$, and we have neglected the expansion of the Universe. Notice from Eq. (8) that the value of P_ϕ/H^2 at the peak (which is the main observable signal) is fixed by the ratio $k_\star/H$, which in turn depends only on the coupling g^2 (there is some dependence also on $m^2 = V_{,\phi\phi}$, however, this is subdominant since $m^2 \ll H^2$ for any inflation model). This observation confirms our previous claim that the dynamics of IR cascading are largely insensitive to the choice of background inflation model.

IV. DISCUSSION OF CURVATURE FLUCTUATIONS FROM IR CASCADING

Any inflaton fluctuations $\delta\phi$, independently on their origin, evolve qualitatively similarly during inflation. When their physical wavelength is smaller than the Hubble radius $1/H$, $\delta\phi$ is oscillating while their amplitude is diluted as $1/a$. As far as the wavelength exceeds the Hubble radius, the amplitude of $\delta\phi$ freezes out. Fluctuations of $\delta\phi$ induce the curvature metric fluctuations. Inflationary expansion of the Universe further stretch the wavelengths of the fluctuations frozen outside the horizon, making them potentially of the cosmological scales, depending on the wavelength. The inflaton fluctuations produced by the IR cascading, therefore, are the potential sources for observable curvature fluctuations. To calculate curvature fluctuations generated by the IR cascading, we have to solve a self-consistent system of linearized Einstein equations for metric and the fields fluctuations. For example, the (0, 0) linearized Einstein equation for our model reads

$$\delta G^0_0 = \frac{8\pi}{M_p^2}(\delta T^0_0(\phi) + \delta T^0_0(\chi)), \tag{9}$$

where δG^0_0 is the perturbed Einstein tensor and in the right-hand side $\delta T^0_0(\phi)$ corresponds to the fluctuations of the inflaton energy density, containing familiar terms linear with respect to $\delta\phi$, like $\dot\phi\delta\dot\phi$, etc. The second term corresponds to the contribution from χ particles

$$\delta T^0_0(\chi) = \frac{1}{2}\dot\chi^2 + \frac{1}{2}(\nabla\chi)^2 + \frac{1}{2}g^2(\phi - \phi_0)^2\chi^2 - \langle T^0_0(\chi)\rangle. \tag{10}$$

Although this expression is bilinear with respect to χ, it turns out taking $\delta T^0_0(\chi)$ into account is important. To begin the investigation the Eq. (9), it is convenient to use its Fourier transformation. While the Fourier components of $\delta T^0_0(\phi)$ contains linear terms of $\delta\phi_k$, the Fourier transform of $\delta T^0_0(\chi)$ contains convolutions like $\sim \int d^3\vec{k}'\dot\chi_{\vec{k}'}\dot\chi^*_{\vec{k}-\vec{k}'}$, etc. As a result, despite the fact that χ particles amplitude is peaked at $k \sim k_\star$, this type of convolution gives significant contribution at small k, which are of interest for the theory of generation of cosmological fluctuations. Preliminary estimations based on the analytical formulas for χ_k involved in the convolution, show that contribution of $\delta T^0_0(\chi)$ is at least the same order of magnitude as $\delta T^0_0(\phi)$. Rigorous treatment of the curvature fluctuations in our model is therefore rather complicated, and will be leave it for separated project.

Since both terms in right-hand side in (9) are the same order of magnitude, here for the crude estimations we will use the simple-minded formula $P_\zeta \sim (\frac{H^2}{\dot\phi})^2$. The curvature fluctuations generated by the IR cascading, are illustrated in the top panel of Fig. 7. The curvature fluctuations from the instance of the IR cascading has the bumplike shape within the interval of the wavelength, roughly corresponding to one e-folding. They are significantly, by orders of magnitude, dominated over the fluctuations generated by the momentary slowing down of $\phi(t)$. If we pick up the background inflationary model to be chaotic inflation with the standard quadratic potential, the ratio of the power spectra from IR cascading and the standard fluctuations estimates as $P_{\rm IR}/P_s \sim 700 \times g^{4.5}$. Thus, depending on the coupling g^2, the IR bump can dominate (for $g^2 > 0.06$) over the standard fluctuations, or just contribute to them for smaller g^2.

Suppose that we have a sequence of the particles creation events at different moments t_{0i}, $i = 1, 2, 3, \ldots$. Each of those events generate, through IR cascading, corresponding bumps in the spectrum, as illustrated in the lower panel of Fig. 7. Depending on the density of t_{0i} moments, superposition of such IR bumps results in broadband contribution to the curvature power spectrum.

We also estimated non-Gaussianity of $\delta\phi$ fluctuations from IR cascading. They are quite significant, we estimate the non-Gaussianity parameter $f_{\rm NL} \sim 2 \times 10^4 g^{2.25}$. Therefore, the non-Gaussian signal from individual bump can be strongly non-Gaussian. In the model with multiple instances of particle creations, the broadband IR cascading fluctuations dominated over the standard fluctuations, apparently, are ruled because of the strong non-Gaussianity. However, the broadband IR cascading fluctuations can be considered as additional subdominant component to the standard fluctuations. In this case the non-Gaussianity of the net curvature fluctuations can be acceptable but different from that of the standard fluctuations alone. We leave a detailed discussion of the non-Gaussianities produced during IR cascading (and their observability) to future studies. Notice that this type of

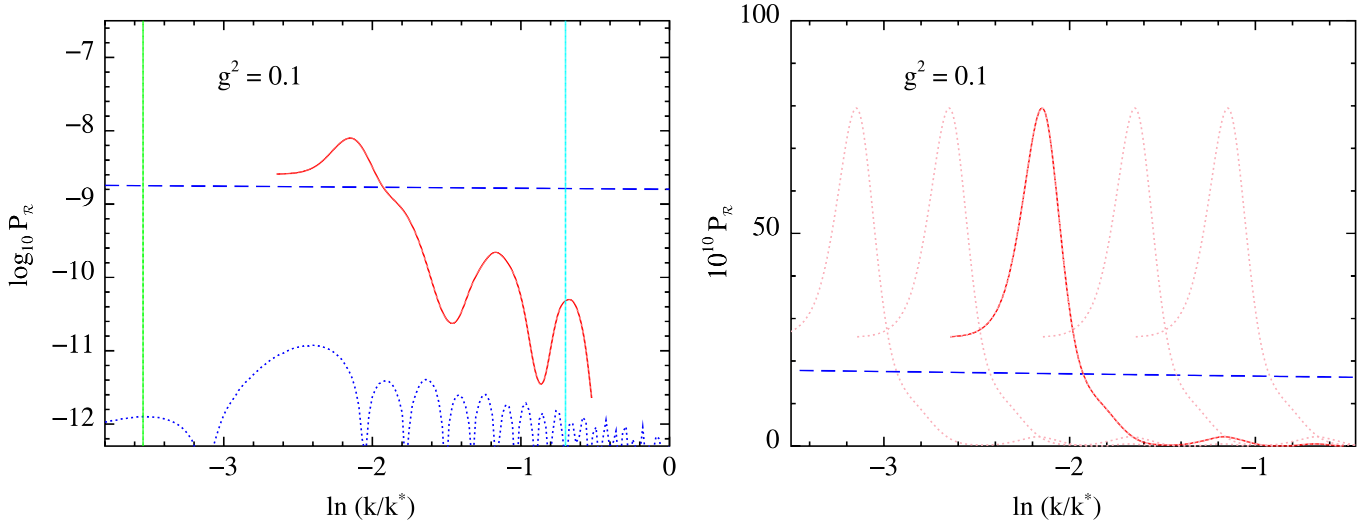

FIG. 7 (color online). The top panel shows a comparison of curvature fluctuations from different effects. We see the dominance of fluctuations produced by IR cascading over the wiggles induced by the momentary slowing down of the inflaton. For illustration we have taken $g^2 = 0.1$, but the dominance is generic for all values of the coupling. The red solid line is the IR cascading curvature power spectrum, while the blue dashed line is the result of a mean-field treatment. (The vertical lines show aH at the beginning of particle production and after ~3 e-foldings.) The bottom panel shows the curvature power spectrum resulting from multiple bursts of particle production and IR cascading. Superposing a large number of these bumps produces a broadband spectrum.

non-Gaussianity, which is localized over a narrow region of scales, is similar to what was obtained in [26].

Another important parameter of the IR cascading fluctuations is the wavelength of the bump, which depend on the value of ϕ_0. there are interesting possibilities to consider them at small CMB angular scales (small-scale non-Gaussianity?), at scales of galaxies, at near the horizon scales (CMB anomalies at large scales?). We leave all of these possibilities for future discussion.

V. SUMMARY AND CONCLUSIONS

We find the following new results for interacting scalars during inflation in the model (1).

(i) In the early stages of rescattering, when the back reaction can be treated linearly, the spectrum of inflaton fluctuations $\delta\phi$ and the corresponding particle number density $n_\phi(k)$ can be rigorously calculated with QFT with the diagram in Fig. 2. We perform such QFT calculations and compare with lattice simulations of the classical field dynamics. The results are highly compatible with each other, even well into the late time nonlinear regime. This signals the dominance of Fig. 2 in the dynamics of rescattering, while the analytic estimate gives a handy fitting formula.

(ii) While the stationary scaling turbulent solution for the scalar fields after preheating was established previously, the way this regime appears dynamically was not traced out in detail. For our example for the first time we show explicitly how this scaling regime is seeded. In the absence of expansion of the Universe, this embryonic scaling behavior will develop into the full turbulent regime that has previously been observed, however, for our purposes only the early stages are relevant because fluctuations freeze outside of the horizon.

(iii) The most unexpected result, which is of interest outside of the inflationary theory, is that even an insignificant amount of out-of-equilibrium particles with $n_\chi(k) \leq 1$, being rescattered off the scalar field condensate, can generate IR cascade of the inhomogeneous condensate fluctuations with a large occupation number $n_\phi(k)$ in the IR region. This is explained by the fact that multiple production of the IR modes is energetically cheap.

(iv) IR fluctuations of the light fields have special significance in the context of inflationary theory. These fluctuations evolve in time similar to the evolution of the usual inflationary fluctuations. Their amplitude is oscillating, while their wavelengths are inside the Hubble radius and are frozen out once their wavelengths exceed the Hubble radius H^{-1}. However, the amplitude of IR cascading fluctuations is different from that of the usual quantum fluctuations. Frozen fluctuations $\delta\phi$, regardless of their origin, will induce cosmological curvature fluctuations. Thus, we get a new mechanism for generating frozen long-wavelength $\delta\phi$ fluctuations from IR cascading. Therefore, IR cascading will lead to observable features in the CMB power spectrum. For generic choices of parameters, these rescattered fluctuations are much more significant than the features induced by the momentary slowing down of the background $\phi(t)$, see the upper panel of Fig. 7.

(v) Since the solution $\delta\phi$ of (5) depends nonlinearly on the Gaussian field χ, the curvature fluctuations induced by IR cascading will be non-Gaussian. This non-Gaussianity is illustrated in Fig. 6. We estimate this non-Gaussianity to be significant. However, this non-Gaussian signal is related to the IR cascading bump of the spectrum and peaks on the range of scales corresponding to roughly one e-folding after $t = t_0$. This type of non-Gaussianity, which is large only over a small range of scales, is not well constrained by observation.

(vi) The strength and location of our effect is model dependent (through g^2 and ϕ_0), however, the very fact that subtle QFT effects of interaction during inflation may lead to an observable effect is intriguing.

(vii) In our analysis we have focused on a single burst of instantaneous particle production during inflation. This scenario is interesting in its own right, however, our results could also be extended in a straightforward manner to study trapped inflation models where there are numerous bursts of particle production; see [12] for more detailed discussion.

(viii) Suppose we have a sequence of points ϕ_{0i} ($i = 1, \cdots, N$) where particles χ_i become massless. In this case the curvature fluctuation profiles generated from individual bursts of IR cascading can superpose to form a smooth spectrum of cosmological fluctuations, see the lower panel of Fig. 7. This provides us with a new mechanism for generating long-wavelength curvature fluctuations during inflation from IR cascading. The amplitude and non-Gaussianity of these curvature fluctuations will depend on the coupling, g^2. These fluctuations are interesting on their own, although they may generate too much non-Gaussianity. They also can be considered as an extra component of the standard vacuum fluctuations, introducing an interesting non-Gaussian signal to the net fluctuations.

(ix) The transfer of energy into fluctuations via successive bursts of particle production can lead to trapped inflation. Our new mechanism of generating cosmological fluctuations from IR cascading can, but need not, be associated with trapped inflation.

(x) Varying the location, strength and non-Gaussianity of the IR cascading bump, it will be interesting to consider other potential implication to the cosmological fluctuations, e.g. their impact on the horizon scale fluctuations or on small-scale fluctuations where they might effect primordial black hole formation or the generation of gravitational waves.

Finally, let us return to the old story of how cosmological fluctuations are affected by phase transition during inflation, which we discussed at the beginning of this paper. We project that our results concerning rescattering and IR cascading will radically change the conventional picture.

ACKNOWLEDGMENTS

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APPENDIX A: ANALYTICAL THEORY OF RESCATTERING

In this appendix we develop an analytical theory of rescattering, which is in good agreement with the result of fully nonlinear lattice field theory simulations. As usual we split the inflaton field into a classical homogeneous component and quantum inhomogeneities as $\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x})$ such that $\langle\phi(t, \mathbf{x})\rangle = \phi(t)$, and we further suppose that $\langle\chi(t, \mathbf{x})\rangle = 0$. Since IR cascading occurs within a single e-folding we can safely neglect the expansion of the Universe. However, there is no obstruction to consistently including this effect [27].

At leading order the physics of rescattering is described by Eq. (5), which corresponds to the diagram in Fig. 2. There is a correction to (5) corresponding to a diagram where two $\delta\phi$ particles interact with two χ particles, however, this effect is subleading [16]. It is understood that one must subtract from (5) the expectation value of the right-hand side in order to consistently define the quantum operators $\delta\phi$ such that $\langle\delta\phi\rangle = \langle\chi\rangle = 0$. Subtracting off this expectation value is equivalent to only considering connected diagrams when we compute correlation functions.

1. Production of χ particles

To solve Eq. (5) we first require explicit expressions for the background field $\phi(t)$ and the wavefunction $\chi(t, \mathbf{x})$. Let us choose the origin of time so that $\phi = \phi_0$ at $t = 0$. Near the moment of particle production we can expand $\phi(t) - \phi_0 \cong \dot{\phi}_0 t$. The interaction term in (1) induces induces a mass for the χ field

$$m_\chi^2 = g^2[\phi(t) - \phi_0]^2 \cong g^2\dot{\phi}^2 t^2 \equiv k_\star^4 t^2, \qquad \text{(A1)}$$

which vanishes at $t = 0$. At this moment particles will be copiously produced by quantum effects.

The mode functions $\chi_k(t)$ obey the following equation:

$$\ddot{\chi}_k(t) + \omega_k^2(t)\chi_k(t) = 0, \tag{A2}$$

where the time-dependent frequency is

$$\omega_k(t) = \sqrt{k^2 + m_\chi^2} = \sqrt{k^2 + k_\star^2(k_\star t)^2}. \tag{A3}$$

The theory of Eq. (A2) is well studied in the literature [10,16]. As long as the frequency (A3) varies adiabatically $|\dot{\omega}_k|/\omega_k^2 \ll 1$ the modes of χ will not be excited and are well described by the adiabatic solution $\chi_k(t) = f_k(t)$, where we have defined

$$f_k(t) \equiv \frac{1}{\sqrt{2\omega_k(t)}} \exp\left[-i\int^t dt'\omega_k(t')\right]. \tag{A4}$$

However, very close to $t = 0$, roughly within the interval $-k_\star^{-1} < t < +k_\star^{-1}$, the parameter $|\dot{\omega}_k|/\omega_k^2$ can become order unity or larger for low momenta $k \lesssim k_\star$, and χ_k modes within this band will be produced. The general solution of (A2) can be written in terms of the adiabatic modes (A4) and the time-dependent Bogoliubov coefficients as

$$\chi_k(t) = \alpha_k(t)f_k(t) + \beta_k(t)f_k(t)^\star, \tag{A5}$$

where the Bogoliubov coefficients obey a set of coupled ordinary differential equations with initial conditions $|\alpha_k(0^-)| = 1$, $\beta_k(0^-) = 0$. Near $t = 0$ the adiabaticity condition is violated, and β_k grows rapidly away from zero as a steplike function. Very shortly after this burst of particle production the frequency again varies adiabatically and α_k, β_k become constant, taking the following values [16]:

$$\alpha_k(t > 0) = \sqrt{1 + n_k}, \tag{A6}$$

$$\beta_k(t > 0) = \sqrt{n_k}e^{i\delta_k}, \tag{A7}$$

where the physical occupation number is defined by (4). The phase δ_k has been computed analytically in [15] and depends nontrivially on k. However, since most of the particle production occurs for momenta $k \lesssim k_\star$ it is an excellent approximation to use the simple result $e^{i\delta_k} \cong -i$. (We have verified that changing the relative phase will at most alter factors order unity in the final results.)

We are now in a position to write out the solution for the χ_k modes in the outgoing adiabatic regime $t \gtrsim k_\star^{-1}$. Since most of our interest is in IR modes with $k \lesssim k_\star$, it is a good approximation to expand the frequency (A3) as $\omega_k(t) \cong k_\star(k_\star t)$. Using the Eqs. (A6) and (A7) we can write the solution (A5) in the region of interest as

$$\chi_k(t) \cong \sqrt{1 + n_k}\frac{e^{-i(k_\star t)^2/2}}{k_\star\sqrt{2t}} - i\sqrt{n_k}\frac{e^{+i(k_\star t)^2/2}}{k_\star\sqrt{2t}}. \tag{A8}$$

2. Equations for rescattering

Having reviewed the solutions for $\phi(t)$ and $\chi_k(t)$ we now turn our attention to solving (5). Let us first briefly discuss our conventions for Fourier transforms and mode functions. We write the q-number valued Fourier transform of χ as

$$\chi(t, \mathbf{x}) = \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \xi_{\mathbf{k}}^\chi(t). \tag{A9}$$

Because χ is Gaussian we can expand $\xi_{\mathbf{k}}^\chi$ into c-number mode functions χ_k (discussed above) and annihilation/creation operators $a_{\mathbf{k}}$, $a_{\mathbf{k}}^\dagger$ as

$$\xi_{\mathbf{k}}^\chi(t) = a_{\mathbf{k}}\chi_k(t) + a_{-\mathbf{k}}^\dagger\chi_k^\star(t). \tag{A10}$$

In the theory of preheating/moduli trapping without rescattering the distinction between q-number Fourier transform and c-number mode functions is not important because both obey the same equation of motion [Eq. (A2) in the case at hand]. However, once rescattering is taken into account this distinction is crucial. To see why, note that the solution $\delta\phi$ of Eq. (5) will not be Gaussian and hence will not admit an expansion of the form (A10).

Finally, we return to the equation for rescattering, Eq. (5). We can solve for the q-number Fourier transform of $\delta\phi$ [defined analogously to (A9)] using the retarded Green function

$$\xi_{\mathbf{k}}^\phi(t) = \frac{g^2\dot{\phi}}{(2\pi)^{3/2}}\frac{1}{\Omega_k}\int_0^t dt't'\sin[\Omega_k(t-t')] \times \int d^3k'\xi_{\mathbf{k}-\mathbf{k}'}^\chi(t')\xi_{\mathbf{k}'}^\chi(t'), \tag{A11}$$

where we have introduced the notations $\Omega_k = \sqrt{k^2 + m^2}$ for the $\delta\phi$-particle frequency and $m^2 = V_{,\phi\phi}$ for the effective $\delta\phi$ mass. Carefully carrying out the Wick contractions yields

$$\langle\xi_{k_1}^\phi(t)\xi_{k_2}^\phi(t)\rangle = \frac{2g^4\dot{\phi}^2}{(2\pi)^3}\frac{1}{\Omega_{k_1}^2}\delta^{(3)}(\mathbf{k_1} + \mathbf{k_2})\int dt'dt''t't'' \times \sin[\Omega_{k_1}(t-t')]\sin[\Omega_{k_1}(t-t'')] \times \int d^3k'\chi_{k_1-k'}(t')\chi_{k_1-k'}^\star(t'')\chi_{k'}(t')\chi_{k'}^\star(t''), \tag{A12}$$

where the χ-particle mode functions χ_k are defined by (A10). Defining the power spectrum in terms of the two-point function in the usual manner

$$\langle 0|\xi_{\mathbf{k}}^\phi(t)\xi_{\mathbf{k}'}^\phi(t)|0\rangle \equiv \delta^{(3)}(\mathbf{k} + \mathbf{k}')\frac{2\pi^2}{k^3}P_\phi, \tag{A13}$$

we can extract the power in rescattered ϕ modes.

Alternatively, one could compute the power spectrum of rescattered inflaton modes using the Schwinger's "in-in" formalism [28], which was implemented to compute cosmological perturbations by Weinberg in [29]. We have verified that the tree level contribution to P_ϕ obtained using this formalism reproduces our result (A12). Our approach is analogous to computing the cosmological perturbation from the field equations using the Seery *et al.* approach [30]. The consistency of this method with the in-in approach at tree level is in accordance with the general theorem of [31].

3. Renormalization

To compute the spectrum of rescattered $\delta\phi$ particles we simply need to insert the solution (A8) into (A12) and evaluate the integrals. However, there is one subtlety. The resulting power spectrum is formally infinite; moreover, it contains the effect of both particle production as well as vacuum fluctuations of the χ field. We are only interested in the rescattered $\delta\phi$, which are due to particle production; thus, we need to subtract off the contribution due to nonlinear $\delta\phi$ production by χ vacuum fluctuations.

To properly define the two-point function of $\delta\phi$ we need to renormalize the four-point function of the Gaussian field χ. As a warmup, let us first consider how to renormalize the two-point function of the Gaussian field χ. We use the following scheme:

$$\langle \xi^\chi_{k_1}(t_1)\xi^\chi_{k_2}(t_2)\rangle_{\rm ren} = \langle \xi^\chi_{k_1}(t_1)\xi^\chi_{k_2}(t_2)\rangle - \langle \xi^\chi_{k_1}(t_1)\xi^\chi_{k_2}(t_2)\rangle_{\rm in}, \tag{A14}$$

where $\langle \xi^\chi_{k_1}(t_1)\xi^\chi_{k_2}(t_2)\rangle_{\rm in}$ is the contribution in the absence of particle production, computed by simply taking the solution (A5) with $\alpha_k = 1$, $\beta_k = 0$. More explicitly, for the case at hand, we have

$$\langle \chi^2(t, \mathbf{x})\rangle_{\rm ren} = \int \frac{d^3k}{(2\pi)^3}\left[|\chi_k^2(t)| - \frac{1}{2\omega_k(t)}\right] \equiv \langle \chi^2(t, \mathbf{x})\rangle - \delta_M, \tag{A15}$$

where δ_M is the contribution from the Coleman-Weinberg potential. This proves that our prescription reproduces the one used in [10].

Having established a scheme for remormalizing the two-point function of the Gaussian field χ it is straightforward to consider higher order correlation functions. We simply rewrite the four-point function as a product of two-point functions using Wick's theorem. Then each Wick contraction is renormalized as above. Applying this prescription to (A12) amounts to

$$\langle \xi^\phi_{k_1}(t)\xi^\phi_{k_2}(t)\rangle_{\rm ren} = \frac{2g^4\dot\phi^2}{(2\pi)^3}\frac{1}{\Omega^2_{k_1}}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})\int dt'dt''t't''\sin[\Omega_{k_1}(t-t')]\sin[\Omega_{k_1}(t-t'')] \times \int d^3k'[\chi_{k_1-k'}(t')\chi^\star_{k_1-k'}(t'') - f_{k_1-k'}(t')f^\star_{k_1-k'}(t'')][\chi_{k'}(t')\chi^\star_{k'}(t'') - f_{k'}(t')f^\star_{k'}(t'')], \tag{A16}$$

where $f_k(t)$ are the adiabatic modes defined in (A4).

4. Spectrum of rescattered modes

Let us now proceed to compute analytically the renormalized spectrum P_ϕ of rescattered inflaton modes by inserting the solutions (A4) and (A8) into (A16) and carrying out the integrations. The computation is tedious but straightforward since the time and phase space integrals factorize. We have relegated the technical details to Appendix B and here we simply state the final result:

$$P_\phi = \frac{g^2}{16\pi^5}\frac{k^3k_\star}{k^2+m^2}\Bigg[\frac{e^{-\pi k^2/(2k_\star^2)}}{2\sqrt{2}}\left(\frac{\pi}{4}|F|^2 + \frac{k_\star^2}{\Omega_k^2}[1-\cos(\Omega_k t)]^2\right) + \left[e^{-\pi k^2/(4k_\star^2)} + \frac{1}{2\sqrt{2}}e^{-3\pi k^2/(8k_\star^2)}\right] \times\left(-\frac{\pi}{4}{\rm Re}[e^{2i\Omega_k t - i\Omega_k^2/(2k_\star^2) - i\pi/2}F] + \frac{k_\star^2}{\Omega_k^2}[1-\cos(\Omega_k t)]^2\right) + \left[\frac{4\sqrt{2}}{3\sqrt{3}}e^{-\pi k^2/(3k_\star^2)} + \frac{2\sqrt{2}}{5\sqrt{5}}e^{-3\pi k^2/(5k_\star^2)}\right]\frac{\sqrt{\pi}k_\star}{\Omega_k} \times[1-\cos(\Omega_k t)]{\rm Im}[e^{i\Omega_k t - i\Omega_k^2/(4k_\star^2) - i\pi/4}F]\Bigg]. \tag{A17}$$

Equation (A17) is the main result of this appendix. The "form factor" $F(k, t)$ is given explicitly in Appendix B.

APPENDIX B: DETAILED COMPUTATION OF P_ϕ

In this appendix we discuss in some detail the technical details associated with the computation of P_ϕ. Inserting the solutions (A4) and (A8) into (A16), we find the result

$$P_\phi = \frac{g^2}{8\pi^5}\frac{k^3}{k^2+m^2}\bigg[\int d^3k' n_{k-k'}n_{k'}\int dt'dt'' \sin[\Omega_k(t-t')]\sin[\Omega_k(t-t'')]\cos^2\bigg[\frac{(k_\star t')^2}{2}-\frac{(k_\star t'')^2}{2}\bigg]$$
$$+\int d^3k'\sqrt{n_{k-k'}n_{k'}}\sqrt{1+n_{k-k'}}\sqrt{1+n_{k'}}\int dt'dt''\sin[\Omega_k(t-t')]\sin[\Omega_k(t-t'')]\sin^2\bigg[\frac{(k_\star t')^2}{2}+\frac{(k_\star t'')^2}{2}\bigg]$$
$$+\int d^3k'[n_{k-k'}\sqrt{n_{k'}}\sqrt{1+n_{k'}}+n_{k'}\sqrt{n_{k-k'}}\sqrt{1+n_{k-k'}}]\int dt'dt''\sin[\Omega_k(t-t')]\sin[\Omega_k(t-t'')]$$
$$\times\sin\bigg[\frac{(k_\star t')^2}{2}+\frac{(k_\star t'')^2}{2}\bigg]\cos\bigg[\frac{(k_\star t')^2}{2}-\frac{(k_\star t'')^2}{2}\bigg]\bigg]. \quad \text{(B1)}$$

We consider the time and phase space integrations separately.

1. Time integrals

All the time integrals appearing in (B1) can be written in terms of two functions, which we call I_1, I_2. These involves are defined as

$$I_1(k,t) = \int_0^t dt' \sin[\Omega_k(t-t')]e^{i(k_\star t')^2}, \quad \text{(B2)}$$

$$I_2(k,t) = \int_0^t dt' \sin[\Omega_k(t-t')]. \quad \text{(B3)}$$

First consider I_1. It is useful to factorize the answer into the product of the stationary phase result (valid for $k_\star t \gg \Omega_k/(2k_\star) \gg 1$) and a "form factor" $F(k,t)$ as follows:

$$I_1(k,t) = \frac{\sqrt{\pi}}{2k_\star}e^{i\Omega_k t - i\Omega_k^2/(4k_\star^2)-i\pi/4}F(k,t), \quad \text{(B4)}$$

$$F(k,t) = \frac{1}{2}\bigg[(1+e^{-2i\Omega_k t})\text{erf}\bigg(\frac{e^{-i\pi/4}}{2}\frac{\Omega_k}{k_\star}\bigg) - \text{erf}\bigg(\frac{e^{-i\pi/4}}{2}\bigg(\frac{\Omega_k}{k_\star}-2k_\star t\bigg)\bigg) - e^{-2i\Omega_k t}\text{erf}\bigg(\frac{e^{-i\pi/4}}{2}\bigg(\frac{\Omega_k}{k_\star}+2k_\star t\bigg)\bigg)\bigg]. \quad \text{(B5)}$$

The form factor $F(k,t)$ has a complicated structure. We have illustrated the qualitative behavior of this function in Fig. 8 taking $\Omega_k/k_\star = 5$ for illustration.

Next, consider the characteristic integral I_2, Eq. (B6). This integration is trivial:

$$I_2(k,t) = \frac{1}{\Omega_k}[1-\cos(\Omega_k t)]. \quad \text{(B6)}$$

Now we will show that all the time integrals appearing in (B1) can be reduced to combinations of the characteristic functions I_1 and I_2. First, consider the first line of (B1) where the following integral appears:

$$\int dt'dt''\sin[\Omega_k(t-t')]\sin[\Omega_k(t-t'')] \times\cos^2\bigg[\frac{(k_\star t')^2}{2}-\frac{(k_\star t'')^2}{2}\bigg] = \frac{|I_1(k,t)|^2}{2}+\frac{I_2(k,t)^2}{2}$$
$$= \frac{\pi}{8k_\star^2}|F(k,t)|^2 + \frac{1}{2\Omega_k^2}[1-\cos(\Omega_k t)]^2. \quad \text{(B7)}$$

Next, consider the time integration on the second line of (B1)

$$\int dt'dt''\sin[\Omega_k(t-t')]\sin[\Omega_k(t-t'')] \times\sin^2\bigg[\frac{(k_\star t')^2}{2}+\frac{(k_\star t'')^2}{2}\bigg] = -\frac{\text{Re}[I_1(k,t)^2]}{2}+\frac{I_2(k,t)^2}{2}$$
$$= -\frac{\pi}{8k_\star^2}\text{Re}[e^{2i\Omega_k t - i\Omega_k^2/(2k_\star^2)-i\pi/2}F(k,t)^2] + \frac{1}{2\Omega_k^2}[1-\cos(\Omega_k t)]^2. \quad \text{(B8)}$$

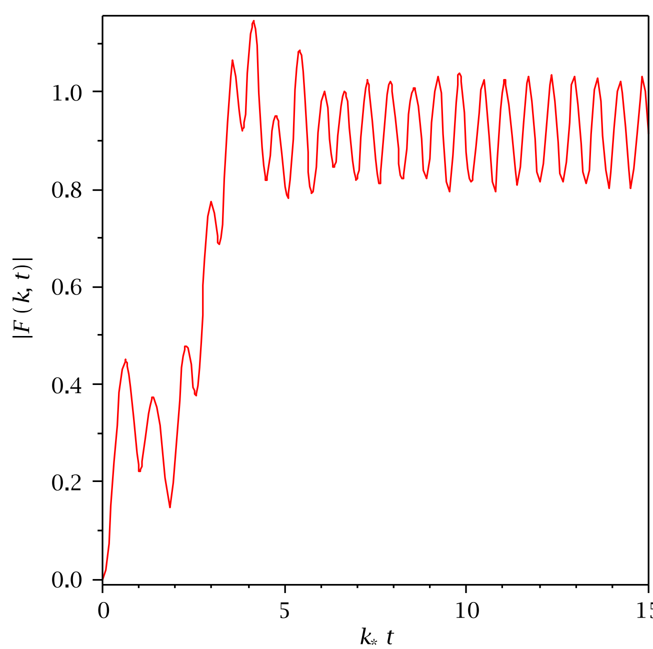

FIG. 8 (color online). The behavior of the function $F(k,t)$ as a function of t. For illustration we have set $\Omega_k = 5k_\star$.

Finally, we consider the time integration on the third line of (B1)

$$\int dt' dt'' \sin[\Omega_k(t-t')]\sin[\Omega_k(t-t'')] \times \sin\left[\frac{(k_\star t')^2}{2}+\frac{(k_\star t'')^2}{2}\right]\cos\left[\frac{(k_\star t')^2}{2}-\frac{(k_\star t'')^2}{2}\right] = \mathrm{Im}[I_1(k,t)I_2(k,t)] = \frac{\sqrt{\pi}}{2k_\star\Omega_k}[1-\cos(\Omega_k t)] \times \mathrm{Im}[e^{i\Omega_k t - i\Omega_k^2/(4k_\star^2)-i\pi/4}F(k,t)]. \quad \text{(B9)}$$

2. Phase space integrals

Throughout the calculation integrals of the following form appears frequently:

$$\int d^3k' n_{k-k'}^a n_{k'}^b = \int d^3k' \exp[-a\pi|\mathbf{k}-\mathbf{k}'|^2/k_\star^2] \times \exp[-b\pi|\mathbf{k}'|^2/k_\star^2] = \frac{k_\star^3}{(a+b)^{3/2}}\exp\left[-\frac{ab}{a+b}\frac{\pi k^2}{k_\star^2}\right]. \quad \text{(B10)}$$

This formula is valid when a, b are positive real numbers. Notice that this expression is symmetric under interchange of a and b.

The phase space integral in the first line of (B1) is computed by a trivial application of the identity (B10)

$$\int d^3k' n_{k-k'}n_{k'} = \frac{k_\star^3}{2\sqrt{2}}e^{-\pi k^2/(2k_\star^2)}. \quad \text{(B11)}$$

The remaining integrals cannot be obtained exactly in closed form because they contain terms like $\sqrt{1+n_{k'}}$ where the Gaussian factors appear under the square root. However, because $n_k \leq 1$ it turns out to be a very good approximation to replace $\sqrt{1+n_{k'}} \cong 1 + n_{k'}/2$. (We have checked numerically that the error induced is less than a few percent.) Let us now proceed in this manner. The phase space integral on the second line of (B1) is

$$\int d^3k' \sqrt{n_{k-k'}n_{k'}}\sqrt{1+n_{k-k'}}\sqrt{1+n_{k'}} \cong \int d^3k'\left[n_{k-k'}^{1/2}n_{k'}^{1/2}+\frac{1}{2}n_{k-k'}^{3/2}n_{k'}^{1/2}+\frac{1}{2}n_{k-k'}^{1/2}n_{k'}^{3/2}\right] = k_\star^3\left[\exp\left(-\frac{\pi k^2}{4k_\star^2}\right)+\frac{1}{2\sqrt{2}}\exp\left(-\frac{3\pi k^2}{8k_\star^2}\right)\right]. \quad \text{(B12)}$$

Finally, consider the phase space integral on the third line of (B1)

$$\int d^3k'[n_{k-k'}\sqrt{n_{k'}}\sqrt{1+n_{k'}}+n_{k'}\sqrt{n_{k-k'}}\sqrt{1+n_{k-k'}}] \cong \int d^3k'[n_{k-k'}n_{k'}^{1/2}+n_{k'}n_{k-k'}^{1/2}+\frac{1}{2}n_{k-k'}n_{k'}^{3/2}+\frac{1}{2}n_{k'}n_{k-k'}^{3/2}] = k_\star^3\left[\frac{4\sqrt{2}}{3\sqrt{3}}\exp\left(-\frac{\pi k^2}{3k_\star^2}\right)+\frac{2\sqrt{2}}{5\sqrt{5}}\exp\left(-\frac{3\pi k^2}{5k_\star^2}\right)\right]. \quad \text{(B13)}$$

Assembling the various results presented in this appendix one arrives straightforwardly at the result (A17).

[1] V. F. Mukhanov and G. V. Chibisov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 549 (1981) [JETP Lett. **33**, 532 (1981)]; S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982); A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[2] L. Kofman, arXiv:astro-ph/0303614; F. Bernardeau, L. Kofman, and J. P. Uzan, Phys. Rev. D **70**, 083004 (2004).

[3] G. Dvali, A. Gruzinov, and M. Zaldarriaga, Phys. Rev. D **69**, 023505 (2004); **69**, 083505 (2004).

[4] D. H. Lyth and D. Wands, Phys. Lett. B **524**, 5 (2002).

[5] L. A. Kofman and A. D. Linde, Nucl. Phys. **B282**, 555 (1987).

[6] L. A. Kofman and D. Y. Pogosian, Phys. Lett. B **214**, 508 (1988).

[7] D. S. Salopek, J. R. Bond, and J. M. Bardeen, Phys. Rev. D **40**, 1753 (1989).

[8] N. Barnaby and J. M. Cline, Phys. Rev. D **73**, 106012 (2006); **75**, 086004 (2007).

[9] D. J. H. Chung, E. W. Kolb, A. Riotto, and I. I. Tkachev, Phys. Rev. D **62**, 043508 (2000).

[10] L. Kofman, A. Linde, X. Liu, A. Maloney, L. McAllister, and E. Silverstein, J. High Energy Phys. 05 (2004) 030.

[11] A. E. Romano and M. Sasaki, Phys. Rev. D **78**, 103522 (2008).

[12] D. Green, B. Horn, L. Senatore, and E. Silverstein, arXiv:0902.1006.

[13] L. A. Kofman and A. D. Linde, Trapped Inflation (unpublished).

[14] E. Silverstein and A. Westphal, Phys. Rev. D **78**, 106003 (2008).

[15] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).

[16] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. D **56**, 3258 (1997).

[17] A. Berera and T. W. Kephart, Phys. Lett. B **456**, 135 (1999).

[18] S. Khlebnikov and I. I. Tkachev, Phys. Rev. Lett. **77**, 219 (1996); **79**, 1607 (1997); Phys. Rev. D **56**, 653 (1997).

[19] G. N. Felder and L. Kofman, Phys. Rev. D **63**, 103503 (2001).
[20] G. N. Felder and I. Tkachev, Comput. Phys. Commun. **178**, 929 (2008).
[21] A. V. Frolov, J. Cosmol. Astropart. Phys. 11 (2008) 009.
[22] R. Micha and I. I. Tkachev, Phys. Rev. Lett. **90**, 121301 (2003).
[23] R. Micha and I. I. Tkachev, Phys. Rev. D **70**, 043538 (2004).
[24] J. Berges, A. Rothkopf, and J. Schmidt, Phys. Rev. Lett. **101**, 041603 (2008).
[25] J. Berges and G. Hoffmeister, Nucl. Phys. **B813**, 383 (2009).
[26] J. R. Bond, A. V. Frolov, Z. Huang, and L. Kofman, arXiv:0903.3407.
[27] N. Barnaby (unpublished).
[28] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **46**, 1401 (1960).
[29] S. Weinberg, Phys. Rev. D **72**, 043514 (2005).
[30] D. Seery, K. A. Malik, and D. H. Lyth, J. Cosmol. Astropart. Phys. 03 (2008) 014.
[31] S. Weinberg, Phys. Rev. D **78**, 063534 (2008).