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Phase space of generalized Gauss-Bonnet dark energy

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The generalized Gauss-Bonnet theory, introduced by Lagrangian $F(R, G)$, has been considered as a general modified gravity for explanation of the dark energy. G is the Gauss-Bonnet invariant. For this model, we seek the situations under which the late-time behavior of the theory is the de Sitter space-time. This is done by studying the two-dimensional phase space of this theory, i.e. the R-H plane. By obtaining the conditions under which the de Sitter space-time is the stable attractor of this theory, several aspects of this problem have been investigated. It has been shown that there exist at least two classes of stable attractors: the singularities of the $F(R, G)$, and the cases in which the model has a critical curve, instead of critical points. This curve is $R = 12H^2$ in R-H plane. Several examples, including their numerical calculations, have been discussed.

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I. INTRODUCTION

Based on various observations, it is believed that our Universe is now in an accelerating phase [1]. Although the origin of this accelerated expansion is not yet known, almost all data indicate that nearly 70% of the present Universe is composed of dark energy, the physical object that induces the negative pressure.

There are two main classes of models that have been introduced as candidates of dark energy. The first class is based on the Einstein cosmology but with extra physical object as the source of dark energy. The scalar field (one-component or multicomponents) models [2], the scalar-tensor theories [3] and the k-essence models [4] are examples in this context.

The second class of the models is based on the assumption that the gravity is being (nowadays) modified. The simplest one is obtained by adding a cosmological constant term to Einstein action. This model suffers two important problems known as cosmological constant and coincidence problems [5]. Also, the cosmological constant model is a static model of dark energy and has not any dynamical behavior. The other modified gravity models are those that are based on the new actions.

The first family of these modified gravity theories are those known as $f(R)$ gravity, with the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}f(R) + \mathcal{L}_m\right]. \quad (1)$$

In $\hbar = c = G = 1$ units, $\kappa^2 = 8\pi$, R is the Ricci scalar and $\mathcal{L}_m$ is the Lagrangian density of dust-like matter. Many features of $f(R)$ gravity models, such as local gravity tests, have been studied [6].

Another well motivated curvature invariant, beyond the Ricci scalar, is the Gauss-Bonnet (GB) term

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}, \quad (2)$$

which is inspired by string theory [7] and is a topological invariant in four dimensions. The second family of modified gravity theories is known as $f(G)$ gravity and is defined through

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + f(G) + \mathcal{L}_m\right]. \quad (3)$$

This model has gained special interest in cosmology [8], and its coupling to scalar fields, as it naturally appears in low-energy string effective actions [7], can introduce extra dynamics to this model. Other aspects of modified GB gravity, such as the possibility of describing the inflationary era, transition from the deceleration phase to the acceleration phase, crossing the phantom divide line, and passing the solar system test have been discussed in [9].

The natural generalization of action (3) is the generalized Gauss-Bonnet dark energy, which have been introduced in [7,10]:

$$S = \int d^4x\sqrt{-g}[F(R, G) + \mathcal{L}_m]. \quad (4)$$

Clearly the $f(R)$ gravity and $f(G)$ gravity are special examples of modified $F(R, G)$ gravity. The hierarchy problem of particle physics and the late-time cosmology have been studied in $F(R, G)$ models [11]. Recently, the behavior of these models in phantom-divide-line crossing and deceleration to acceleration transition, including the contribution of quantum effects to those phenomena, have been studied in [12]. It has been shown that the quantum effects can induce these transitions, when they are classically forbidden.

One of the important characteristics of all dynamical systems, including the dynamical models of dark energy, is their late-time behaviors, studied in a framework known as the attractor solution of dynamical systems, which has been deeply investigated in mathematics. For dark energy

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models, the attractor solutions of scalar theories and some of the modified gravity theories have been studied in [13–15].

The main step in studying the attractor solution of a dynamical system is considering a set of suitable dynamical variables $x_1(t), \ldots, x_n(t)$, such that their first time derivatives dx_i/dt do not depend explicitly on time:

$$\frac{dx_1}{dt} = F_1(x_1, \ldots, x_n), \quad \vdots \quad \frac{dx_n}{dt} = F_n(x_1, \ldots, x_n). \tag{5}$$

The space constructed by variables $x_1, \ldots, x_n$ is called the phase space of the system and the system of equations (5) is said to be autonomous. The functions $x_1(t), \ldots, x_n(t)$ define a path in the phase space, and there is a unique path that passes any specific initial values $x_1(t_0) = x_{1,0}, \ldots, x_n(t_0) = x_{n,0}$, i.e. the paths do not intersect one another. The only exception to this statement occur at points $(x_{1,c}, \ldots, x_{n,c})$, where

$$F_1(x_{1,c}, \ldots, x_{n,c}) = 0, \ldots, F_n(x_{1,c}, \ldots, x_{n,c}) = 0. \tag{6}$$

These points are called critical points, and any paths near these points, under specific conditions, will lead them at $t \to \infty$. In these cases, the critical points are called the stable attractors.

The present paper is devoted to the study of the phase space and attractor solutions of generalized Gauss-Bonnet dark energy models. We will consider the R-H space (R is the Ricci scalar and H is the Hubble parameter) as the phase space of these models and show that in the special case of $F(R, G) = f(R)$, the results of [16], in which some features of phase space of $f(R)$ have been studied, are reproduced. The choice of this phase space, which is the only possible choice in the general $F(R, G)$ model, has an important property. Since the attractor solutions are those that asymptotically lead to $\dot{R} = 0$ and $\dot{H} = 0$, or $R = R_c$ and $H = H_c$ (R_c and H_c are some constant values), our phase-space study is in fact the study of possible de Sitter solutions of modified generalized GB gravity. The scheme of the paper is as follows:

In Sec. II, the set of autonomous equations of $F(R, G)$ models is obtained, and the condition of the existence of stable attractors is discussed. Some specific examples of $F(R, G)$ models that admit the stable attractors are investigated in Sec. III, and it is shown that the numerical studies confirm our results. Section IV is devoted to the cases where the standard linear approximation method, used in obtaining the stability behavior of solutions, does not work. In Sec. V, it is shown that the singular points of the Lagrangian are always the stable attractors, and finally in Sec. VI, the interesting cases where the critical points replaced by critical curves are studied. It is shown that all these critical curves always behave as stable attractor curves. We end the paper with a conclusion in Sec. VII.

II. CRITICAL POINTS OF $F(R, G)$ GRAVITY

Consider the generalized GB dark energy model with action (4). Variation of this action with respect to the metric $g_{\mu\nu}$ results in [11]

$$\begin{aligned} &\tfrac{1}{2}g^{\mu\nu}F(R,G) - 2F_G(R,G)RR^{\mu\nu} + 4F_G(R,G)R^{\mu}{}_{\rho}R^{\nu\rho} - 2F_G(R,G)R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4F_G(R,G)R^{\mu\rho\sigma\nu}R_{\rho\sigma} \\ &\quad + 2(\nabla^\mu\nabla^\nu F_G(R,G))R - 2g^{\mu\nu}(\nabla^2F_G(R,G))R - 4(\nabla_\rho\nabla^\mu F_G(R,G))R^{\nu\rho} - 4(\nabla_\rho\nabla^\nu F_G(R,G))R^{\mu\rho} \\ &\quad + 4(\nabla^2F_G(R,G))R^{\mu\nu} + 4g^{\mu\nu}(\nabla_\rho\nabla_\sigma F_G(R,G))R^{\rho\sigma} - 4(\nabla_\rho\nabla_\sigma F_G(R,G))R^{\mu\rho\nu\sigma} - F_R(R,G)R^{\mu\nu} + \nabla^\mu\nabla^\nu F_R(R,G) \\ &\quad - g^{\mu\nu}\nabla^2F_R(R,G) = 0. \end{aligned} \tag{7}$$

Here, for simplicity, we do not consider the background matter field, i.e. $\mathcal{L}_m = 0$. In Eq. (7), F_R and F_G are defined as follows:

$$F_R(R,G) = \frac{\partial F(R,G)}{\partial R}, \qquad F_G(R,G) = \frac{\partial F(R,G)}{\partial G}. \tag{8}$$

For the background metric, we consider, as usual, the spatially flat Friedmann-Robertson-Walker metric in comoving coordinates (t, x, y, z) as follows:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{9}$$

in which $a(t)$ is the scale factor. The (t, t)-component of the evolution equation (7) then becomes

$$\begin{aligned} -6H^2F_R(R,G) &= F(R,G) - RF_R(R,G) + 6H\dot{F}_R(R,G) \\ &\quad + 24H^3\dot{F}_G(R,G) - GF_G(R,G). \end{aligned} \tag{10}$$

$H = \dot{a}(t)/a(t)$ is the Hubble parameter. For this metric, the Ricci scalar R and the Gauss-Bonnet invariant G are

$$R = 6(\dot{H} + 2H^2), \tag{11}$$

and

$$G = 24H^2(\dot{H} + H^2), \tag{12}$$

respectively. The Eqs. (10)–(12) are the Friedmann equations of $F(R, G)$ gravity. The sum of (i, i) components of

Eq. (7) is obtained by using the time derivative of Eq. (10) and Eqs. (11) and (12).

From Eq. (11), one has

$$\dot{H} = \frac{R}{6} - 2H^2, \tag{13}$$

which can be used to express G from Eq. (12) as follows:

$$G = 4H^2(R - 6H^2), \tag{14}$$

from which

$$\dot{G} = \frac{4}{3}HR^2 + 192H^5 - 32RH^3 + 4H^2\dot{R}. \tag{15}$$

Using

$$\frac{d}{dt}f(R, G) = f_R\dot{R} + f_G\dot{G}, \tag{16}$$

and Eq. (15), $\dot{R}$ can be extracted from Eq. (10) as follows:

$$\dot{R} = \frac{(R - 6H^2)F_R + GF_G - F - 288H^2(R/6 - 2H^2)^2(F_{RG} + 4H^2F_{GG})}{6H(F_{RR} + 8H^2F_{RG} + 16H^4F_{GG})}, \tag{17}$$

$$\dot{H} = \frac{R}{6} - 2H^2, \tag{18}$$

where the second equation is the same as Eq. (13). The set of the above equations are the autonomous equations of $F(R, G)$ gravity. The phase space of this problem is the two-dimensional (R-H) space. In the right-hand side of Eq. (17), the expression (14) must be used for the Gauss-Bonnet invariant G. Therefore, the above equations are in the form

$$\dot{H} = f_1(R, H), \qquad \dot{R} = f_2(R, H). \tag{19}$$

The critical points are found by setting Eqs. (17) and (18) equal to zero. The result is

$$\tfrac{1}{2}RF_R + GF_G - F = 0, \tag{20}$$

$$R = 12H^2. \tag{21}$$

Equation (14) also results in

$$G = 24H^4 = \frac{R^2}{6} \tag{22}$$

at critical points. In obtaining Eq. (20), it is assumed that the denominator of Eq. (17) has finite value at critical points. We will return to this assumption in Sec. V. Note that in Eq. (20) G must be replaced by Eq. (22). In the case of $f(R)$ gravity, i.e. $F(R, G) = f(R)/2\kappa^2$, Eqs. (17)–(21) are reduced to the corresponding relations in Ref. [15].

Since the effective equation of state parameter is defined through

$$\omega_{\rm eff} = \frac{p}{\rho} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}, \tag{23}$$

at critical points where $\dot{H} = 0$, one has

$$\omega_{\rm eff} \rightarrow \omega_c = -1, \tag{24}$$

which is a characteristic of de Sitter space-time.

To study the stability of each critical point, one must evaluate the eigenvalues of matrix

$$M = \begin{pmatrix} \partial f_1/\partial H & \partial f_1/\partial R \\ \partial f_2/\partial H & \partial f_2/\partial R \end{pmatrix}_{R=R_c, H=H_c}. \tag{25}$$

R_c and H_c denote the values of R and H at the considered critical point. The critical point is a stable attractor only when the real parts of all the eigenvalues of matrix M are negative. For negative real eigenvalues, the stable critical point is called a node, and for complex eigenvalues with negative real parts, the stable attractor is called a spiral.

For autonomous Eqs. (17) and (18), the matrix M becomes

$$M = \begin{pmatrix} -4H & 1/6 \\ -2F_R/A & H \end{pmatrix}_{R=R_c, H=H_c}, \tag{26}$$

where

$$A = F_{RR} + 8H^2F_{RG} + 16H^4F_{GG}. \tag{27}$$

Therefore, the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[-3H \pm \sqrt{(3H)^2 - 4\left(\frac{F_R}{3A} - 4H^2\right)}\right]_{R=R_c, H=H_c}. \tag{28}$$

It is clear that the real part of eigenvalues λ_1 and λ_2 are negative, if and only if

$$\eta = \frac{F_R}{3A} - 4H^2|_{R=R_c, H=H_c} > 0. \tag{29}$$

This is the condition of stability of the attractors of $F(R, G)$ gravity. The attractors are node if $(3H_c)^2 > 4\eta$ and are spiral if $(3H_c)^2 < 4\eta$.

An interesting observation is that for R-independent Lagrangian

$$F(R, G) = F(G), \tag{30}$$

$\eta = -4H_c^2 < 0$ and therefore $\lambda_1 > 0$. So all the critical points of these models are unstable.

III. SOME EXAMPLES OF STABLE ATTRACTORS

In this section we will discuss two classes of $F(R, G)$ models that lead to stable attractors.

A. $F(R, G) = F(RG)$ models

For these models, Eq. (20) results in

$$\frac{3}{2}xF'(x) - F(x) = 0, \tag{31}$$

where $x = RG$ and $F'(x) = \frac{d}{dx}F(x)$. This relation determines the critical values $x = x_c$. The stability condition (29) leads to

$$\frac{F'(x)}{9xF''(x) + 4F'(x)} - 1 > 0 \tag{32}$$

at $x = x_c$. For $F(x) = x^n$ cases, it can be shown that Eq. (31) does not have a nontrivial solution, except for a very special case, which we will discuss it later. For $F(x) = x^n - c$, where c is a positive constant, Eqs. (31) and (32) result in

$$x_c = \left(\frac{c}{1 - 3n/2}\right)^{1/n}, \tag{33}$$

and

$$\frac{10}{18} < n < \frac{12}{18}, \tag{34}$$

respectively. As an explicit example, we consider $n = 11/18$ and $c = 1$:

$$F(RG) = (RG)^{11/18} - 1. \tag{35}$$

Equation (33) then results in

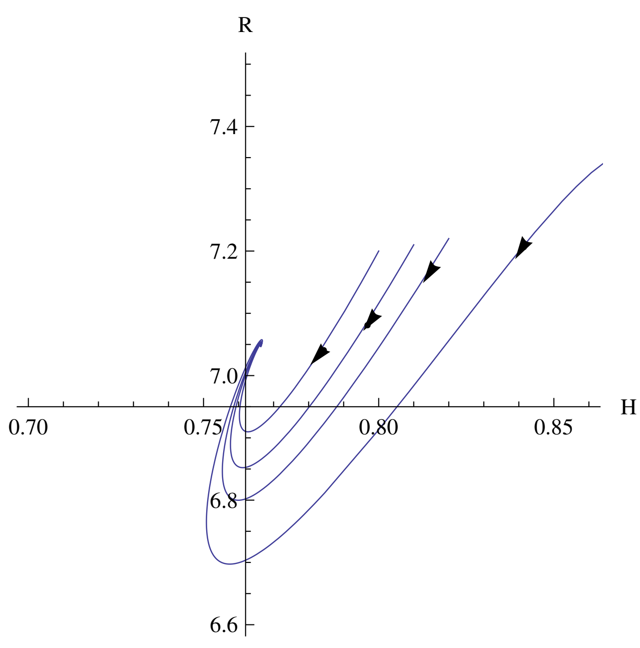

FIG. 1 (color online). The spiral paths in R-H plane of the $F(R, G) = (RG)^{11/18} - 1$ model.

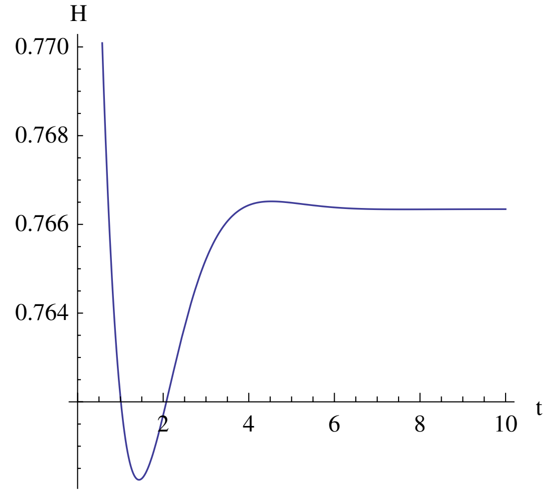

FIG. 2 (color online). The plot of $H(t)$ of the $F(R, G) = (RG)^{11/18} - 1$ model.

$$x_c = R_c G_c = \frac{R_c^3}{6} = (12)^{18/11}. \tag{36}$$

So

$$(R_c, H_c) = (7.047, 0.766). \tag{37}$$

Numerical calculation of Eqs. (10)–(12) results in Figs. 1–4 for phase space, $R(t)$, $H(t)$, and $\omega(t)$ behaviors, respectively. These figures show that point (37) is a stable attractor of spiral type.

B. $F(R, G) = R + Gf(R)$ models

For this functional form of $F(R, G)$, the critical point equation (20), using Eq. (22), results in

$$R^2 f'(R) = 6. \tag{38}$$

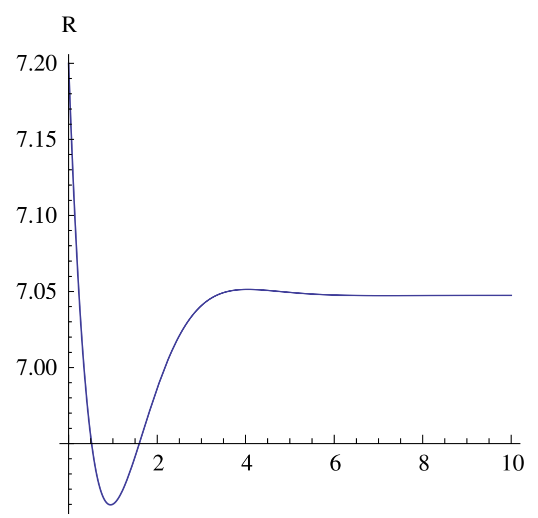

FIG. 3 (color online). The plot of $R(t)$ of the $F(R, G) = (RG)^{11/18} - 1$ model.

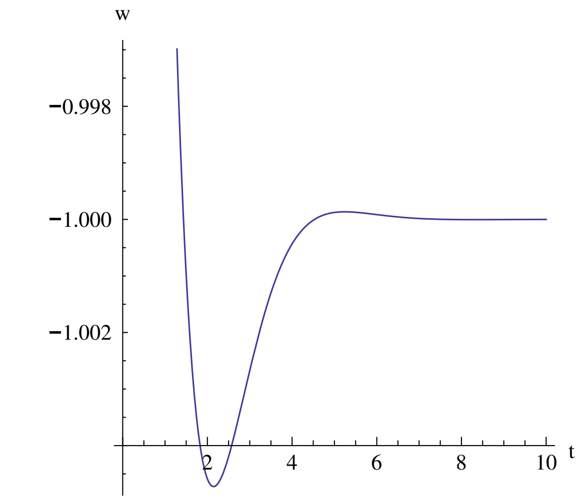

FIG. 4 (color online). The plot of $\omega(t)$ of the $F(R,G) = (RG)^{11/18} - 1$ model.

This equation specifies R_c. The stability condition (29) reduces to

$$\frac{1}{R^3 f''(R)/12 + 2} > 1, \tag{39}$$

which must be calculated at $R = R_c$. For example, for $f(R) = mR^n$ functions, where m and n are some constants, Eqs. (38) and (39) result in

$$R_c = \left(\frac{6}{mn}\right)^{1/(n+1)}, \tag{40}$$

and

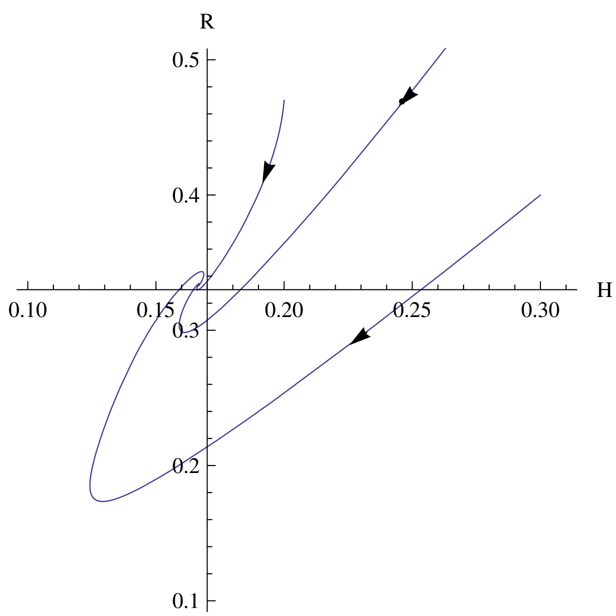

FIG. 5 (color online). The spiral paths of the $F(R,G) = R - G/R^2$ model in the R-H plane.

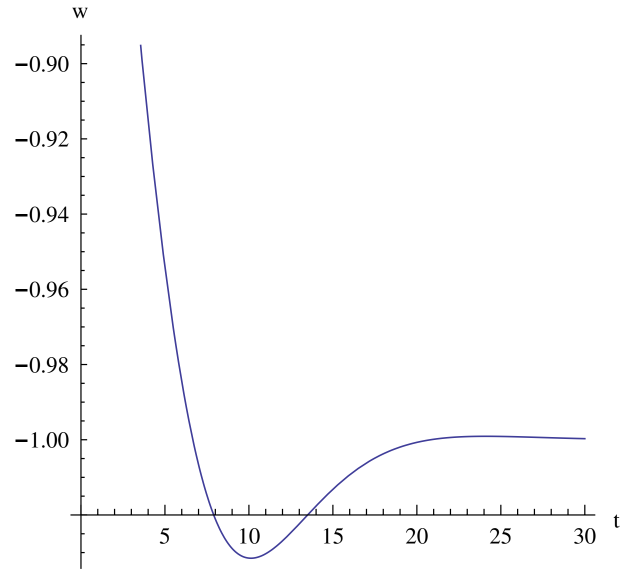

FIG. 6 (color online). The plot of $\omega(t)$ for the $F(R,G) = R - G/R^2$ model.

$$-3 < n < -1, \tag{41}$$

respectively. Since n is a negative number, m must be chosen negative so that R_c in (40) becomes a real positive number. As a specific example, we consider $m = -1$ and $n = -2$, or

$$F(R,G) = R - G/R^2. \tag{42}$$

R_c and H_c then become

$$(R_c, H_c) = (\tfrac{1}{3}, \tfrac{1}{6}). \tag{43}$$

Numerical results for Lagrangian (42) are given by Figs. 5 and 6, which are the paths in the R-H plane and $\omega(t)$, respectively. Figure 5 verifies that the critical point (43) is a stable attractor.

IV. CRITICAL POINTS WITH ZERO EIGENVALUE

As was mentioned previously, an attractor is stable if, and only if, the real parts of all eigenvalues are negative. Now we want to consider the situations in which one or more of the eigenvalues are zero. The appearance of zero eigenvalues may be rooted in the nonindependency of the chosen dynamical variables. For instance, if we consider the phase space of $F(R,G)$ models by three-dimensional (R,H,G) space, one can show that besides the two eigenvalues λ_1 and λ_2 of Eq. (28), we have a third eigenvalue $\lambda_3 = 0$. This indicates that we can reduce the dimensionality of our phase space.

For the cases where the dimension of the phase space can not be reduced, the appearance of zero eigenvalue means that the linear approximation, which leads to the matrix (25), is not adequate, and we must consider the higher order approximations to determine the behavior of the critical points. See for example [17].

In our $F(R, G)$ models, for the cases where

$$\eta = \frac{F_R}{3A} - 4H^2|_{R=R_c, H=H_c} = 0, \tag{44}$$

one has

$$\lambda_1 = 0, \qquad \lambda_2 = -3H, \tag{45}$$

and the higher order approximations must be used to determine whether the considered critical point is stable or not. This is done by a standard method, which has been discussed, for example, in [17].

At first, the eigenvectors of matrix M for the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3H$ must be calculated, which result in

$$\psi_1 = \begin{pmatrix} 1 \\ 24H_c \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 1 \\ 6H_c \end{pmatrix}, \tag{46}$$

respectively. Using the transformation matrix $T = (\psi_1, \psi_2)$, the new phase-space basis (U, V), i.e. the normal basis, can be found from (R, H) as follows:

$$\begin{pmatrix} U \\ V \end{pmatrix} = T^{-1}\begin{pmatrix} R \\ H \end{pmatrix}, \tag{47}$$

which results in

$$U = \frac{1}{18H_c}(R - 6H_cH), \qquad V = \frac{1}{18H_c}(24H_cH - R). \tag{48}$$

To translate the critical point from (R_c, H_c) to the origin of the phase space, we introduce $\mathcal{R} = R - R_c$ and $\mathcal{H} = H - H_c$. $\dot{\mathcal{R}}$ and $\dot{\mathcal{H}}$, up to second order, then become

$$\dot{\mathcal{R}} = H_c\mathcal{R} - 24H_c^2\mathcal{H} + D\mathcal{R}^2 + B\mathcal{H}^2 + C\mathcal{R}\mathcal{H},$$
$$\dot{\mathcal{H}} = \tfrac{1}{6}\mathcal{R} - 4H_c\mathcal{H} - 2\mathcal{H}^2. \tag{49}$$

In above equations, the Eqs. (17) and (18) have been used, and the coefficients D, B and C are defined by

$$D = \frac{1}{2}\left(\frac{\partial^2 f_2(R, H)}{\partial R^2}\right), \qquad B = \frac{1}{2}\left(\frac{\partial^2 f_2(R, H)}{\partial H^2}\right),$$
$$C = \left(\frac{\partial^2 f_2(R, H)}{\partial R \partial H}\right). \tag{50}$$

$f_2(R, G)$ is one introduced in Eq. (19), and all derivatives are calculated at critical point (R_c, H_c). Note that the linear terms in the right-hand side of Eq. (49), result in the matrix elements of M in Eq. (26) for the case where $\eta = 0$, or $-2F_R/A = -24H^2$.

For the new phase space, introducing $\mathcal{U} = U - U_c$ and $\mathcal{V} = V - V_c$, the Eqs. (48) and (49) then result in

$$\dot{\mathcal{U}} = \left(\frac{2}{3} + \frac{B}{18H_c}\right)(\mathcal{U} + \mathcal{V})^2 + 2DH_c(4\mathcal{U} + \mathcal{V})^2 + \frac{C}{3}(\mathcal{U} + \mathcal{V})(4\mathcal{U} + \mathcal{V}), \tag{51}$$

$$\dot{\mathcal{V}} = -\left(\frac{8}{3} + \frac{B}{18H_c}\right)(\mathcal{U} + \mathcal{V})^2 - 2DH_c(4\mathcal{U} + \mathcal{V})^2 - \frac{C}{3}(\mathcal{U} + \mathcal{V})(4\mathcal{U} + \mathcal{V}). \tag{52}$$

Now, taking $\mathcal{V} = h(\mathcal{U}) = a\mathcal{U}^2 + b\mathcal{U}^3 + \cdots$, the coefficients a and b can be found by using the chain rule

$$\dot{\mathcal{V}} = h'(\mathcal{U})\dot{\mathcal{U}}. \tag{53}$$

By this way, the problem effectively becomes one dimensional. Using Eqs. (51) and (52), the coefficients of $\mathcal{U}^2$ terms of Eq. (53) result in the parameter a as follows:

$$a = -\frac{1}{3H_c}\left(\frac{8}{3} + 32H_cD + \frac{B}{18H_c} + \frac{4}{3}C\right). \tag{54}$$

The coefficient b can be also found by the $\mathcal{U}^3$ terms. Using the expansion $\mathcal{V} = a\mathcal{U}^2 + \ldots$, Eq. (51) leads to

$$\begin{aligned}\dot{\mathcal{U}} &= \left(\frac{2}{3} + 32H_cD + \frac{B}{18H_c} + \frac{4}{3}C\right)\mathcal{U}^2 \\ &\quad + \left(\frac{4}{3} + 16H_cD + \frac{B}{9H_c} + \frac{5}{3}C\right)a\mathcal{U}^3 + \cdots \\ &= \alpha\mathcal{U}^2 + \beta\mathcal{U}^3 + \cdots.\end{aligned} \tag{55}$$

So for the cases with zero eigenvalues, the higher order terms, through Eq. (55), must be considered in studying the stability behavior of attractors. Note that the absence of the linear terms in Eq. (55) reflects the fact that λ_1 is zero, and we must focus on the next-leading terms. The attractor is then stable if $\alpha < 0$. For the cases where $\alpha = 0$, we must look at the sign β. $\beta < 0$ leads to stable attractors. If again β becomes zero, we must go to higher orders.

As an explicit example, we consider

$$F(R, G) = R + \tfrac{1}{3}R^3 - \tfrac{1}{3}. \tag{56}$$

The critical points are obtained by solving Eqs. (20) and (21), which result in

$$(R_c, H_c) = \left(1, \frac{1}{\sqrt{12}}\right). \tag{57}$$

Note that we have not considered the unphysical solution $R_c = -2$. It can be easily seen that $\eta = 0$ [using (29)], therefore $\lambda_1 = 0$ and $\lambda_2 = -3H$. Despite a negative eigenvalue, the numerical calculations show that the critical point (57) is not a stable attractor (see Fig. 7)

This can be justified by calculating the parameter α and β of Eq. (55), which leads to

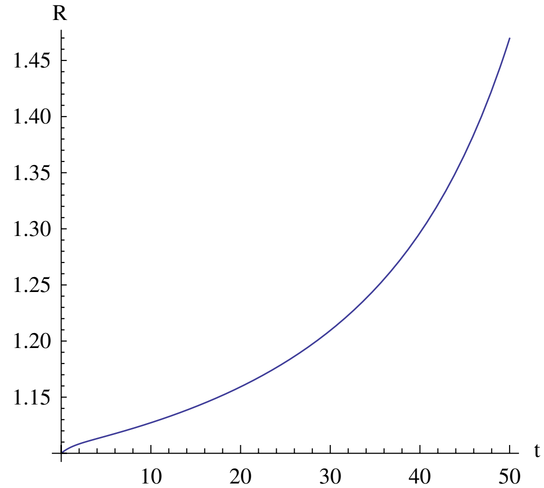

FIG. 7 (color online). The plot of $R(t)$ of $F(R, G) = R + 1/3R^3 - 1/3$. The initial values are $(R_0, H_0) = (1.1, 0.3)$. It is clear that system does not approach $R_c = 1$.

$$\dot{\mathcal{U}} = \frac{4}{3}\mathcal{U}^2 - \frac{100}{9\sqrt{3}}\mathcal{U}^3 + \ldots. \tag{58}$$

Since $\alpha > 0$, the critical point is not stable, in accordance with Fig. 7.

V. SINGULAR POINTS OF $F(R, G)$

As pointed out after Eq. (22), in deriving the critical point equation (20), it has been assumed that the denominator of Eq. (17) is finite at the critical values, so $\dot{R} = 0$ leads us to set the numerator of Eq. (17) equal to zero. But, as we will show, for the cases where the Lagrangian $F(R, G)$ has some singularities, this assumption, i.e. the finiteness of the denominator of Eq. (17), is not right, and we must carefully reinvestigate our results.

Take the function $F(R, G)$ as follows:

$$F(R, G) = \frac{P(R, G)}{Q(R, G)}, \tag{59}$$

where $Q(R, G)$ has a root of order n at $R = \alpha$, i.e.

$$Q(R, G) = (R - \alpha)^n g(R, G). \tag{60}$$

It can be easily seen that in this case, the denominator of Eq. (17) diverges at $R \to \alpha$, with the power greater than numerator. Substituting Eq. (59) into Eq. (17), results in

$$\dot{R} = \frac{(6H^2 - R)PQQ_R + 8H^2(R - 12H^2)^2(2PQ_RQ_G - PQQ_{RG} - QP_GQ_R) + \cdots}{6H(2Q_R^2 - QQ_{RR})P + \cdots}, \tag{61}$$

where the dots denote the higher order terms of $R - \alpha$. Power counting of Eq. (61) shows that

$$\dot{R} = \frac{O((R - \alpha)^{2n-1})}{O((R - \alpha)^{2n-2})}, \tag{62}$$

which results an extra solution for $\dot{R} = 0$ as follows:

$$(R_c, H_c) = \left(\alpha, \sqrt{\frac{\alpha}{12}}\right). \tag{63}$$

H_c is found from Eq. (21).

To study the stability of this critical point, we need to calculate the eigenvalues of the matrix M [in Eq. (25)]. A lengthy calculation shows that

$$\delta\dot{R} = -H\left[\frac{Q_R^2 + QQ_{RR}}{2Q_R^2 - QQ_{RR}} - \frac{QQ_R(3Q_RQ_{RR} - QQ_{RRR})}{(2Q_R^2 - QQ_{RR})^2} + \cdots\right]\delta R, \tag{64}$$

where using the expression (60) for $Q(R, G)$, leads to

$$\delta\dot{R} = -\frac{H_c}{n+1}\delta R + \ldots, \tag{65}$$

at $R = \alpha$. So the matrix M becomes

$$M = \begin{pmatrix} -4H_c & -1/6 \\ 0 & -H_c/(n+1) \end{pmatrix}, \tag{66}$$

with eigenvalues

$$\lambda_1 = -4H_c, \qquad \lambda_2 = -\frac{H_c}{n+1}, \tag{67}$$

where both of them are real negative numbers. So we lead to an important general consequence: *Any singularity of the function $F(R, G)$ is a stable attractor solution.*

The same is true for the cases where $Q(R, G)$ has a root of order n at $G = \beta$:

$$Q(R, G) = (G - \beta)^n g(R, G). \tag{68}$$

The same procedure results in a critical point at $G_c = \beta$, or

$$(R_c, H_c) = \left(\sqrt{6\beta}, \left(\frac{\beta}{24}\right)^{1/4}\right), \tag{69}$$

in which Eq. (22) has been used. The matrix M becomes the same as Eq. (66), which proves that this critical point is a stable attractor.

So generally for

$$Q(R, G) = (R - \alpha)^n(G - \beta)^m g(R, G), \tag{70}$$

the model has the stable attractor points (63) and (69).

As an example, we consider the model discussed in Sec. III B, that is $F(R, G) = R + Gf(R)$. The regular critical points (nonsingular type) can be found by solving the relation (38), and the stability condition is Eq. (39). Now consider the explicit example

$$F(R, G) = R + \frac{mG}{R^2 - 1}, \tag{71}$$

where m is a constant. Equation (38) leads to

$$3(R^2 - 1)^2 + mR^3 = 0 \Rightarrow R_c = R_c(m), \tag{72}$$

and the inequality (39) results in

$$\frac{2(R_c^2 - 1)}{R_c^2 - 5} > 1. \tag{73}$$

For $R_c^2 - 5 > 0$, Eq. (73) leads to $R_c^2 > -3$, which is always true, and for $R_c^2 - 5 < 0$, it results in $R_c^2 < -3$, which is never true. So the condition (73) holds if

$$R_c^2 > 5. \tag{74}$$

Now if we choose $m = -10$, Eq. (72) gives two following real solutions:

$$R_{1c} = 0.534, \qquad R_{2c} = 3.837. \tag{75}$$

It is clear that R_{1c} does not satisfy (74), while R_{2c} does. Explicit calculation of η [in Eq. (29)] shows that $\eta|_{R=R_{1c}} < 0$ and $\eta|_{R=R_{2c}} > 0$. So we expect that the stable critical point of

$$F(R, G) = R - \frac{10G}{R^2 - 1}, \tag{76}$$

is

$$(R_c, H_c) = (3.837, 0.565). \tag{77}$$

Numerical calculation verifies this. See Fig. 8.

Until now, we find the regular attractor of (76). But it is clear that the $F(R, G)$ in Eq. (76) is singular at $R = 1$. So we expect another stable attractor at the point

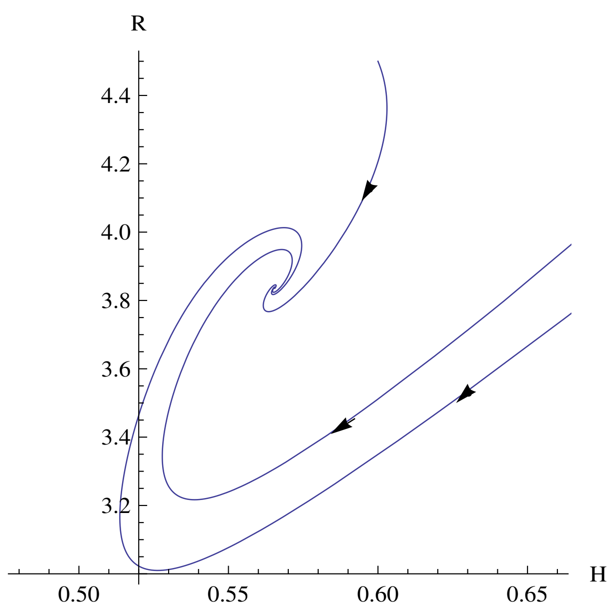

FIG. 8 (color online). The spiral paths leading to attractor (77) of the Lagrangian $F(R, G) = R - 10G/(R^2 - 1)$.

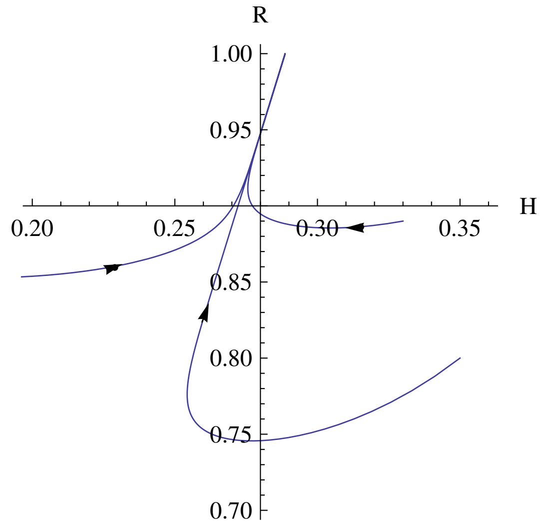

FIG. 9 (color online). The paths of $F(R, G) = R - 10G/(R^2 - 1)$ leading to the attractor $(R_c, H_c) = (1, \sqrt{1/12})$

$$(R_c, H_c) = \left(1, \sqrt{\frac{1}{12}}\right). \tag{78}$$

This new attractor is also verified by the numerical method. See Fig. 9.

VI. THE CRITICAL CURVES

There are other interesting cases in which the critical points are replaced by critical curves. In this case, each of the infinite points on this critical curve are in fact a critical point, and besides, as we will show, they are stable attractors. This situation occurs when the criticality condition (20) holds for any R and H values.

Before introducing some special examples, we first prove a general statement:

If a $F(R, G)$ function satisfies (20) and R and G satisfy Eqs. (21) and (22), respectively, then $\eta = F_R/(3A) - 4H^2$ is equal to zero. *Proof*: Since condition (20) holds for any R and H, it can be differentiated with the result

$$(\tfrac{1}{2}RF_{RR} - \tfrac{1}{2}F_R + GF_{RG})dR + (\tfrac{1}{2}RF_{RG} + GF_{GG})dG = 0. \tag{79}$$

But from Eq. (14) we have

$$dG = 8H(R - 12H^2)dH + 4H^2 dR, \tag{80}$$

so

$$[\tfrac{1}{2}RF_{RR} - \tfrac{1}{2}F_R + GF_{RG} + 4H^2(\tfrac{1}{2}RF_{RG} + GF_{GG})]dR + 8(\tfrac{1}{2}RF_{RG} + GF_{GG})H(R - 12H^2)dH = 0. \tag{81}$$

Using Eq. (21), the coefficient of dH becomes zero. The coefficient of dR, which now must be set to zero, specifies F_R as follows:

$$F_R = R(F_{RR} + 8H^2 F_{RG} + 16H^4 F_{GG}) = RA, \qquad (82)$$

in which the function A, introduced in (27), has been used. Therefore, η becomes

$$\eta = \frac{F_R}{3A} - 4H^2 = \frac{1}{3}R - 4H^2 = 0, \qquad (83)$$

where $R = 12H^2$ has been used. This completes our proof.

Now we note that if $F(R, G)$ satisfies (20), this equation does not impose any extra constraint on $F(R, G)$ and therefore does not specify any critical values for R and H. So the only remaining relation in the R-H phase-space plane is the second equation, i.e. Eq. (21), which defines a critical curve. The eigenvalues of this critical curve, as a result of the above statement, which leads to Eq. (83), are

$$\lambda_1 = 0, \qquad \lambda_2 = -3H. \qquad (84)$$

But it can be shown that in the case of the emergence of the critical curve, the stability of any particular point on this curve can be determined by the nonzero eigenvalues [15]. Since in our case, $\lambda_2 = -3H < 0$, therefore *any points on the critical curve $R = 12H^2$ of $F(R, G)$ models is a stable attractor*. This is a general result.

Now let us consider some explicit examples.

Example 1: Let us first consider the class of models introduced in Sec. III A. For $F(R, G) = F(RG)$, it is obtained that Eq. (20) results in

$$\tfrac{3}{2}xF'(x) - F(x) = 0, \qquad (85)$$

where $x = RG$. If we demand that the Eq. (85) satisfies for all x, then it can be viewed as a differential equation with solution $F(x) = x^{2/3}$. So the $F(R, G)$ model

$$F(R, G) = (RG)^{2/3} \qquad (86)$$

has a critical curve $R = 12H^2$. All the points on this curve are stable attractors. Figures 10 and 11 show that both $H_{1c} = 0.977$ and $H_{2c} = 1.99$ points (as two arbitrary points), with $R_{1c} = 11.454$ and $R_{2c} = 47.52$, respectively, are stable attractors of this model.

Example 2: Consider the following $F(R, G)$ model:

$$F(R, G) = R^n f(G^k R^m) \qquad (87)$$

with arbitrary constants n, k and m. Substituting (87) into Eq. (20), results in

$$\left(\frac{n}{2} - 1\right)R^n f(x) + \left(\frac{m}{2} + k\right)R^{n+m}G^k f'(x) = 0, \qquad (88)$$

where $x = G^k R^m$. If one demands the above equation satisfies for all R and Gs and for any arbitrary function $f(x)$, then the constants n, k and m must satisfy

$$n = 2, \qquad m = -2k. \qquad (89)$$

So $F(R, G) = R^2 f(G^k/R^{2k}) = R^2 g(G/R^2)$ satisfies (20), and the curve $R = 12H^2$ is its critical curve. It is interesting to note that the case considered in example 1, i.e. the Eq. (86), is in fact of this type: $(RG)^{2/3} = R^2(G/R^2)^{2/3}$.

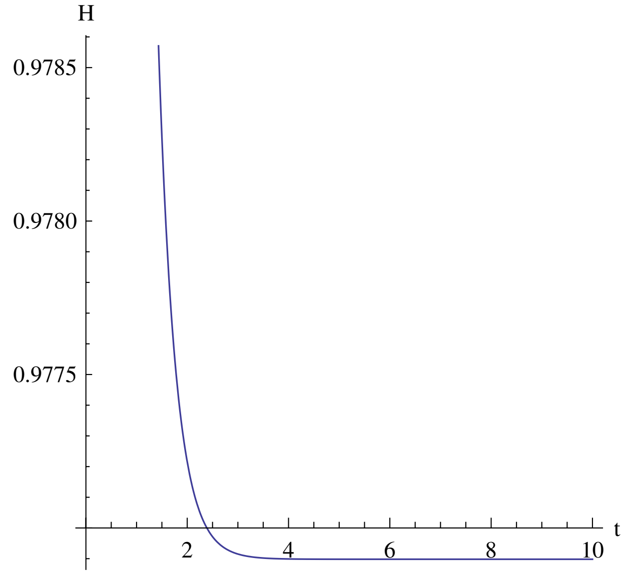

FIG. 10 (color online). The plot of $H(t)$ for the $F(R, G) = (RG)^{2/3}$ model. The point $(H_c, R_c) = (0.977, 11.454)$ is a stable attractor.

Example 3: Consider the following model:

$$F(R, G) = \alpha G + f(R), \qquad (90)$$

then Eq. (20) results in

$$Rf' = 2f, \qquad (91)$$

which its solution, as a differential equation, is $f(R) = \beta R^2$. So all models of the type

$$F(R, G) = \alpha G + \beta R^2 \qquad (92)$$

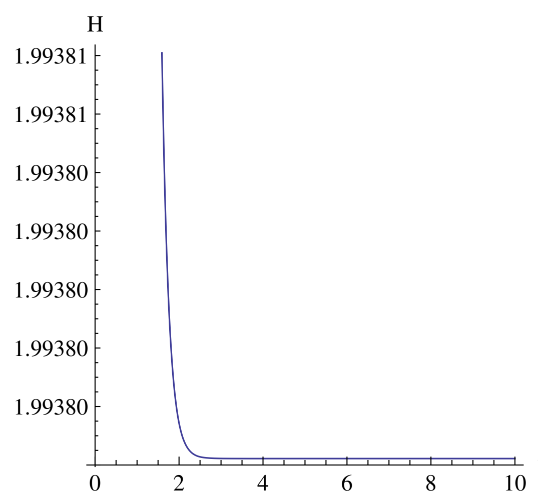

FIG. 11 (color online). The plot of $H(t)$ for the $F(R, G) = (RG)^{2/3}$ model. The point $(H_c, R_c) = (1.99, 47.52)$ is a stable attractor.

have $R = 12H^2$ as their critical curve, with an infinite number of stable attractors.

This example also shows that in the case of $f(R)$ gravity theories, which is the $\alpha = 0$ case of Eq. (90), the only model that leads to the critical curve $R = 12H^2$ is the $f(R) = R^2$ model.

Example 4: As the last example, consider the model

$$F(R, G) = R + Gf(R). \tag{93}$$

Substituting (93) into Eq. (20), results in

$$R^2 f'(R) = 6 \Rightarrow f(R) = -\frac{6}{R}. \tag{94}$$

So $F(R, G) = R - 6G/R$ also has the critical curve $R = 12H^2$.

The above-mentioned procedure can be applied to some other functional forms, such as $F(R, G) = f(G/R)$, with the result $F = (G/R)^2$, etc.

VII. CONCLUSION

As a candidate of dark energy, we consider the generalized Gauss-Bonnet dark energy models looking for the situations where the late-time behavior of this modified gravity theory is the de Sitter space-time. We describe the phase space of this theory by the two-dimensional R-H space. This dimensionality verifies by the fact if the three-dimensional R-H-G space has been chosen, one of the eigenvalues of stability matrix is always zero, which indicates that the number of independent variables is two.

The eigenvalues of stability matrix show that the critical points, i.e. the de Sitter space-times, are, in general, the stable attractor if $\eta = F_R/(3A) - 4H^2 > 0$, a fact that has been verified by several examples. The emergence of critical points with $\eta = 0$, in which one of the eigenvalues is zero, $\lambda_1 = 0$, forces us to consider the higher order terms in normal basis in order to have a correct judgment about the stability of these kinds of critical points.

We also find two classes of stable attractors: the singular points of the Lagrangian $F(R, G)$ and the cases where the critical points are replaced by the critical curve $R = 12H^2$ (in R-H plane). In the latter case, all the points on this curve are stable attractors.

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