Hard thermal loops, to quadratic order, in the background of a spatial 't Hooft loop

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We compute the simplest hard thermal loops for a spatial 't Hooft loop in the deconfined phase of a SU(N) gauge theory. We expand to quadratic order about a constant background field $A_0 = Q/g$, where Q is a diagonal, color matrix and g is the gauge coupling constant. We analyze the problem in sufficient generality that the techniques developed can be applied to compute transport properties in a ''semi''-quark gluon plasma. Notably, computations are done using the double line notation at finite N. The quark self-energy is a Q-dependent thermal mass squared $\sim g^2T^2$, where T is the temperature, times the same hard thermal loop as at $Q = 0$. The gluon self-energy involves two pieces: a Q-dependent Debye mass squared, $\sim g^2T^2$, times the same hard thermal loop as for $Q = 0$, plus a new hard thermal loop $\sim g^2T^3$, due to the color electric field generated by a spatial 't Hooft loop.

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I. INTRODUCTION

The spectacular success of the experiments at the Relativistic Heavy Ion Collider at Brookhaven has invigorated the study of gauge theories at a nonzero temperature, T [1,2]. Experiment clearly shows that an understanding of quantities in thermal equilibrium are not sufficient to understand the data, and that one also needs quantities near thermal equilibrium, especially transport coefficients.

In this paper we compute for what appears to be an unrelated problem: the real time response functions for a spatial 't Hooft loop [3,4]. A Wilson loop, $\mathrm{tr}\mathcal{P}\exp(ig\oint_C A_\mu dx^\mu)$, represents the propagation of a test electric charge along the path C, and measures the response to magnetic flux. Similarly, a 't Hooft loop introduces a test magnetic charge along a given path, and measures the response to electric flux. Their behavior is dual to one another. At zero temperature in a SU(N) gauge theory without quarks, the condensation of magnetic charges confines electric charge, so the Wilson loop has area behavior, and the 't Hooft loop, perimeter. Conversely, at temperatures above that for deconfinement, magnetic charges do not condense, and electric charge is not confined; hence, a (thermal) Wilson loop has perimeter behavior, and the (spatial) 't Hooft loop, area [5].

The 't Hooft loop does not have a simple representation in terms of the vector potential for the gauge field, but in the deconfined phase, the area behavior of the spatial 't Hooft loop can be simply understood [5] as a Z(N) interface [6–11]. Without dynamical quarks, a SU(N) gauge theory has N equivalent vacua, which differ by global Z(N) transformations from one another [3,4]. To probe this, take a box that is long in one spatial direction, say of length L in z, and let the two ends of the box differ by a Z(N) transformation. Thus, at one end of the box, $z = 0$, the Wilson line in the imaginary time direction, $\mathcal{P}\exp(ig\int A_0 d\tau)$ equals the unit matrix; at the other end of the box, $z = +L$, the Wilson line is a Z(N) phase, $\exp(2\pi i/N)$, times the unit matrix. These boundary conditions can be imposed by introducing a background field for the timelike component of the vector potential [6–16]:

$$A_0(z) = \frac{1}{g}Q(z), \tag{1}$$

where Q is a diagonal matrix in color space, and g is the gauge coupling constant. The matrix $Q(z)$ is then chosen to vary so that a Z(N) interface, centered at $z = +L/2$, forms. Only the ends of the box represent allowable vacua, so a nonzero color electric field is generated along the z direction, $E_z \sim \partial_z Q(z)$, and the configuration has nonzero action. By construction, the interface is independent of the x and y directions, and so the action is proportional to the transverse area. This Z(N) interface is then equivalent to a spatial 't Hooft loop, in the plane of x and y, at $z = +L/2$ [5].

The action for the Z(N) interface can be computed in weak coupling, and reduces to a tunneling problem in one dimension [6]. There is no potential for Q classically, but one is generated at one-loop order, and so the action for the associated instanton is not $\sim 1/g^2$, but $\sim 1/\sqrt{g^2}$ [6,17]. A derivative expansion can be used to compute, because along the z direction, the width of the interface is proportional to the inverse Debye mass, $\sim 1/(gT)$, which is large relative to the typical thermal correlation length for massless fields, $\sim 1/(2\pi T)$. Consequently, at the outset one computes for a field $Q(z)$, which is constant in z, as effects from the variation in z enter through corrections that are of higher order in g. Corrections to the interface tension have been carried out to $\sim g^3$ [7], and are underway to $\sim g^4$ [8]. This is to be compared with the free energy (where $Q = 0$), which has been computed to $\sim g^6$ [18–22].

The interface tension is measurable through numerical simulations on the lattice [10]. This includes simulations that model the behavior in real time [11]. In this paper, we

address a related problem semiclassically, by computing the simplest real time response functions, for a spatial 't Hooft loop, in weak coupling. To do this, we expand to quadratic order about the background field in Eq. (1), taking Q to be a constant matrix, and analytically continue to real time. As for quantities computed in equilibrium, near equilibrium there is a natural division between momenta that are "soft," with components of order the Debye mass, $\sim gT$, and "hard," $\sim T$. The simplest real time response functions are the quark and gluon self-energies, computed in the hard thermal loop approximation, for soft external momenta [23–25].

We perform this computation in order to develop techniques that will enable us to address a problem of much broader interest. Resummations of perturbation theory appear to break down by temperatures several times the critical temperature [20,21]. The obvious guess is to assume that since $T_c \sim 150$–200 MeV in a gauge theory, that the theory has entered a nonperturbative regime by this point, where the QCD coupling is large [2]. While the former must be true—confinement cannot be seen in perturbation theory—computations of an effective theory for the pressure find that the coupling is *moderate* even at T_c, with $\alpha_s^{\rm eff}(T_c) = g_{\rm eff}^2(T_c)/(4\pi) \sim 0.3$ [22]. This is because in imaginary time, the typical "energies" are large, multiples of $2\pi T$ [18].

Why, then, does deconfinement occur at moderate coupling? Deconfinement is an ordering of global Z(N) spins, and is measured by the trace of the thermal Wilson line, which is the Polyakov loop [3,4]. In the fundamental representation, without quarks the expectation value of the Polyakov loop vanishes below T_c, and approaches one at high temperature. Since the Polyakov loop is not equal to one whenever $Q \neq 0$, one way to model the region where the Polyakov loop is not near one is to assume a nontrivial distribution of Q's [11,26–31], which in Ref. [32] we term a semi quark-gluon plasma (semi-QGP). At least at infinite N, it is easy to model the confined phase, as a distribution that is flat in Q. This implies that the expectation value of the Polyakov loop in any nontrivial representation vanishes, whether or not the loop carries Z(N) charge. This is also consistent with how the 't Hooft loop must change near T_c. At high temperature, Z(N) interfaces are rare, and in infinite volume the theory lies in one Z(N) domain. As $T \to T_c^+$, though, the interface tension decreases, Z(N) domains become plentiful, and the Z(N) spins are disordered. The decrease of the Z(N) interface tension near T_c has been confirmed on the lattice [10].

A semi-QGP can be shown to occur in one unphysical limit. Let the spatial volume be a sphere of hadronic dimensions, so small that by asymptotic freedom, the coupling constant runs to a very small value [33]. Although systems at finite volume cannot have phase transitions, they can if the number of colors is infinite. If R is the radius of the sphere, then even when $g^2 = 0$, at infinite N there is a deconfining phase transition, of first order, when $T_c = c/R$, where $c = 1/\log(2+\sqrt{3})$ [33]. At T_c^+, the expectation value of the (renormalized) Polyakov loop, in the fundamental representation, is exactly $1/2$. The Q distribution for the constant mode on the sphere reduces to a type of matrix model, which can be computed analytically about T_c [33]. Further, since all resonances are of zero width at infinite N, the Hagedorn temperature is a precise quantity. On a small sphere at $N = \infty$, at zero coupling the Hagedorn temperature coincides with T_c. Perturbative corrections move T_c below the Hagedorn temperature, and the loop at T_c^+ away from $1/2$, by an amount $\sim (g^2(R)N)^2$ [34]. Presumably, the expectation value of the (renormalized) Polyakov loop, in the fundamental representation, goes from $1/2$ at T_c^+, to near one, by temperatures that are a few times T_c. This is then the semi-QGP on a small sphere.

Of course this might be an artifact of working on a small sphere at infinite N. In a large spatial volume, one must look to numerical simulations on a lattice. In any volume, the Wilson line, and so the Polyakov loop has ultraviolet divergences, so that the bare loop vanishes in the continuum limit. A nonzero value in the continuum limit is obtained after a type of mass renormalization [27,35–40]. On a small sphere, when $g^2(R) \ll 1$, one can renormalize the loop perturbatively. In a large volume, on the lattice there are two methods of renormalizing the Polyakov loop [27,35], which now agree up to the numerical accuracy [40]. The most precise measurements are for a SU(3) gauge theory without quarks [40]. From Fig. 1 of [40], in the triplet representation the expectation of the renormalized Polyakov loop vanishes below T_c, and is ~ 0.5 at T_c. It then rises to it rises rapidly, and is ~ 0.9 by $2.0T_c$. It then rises slowly, reaching ~ 1.1 by $\sim 4T_c$. From 4 to $12T_c$, its expectation value is flat. This suggests that there is a nontrivial Q distribution about T_c, which is relevant up to temperatures that are 2–$4T_c$. Above $4T_c$, any nonperturbative effects from the Q distribution appear negligible, consistent with the success of resummations of perturbation theory [21,22]. In a SU(3) gauge theory with quarks, the simulations are of more limited accuracy, but a similar picture emerges [39]. The principal difference is that the expectation value of the renormalized triplet loop is nonzero even below T_c [39].

At present, it is not known what the Q distribution is in a gauge theory, even without quarks. This would correlate the pressure with the expectation value of the (renormalized) Polyakov loop(s). Such a distribution might be obtained from numerical simulations, in both the original and an effective theory [30].

Even without knowing the full Q distribution, though, one can take the first steps toward the response functions in real time, which we do here. For either a Q distribution, or a 't Hooft loop, one begins by computing the quadratic fluctuations about the background field in Eq. (1). We then

analytically continue the fluctuations, computed in imaginary time, to real time [25,41]. (For a numerical approach to quantities in real time, see [11].)

We find that the quark self-energy is a trivial extension of that for $Q = 0$: it is equal to a thermal mass squared, $\sim g^2T^2$, times the same hard thermal loop as in zero field. The gluon self-energy is different, though. Besides the usual hard thermal loop, proportional to a thermal mass squared, $\sim g^2T^2$, there is a new piece, $\sim g^2T^3$. The function is similar to other hard thermal loops, but is novel. We suggest that it arises because of the background color electric field in a 't Hooft loop.

The paper is organized as follows: In Sec. II, we discuss how the double line notation, which is standard at large N [42], can also be used easily at finite N [43]. Our discussion is elementary, but is absolutely essential to being able to compute in an arbitrary background field with $A_0 \neq 0$, Eq. (1).

Section III introduces the background field calculation for constant A_0. The perturbative rules in the 't Hooft basis are given in Sec. III A. These are nothing more involved than the usual perturbative rules, with a simple "shift" in the energies, $\sim Q$. Section III B gives expressions, useful for calculation, in terms of a "mixed" representation, working with spatial momenta and Euclidean time. In Sec. III C, we follow Furuuchi [41] and discuss how to obtain scattering amplitudes, in real time, from those computed in imaginary time.

The computation of the hard thermal loop in the quark self-energy is given in Sec. IV. We go through this example in some detail to develop familiarity with computing diagrams when $Q \neq 0$. The hard thermal loop in the gluon self-energy is computed in Sec. V. In Sec. V A, we discuss the Q dependence of the hard thermal loops in two tadpole diagrams, which are independent of the external momenta. In Sec. V B, we consider the hard thermal loops that arise from diagrams that exhibit with Landau damping. With these examples in hand, computing the one point gluon function, in Sec. V C, the quark contribution to the gluon self-energy, Sec. V D, and that of ghosts and gluons to the gluon self-energy, Sec. V E, is relatively straightforward.

In an appendix we draw some distinctions on the differences between Z(N) and U(1) interfaces [29].

II. THE DOUBLE LINE NOTATION AT FINITE N

In order to compute efficiently, it is useful to have a convenient basis for the generators of SU(N). In this section we follow Cvitanovíc [43] to show how the usual double line notation, which is familiar at large N [42], is also natural at small N.

Our purpose here is to establish the notation that we need to compute in the presence of a constant, background field for A_0, Eq. (1). We note that at $N = \infty$, Aharony *et al.* computed the free energy with $A_0 \neq 0$ to three-loop order, $\sim g^4$, on a small sphere [33]. At finite N, recently Korthals Altes used the double line notation to compute the free energy for $A_0 \neq 0$ to $\sim g^2$ in supersymmetric theories [8].

The standard choice for the generators of a gauge group, λ^A, is to take a complete and orthonormal basis,

$$\mathrm{tr}\,(\lambda^A \lambda^B) = \tfrac{1}{2}\delta^{AB}; \tag{2}$$

A and B refer to adjoint indices, which for SU(N) run from $A, B = 1, 2, \ldots, (N^2 - 1)$.

We denote indices in the fundamental representation by $a, b \ldots = 1, 2, \ldots, N$. Indices in the adjoint representation are then denoted by a pair of fundamental indices, (ab). The basic quantity that we need is a projection operator that ties together upper and lower adjoint indices

$$\mathcal{P}^{\,ab}_{\,cd} = \delta^a_c \delta^b_d - \frac{1}{N}\delta^{ab}\delta_{cd}. \tag{3}$$

For an arbitrary matrix M, adjoint indices are raised and lowered by flipping the order in the pair: $M^{ab} = M_{ba}$. The indices flip because off-diagonal generators are ladder operators. With this convention, the projection operators between a pair of upper indices, or a pair of lower indices, are

$$\mathcal{P}^{\,ab}_{\,cd} = \mathcal{P}^{ab,dc} = \mathcal{P}_{ba,cd}. \tag{4}$$

It is obvious that Eq. (3) represents a projection operator

$$\mathcal{P}^{\,ab}_{\,ef}\mathcal{P}^{\,ef}_{\,cd} = \mathcal{P}^{\,ab}_{\,cd}. \tag{5}$$

While we have to be careful in lowering and raising adjoint indices, we can raise or lower single indices without concern, $\delta^{ab} = \delta^a_b = \delta_{ab}$. The second term in Eq. (3) ensures that it is traceless in either of the two pairs of adjoint indices,

$$\delta_{ab}\mathcal{P}^{\,ab}_{\,cd} = \delta^{cd}\mathcal{P}^{\,ab}_{\,cd} = 0. \tag{6}$$

While we call (ab) an adjoint index, this terminology is somewhat misleading. SU(N) has $N^2 - 1$ independent generators, but there are obviously N^2 values for the index (ab). By using projection operators (or in the terminology of [43], invariant tensors), the resulting basis is overcomplete, with one extra generator.

While we give explicit expressions for all quantities, it is handy to use a diagramatric notation [42,43]. For SU(N), lines always carry an arrow, with fields in the fundamental representation represented by a single line, and those in the adjoint, by a double line. For an adjoint index (ab), we adopt the notation that for upper indices, a is outgoing and b ingoing; for lower indices, a is ingoing and b outgoing. This reversal is necessary so that upper and lower indices are contracted accordingly.

Generally, any confusion with indices is dispelled by drawing the corresponding diagram. For example, the projection operator of Eq. (3) is illustrated in Fig. 1, and is drawn like a gluon propagator.

With these conventions, the generators of the fundamental representation are just projection operators,

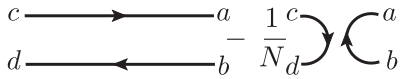

FIG. 1. Projection operator, $\mathcal{P}^{ab,dc}$.

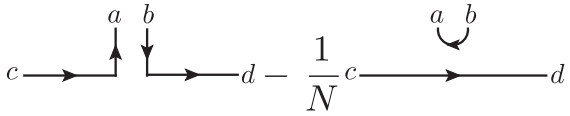

FIG. 2. Generator for SU(N), times $\sqrt{2}$.

$$(t^{ab})_{cd} = \frac{1}{\sqrt{2}}\mathcal{P}^{ab}_{cd}. \tag{7}$$

Note that the upper pair, (ab), refers to the index for the adjoint representation, while the lower pair, (cd), refers to the components of this matrix in the fundamental representation. This is illustrated in Fig. 2. While this is also a projection operator, we draw it differently from Fig. 1, distinguishing between the adjoint indices, on top of the diagram, and the matrix indices for the fundamental representation, on the two sides.

As each generator is a projection operator, the trace of two generators is again a projection operator

$$\mathrm{tr}\,(t^{ab}t^{cd}) = \tfrac{1}{2}\mathcal{P}^{ab}_{ef}\mathcal{P}^{cd}_{fe} = \tfrac{1}{2}\mathcal{P}^{ab,cd}. \tag{8}$$

We now make an extended comment about the normalization of generators, which is implied by Eq. (8). While mathematically elementary, at least we found it confusing at first.

The off-diagonal generators are the customary ladder operators of the Cartan basis. That is, for $a \neq b$, they are normalized as in Eq. (2), i.e.,

$$\mathrm{tr}\,(t^{ab}t^{ba}) = \tfrac{1}{2}; \tag{9}$$

here, a and b are fixed indices, with no summation convention. In SU(N), there are $N(N-1)$ off-diagonal generators.

The only difference lies in the choice of the diagonal generators (which is the Cartan subalgebra, the space of mutually commuting generators). The Cartan basis includes one generator proportional to

$$t^{NN} = \frac{-1}{N\sqrt{2}}\begin{pmatrix}\mathbf{1}_{N-1} & 0\\ 0 & -(N-1)\end{pmatrix}, \tag{10}$$

where $\mathbf{1}_{N-1}$ is the unit matrix in $N-1$ dimensions. In the Cartan basis, the corresponding matrix is $\lambda^N = -\sqrt{N/(N-1)}t^{NN}$, where the overall constant is required so that λ^N obeys Eq. (2). In the Cartan basis, the other diagonal generators are like t^{NN}, but for smaller N. For example,

$$\lambda^{N-1} = \frac{1}{\sqrt{2(N-1)(N-2)}}\begin{pmatrix}\mathbf{1}_{N-2} & 0 & 0\\ 0 & -(N-2) & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{11}$$

and so on. We denote the $N-1$ diagonal generators in the Cartan basis as λ^a, $a = 2, 3, \ldots, N$. While orthonormal, this basis clearly treats the different diagonal elements on an unequal footing, with the Nth element occuping a privileged position.

In contrast, for the double line basis the diagonal generators are just permutations of one another: start with Eq. (10), and simply shuffle where the factor of $-(N-1)$ lies along the diagonal, e.g.,

$$t^{11} = \frac{-1}{N\sqrt{2}}\begin{pmatrix}-(N-1) & 0\\ 0 & \mathbf{1}_{N-1}\end{pmatrix}, \tag{12}$$

and so on. The $t^{11}, \ldots, t^{NN}$ are a set of N diagonal generators, which manifestly do not treat any diagonal element different from any other. This is only possible only because they are not independent, that is, their sum vanishes,

$$\sum_{a=1}^{N} t^{aa} = 0. \tag{13}$$

Consider the example of two colors, where the double line basis has four generators. There are two off-diagonal, ladder generators,

$$t^{12} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad t^{21} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{14}$$

For the diagonal generators, from Eqs. (3) and (7) there are two contributions,

$$t^{11} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = -t^{22}. \tag{15}$$

As expected, t^{11} is proportional to the Pauli matrix σ^3, but the constant appears wrong. While the ladder generators are normalized as in Eq. (9), $\mathrm{tr}(t^{12}t^{21}) = 1/2$, the diagonal generators satisfy $\mathrm{tr}(t^{11})^2 = \mathrm{tr}(t^{22})^2 = 1/4$, instead of $1/2$; further, $\mathrm{tr}(t^{11}t^{22}) = -1/4$.

This normalization is correct, and arises because the basis is overcomplete. From Eq. (8), the trace of a product of two generators is itself a generator. Thus, the trace of a given diagonal generator, squared, is

$$\mathrm{tr}\,(t^{aa})^2 = \frac{1}{2}\left(1 - \frac{1}{N}\right), \tag{16}$$

with no summation over a; for $N = 2$, this $= 1/4$. Further, the trace between two different diagonal generators is nonzero:

$$\mathrm{tr}\,t^{aa}t^{bb} = \frac{1}{2}\left(-\frac{1}{N}\right); \tag{17}$$

with no summation over a or b, and $a \neq b$; for $N = 2$, this $= -1/4$. Thus, in this basis, the (peculiar) normalization of the diagonal generators arises because the generators are projection operators. Of course the diagonal and off-diagonal generators are orthogonal to one another in the usual manner.

In the absence of a background field, that the Cartan basis chooses a preferred direction amongst the diagonal generators is of no concern. The Cartan basis is also convenient when computing the properties of a Z(N) interface [6], since then the background field is along t^{NN}, and treating the Nth diagonal element as special is natural.

In the presence of an *arbitrary* background field, Eq. (1), though, where Q is a diagonal matrix, $(Q)_{ab} = Q^a \delta_{ab}$, Eq. (32), the double line basis is more useful. In particular, all covariant derivatives are simple. In the fundamental representation, Q acts linearly upon fields ψ. Then ψ is like a column vector, so if ψ_a is the ath element,

$$Q\psi_a = Q^a \psi_a. \tag{18}$$

In the adjoint representation, the covariant derivative involves a commutator. The commutator of Q with any generator, though, is just that generator times a difference of Q's:

$$[Q, t^{ab}] = (Q^a - Q^b)t^{ab} \equiv Q^{ab} t^{ab}. \tag{19}$$

This is clear if $a = b$, t^{ab} is a diagonal matrix, and the commutator of t^{aa} with another diagonal matrix, Q, vanishes. If $a \neq b$, only the first term in $(t^{ab})_{cd}$, $\sim \delta^a_c \delta^b_d$, Eqs. (3) and (7), contributes, to give Eq. (19). We introduce the notation $Q^{ab} = Q^a - Q^b$, which we shall use extensively.

As a consequence of Eqs. (18) and (19), we find in Sec. III A that with the double line basis, perturbation theory for $Q \neq 0$ is a *trivial* generalization of that for $Q = 0$: just a constant, albeit color dependent, shift in the energies. Energies that carry color indices were introduced when the determinant in a background constant A_0 field was first computed, in Appendix D of Ref. [12].

The double line basis is useful in other ways. While admittedly perverse for two colors, for three or more colors it is a very efficient means of deriving various identities amongst generators of the gauge group. For example, the product of two generators is

$$\begin{aligned}(t^{ab}t^{cd})_{ef} &= \frac{1}{2}\mathcal{P}^{ab}_{eg}\mathcal{P}^{cd}_{gf} \\ &= \frac{1}{2}\Big(\delta^a_e \delta^{bc}\delta^d_f - \frac{1}{N}(\delta^a_e\delta^b_f\delta^{cd} + \delta^{ab}\delta^c_e\delta^d_f) \\ &\quad + \frac{1}{N^2}\delta^{ab}\delta^{cd}\delta_{ef}\Big),\end{aligned} \tag{20}$$

as illustrated in Fig. 3. This certainly shows how writing down all of the indices is more tedious than just drawing the corresponding diagram. By tying the sides of the diagram together, representing summation over the matrix

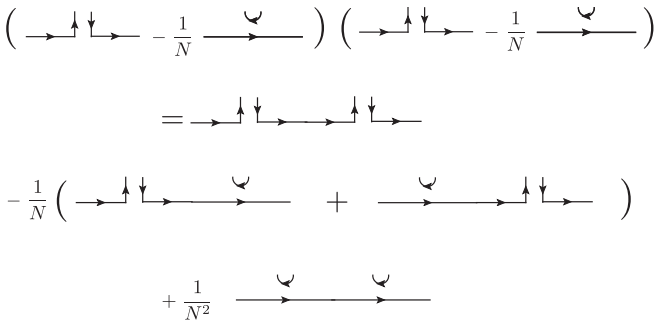

FIG. 3. The product of two generators, times two.

indices, we obtain the normalization conditions above, Eqs. (16) and (17). By tying the adjoint indices together on top of the diagram, we also obtain the familiar identity

$$\sum_{a,b=1}^{N} (t^{ab}t^{ba})_{cd} = \frac{N^2 - 1}{2N}\delta_{cd}. \tag{21}$$

In principle, we really should contract the adjoint indices with a projection operator. Since the generators are traceless, though, Eq. (13), the projection operator reduces to ordinary Kronecker deltas.

The product of three arbitrary generators can be written out, similar to Fig. 3. By drawing diagrams, it is easy obtaining the standard relation

$$\sum_{c,d=1}^{N} t^{cd}t^{ab}t^{dc} = -\frac{1}{2N}t^{ab}, \tag{22}$$

where we leave the matrix indices implicit.

More useful is to take a trace of the product of three generators. The antisymmetric combination is proportional to the structure constant of the group

$$[t^{ab}, t^{cd}] = i\sum_{e,f=1}^{N} f^{(ab,cd,ef)}t^{fe}, \tag{23}$$

and is simple,

$$f^{(ab,cd,ef)} = \frac{i}{\sqrt{2}}(\delta^{ad}\delta^{cf}\delta^{eb} - \delta^{af}\delta^{cb}\delta^{ed}), \tag{24}$$

as illustrated in Fig. 4. In this basis the structure constants satisfy the identity

$$\sum_{e,f,g,h=1}^{N} f^{(ab,ef,gh)}f^{(cd,fe,hg)} = N\mathcal{P}^{ab,cd}. \tag{25}$$

It is also easy to draw the diagram for the symmetric structure constant, $d^{(ab,cd,ef)}$,

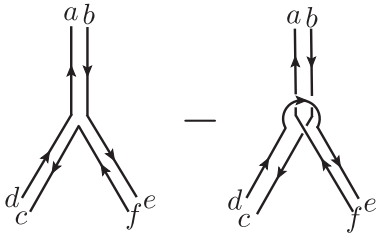

FIG. 4. Structure constant for SU(N), times $-i\sqrt{2}$.

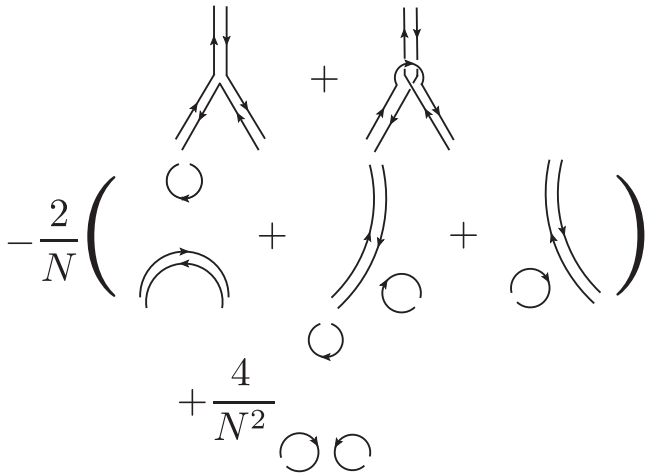

FIG. 5. Symmetric structure constant, times $\sqrt{2}$.

$$d^{(ab,cd,ef)} = 2\,\mathrm{tr}(t^{ab}\{t^{cd}, t^{ef}\}) = \frac{1}{\sqrt{2}}\Big(\delta^{ad}\delta^{cf}\delta^{eb} + \delta^{af}\delta^{cb}\delta^{ed} - \frac{2}{N}(\delta^{ab}\delta^{cf}\delta^{ed} + \delta^{ad}\delta^{cb}\delta^{ef} + \delta^{af}\delta^{cd}\delta^{eb}) + \frac{4}{N^2}\delta^{ab}\delta^{cd}\delta^{ef}\Big), \tag{26}$$

as illustrated in Fig. 5.

For higher representations of the group, instead of diagrams with just two lines, one obtains diagrams with many lines, or ''birdtracks.'' For a careful discussion of the classification of arbitrary representations of Lie groups by means of birdtrack diagrams, see Cvitanovíc [43].

III. COMPUTING IN BACKGROUND FIELD GAUGE

A. Propagators in a background A_0 field

In this section we develop the perturbative rules in the appropriate background field [16]. At tree level the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\mathrm{tr}(G_{\mu\nu}^2) + \bar{\psi}(\not{D} + m)\psi. \tag{27}$$

We assume there are N_f flavors of quarks, ψ, in the fundamental representation of the gauge group; the covariant derivative in that representation is $D_\mu = \partial_\mu - igA_\mu$. The field strength tensor $G_{\mu\nu} = [D_\mu, D_\nu]/(-ig) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. The covariant derivative in the adjoint representation is $D_\mu = \partial_\mu - ig[A_\mu, .]$. We work in Euclidean spacetime, with a positive metric. The gamma matrix is Hermitian and satisfies $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

We expand about a background field A_μ^{cl},

$$A_\mu = A_\mu^{\mathrm{cl}} + B_\mu, \tag{28}$$

where B_μ denotes the fluctuation. The classical covariant derivative is then $D_\mu^{\mathrm{cl}} = \partial_\mu - igA_\mu^{\mathrm{cl}}$, etc. The gauge fixing and ghost terms are chosen to be those for background field gauge, with gauge fixing parameter ξ:

$$\mathcal{L}_{\mathrm{gauge}} = \frac{1}{\xi}\,\mathrm{tr}(D_\mu^{\mathrm{cl}} B_\mu)^2 - 2\,\mathrm{tr}(\bar{\eta} D_\mu^{\mathrm{cl}} D_\mu \eta), \tag{29}$$

with η the ghost field.

The inverse propagators follow directly. That for the quark field is $\not{D}^{\mathrm{cl}} + m$. The inverse propagator for the ghost is $-(D_\mu^{\mathrm{cl}})^2$, while that for gluon fluctuations, B_μ, is

$$(\Delta_{\mu\nu}^{\mathrm{cl}})^{-1} = -(D_\lambda^{\mathrm{cl}})^2\delta_{\mu\nu} + D_\nu^{\mathrm{cl}} D_\mu^{\mathrm{cl}} - \frac{1}{\xi} D_\mu^{\mathrm{cl}} D_\nu^{\mathrm{cl}} + ig[G_{\mu\nu}^{\mathrm{cl}}, .] = -(D_\lambda^{\mathrm{cl}})^2\delta_{\mu\nu} + \Big(1 - \frac{1}{\xi}\Big) D_\mu^{\mathrm{cl}} D_\nu^{\mathrm{cl}} + 2ig[G_{\mu\nu}^{\mathrm{cl}}, .]. \tag{30}$$

Most of our calculations are done assuming a background field that is constant in spacetime. Notice, however, that the last term in the inverse gluon propagator is proportional to the field strength tensor of the background field. This will be important in understanding novel terms for gluon hard thermal loops in the presence of an interface.

The covariant derivative in the fundamental representation enters into the quark inverse propagator, while that in the adjoint representation enters into the ghost and gluon inverse propagators. We now compute at a nonzero temperature T in the imaginary time formalism, where the Euclidean time τ: $0 \longrightarrow 1/T$. The energies are then

$$p_0 = 2n\pi T, \quad \text{bosons}; \qquad \tilde{p}_0 = (2n+1)\pi T, \quad \text{fermions}. \tag{31}$$

We use a tilde for the energies and momenta of fermions, to distinguish them from bosons.

We take the background field as a constant, diagonal matrix for the timelike component of the vector potential

$$A_0^{\mathrm{cl}} = \frac{1}{g} Q; \qquad (Q)_{ab} = Q^a \delta_{ab}; \tag{32}$$

as an SU(N) matrix, the sum of the Q^a's vanishes,

$$\sum_{a=1}^{N} Q^a = 0. \tag{33}$$

As discussed in Sec. II, the great virtue of the double line notation is that the covariant derivatives in a field like Eq. (32) are trivial. For fields in the fundamental representation, the covariant derivative in the background field is $D^{\rm cl}_\mu \psi_a = -i\tilde{P}^a_\mu \psi_a$, Eq. (18). If this covariant derivative acts upon a quark field, $\tilde{P}^a_\mu$ is a momenta with one color index,

$$\tilde{P}^{\ a}_{\mu} = (\tilde{p}_0 + Q^a, \vec{p}). \tag{34}$$

Since the background field shifts the Euclidean energies, it is convenient to write

$$\tilde{P}^{\ a}_{\mu} = P_\mu + \tilde{Q}^a, \qquad \tilde{Q}^a = Q^a + \pi T. \tag{35}$$

That is, we treat all momenta as bosonic, which we can easily do by just putting the change in the boundary condition for fermions, versus bosons, into part of the background field.

The covariant derivative acts upon fields in the adjoint representation as $D^{\rm cl}_\mu t^{ab} = -iP^{ab}_\mu t^{ab}$, Eq. (19). For bosonic fields,

$$P^{ab}_\mu = (p_0 + Q^a - Q^b, \vec{p}) = (p^{ab}_0, \vec{p}), \tag{36}$$

and involves an adjoint color index, (ab).

To be explicit, the quark propagator is

$$\langle \psi^a(P)\bar{\psi}^b(-P)\rangle = \frac{\delta^{ab}}{-i\tilde{\not{P}}^a + m}; \tag{37}$$

the ghost propagator,

$$\langle \eta^{ab}(P)\bar{\eta}^{cd}(-P)\rangle = \frac{1}{(P^{ab})^2}\mathcal{P}^{ab,cd}; \tag{38}$$

and the gluon propagator,

$$\langle B^{ab}_\mu(P)B^{cd}_\nu(-P)\rangle = \left(\delta_{\mu\nu} - (1-\xi)\frac{P^{ab}_\mu P^{ab}_\nu}{(P^{ab})^2}\right) \times \frac{1}{(P^{ab})^2}\mathcal{P}^{ab,cd}. \tag{39}$$

These are illustrated in Fig. 6.

There are several matters of notation to attend to. All of these sound more complicated than is true after drawing the corresponding double line diagram. In Eqs. (34) to (39), we adopt the convention that color indices shared between momenta and projection operators are *not* summed over.

Implicitly, a quark line carries two arrows: one as a Dirac particle, and one for color. Either $P^a_0 = p_0 + Q^a$, if the directions coincide, or $(P^a)_0 = p_0 - Q^a$, if they do not. As long as one is consistent with directions, though, this does not really matter. For the quark propagator, this leads to the obvious identity

$$\frac{\delta^{ab}}{-i\tilde{\not{P}}^a + m} = \frac{\delta^{ab}}{-i\tilde{\not{P}}^b + m}. \tag{40}$$

Similarly, for fields in the adjoint representation, where $P^{ab}_0 = p_0 + Q^a - Q^b$, we define the left index so that it is in the direction of the momentum. Thus, if we change the direction, $P' = -P$, then we must also reverse the order of indices, and $(P')^{ba} = -P^{ab}$. For the ghost propagator, for example,

$$\frac{1}{(P^{ab})^2}\mathcal{P}^{ab,cd} = \frac{1}{(P^{dc})^2}\mathcal{P}^{ab,cd}. \tag{41}$$

The same is true for the gluon propagator.

The vertices between quantum fluctuations, B_μ and ψ, are also simple, taking care of indices and the like. The vertex between a quark, antiquark, and a gluon is obtained by taking the derivative of the action, $\mathcal{S} = \int d^4x\mathcal{L}$, as

$$-\frac{\delta\mathcal{S}}{\delta\psi^b(R)\delta B^{dc}_\mu(Q)\delta\bar{\psi}^a(P)} = ig(t^{cd})_{ab}\gamma_\mu. \tag{42}$$

The order of the gluon indices flip, from the left- to the right-hand side, because $B_\mu = t^{cd}B^{dc}_\mu$. The vertex between a ghost, antighost, and a gluon is

$$-\frac{\delta\mathcal{S}}{\delta\eta^{fe}(R)\delta B^{dc}_\mu(Q)\delta\bar{\eta}^{ba}(P)} = igf^{(ab,cd,ef)}(P^{ab})_\mu. \tag{43}$$

The three gluon vertex is

$$-\frac{\delta\mathcal{S}}{\delta B^{fe}_\lambda(R)\delta B^{dc}_\nu(Q)\delta B^{ba}_\mu(P)} = -igf^{(ab,cd,ef)}\Gamma_{\mu\nu\lambda}(P^{ab}, Q^{cd}, R^{ef}), \tag{44}$$

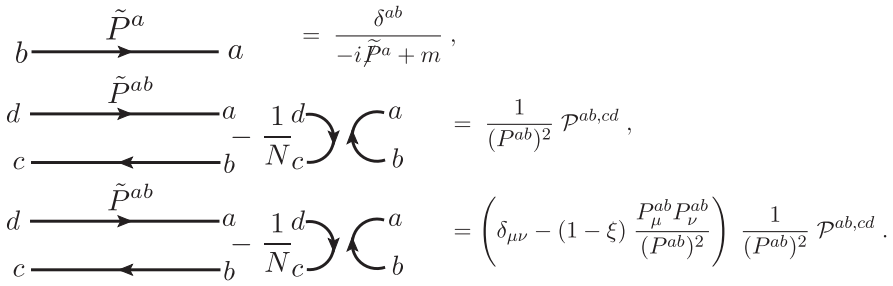

FIG. 6. Quark, ghost, and gluon propagators.

where

$$\Gamma_{\mu\nu\lambda}(P^{ab}, Q^{cd}, R^{ef}) = (P^{ab}_\lambda - Q^{cd}_\lambda)\delta_{\mu\nu} + (Q^{cd}_\mu - R^{ef}_\mu)\delta_{\nu\lambda} + (R^{ef}_\nu - P^{ab}_\nu)\delta_{\lambda\mu}. \quad (45)$$

These are all the usual vertices, with the replacement of ordinary momenta by momenta that carry color indices. Again, in Eqs. (43) and (45), color indices shared by momenta are not summed over.

Momenta with colored indices satisfy momentum conservation as usual, so in Eq. (45),

$$P^{ab}_\mu + Q^{cd}_\mu + R^{ef}_\mu = 0. \quad (46)$$

Now consider one of the external momenta, say R^{ef}_λ, contracted with the three gluon vertex. This satisfies the identity

$$R^{ef}_\lambda \Gamma_{\mu\nu\lambda}(P^{ab}, Q^{cd}, R^{ef}) = \Delta^{-1}_{\mu\nu}(Q^{cd}) - \Delta^{-1}_{\mu\nu}(P^{ab}), \quad (47)$$

where Δ^{-1} is the transverse piece of the inverse gluon propagator

$$\Delta^{-1}_{\mu\nu}(P) = \delta_{\mu\nu}P^2 - P_\mu P_\nu. \quad (48)$$

This is the same identity as for $Q = 0$, Eq. (2.50) of Ref. [24]. This is useful because as for $Q = 0$ [24], it can be used to show that hard thermal loops are independent of the gauge fixing condition.

The four gluon vertex has the usual form, a sum over products of structure constants

$$\begin{aligned} &-\frac{\delta\mathcal{S}}{\delta B^{hg}_\sigma(S)\delta B^{fe}_\lambda(R)\delta B^{dc}_\nu(Q)\delta B^{ba}_\mu(P)} \\ &= -g^2\sum_{i,j=1}^{N}\left(f^{(ab,cd,ij)}f^{(ef,gh,ji)}(\delta_{\mu\lambda}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\lambda})\right. \\ &\quad + f^{(ab,ef,ij)}f^{(gh,cd,ji)}(\delta_{\mu\sigma}\delta_{\lambda\nu} - \delta_{\mu\nu}\delta_{\lambda\sigma}) \\ &\quad \left. + f^{(ab,gh,ij)}f^{(cd,ef,ji)}(\delta_{\mu\nu}\delta_{\sigma\lambda} - \delta_{\mu\lambda}\delta_{\sigma\nu})\right). \end{aligned} \quad (49)$$

We conclude by noting that Eq. (41) can be used to simplify insertions of ghost or gluon lines in loop diagrams. If a gluon ties onto a quark line, then in Feynman gauge this enters as

$$t^{ba}\frac{\mathcal{P}^{ab,cd}}{(P^{ab})^2}t^{dc}, \quad (50)$$

where we neglect the rest of the diagram. The projection operator in the gluon propagator is a sum of two terms, Eq. (3). Since the gluon appears inside the loop, the c and d indices are summed over. Unlike the a and b indices, which also enter through P^{ab}, this is the *only* place where c and d indices enter. Since the generators are traceless, though, Eq. (13), any contribution from the second term in the projection operator, $-\delta^{ab}\delta^{cd}/N$, vanishes. Hence, in the gluon line, we can replace the projection operator by the first term, which is just a Kronecker delta, $\mathcal{P}^{ab,cd} \to \delta^{ad}\delta^{bc}$, so Eq. (50) becomes

$$t^{ba}\frac{1}{(P^{ab})^2}t^{ab}. \quad (51)$$

The same is true in any gauge. It is also true for a gluon tied to either a three gluon or four gluon vertex, since in each case, Eqs. (45) and (49), the projection operator in the gluon propagator ties onto a factor $f^{(cd,ef,gh)}$, and that any such structure constant involves a commutator of t^{cd}, Eq. (23). The same holds for a ghost propagator tied onto a ghost antighost gluon vertex. This useful simplification was first seen in Eq. (21) in Sec. II: generators can be contracted not with projection operators, but just with ordinary Kronecker deltas.

B. Propagators in a mixed representation

In this section we discuss a useful trick for computing scattering amplitudes, starting in the imaginary time. To avoid unnecessary complication, we replace the color matrices, either Q^a or Q^{ab}, by a single background field, Q. This is identical to considering the propagation of an electron in QCD, in the presence of a background field $A_0 \sim Q/e$. The extension to QCD is automatic, as will be clear from the examples which follow in later sections.

We then introduce a mixed representation for the propagators [23–25]. For the spatial directions, one works as usual in momentum space, but for the time direction, instead one stays in coordinate space. For example, consider a propagator,

$$\Delta_Q(\tau, E) = T\sum_{n=-\infty}^{+\infty}\frac{e^{-i(p_0+Q)\tau}}{(p_0+Q)^2 + E^2}; \quad (52)$$

E is the energy, typically $E = \sqrt{\vec{p}^2 + m^2}$. We assume the field is bosonic, so the Euclidean energy $p_0 = 2\pi nT$, for integral n. For the time being, we also assume that $0 \le \tau \le 1/T$. The sum is performed by contour integration in the complex p_0 plane. There are two poles, for $p_0 = -Q \pm iE$, which give

$$\Delta_Q(\tau, E) = \sum_{s=\pm}\frac{s}{2E}(1 + n(sE - iQ))e^{-sE\tau}. \quad (53)$$

Here, $n(E)$ is the usual Bose-Einstein statistical distribution function,

$$n(E) = \frac{1}{e^{E/T} - 1}, \quad (54)$$

so the only change for $Q \neq 0$ is the change in the statistical distribution function,

$$n(E \mp iQ) = \frac{1}{e^{(E\mp iQ)/T} - 1}. \quad (55)$$

Notice that this is the only place where Q enters into Eq. (53): the propagators in Euclidean time, $\sim \exp(\mp E\tau)/2E$, are *identical* to that for $Q = 0$.

This can be understood by recognizing that the parameter iQ enters exactly like a chemical potential, albeit one that is imaginary. Because of this, the associated statistical distribution functions, $n(E \mp iQ)$, are complex valued. It helps to rewrite Eq. (53) in a less compact form,

$$\Delta_Q(\tau, E) = \frac{1}{2E}((1 + n(E - iQ))\mathrm{e}^{-E\tau} + n(E + iQ)\mathrm{e}^{+E\tau}). \tag{56}$$

This form is physically more transparent. The first term, with propagator $\mathrm{e}^{-E\tau}/2E$, is proportional to $1 + n(E - iQ)$. The 1 is the contribution in vacuum, while $n(E - iQ)$ represents the induced emission of a particle, with energy E and chemical potential $+iQ$, into the thermal bath. The second term, with propagator $\exp(+E\tau)/2E$, is proportional to $\sim n(E + iQ)$. This represents absorption of a field with energy E, and chemical potential for the antiparticle, $-iQ$, from the thermal bath.

This expression for the propagator is the same as for a real chemical potential, μ, except that μ is replaced by iQ. As when $\mu \neq 0$, the statistical distribution functions are modified, but the energies of the system remain unchanged. The same is then true for an imaginary chemical potential, iQ. This is why the form of the propagators in imaginary time, $\exp(\pm E\tau)/2E$, are unaffected by Q. We argue in the next subsection that this remains valid for propagation in real time as well.

We make some remarks to help illuminate the meaning of the statistical distribution functions when $0 < Q < \pi T$. First, when $Q = 0$, the Bose-Einstein distribution function is singular as $E \to 0$, $n(E) \sim T/E$. This singularity is related to the phenomenon of Bose-Einstein condensation at low temperature. In contrast, whenever $Q \neq 0$, the distribution function is regular as the energy vanishes, $n(0 - iQ) = 1/(\exp(-iQ/T) - 1)$. This includes ordinary fermions, when $Q = \pi T$, and $n(-i\pi T) = -1/2$.

It is also helpful to consider adding a real chemical potential, μ, in addition to iQ. For ordinary fermions, $Q = \pi T$, the Fermi-Dirac distribution function with $\mu \neq 0$ is

$$\tilde{n}(E - \mu) = -n(E - i\pi T - \mu) = \frac{1}{\mathrm{e}^{(E-\mu)/T} + 1}. \tag{57}$$

In the limit of zero temperature, if $E > \mu$, $\tilde{n} = 0$, while if $E < \mu$, $\tilde{n} = 1$. For antiparticles, $\tilde{n}(E + \mu) = 0$ for any E as $T \to 0$. This is just a Fermi sea, and represents a net excess of particles over antiparticles.

When $Q \neq \pi T$, if there is also a real chemical potential, μ, the associated statistical distribution function is

$$n(E - \mu - iQ) = \frac{1}{\mathrm{e}^{(E-\mu-iQ)/T} - 1}. \tag{58}$$

Taking $Q = 2\pi Tq$, as $T \to 0$, if $E > \mu$, $n = 0$. However, if $E < \mu$, $n = -1$; the negative sign of n is natural, see Eq. (57). For antiparticles, $n(E + \mu + iQ)$, one finds $n = 0$ for all energies. Thus, a real chemical potential introduces an asymmetry between particles and antiparticles for all $Q \neq 0$.

Speaking loosely, when $0 < Q < \pi T$ particles behave with something like fractional statistics. This analogy is not precise, though, merely suggestive. In particular, for both cases in which Q arises, a given Q is not physical. For a Z(N) interface, Q is a function of z, and one integrates over all $Q(z)$. In the semi-QGP, there is a distribution of Q's, and it is only integrals over the distribution that are physically meaningful. In both cases, after summing over all Q's, the usual relationship between spin and statistics is recovered.

We conclude with some useful identities. The first is

$$1 + n(E - iQ) = \mathrm{e}^{(E-iQ)/T} n(E - iQ). \tag{59}$$

This is well known for $Q = 0$, and by construction, must then be true for $Q \neq 0$, by simply replacing $E \to E - iQ$. It applies for either sign of Q.

The propagator in Eq. (52) is defined for positive τ, $0 \leq \tau \leq 1/T$. The extension to negative values, $-1/T \leq \tau \leq 1/T$, is

$$\Delta_Q(\tau, E) = \sum_{s=\pm} \frac{s}{2E}(1 + n(sE - iQ\,\mathrm{sign}(\tau)))\mathrm{e}^{-sE|\tau|}. \tag{60}$$

From this, or directly from Eq. (52),

$$\Delta_Q(-\tau, E) = \Delta_{-Q}(\tau, E). \tag{61}$$

From this it also follows that

$$\Delta_Q(\tau - 1/T, E) = \mathrm{e}^{iQ/T}\Delta_Q(\tau, E). \tag{62}$$

This is the generalization of the Kubo-Martin-Schwinger condition [25] to a background field Q.

In practice we will start with diagrams in momentum space, and then transform a sum over p_0 to an integrals over τ's. This is done by using the Fourier transform of Eq. (52), which is

$$\frac{1}{(p_0 + Q)^2 + E^2} = \int_0^{1/T} d\tau \frac{e^{i(p_0+Q)\tau}}{2E}((1 + n(E - iQ))\mathrm{e}^{-E\tau} + n(E + iQ)\mathrm{e}^{+E\tau}). \tag{63}$$

In computation we also require

$$\frac{p_0 + Q}{(p_0 + Q)^2 + E^2} = \int_0^{1/T} d\tau \frac{-i}{2E}\left(\frac{\partial}{\partial\tau} e^{i(p_0+Q)\tau}\right) \times ((1 + n(E - iQ))\mathrm{e}^{-E\tau} + n(E + iQ)\mathrm{e}^{+E\tau}). \tag{64}$$

Integrating by parts, this equals

$$-i(\mathrm{e}^{i(p_0+Q)/T}\Delta_Q(1/T, E) - \Delta_Q(0, E)) + \int_0^{1/T} d\tau \mathrm{e}^{i(p_0+Q)\tau} i\frac{\partial}{\partial\tau}\Delta_Q(\tau, E). \tag{65}$$

The first term vanishes, since p_0 is bosonic, so $e^{ip_0/T} = 1$, and by the condition of Eq. (62), for $\tau = 1/T$. The second term is easy to evaluate, and gives

$$\frac{p_0 + Q}{(p_0 + Q)^2 + E^2} = \int_0^{1/T} d\tau \frac{e^{i(p_0+Q)\tau}}{2E}((1 + n(E - iQ)) \times (-iE)e^{-E\tau} + n(E + iQ)(+iE)e^{+E\tau}). \tag{66}$$

Integrals with higher powers of $p_0 + Q$ are not required, since they can be reduced as

$$\frac{(p_0 + Q)^2}{(p_0 + Q)^2 + E^2} = 1 - \frac{E^2}{(p_0 + Q)^2 + E^2}, \tag{67}$$

which can be handled by previous results.

C. Amplitudes in real time

In this section we follow Furuuchi [41] and discuss how to proceed from amplitudes, computed in imaginary time, to scattering amplitudes.

We remark that Smilga [15] has argued that Z(N) domain walls are entirely a construction valid only in imaginary time, and have no relevance for scattering amplitudes. If Z(N) domain structure is a natural consequence of the system in thermal equilibrium—i.e., in imaginary time—then it is difficult to see how they could not be relevant for small fluctuations about thermal equilibrium, which is what amplitudes in real time represent. Notably, lattice simulations of the dynamical evolution of Z(N) domains have been performed [11], and as one might expect, are very similar to the evolution in Potts models, to which they are closely analogous. We do acknowledge, however, that the lattice time step is not immediately related to a physical time.

Consider the usual contour in the plane of real and imaginary time. The imaginary time variable τ runs from 0 to $1/T$, and represents a thermal ensemble in thermal equilibrium. The contour also runs in real time, t, from 0 to ∞, and then back again, representing fluctuations about the thermal ensemble. This is illustrated in a standard figure, Fig. 7. The exact shape of the contour [25] will not matter for our purposes.

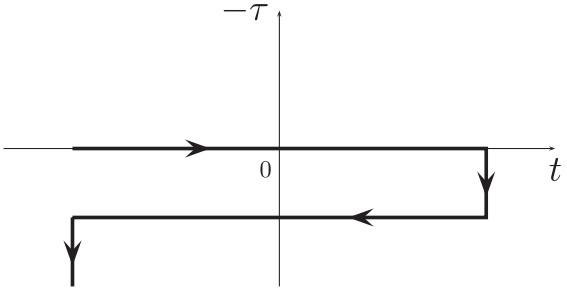

FIG. 7. Contour in real and imaginary time.

The essential question is on what parts of the contour the background field A_0 is nonzero. The answer is to take the background A_0 field *only* for the part of the contour in imaginary time, and *not* for the part of the contour in real time. This is absolutely necessary for the integrals to be well defined. Since the real time runs from $-\infty$ to $+\infty$, and then back again, if there was such a background field, it would affect the behavior at large times.

This is clearest in considering the background Q field as a chemical potential for color charge [14]. A real chemical potential alters the initial statistical distribution of the particles: for fermions, for example, it represents a net excess of particles over antiparticles, or a Fermi sea. While the canonical momenta are shifted by a chemical potential, the evolution in real time is by the usual Hamiltonian of the system. Thus, the evolution, in real time t, of some operator ϕ proceeds by the usual Heisenberg relation, $\phi(t) = e^{+iHt}\phi(0)e^{-iHt}$.

While the Q field represents an imaginary chemical potential, its effect is only to alter the initial color distribution of particles. The canonical momentum is shifted, but not the Hamiltonian.

In practice, one computes an amplitude with Q dependent momenta: p_0^{ab} for gluons, p_0^a for quarks. The above implies that one amplitudes are constructed by taking

$$p_0^{ab} = p_0 + (Q^a - Q^b) \to -i\omega^{ab}, \tag{68}$$

for gluons, and similarly for quarks. Here, ω is an energy in real time, and as such, can take arbitrary values.

The division into hard momenta, of order $\sim T$, and soft momenta, $\sim gT$, is preserved by this procedure. The usual Euclidean p_0 is a multiple of $2\pi T$ for bosons. Thus, the shift by the fractional amount, $\sim Q$, does not change this. In contrast, the Minkowski energy ω^{ab} is a continuous variable, and it is consistent to assume that it is soft.

There is an important subtlety that we ignore. Usual scattering amplitudes are invariant under arbitrary reparametrizations of the fields. It is far from clear that this is true for scattering in a *fixed* Q field. The point is that it is necessary to compute a physical process. For a Z(N) interface, this would be scattering not at a given point, but integrated over the entire spatial extent of the interface. Similarly, for the semi-QGP, scattering in a fixed Q field could well exhibit unphysical behavior. The physical quantity there are amplitudes in which one integrates over the entire distribution of Q's, representing the thermal equilibrium state.

IV. QUARK SELF-ENERGY

With the formalism in place, we proceed to computing the self-energy for a quark in a background field, $Q \neq 0$. The computations are relatively straightforward, and do not exhibit complications, which will arise for the gluon self-energy in the next section. We go through this example in some detail, so that the reader can develop familiarity

with computing in the presence of a background field. It also helps to understand the novelty of the new terms in the gluon self-energy.

At one-loop order, the standard diagram is, in our notation,

$$-\Sigma(\tilde{P}^a)_{ab} = -g^2 (t^{de})_{ac} \mathcal{P}_{de,fg} (t^{fg})_{cb} \int \frac{d^4K}{(2\pi)^4} \times \frac{\gamma_\mu i\tilde{\not{K}}^c \gamma^\mu}{(\tilde{P}^a - \tilde{K}^c)^2 (\tilde{K}^c)^2}. \quad (69)$$

Here, $\tilde{K}^c$ is the momentum of the quark in the loop, and $\tilde{P}^a - \tilde{K}^c$ the momentum of the gluon. Thus $\tilde{K}^c$ is a fermionic momentum, and $\tilde{P}^a - \tilde{K}^c = P^a - K^c$ is bosonic.

In Eq. (69) the integral is that appropriate for a bosonic field at nonzero temperature,

$$\int \frac{d^4K}{(2\pi)^4} = T \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3}, \qquad k_0 = 2\pi n T. \quad (70)$$

Remember that we can take p_0 and k_0 to be bosonic, by using a background field that is $\tilde{Q}^a = Q^a + \pi T$.

The color structure reduces immediately. By Eqs. (50) and (51), the gluon projection operator can be replaced by an ordinary Kronecker delta

$$-\Sigma(\tilde{P}^a)_{ab} = i g^2 \mathcal{P}_{ac,cb} \int \frac{d^4K}{(2\pi)^4} \frac{\tilde{\not{K}}^c}{(P^a - K^c)^2 (\tilde{K}^c)^2}. \quad (71)$$

The color structure is illustrated by the diagram of Fig. 8. This is a sum of the planar diagram, minus $1/N$ times a diagram in which all indices are equal.

We wish to extract the hard thermal loop, $\sim T^2$, from Eq. (71). Instead, to simplify the discussion, we consider the integral

$$\mathcal{I}(\tilde{P}^a) = \int \frac{d^4K}{(2\pi)^4} \frac{1}{(P^a - K^c)^2 (\tilde{K}^c)^2}, \quad (72)$$

in full generality.

To perform the sum over n, it is useful to use the mixed representation, Sec. III B. Using Eq. (63), we write

$$\frac{1}{(\tilde{K}^c)^2} = \int_0^{1/T} d\tau \frac{e^{i(k_0 + \tilde{Q}^c)\tau}}{2E_k} ((1 + n(E_k - i\tilde{Q}^c))e^{-E_k\tau} + n(E_k + i\tilde{Q}^c)e^{+E_k\tau}) \quad (73)$$

for the quark-like propagator, $E_k = \sqrt{\vec{k}^2}$, and

$$\frac{1}{(P^a - K^c)^2} = \int_0^{1/T} d\tau' \frac{e^{i(p_0 - k_0 + Q^{ac})\tau'}}{2E_{p-k}} \times ((1 + n(E_{p-k} - iQ^{ac}))e^{-E_{p-k}\tau'} + n(E_{p-k} + iQ^{ac})e^{+E_{p-k}\tau'}) \quad (74)$$

for the gluon propagator, $E_{p-k} = \sqrt{(\vec{p} - \vec{k})^2}$.

The sum over n, where $k_0 = 2\pi n T$, can then be performed immediately, and gives a delta function in time,

$$T \sum_{-\infty}^{+\infty} e^{ik_0(\tau - \tau')} = \delta(\tau - \tau'). \quad (75)$$

We divide this up into four integrals,

$$\mathcal{I} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{(2E_k)(2E_{p-k})} (\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4). \quad (76)$$

All of the terms are integrals over τ. The first is

$$\mathcal{I}_1(\tilde{Q}^c, Q^{ac}) = \int_0^{1/T} d\tau e^{(ip_0 + iQ^{ac} + i\tilde{Q}^c - E_k - E_{p-k})\tau} \times (1 + n(E_k - i\tilde{Q}^c))(1 + n(E_{p-k} - iQ^{ac})), \quad (77)$$

with the other three of a similar form. The integral is easy to do,

$$\mathcal{I}_1(\tilde{Q}^c, Q^{ac}) = \left(\frac{e^{(ip_0 + iQ^{ac} + i\tilde{Q}^c - E_k - E_{p-k})/T} - 1}{ip_0 + i\tilde{Q}^a - E_k - E_{p-k}}\right) \times (1 + n(E_k - i\tilde{Q}^c))(1 + n(E_{p-k} - iQ^{ac})). \quad (78)$$

In the energy denominator, we rewrite

$$Q^{ac} + \tilde{Q}^c = Q^a - Q^c + Q^c + \pi T = \tilde{Q}^a. \quad (79)$$

For the other terms, though, it is better not to use this. Since p_0 is a bosonic momentum, $\exp(ip_0/T) = 1$, and we group

$$e^{(ip_0 + iQ^{ac} + i\tilde{Q}^c - E_k - E_{p-k})/T} = e^{-(E_k - i\tilde{Q}^c)/T} e^{-(E_{p-k} - iQ^{ac})/T}. \quad (80)$$

This is useful because the identity of Eq. (59) can now be brought to bear, so that

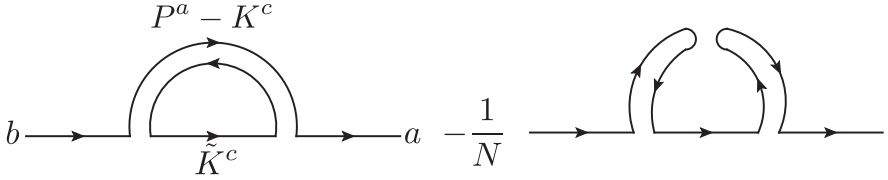

FIG. 8. One-loop diagram for the quark self-energy.

$$\mathcal{I}_1(\tilde{Q}^c, Q^{ac}) = \frac{n(E_k - i\tilde{Q}^c)n(E_{p-k} - iQ^{ac}) - (1 + n(E_k - i\tilde{Q}^c))(1 + n(E_{p-k} - iQ^{ac}))}{ip_0 + i\tilde{Q}^a - E_k - E_{p-k}}$$
$$= \frac{-1}{i\tilde{p}_0^a - E_k - E_{p-k}}(1 + n(E_k - i\tilde{Q}^c) + n(E_{p-k} - iQ^{ac})). \quad (81)$$

This is the exact same expression as for $Q = 0$, with the replacement

$$E_k \to E_k - i\tilde{Q}^c; \qquad E_{p-k} \to E_{p-k} - iQ^{ac}. \quad (82)$$

This holds for the energy denominator as well: that for $Q = 0$, $ip_0 - E_k - E_{p-k}$, becomes $i\tilde{p}_0^a - E_k - E_{p-k}$ after the shift of Eq. (82), using Eq. (79). Note, however, that this substitution is not universal, and holds only for the $\mathcal{I}$'s: in Eq. (76), the residues for the propagators remain $1/(2E_k)$ and $1/(2E_{p-k})$, respectively.

We went through this derivation in detail, because the chemical potentials are imaginary, so that some care is in order. Even so, the manipulations for $Q \neq 0$ are very similar to those for $Q = 0$. For example, in going from the first line in Eq. (81) to the second line, that terms involving two statistical distribution functions, $n(E_k - i\tilde{Q}^c)n(E_{p-k} - iQ^{ac})$, drop out is just the usual cancellation between stimulated emission and absorption in a thermal bath.

The other integrals can be done similarly, and follow from the result for $Q = 0$, by shifting the energies, as in Eq. (82), using the identity of Eq. (79):

$$\mathcal{I}_2(\tilde{Q}^c, Q^{ac}) = \frac{1}{i\tilde{p}_0^a - E_k + E_{p-k}} \times (n(E_k - i\tilde{Q}^c) - n(E_{p-k} + iQ^{ac})), \quad (83)$$

$$\mathcal{I}_3(\tilde{Q}^c, Q^{ac}) = \frac{-1}{i\tilde{p}_0^a + E_k - E_{p-k}} \times (n(E_k + i\tilde{Q}^c) - n(E_{p-k} - iQ^{ac})), \quad (84)$$

$$\mathcal{I}_4(\tilde{Q}^c, Q^{ac}) = \frac{1}{i\tilde{p}_0^a + E_k + E_{p-k}} \times (1 + n(E_k + i\tilde{Q}^c) + n(E_{p-k} + iQ^{ac})). \quad (85)$$

The extension to the original integral of Eq. (71) is immediate. The term $\sim\vec{k}\cdot\vec{\gamma}$ is the same as above. That $\sim\tilde{k}_0^c\gamma_0$ is evaluated using the identity of Eq. (66). In this context, this tells us to replace $\tilde{k}_0^c \to \mp iE_k$. For the terms with positive energy, $\mathcal{I}_1$ and $\mathcal{I}_2$, $-iE_k$ enters; for those with negative energy, $\mathcal{I}_3$ and $\mathcal{I}_4$, $+iE_k$.

We next extract the hard thermal loop from Eq. (71). These are terms where the external momenta are soft, $p \sim gT$, and are as large as the corresponding term at tree level. In the quark self-energy, then, the hard thermal loops are $\sim g^2T^2/p$.

The loop momenta in a hard thermal loop are hard, $k \sim T$. In the statistical distribution functions we can then approximate $E_{p-k} \sim E_k = k$. The dominant terms arise from the energy denominators with Landau damping, $\mathcal{I}_2$ and $\mathcal{I}_3$. These terms are dominant because the energy denominators are a difference of large energies, and thus are small:

$$i\tilde{p}_0^a \pm (E_k - E_{p-k}) \approx i\tilde{p}_0^a \pm p\cos\theta, \quad (86)$$

where $\cos\theta = \hat{k}\cdot\hat{p}$.

Notice that these terms are only dominant if—and only if—the energy $i\tilde{p}_0^a$ is small after analytic continuation, Sec. III C. If only ip_0, and not $i\tilde{p}_0^a$, were small after analytic continuation, then $i\tilde{Q}^a$ would be a hard momentum, and there would be no hard thermal loop in the diagram: everything would be a correction that is suppressed by at least $\sim g$ relative to the propagator at tree level.

Introducing the vector $\hat{K} = (i, \hat{k})$, $\hat{k}^2 = 1$, the hard thermal loop in the quark self-energy becomes

$$-\Sigma(\tilde{P}^a)_{ab} \overset{\text{HTL}}{\approx} (m_{\text{qk}}^2)_{ab}\delta\Sigma(\tilde{P}^a), \quad (87)$$

where the $\approx^{\text{HTL}}$ sign indicates that the hard thermal loops are equal, but not (necessarily) terms beyond that order, and

$$\delta\Sigma(P) = \int\frac{d\Omega}{4\pi}\frac{i\hat{K}\!\!\!/}{P\cdot\hat{K}}; \quad (88)$$

the angular integral is over all directions of the unit vector $\hat{k}$. The function $\delta\Sigma(P)$ is identical to that for $Q = 0$. In the quark self-energy, this function is multiplied by a thermal quark "mass," which is a function of Q:

$$(m_{\text{qk}}^2(Q))_{ab} = \frac{g^2T^2}{24}\sum_{c=1}^{N}\mathcal{P}_{ac,cb}(\mathcal{A}(Q^{ac}) - \mathcal{A}(\tilde{Q}^c))$$
$$= \delta_{ab}\frac{g^2T^2}{24}\Big(\sum_{c=1}^{N}(\mathcal{A}(Q^{ac}) - \mathcal{A}(\tilde{Q}^c)) - \frac{1}{N}(\mathcal{A}(0) - \mathcal{A}(\tilde{Q}^a))\Big), \quad (89)$$

where

$$\mathcal{A}(Q) = \frac{3}{\pi^2T^2}\int_0^\infty dk k(n(k - iQ) + n(k + iQ)). \quad (90)$$

We normalize $\mathcal{A}(Q)$ in anticipation of the final result. For ordinary hard thermal loops, the integrals over the hard, loop momenta decouple into an angular integral times an integral over the statistical distribution functions. Equation (87) shows that this remains true for the quark hard thermal loop when $Q \neq 0$. The same is also true for the gluon hard thermal loop.

For the quark self-energy, in the end the color structure is no different than the propagator at tree level, $\sim\delta_{ab}$. New color structures do arise for the hard thermal loop in the gluon self-energy.

The integrals over statistical distribution functions when $Q \neq 0$ are not much more difficult than for $Q = 0$ [6]. We write

$$Q = 2\pi T q. \tag{91}$$

Then

$$\begin{aligned}\frac{6}{\pi^2 T^2}\int_0^\infty dk k n(k - iQ) &= \frac{6}{\pi^2}\int_0^\infty dk k \frac{e^{-k+2\pi i q}}{1 - e^{-k+2\pi i q}} \\ &= \frac{6}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}e^{2\pi i q j} \\ &= 1 - 6q(1-q) + i\mathrm{Cl}_2(2\pi q),\end{aligned} \tag{92}$$

where $\mathrm{Cl}_n(\theta)$ is the Clausen function. As is typical of similar expressions for a Z(N) interface, this is valid only for $0 < q < 1$, but the extension to other values is direct. The imaginary term $\mathrm{Cl}_2(2\pi q)$ cancels in the sum, which enters into $\mathcal{A}$, so that

$$\mathcal{A}(Q) = 1 - 6q(1-q). \tag{93}$$

The hard thermal loop in the quark self-energy is identical to that for $Q = 0$, up to the change in the thermal quark mass. To check that one obtains the usual value for $Q = 0$, remember that when all $Q = 0$, $\tilde{Q}^c = \pi T$, or $q = 1/2$. Since $\mathcal{A}(0) = 1$ and $\mathcal{A}(1/2) = -1/2$,

$$m_{qk}^2(0)_{ab} = \frac{N^2 - 1}{2N}\frac{g^2T^2}{8}\delta_{ab}. \tag{94}$$

There is a simple interpretation of the function $\mathcal{A}(Q)$. Although classically there is no potential for Q, in the presence of a background field Q, a potential is generated at one-loop order. For convenience we normalize this potential as

$$\mathcal{V}(Q) = \tfrac{1}{2}q^2(1-q)^2. \tag{95}$$

Then

$$\mathcal{A}(Q) = \frac{d^2}{dq^2}\mathcal{V}(Q). \tag{96}$$

Hence, the thermal quark mass is naturally the second derivative of a potential, as one would expect.

V. GLUON SELF-ENERGY

A. Gluonic hard thermal loops: tadpoles

From the example of the quark self-energy, one might expect that the hard thermal loop in the gluon self-energy when $Q \neq 0$ is just like that when $Q = 0$, with the same functional form, and the only change a relatively trivial redefinition of the thermal mass.

We shall see that this is not true: there are new terms that arise uniquely for non-Abelian gauge fields. These are not present either for fermions, coupled to either Abelian or non-Abelian gauge fields, nor for Abelian gauge fields.

Before delving into the details of the computation in a non-Abelian gauge theory, in the next two subsections we discuss the differences between hard thermal loops in the gluonic self-energy when $Q \neq 0$, versus $Q = 0$. Hopefully this will make the origin of the new terms less obscure.

Hard thermal loops are one-loop diagrams that are as large as the corresponding terms at tree level when the external momenta are soft, $P \sim gT$. For the gluon self-energy in zero field, at tree level the inverse propagator is $\sim P^2 \sim (gT)^2$ for soft P. Thus, the hard thermal loops are diagrams that are $\sim g^2T^2$ times a dimensionless function of the energy, $ip_0 = \omega$, divided by the spatial momentum, p.

The simplest hard thermal loop is present in (massless) scalar field theories, and is just a tadpole integral,

$$\int \frac{d^4K}{(2\pi)^4}\frac{1}{K^2} = \frac{T^2}{12}. \tag{97}$$

Consider the extension of this integral to $Q \neq 0$:

$$\int \frac{d^4K}{(2\pi)^4}\frac{1}{(k_0 + Q)^2 + \vec{k}^2}. \tag{98}$$

We use the representation of the propagator in Eq. (66). The sum over k_0 generates $\delta(\tau)$, so the τ integral is trivial, and we are left with a single integral over $k = \sqrt{\vec{k}^2}$, so that Eq. (98) becomes

$$\frac{1}{4\pi^2}\int_0^\infty dk k(n(k - iQ) + n(k + iQ)) = \frac{T^2}{12}\mathcal{A}(Q). \tag{99}$$

By comparison with Eq. (90), this is the function $\mathcal{A}(Q)$ we found for the quark self-energy, Eq. (93). [An ultraviolet divergent term at zero temperature, $\sim\int k dk$ has been dropped. For a scalar field, this is part of a mass divergence. For a gauge field, the sum of all such terms vanishes at zero temperature by gauge invariance.]

A less trivial example is given by the integral

$$\int \frac{d^4K}{(2\pi)^4}\frac{k_0 + Q}{(k_0 + Q)^2 + \vec{k}^2} = \frac{\pi T^3}{3}\mathcal{A}_0(Q). \tag{100}$$

As we shall see in Sec. V C, there is a contribution to the *one* point function of the gluon from such a term, where $k_0 + Q$ arises from the three gluon vertex. We stress that

such a diagram does *not* arise for the two point gluon function.

This integral vanishes when $Q = 0$ because it is odd in k_0, and one sums over both negative and positive values of k_0. Using the same tricks as above,

$$\mathcal{A}_0(Q) = \frac{3}{4i\pi^3 T^3}\int_0^\infty dk k^2 (n(k - iQ) - n(k + iQ)). \tag{101}$$

A term $\sim \int d^3k$ at zero temperature is dropped, and certainly vanishes when all such terms are summed together. Expanding as in Eq. (92), with $Q = 2\pi T q$,

$$\mathcal{A}_0(Q) = \frac{3}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin(2\pi q j) = q(1-q)(1-2q). \tag{102}$$

We recognize this as the *first* derivative of the potential in Eq. (95),

$$\mathcal{A}_0(Q) = \frac{d}{dq}\mathcal{V}(Q). \tag{103}$$

Notice that this term manifestly vanishes when $Q = q = 0$. It also vanishes for $q = 1$, because this is a vacuum equivalent to $q = 0$, and for $q = 1/2$, because this is an extremal point of the potential.

B. Gluonic hard thermal loops: Landau damping

We next turn to the nontrivial hard thermal loops, which have discontinuities, as a function of the external momenta. These arise from energy denominators with Landau damping.

We start by deriving, briefly, how such hard thermal loops arise for the gluon (or photon) self-energy when $Q = 0$. Consider, as a prototype, the integral

$$\int \frac{d^4K}{(2\pi)^4}\frac{k^i k^j}{K^2(P-K)^2}. \tag{104}$$

For the quark contribution to the gluon self-energy, the factors k^i and k^j arise from the quark propagators. For the gluon contribution, these momenta arise from the momentum dependence of the three gluon vertices. There are, of course, other contributions, with momenta $(k_0)^2$ and $k_0 k_i$, but these can be treated similarly. We take the momenta in the numerator to be $\sim k$ because the loop momenta for hard thermal loops are hard momenta, $k \sim T$, and dominate over soft loop momenta, $\sim gT$.

This integral is done as for Eq. (72) in Sec. IV. There are two terms that persist at zero temperature, $\mathcal{I}_1$ and $\mathcal{I}_4$, with energy denominators $ip_0 \mp (E_k + E_{p-k})$. If $k \sim T$ is hard, and $p \sim gT$ is soft, then these energy denominators are hard, $\sim 2k$. Such terms are independent of p_0 and p, so that $\mathcal{I}_1 + \mathcal{I}_4 \approx 2n(k)/k$, which just produce a tadpole term as in Eq. (97):

$$\frac{1}{8\pi^2}\int\frac{d\Omega}{4\pi}\hat{k}^i\hat{k}^j\int_0^\infty dk k^2\frac{2n(k)}{k} = \frac{\delta^{ij}T^2}{72}. \tag{105}$$

Instead, concentrate on the terms $\mathcal{I}_2$ and $\mathcal{I}_3$, which arise from the denominators with Landau damping:

$$\frac{1}{8\pi^2}\int\frac{d\Omega}{4\pi}\hat{k}^i\hat{k}^j\int_0^\infty dk k^2(n(E_k) - n(E_{p-k})) \times\left(\frac{1}{ip_0 - E_k + E_{p-k}} - \frac{1}{ip_0 + E_k - E_{p-k}}\right). \tag{106}$$

For the residues of the propagators we have taken $E_{p-k} \sim k$, but this approximation cannot be made so cavalierly in the rest of the expression. Indeed, notice that without any cancellation, these terms are nominally larger than we expect, $\sim \int dk k^2 n(k) \sim T^3$, and not $\sim T^2$. However, what enters into Eq. (106) is only the *difference* of the statistical distribution functions. For $k \sim T$ and $p \sim gT$,

$$n(E_{p-k}) - n(E_k) \approx \frac{\vec{p}\cdot\hat{k}}{T}n(k)(1 + n(k)). \tag{107}$$

Because of this cancellation in the statistical distribution functions, the diagram is not $\sim T^3$, but only $\sim T^2$. It is a product of an integral over k,

$$\frac{1}{4\pi^2 T}\int_0^\infty dk k^2 n(k)(1 + n(k)), \tag{108}$$

and an angular integral,

$$-\frac{1}{2}\int\frac{d\Omega}{4\pi}\hat{k}^i\hat{k}^j(\hat{p}\cdot\hat{k})\left(\frac{1}{ip_0/p - \hat{p}\cdot\hat{k}} - \frac{1}{ip_0/p + \hat{p}\cdot\hat{k}}\right). \tag{109}$$

This is the usual hard thermal loop. The integral over k generates the thermal mass for the gluon. This is multiplied times an angular integral, which generates a dimensionless function of ip_0/p. This function has discontinuities on the light cone from the Landau damping of massless particles.

We write the angular integral in Eq. (109) as we do to emphasize its behavior as a function of

$$x = \cos\theta = \hat{p}\cdot\hat{k}. \tag{110}$$

The integral over x is from -1 to $+1$, so a nonzero result must be even in x. In Eq. (109), this happens because the difference of statistical distribution functions, and the difference of energy denominators, are each odd in x, so the product is even.

Now consider the analogous integral when $Q \neq 0$. As a typical example, consider

$$\mathcal{J}^{ij}(P, Q_1, Q_2) = \frac{1}{4}\int\frac{d^4K}{(2\pi)^4}\frac{(2k^i - p^i)(2k^j - p^j)}{(K+Q_1)^2(P-K+Q_2)^2}. \tag{111}$$

Here, k_0 and p_0 are both taken to be bosonic momenta, while Q_1 and Q_2 are arbitrary background fields. The numerator is chosen to $(2k^i - p^i)(2k^j - p^j)$, which differs

terms proportional to p^i and p^j from Eq. (104). We need this term to keep a symmetry, $Q_1 \leftrightarrow Q_2$. This symmetry can be checked by shifting of integration variables, $K \to P - K$, since both k_0 and p_0 are, by assumption, bosonic momenta. In the numerator we take $\sim k^i k^j$ as a term which generates the largest terms for hard k; terms $\sim k^i p^j$ are down by $\sim g$ to this term, those $\sim p^i p^j$ by $\sim g^2$. We keep the numerator of order p because the existence of the background field makes the new leading order, which we will see in the following, of order g^2T^3/p. If one neglects the terms proportional to p^i and p^j in the numerator of Eq. (111), one find a new term that is odd under $Q_1 \leftrightarrow Q_2$. To see this symmetry in another way, note that the relevant external momentum is $P^{12} = P + Q_1 + Q_2$. Then the hard thermal loops in Eq. (111) can be rewritten as

$$\frac{1}{4}\int \frac{d^4K}{(2\pi)^4}\frac{(2k^i - p^i)(2k^j - p^j)}{(K^1)^2(P^{12} - K^1)^2} = \frac{1}{4}\int \frac{d^4K}{(2\pi)^4}\frac{(2k^i - p^i)(2k^j - p^j)}{(K^2)^2(P^{12} - K^2)^2}, \tag{112}$$

where $K^1 = K + Q_1$ and $K^2 = K + Q_2$. This is not a Q-dependent shift of momenta, which would be invalid for arbitrary Q. Instead, $K^1 \to P^{12} - K^2$ is just a shift of purely bosonic momenta, $K + Q_1 \to P - K + Q_1$.

In Eq. (111) there are terms that persist at zero temperature, $\mathcal{I}_1$ and $\mathcal{I}_4$, with energy denominators $ip_0^{12} \mp (E_k + E_{p-k})$. These are really no different than for $Q = 0$, Eq. (105); the dependence on the external momenta drops out, and one is left with integrals

$$\sim \int \frac{d^3k}{(2\pi)^3}\frac{k^i k^j}{4k^2}\frac{1}{2k}(n(k - iQ_1) + n(k + iQ_1) + n(k - iQ_2) + n(k + iQ_2)) = \frac{T^2\delta^{ij}}{144}(\mathcal{A}(Q_1) + \mathcal{A}(Q_2)), \tag{113}$$

where we have dropped p^i on the numerator, which is a higher order of the hard thermal loop approximation. This is just like the integral of Eq. (98). The angular integral is trivial, $\sim\delta^{ij}$, and the Q dependence is only through terms $\sim T^2\mathcal{A}(Q)$. [As noted before, tadpole integrals such as Eq. (100) arise only for the one point gluon function, and not in the gluon propagator. This is clear just on dimensional grounds, as that diagram has dimensions of (mass)^3.]

We then turn to the terms with energy denominators that correspond to Landau damping, $\mathcal{I}_2$ and $\mathcal{I}_3$. This part of the integral is

$$\mathcal{J}^{ij}(P, Q_1, Q_2) = \frac{1}{8\pi^2}\int_0^\infty dk\frac{k^4}{E_k E_{p-k}}\int\frac{d\Omega}{4\pi}\left(\hat{k}^i - \frac{p^i}{2k}\right)\times\left(\hat{k}^j - \frac{p^j}{2k}\right)(\mathcal{I}_2 + \mathcal{I}_3). \tag{114}$$

The statistical distribution functions, and energy denominators, which represent Landau damping are the generalization of that for $Q = 0$, Eq. (106), to $Q \neq 0$. These are just modifications of Eqs. (83) and (84), replacing the Q's there by Q_1 and Q_2:

$$\mathcal{I}_2 + \mathcal{I}_3 = \frac{n(E_k - iQ_1) - n(E_{p-k} + iQ_2)}{ip_0^{12} - E_k + E_{p-k}} + \frac{n(E_{p-k} - iQ_2) - n(E_k + iQ_1)}{ip_0^{12} + E_k - E_{p-k}}; \tag{115}$$

$p_0^{12} = p_0 + Q_1 + Q_2$. Now we symmetrize each term with respect to then interchange of Q_1 and Q_2:

$$\mathcal{I}_2 = \frac{1}{2}\frac{n(E_k - iQ_1) - n(E_{p-k} + iQ_1) + n(E_k - iQ_2) - n(E_{p-k} + iQ_2)}{ip_0^{12} - E_k + E_{p-k}}, \tag{116}$$

$$\mathcal{I}_3 = \frac{1}{2}\frac{n(E_{p-k} - iQ_1) - n(E_k + iQ_1) + n(E_{p-k} - iQ_2) - n(E_k + iQ_2)}{ip_0^{12} + E_k - E_{p-k}}. \tag{117}$$

After symmetrization it is then easy to pick out both the leading, and next to leading terms from (116) and (117).

The leading term is easy. In every term, we approximate $E_{p-k} \approx E_k = k$, and neglect p^i and p^j in the numerator, so we find that $\mathcal{J}^{ij}(P, Q_1, Q_2)$ factorizes into a product of an integral over $\int dk$, and an angular integral. The former is

$$\frac{1}{16\pi^2}\int_0^\infty dk k^2(n(k - iQ_1) - n(k + iQ_1) + n(k - iQ_2) - n(k + iQ_2)) = \frac{\pi T^3}{12}(\mathcal{A}_0(Q_1) + \mathcal{A}_0(Q_2)), \tag{118}$$

using the function $\mathcal{A}_0$ of Eq. (102). In all,

$$\mathcal{J}^{ij}(P, Q_1, Q_2) = \frac{\pi T^3}{12p}(\mathcal{A}_0(Q_1) + \mathcal{A}_0(Q_2)) \int \frac{d\Omega}{4\pi} \hat{k}^i \hat{k}^j \times \left(\frac{1}{ip_0^{12}/p - \hat{p}\cdot\hat{k}} + \frac{1}{ip_0^{12}/p + \hat{p}\cdot\hat{k}} \right). \tag{119}$$

In the angular integral, $\hat{k}^i\hat{k}^j$ produce terms ~ 1 and x^2, where $x = \hat{p}\cdot\hat{k}$, Eq. (110). The angular integral is manifestly even in x, and so does not vanish. This expression can be rewritten in a form similar to that of Eq. (87),

$$\mathcal{J}^{ij}(P, Q_1, Q_2) \overset{\text{HTL}}{\approx} \frac{\pi T^3}{6}(\mathcal{A}_0(Q_1) + \mathcal{A}_0(Q_2))\delta\Gamma^{ij}(P^{12}) + \ldots. \tag{120}$$

We introduce the function

$$\delta\Gamma^{\mu\nu}(P) = \int \frac{d\Omega}{4\pi} \left(\frac{\hat{K}^\mu \hat{K}^\nu}{P\cdot\hat{K}} \right); \tag{121}$$

$\hat{K} = (i, \hat{k})$, with $d\Omega$ the integral over $\hat{k}$. The hard thermal loop is g^2 times $\mathcal{J}^{ij}$. Since $\delta\Gamma \sim 1/P$, this is $\sim g^2T^3/p$ times a dimensionless function of ip_0^{12}/p. This is rather different from the hard thermal loops for $Q = 0$, which are $\sim g^2T^2$ times a dimensionless function of ip_0/p.

The origin for this difference is natural, when one considers the propagator in a background field, Eq. (30). There are the ordinary terms, $\sim(D_\mu^{\text{cl}})^2$, which becomes $(P_\mu^{ab})^2$ in momentum space. In addition, however, there is also a term $\sim 2ig[G_{\mu\nu}^{\text{cl}}, .]$. For a Z($N$) interface,

$$gG_{0z}^{\text{cl}} \sim g\partial_z \left(\frac{Tq(z)}{g} \right) \sim T\partial_z q(z) \sim gT^2, \tag{122}$$

where we use the fact that the typical spatial momenta for a Z(N) interface is small, $\sim gT$. Thus, this term is *larger* by $\sim 1/g$ than the ordinary terms, $\sim P^2 \sim g^2T^2$.

The hard thermal loop in Eq. (119) is $\sim g^2T^3/p$, which for soft p is $\sim gT^2$. Thus, the new hard thermal loop can be viewed as a modification of the term in the background field propagator for a gluon. Like any other hard thermal loop, it is as large as the term at tree level for soft external momenta. It is just that in a background field, this term is larger than expected.

The term $\sim 2ig[G_{\mu\nu}^{\text{cl}}, .]$ is special to a non-Abelian gauge field, since it involves the commutator in group space. It also has no analogy for a fermion field, either Abelian or non-Abelian. This explains why it did not appear in previous examples.

Having such a term is special to computing for a background field with a nonzero color field; for a Z(N) interface, it is a nonzero color electric field. Thus, for the semi-QGP, one does not expect a nonzero color field in vacuum, and such terms should not appear. We admit that at present, we do not have a fully self-consistent theory of the semi-QGP, which would allow us to demonstrate this.

We turn to subleading terms in the gluon self-energy, $\sim T^2$. In this, it helps greatly to recognize that the integral must be even in $x = \cos\theta$, Eq. (110). The measure is even in x, as are terms $\sim k^i k^j$, which produce contributions ~ 1 or $\sim x^2$.

There are several ways that corrections $\sim T^2$ can arise. The first is a numerator proportional to p^i and p^j:

$$-\frac{1}{32\pi^2}\int_0^\infty dk k(n(k - iQ_1) - n(k + iQ_1) + n(k - iQ_2) - n(k + iQ_2)) \int \frac{d\Omega}{4\pi}(\hat{k}^i p^j + p^i \hat{k}^j) \times \left(\frac{1}{ip_0^{12}/p - \hat{p}\cdot\hat{k}} + \frac{1}{ip_0^{12}/p + \hat{p}\cdot\hat{k}} \right). \tag{123}$$

This vanishes, because the integrand is odd in $\hat{k}$. The second is expanding $1/E_{p-k}$, which arises in the measure of the integral, as the residue of the propagator,

$$\frac{1}{E_{p-k}} \sim \frac{1}{k} + \frac{\vec{p}\cdot\hat{k}}{k^2} + \ldots. \tag{124}$$

Again, since $\vec{p}\cdot\hat{k} = px$, this is odd in x, and so vanishes.

The third is by expanding E_{p-k} in the energy denominators

$$\frac{1}{ip_0^{12} \pm (E_k - E_{p-k})} \approx \frac{1}{ip_0^{12} \pm \vec{p}\cdot\hat{k}} \mp \frac{\vec{p}^2 - (\vec{p}\cdot\hat{k})^2}{2k(ip_0^{12} \pm \vec{p}\cdot\hat{k})^2} + \ldots \tag{125}$$

The numerator of the second term on the right-hand side is $p^2(1 - x^2)$, which is even in x. Because of the $\mp$ sign in front of the second term, though, this is in all odd in x, and so vanishes.

Thus, the *only* way that corrections $\sim T^2$ arise is by expanding E_{p-k} in the statistical distribution functions

$$n(E_{p-k} - iQ) \approx n(k - iQ) + \frac{\vec{p}\cdot\hat{k}}{T} n(k - iQ)(1 + n(k - iQ)) + \ldots \tag{126}$$

This is exactly the same sort of expression as at $Q = 0$. Now this term is odd in x, but note that $-n(E_{p-k} + iQ_{1,2})$ enters in $\mathcal{I}_2$, Eq. (116), and $+n(E_{p-k} - iQ_{1,2})$ enters in $\mathcal{I}_3$, Eq. (117). Thus, the result is even in x, as it must be if not to vanish. Explicitly, the terms $\sim T^2$ in $\mathcal{J}^{ij}$ are a product of an integral over k,

$$\frac{1}{16\pi^2 T}\int_0^\infty dk k^2 (n(k - iQ_1)(1 + n(k - iQ_1)) + n(k + iQ_1)(1 + n(k + iQ_1)) + (Q_1 \leftrightarrow Q_2)), \tag{127}$$

and an integral over the angular variables,

$$\int \frac{d\Omega}{4\pi}\left(\frac{\hat{k}^i\hat{k}^j\vec{p}\cdot\hat{k}}{P\cdot\hat{K}}\right). \tag{128}$$

The angular integral in Eq. (128) is identical to that for ordinary hard thermal loops. The momentum integral is also a minor modification. Consider

$$\frac{1}{T}\int_0^\infty dk k^2 (n(k-iQ)(1+n(k-iQ)) + n(k+iQ)(1+n(k+iQ)), \tag{129}$$

which we can rewrite as

$$-i\frac{\partial}{\partial Q}\int_0^\infty dk k^2 (n(k-iQ)-n(k+iQ)). \tag{130}$$

This integral arose previously in Eq. (102), and involves the function $\mathcal{A}_0(Q)$, which is the first derivative of the potential $\mathcal{V}(Q)$, Eq. (95). Since in Eq. (130) we take a derivative of this function with respect to Q, however, the momentum integral in Eq. (129) involves not the first derivative of $\mathcal{V}(Q)$, but the second, through the function $\mathcal{A}(Q)$, Eq. (96):

$$\frac{2\pi^2T^2}{3}\mathcal{A}(Q). \tag{131}$$

In summary, the terms $\sim T^3$ in $\mathcal{J}^{ij}(P,Q_1,Q_2)$ are those of Eq. (120); the terms $\sim T^2$ are

$$\frac{T^2}{24}(\mathcal{A}(Q_1)+\mathcal{A}(Q_2))\left(\frac{\delta^{ij}}{6}+\int\frac{d\Omega}{4\pi}\frac{\hat{k}^i\hat{k}^j\vec{p}\cdot\hat{k}}{P^{12}\cdot\hat{K}}\right). \tag{132}$$

For completeness, we have added the tadpole terms, $\sim\delta^{ij}/6$. This expression can be rewritten as

$$\frac{T^2}{24}(\mathcal{A}(Q_1)+\mathcal{A}(Q_2))\left(\frac{\delta^{ij}}{2}-ip_0^{12}\int\frac{d\Omega}{4\pi}\frac{\hat{k}^i\hat{k}^j}{P^{12}\cdot\hat{K}}\right). \tag{133}$$

It is direct to show that these are the only terms $\sim T^3$ or $\sim T^2$. The one concern is terms where the numerator is $\sim k^i p^j$ or $\sim p^i k^j$: while the single power of p^i brings in a suppression by a soft momenta, $\sim p/T$, this times a term $\sim T^3/p$ could produce a result $\sim T^2$. However, a single power of k^i or k^j is manifestly odd in the angular variable x, Eq. (110). In contrast, the terms that produce contributions $\sim T^3$, Eq. (109), are even in x. Thus, these possible terms $\sim T^2$ vanish when integrated over x. Therefore, in the hard thermal loop equation, terms proportional to p^i and p^j in Eq. (111) may simply be dropped

$$\mathcal{J}^{ij}(P,Q_1,Q_2)\overset{\text{HTL}}{\approx}\int\frac{d^4K}{(2\pi)^4}\frac{k^ik^j}{(K+Q_1)^2(P-K+Q_2)^2}. \tag{134}$$

For the gluon self-energy, the general integral required is

$$\tilde{\mathcal{J}}^{\mu\nu}(P^{12},Q_1,Q_2)=\frac{1}{4}\int\frac{d^4K}{(2\pi)^4}\left(-\delta^{\mu\nu}\left(\frac{1}{(K+Q_1)^2}+\frac{1}{(K+Q_2)^2}\right)+\frac{(2K-P+Q_1-Q_2)^\mu(2K-P+Q_1-Q_2)^\nu}{(K+Q_1)^2(P-K+Q_2)^2}\right), \tag{135}$$

where $Q^\mu=u^\mu Q$, $u^\mu=1$ if $\mu=0$, and zero otherwise. The first term on the left-hand side, $\sim\delta^{\mu\nu}$, is added in anticipation of the integrals which arise for hard thermal loops, and is obviously independent of the external momentum, P.

Momentum dependence arises from the second term on the left-hand side. Its numerator involves $2K-P+Q_1-Q_2=2K^1-P^{12}$; under a shift of the loop momentum, $K\to P-K$, this becomes $-(2K^2-P^{12})$. Comparing with Eq. (112) shows that Eq. (135) is symmetric under interchange of Q_1 and Q_2.

When $Q=0$, the computation of hard thermal loops is simplified by keeping only powers of the hard loop momentum K^μ in the numerator, and dropping those of the soft external momenta, P^μ [23–25]. When $Q\neq 0$, the analogous approximation is to keep powers of K^1 or K^2, and drop powers of P^{12}, as we saw in $\mathcal{J}^{ij}$. It is essential to remember that while Q_1 and Q_2 are each hard, that the real energy P^{12} is soft, Sec. III C. Thus, we can write the second term on the left-hand side of Eq. (135) as

$$\int\frac{d^4K}{(2\pi)^4}\frac{(K^1)^\mu(K^1)^\nu}{(K^1)^2(P^{12}-K^1)^2}\overset{\text{HTL}}{\approx}\int\frac{d^4K}{(2\pi)^4}\frac{(K^2)^\mu(K^2)^\nu}{(K^2)^2(P^{12}-K^2)^2}. \tag{136}$$

This simplification will help greatly in computing the gluon self-energy in Secs. V D and V E.

Computations similar to those above show that this integral equals

$$\tilde{\mathcal{J}}^{\mu\nu}(P^{12},Q_1,Q_2)\overset{\text{HTL}}{\approx}\frac{\pi T^3}{6}(\mathcal{A}_0(Q_1)+\mathcal{A}_0(Q_2))\delta\Gamma^{\mu\nu}(P^{12}) + \frac{T^2}{24}(\mathcal{A}(Q_1)+\mathcal{A}(Q_2))\delta\Pi^{\mu\nu}(P^{12}), \tag{137}$$

where the momentum dependence enters through the functions $\delta\Gamma^{\mu\nu}(P^{12})$, Eq. (121), and

$$\delta\Pi^{\mu\nu}(P)=\left(-u^\mu u^\nu-ip_0\int\frac{d\Omega}{4\pi}\frac{\hat{K}^\mu\hat{K}^\nu}{P\cdot\hat{K}}\right). \tag{138}$$

Again, $\hat{K} = (i, \hat{k})$, so $\hat{K}^2 = 0$, and the angular integral is over all directions of $\hat{k}$.

Previously, we computed $\delta\Gamma^{ij}$ and $\delta\Pi^{ij}$. Given this result, it is immediate to show that $\delta\Gamma^{i0}$ and $\delta\Gamma^{00}$ are correct. The only effort necessary is to establish the correctness of $\delta\Pi^{i0}$ and $\delta\Pi^{00}$. However, the derivation of $\delta\Pi^{ij}$ above shows that the terms $\sim T^2$ arise in precisely the same way when $Q \neq 0$ as for $Q = 0$, entirely from the change in the statistical distribution functions, Eq. (126). Thus, we can be certain that the only change is that of the thermal gluon mass in a background field. That is, the momentum dependence, which enters through $\delta\Pi^{\mu\nu}(P)$, is unchanged. For this, the previous computation suffices.

The function $\delta\Pi^{\mu\nu}(P)$ is the same function as arises in the hard thermal loops when $Q = 0$. The function $\delta\Gamma^{\mu\nu}(P)$ enters only when $Q \neq 0$, since it is multiplied by $\mathcal{A}_0$, and $\mathcal{A}_0(0) = 0$. From Eqs. (121) and (138), the two functions are simply related to one another,

$$\delta\Gamma^{\mu\nu}(P) = \frac{-1}{ip_0}(\delta\Pi^{\mu\nu}(P) + u^\mu u^\nu). \tag{139}$$

The function $\delta\Pi^{\mu\nu}(P)$ is transverse,

$$P^\mu \delta\Pi^{\mu\nu}(P) = 0. \tag{140}$$

Thus, the new hard thermal loop is not,

$$P^\mu \delta\Gamma^{\mu\nu}(P) = iu^\nu. \tag{141}$$

As discussed following Eq. (122), the new hard thermal loop is the modification of the propagator in a background field. Thus, it is not necessary for the term to be transverse. We shall also see this in the next section, where we compute the one point function for a gluon.

C. One point gluon function

With these results in hand, the diagrams can be computed directly. In this subsection we begin with the one point function for a gluon. The integral which enters is that of Eq. (100), and generates the function $\mathcal{A}_0$, Eq. (102). As can be seen from Eq. (101), this function vanishes when $Q = 0$, as then the integral is odd in k_0.

Consider a gluon loop tied onto a gluon line. By an argument similar to that which lead to Eqs. (50) and (51), if the gluon propagator in the loop is $\mathcal{P}^{cd,ef}/(K^{cd})^2$, Eq. (39), we can replace this by $\delta^{cf}\delta^{de}/(K^{cd})^2$. Since the external gluon has zero momentum, the three gluon vertex involves one momentum, which we choose as K^{cd}; from Eq. (45), the three gluon vertex is

$$-igf^{(ab,cd,ef)}(-K^{cd}_\lambda \delta_{\mu\nu} + 2K^{cd}_\mu \delta_{\nu\lambda} - K^{cd}_\nu \delta_{\lambda\mu}). \tag{142}$$

This is transverse in K^{cd}_ν and K^{cd}_λ, so the gauge dependent term in the gluon propagator, $\sim(\xi - 1)K^{cd}_\nu K^{cd}_\lambda$, Eq. (39), drops out. The term in the gluon propagator $\sim\delta^{\mu\nu}$ acting upon Eq. (142) gives $2(d-1)$ in $d = 4$ spacetime dimensions. Including an overall $1/2$ for the gluon loop,

$$-\frac{6}{2}igf^{ab,cd,dc}\int\frac{d^4K}{(2\pi)^4}\frac{(K^{cd})^\mu}{(K^{cd})^2} = -u^\mu ig\pi T^3\sum_{c,d=1}^{N} f^{ab,cd,dc}T^3\mathcal{A}_0(Q^c - Q^d). \tag{143}$$

Here, $u^\mu = (1, \vec{0})$. The contribution from a ghost loop is similar, with a coefficient of -1 instead of $+3$ in Eq. (143); thus, the sum is $+2$, which reflects for the two transverse degrees of freedom of a gluon.

The contribution of N_f flavors of massless quarks is

$$(-)igN_f(t^{ab})_{cc}\int\frac{d^4K}{(2\pi)^4}\,\mathrm{tr}\frac{\gamma^\mu}{-i\tilde{\not{K}}^c} = \frac{4\pi}{3}u^\mu gN_fT^3(t^{ab})_{cc}\mathcal{A}_0(\tilde{Q}^c). \tag{144}$$

Using Eqs. (7) and (24), the sum of the gluon, ghost, and quark contributions is

$$\langle J^{ab}_\mu\rangle \overset{\mathrm{HTL}}{\approx} -u^\mu\delta^{ab}\frac{4\pi gT^3}{3\sqrt{2}}\Big(\sum_{c=1}^{N}\Big[\mathcal{A}_0(Q^{ac}) + \frac{N_f}{N}\mathcal{A}_0(\tilde{Q}^c)\Big] - N_f\mathcal{A}_0(\tilde{Q}^a)\Big). \tag{145}$$

In obtaining this, we have used the fact that $\mathcal{A}_0$ is odd in $Q \to -Q$. This is clear from its definition in Eq. (101); its form in Eq. (102) is only valid for $0 < q < 1$.

For a Z(N) interface, this current is part of the equation of motion for the gluon, corrected to one-loop order. There it is natural: the interface arises from a balance of the Lagrangian at tree level, and a potential for the Q's at one-loop order. Equation (145) is precisely the derivative of the one-loop potential.

For the semi-QGP, a term must be added to the Lagrangian to cancel the contribution of this current. This is natural; if such a term is not added, the minimum would be at $Q = 0$, or equivalent points; i.e., the usual Z(N) minima.

D. Quark contribution to the gluon self-energy

With the previous examples in hand, the computation of the gluon self-energy is mainly a matter of putting things together. Even so, because the Q's are nontrivial, what is of interest is to see how new color structures arise at one-loop order from quantum corrections to the propagators at tree level. This is unlike the case of the quark self-energy, Eq. (87), where the only change from $Q = 0$ was the value of the thermal quark mass.

We start with the contribution of the quark loop. For N_f flavors, this is

$$-(\Pi^{ab,cd}_{\mu\nu})_{\rm qk}(P^{ab}) = (-)(ig)^2 N_f \int \frac{d^4K}{(2\pi)^4}\,{\rm tr}\gamma^\mu (t^{ab})_{ef} \times \frac{1}{-i(\tilde{\not K}^e)}\gamma^\nu (t^{cd})_{fe}\frac{1}{-i(\tilde{\not K}^e - \not P^{ab})}. \quad (146)$$

Keeping only the terms $\sim \tilde{\not K}^e$ in the numerators, the hard thermal loop in this contribution is

$$(\Pi^{ab,cd}_{\mu\nu})_{\rm qk}(P^{ab}) \overset{\rm HTL}{\approx} 8g^2 N_f \sum_{e,f=1}^{N} (t^{ab})_{ef}(t^{cd})_{fe} \times \tilde{\mathcal{J}}^{\mu\nu}(P^{ab}, \tilde{Q}^e, Q^{ab} - \tilde{Q}^e). \quad (147)$$

Because $\tilde{Q}^e$ appears in the function $\tilde{\mathcal{J}}^{\mu\nu}$, what enters is not simply the trace of two projection operators, Eq. (8). The sum over the color index e and f is easy, and gives

$$(\Pi^{ab,cd}_{\mu\nu})_{\rm qk}(P^{ab}) \overset{\rm HTL}{\approx} 4g^2 N_f \Big[\delta^{ad}\delta^{bc}\tilde{\mathcal{J}}^{\mu\nu}(P^{ab}, \tilde{Q}^a, -\tilde{Q}^b) - \frac{1}{N}\delta^{ab}\delta^{cd}\Big(\tilde{\mathcal{J}}^{\mu\nu}(P, \tilde{Q}^a, -\tilde{Q}^a) + \tilde{\mathcal{J}}^{\mu\nu}(P, \tilde{Q}^c, -\tilde{Q}^c) - \frac{1}{N}\sum_{e=1}^{N}\tilde{\mathcal{J}}^{\mu\nu}(P, \tilde{Q}^e, -\tilde{Q}^e)\Big)\Big]. \quad (148)$$

This is illustrated in Fig. 9. There is the usual planar diagram, plus three contributions from diagrams in which one or both of the gluon indices are traced. Notice that as usual, the color structure is far more clear from the diagram, than from the detailed expression in Eq. (148).

The last three terms, $\sim\delta^{ab}\delta^{cd}$, are like the photon propagator for QED, where the fermions propagate in a background Q field. In all, the quark contribution is traceless,

$$\sum_{a=1}^{N}(\Pi^{aa,cd}_{\mu\nu})_{\rm qk}(P) = \sum_{c=1}^{N}(\Pi^{ab,cc}_{\mu\nu})_{\rm qk}(P) = 0. \quad (149)$$

which is necessary for self-consistency. Note that the quark contribution is not simply proportional to a projector operator, $\sim\mathcal{P}^{ab,cd}$. Instead, because the Q's can be unequal, at one-loop order the color structure is more complicated, Eq. (148).

E. Ghost and gluon contributions to the gluon self-energy

To compute the contributions of gluons and ghosts to the hard thermal loop in the gluon self-energy, we first compute in background Feynman gauge, $\xi = 1$ in Eq. (39). We then show that the results are independent of ξ, in complete analogy to when $Q = 0$.

The simplest contribution is the tadpole diagram, which is independent of the external momentum. This involves the four gluon vertex of Eq. (49). For the gluon in the loop, we can use the simplification of Eq. (51) to take the gluon propagator as a Kronecker delta. The result is

$$-\frac{g^2}{2}6f^{(ab,ef,gh)}f^{(cd,fe,hg)}\int\frac{d^4K}{(2\pi)^4}\delta^{\mu\nu}\frac{1}{(K^{fe})^2}. \quad (150)$$

The coefficient $6 \to 2(d-1)$ in d spacetime dimensions; the $1/2$ is for a bosonic loop.

The gluon and ghost loops are, generally, involved. However, we can use the simplification of Eqs. (136): write the momenta in terms of the external momentum, P^{ab}, and a loop momentum, which we can define as K^{fe}. Then although the Q's are hard to begin with, we can consistently treat P^{ab} as soft, and K^{fe} as hard. This allows us to

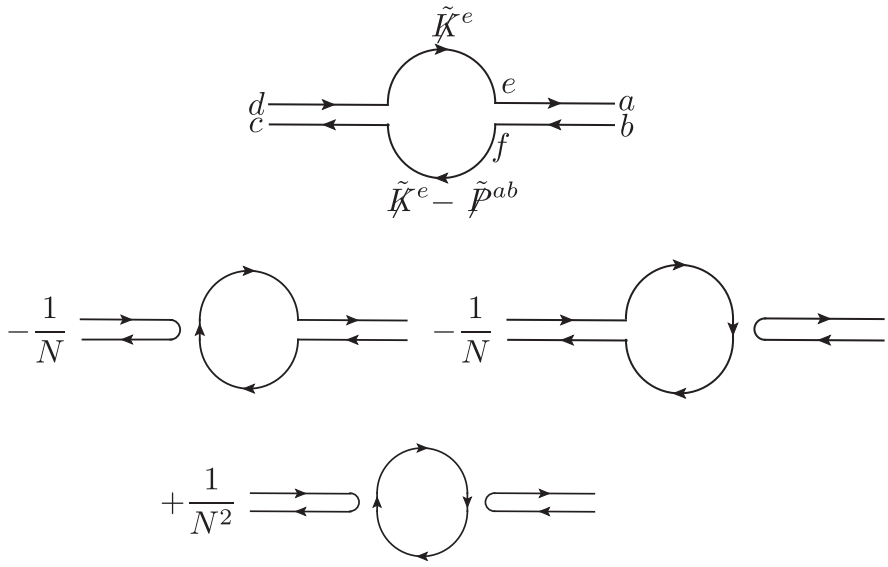

FIG. 9. One-loop diagram for the quark loop in the gluon self-energy.

drop powers of P^{ab} uniformly. (This is only valid in Feynman and Coulomb gauges [23–25]).

With this approximation it is then easy to read off the hard thermal loops in the gluon self-energy. The ghost loop is given by taking K^{fe} at each vertex, and so is

$$(-)g^2 f^{(ab,ef,gh)} f^{(cd,fe,hg)} \int \frac{d^4K}{(2\pi)^4} \frac{(K^{fe})^\mu (K^{fe})^\nu}{(K^{fe})^2 (P^{ab} - K^{fe})^2}. \tag{151}$$

For the contribution to the gluon self-energy from the diagram with two three gluon vertices, in each vertex we can neglect the external momentum, as in Eq. (142). This gives

$$\frac{g^2}{2} f^{(ab,ef,gh)} f^{(cd,fe,hg)} \int \frac{d^4K}{(2\pi)^4} \times \frac{10(K^{fe})^\mu (K^{fe})^\nu + 2(K^{fe})^2 \delta^{\mu\nu}}{(K^{fe})^2 (P^{ab} - K^{fe})^2}. \tag{152}$$

The coefficient $10 \longrightarrow 2(2d-3)$ in d spacetime dimensions.

In all, the sum of Eqs. (150)–(152) is

$$-(\Pi^{ab,cd}_{\mu\nu})_{\rm gl}(P^{ab}) \overset{\rm HTL}{\approx} 4g^2 f^{(ab,ef,gh)} f^{(cd,fe,hg)} \times \tilde{\mathcal{J}}^{\mu\nu}(P^{ab}, Q^{fe}, Q^{hg}). \tag{153}$$

From Eq. (135),

$$(\Pi^{ab,cd}_{\mu\nu})_{\rm gl}(P^{ab}) \overset{\rm HTL}{\approx} -4g^2 \Big(\delta^{ad}\delta^{bc} \sum_{e=1}^{N} \tilde{\mathcal{J}}^{\mu\nu}(P^{ab}, Q^{ae}, Q^{eb}) - \delta^{ab}\delta^{cd} \tilde{\mathcal{J}}^{\mu\nu}(P^{ab}, Q^{ca}, Q^{ac}) \Big). \tag{154}$$

Like the quark self-energy, Eq. (149), this is traceless, as it must be, to represent a matrix in SU(N).

The color structure is illustrated in the diagram of Fig. 10. There is the planar diagram, minus a diagram in which the indices are summed over.

In all, the hard thermal loop in the gluon self-energy is the sum of Eqs. (148) and (154). Each term is a product of a color dependent factor, times a function of the soft momentum

$$\Pi^{ab,cd}_{\mu\nu}(P^{ab}) \overset{\rm HTL}{\approx} -\mathcal{K}^{ab,cd}(Q)\delta\Gamma^{\mu\nu}(P^{ab}) - (m^2_{\rm gl})^{ab,cd}(Q) \times \delta\Pi^{\mu\nu}(P^{ab}). \tag{155}$$

The thermal gluon mass in a background field is

$$(m^2_{\rm gl})^{ab,cd}(Q) = \frac{g^2T^2}{6} \Big(\delta^{ad}\delta^{bc} \Big[\sum_{e=1}^{N} (\mathcal{A}(Q^{ae}) + \mathcal{A}(Q^{eb})) - N_f(\mathcal{A}(\tilde{Q}^a) + \mathcal{A}(\tilde{Q}^b)) \Big] - 2\delta^{ab}\delta^{cd} \Big[\mathcal{A}(Q^{ac}) - \frac{N_f}{N} \Big(\mathcal{A}(\tilde{Q}^a) + \mathcal{A}(\tilde{Q}^c) - \frac{1}{N} \sum_{e=1}^{N} \mathcal{A}(\tilde{Q}^e) \Big) \Big] \Big). \tag{156}$$

In zero field, $\mathcal{A}(0) = 1$, $\mathcal{A}(\tilde{Q}^a) = \mathcal{A}(\pi T) = -1/2$, and

$$(m^2_{\rm gl})^{ab,cd}(0) = \mathcal{P}^{ab,cd} \Big(N + \frac{N_f}{2} \Big) \frac{g^2T^2}{3}. \tag{157}$$

The new hard thermal loop in the gluon self-energy involves the color matrix

$$\mathcal{K}^{ab,cd}(Q) = 2g^2T^3 \delta^{ad}\delta^{bc} \Big[\sum_{e=1}^{N} (\mathcal{A}_0(Q^{ae}) + \mathcal{A}_0(Q^{eb})) - N_f(\mathcal{A}_0(\tilde{Q}^a) + \mathcal{A}_0(-\tilde{Q}^b)) \Big]. \tag{158}$$

The terms proportional to $\delta^{ab}\delta^{cd}$ vanish because $\mathcal{A}_0(Q)$ is odd in Q, Eq. (102). As necessary, the matrix $\mathcal{K}^{ab,cd}(Q)$ is traceless.

Comparing Eq. (158) with Eq. (145), we find

$$\mathcal{K}^{ab,cd}(Q) = -igf^{ab,cd,ef}\langle J^{fe}_0 \rangle. \tag{159}$$

This is natural; the background field induces a color current, which couples to the gluon. The self-energy obeys the Ward-Takahashi identity in the background field

$$P^{ab}_\mu \Pi^{ab,cd}_{\mu\nu} = -gf^{ab,cd,ef}\langle J^{fe}_\nu \rangle. \tag{160}$$

These expressions were computed in Feynman gauge, but the results are independent of the gauge fixing parameter, ξ, Eq. (29). Except for the gluon self-energy, where the tadpole diagram enters, the hard thermal loops in any gluon amplitude only involve three gluon vertices. This vertex satisfies an Abelian-type Ward identity, Eq. (47). This

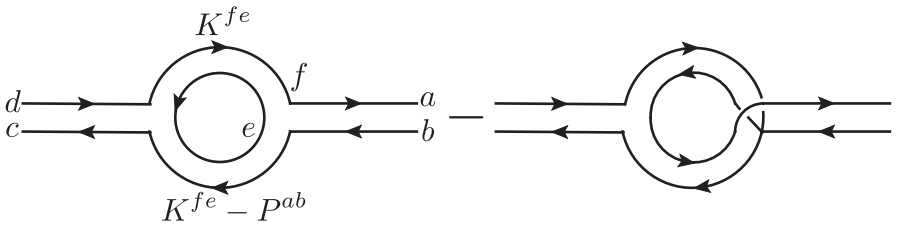

FIG. 10. One-loop diagram for the gluon loop in the gluon self-energy.

identity can be used to show that all hard thermal loops are independent of ξ, up to possibly gauge dependence in terms that are independent of the static momentum. However, it is known that at one-loop order that the potential in a background Q field is independent of ξ [6–8]. Thus, the hard thermal loops are independent of ξ when $Q \neq 0$, as for $Q = 0$.

VI. CONCLUSIONS

In this paper we developed techniques to analyze the real time response functions for a 't Hooft loop, or Z(N) interface. By introducing the double line notation in Sec. II, we are able to analyze a much more general problem, as is appropriate for the semi-QGP phase of a gauge theory [32].

While the final expressions that we obtain appear involved, in fact the physics for $Q \neq 0$ is very similar to that for $Q = 0$. For the quark self-energy, Eq. (87), the hard thermal loop is a thermal quark mass times a function of momentum. The function of momentum, $\delta\Sigma(P)$ in Eq. (88), is unchanged from $Q = 0$. What does change with Q is the thermal quark mass, Eq. (89). It is most natural that in the presence of a nonzero background field, that the curvature about the minimum changes with the background field.

For the gluon self-energy, Eq. (155), there is a piece very similar to that for $Q = 0$. There is the same function of momentum, $\delta\Pi^{\mu\nu}(P)$ in Eq. (138), as in zero field. This function is multiplied by a thermal gluon mass, Eq. (156), which is of course Q dependent.

The surprise is that there is a new function in the gluon self-energy, $\delta\Gamma^{\mu\nu}(P)$ in Eq. (121). [Note, however, that this function is linearly related to $\delta\Pi^{\mu\nu}(P)$, Eq. (139)]. The usual hard thermal loop in the gluon self-energy is $\delta\Pi^{\mu\nu} \sim g^2T^2$, and is smaller than the new hard thermal loop, $\delta\Gamma^{\mu\nu}(P) \sim g^2T^3/p$. The general principle of hard thermal loops, however, is that for soft external momentum, they are as large as the terms at tree level. This remains valid, since for a non-Abelian gauge field, the propagator in a background field has a new term, $\sim[G^{\mu\nu}, \,]$, Eq. (30).

We also note that surely our entire derivation would be much simpler if we had used kinetic theory in the presence of a nonzero background field. However, we preferred to use an ordinary perturbative analysis, since computing at $Q \neq 0$ is an unfamiliar exercise. We do expect that the derivation of a complete action for all hard thermal loops for nonzero Q would be much simpler with kinetic theory, as it is when $Q = 0$.

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APPENDIX: Z(N) AND U(1) INTERFACES

In this appendix we use this opportunity to make some comments about Z(N) [3–7] and U(1) [29] interfaces in both Abelian and non-Abelian gauge theories.

As discussed in the Introduction, Sec. I, to define an interface we pick out one of the three spatial directions, say that in the z direction, and consider it separately from the two other spatial directions, and from that for imaginary time, τ. It is also necessary to assume that the length in the z direction, L, is much larger than that for the two transverse dimensions, of size L_t [6]. Lastly, we assume that both L and L_t are much larger than any physical mass scale, such as the inverse Debye mass, $\sim 1/(gT)$.

For a pure gauge theory, without dynamical quarks the gauge group is SU(N)/Z(N). Thus, we can require that the gauge field at $z = L$ is a gauge transformation of that at $z = 0$. We take this gauge transformation to be a constant element of Z(N) [3]:

$$\Omega = \exp\left(\frac{2\pi ik}{N}\right)\mathbf{1}_N. \tag{A1}$$

Since this Ω commutes with all group elements, the gauge field at $z = L$ is identical to that at $z = 0$. In going from $z = 0$ to $z = L$, though, one winds, in a topologically nontrivial manner, in the configuration space of gauge potentials [44]. One can show that these boundary conditions are equivalent to inserting a 't Hooft loop, at $z = L/2$, at the boundary of the two transverse dimensions [3,7].

There are k distinct transformations possible, where $k = 1, 2, \ldots (N-1)$. This generates the cyclic group, Z(N), where $k = N$ is equivalent to the identity. At tree level, this transformation is implemented by a gauge transformation which is linear in z,

$$\Omega(z) = \exp\left(2\pi ik(\sqrt{2}t^{NN})\frac{z}{L}\right). \tag{A2}$$

The factor of $\sqrt{2}$ is because of the normalization for t^{NN} in (10). This gauge transformation is generated by a constant vector potential in the imaginary time direction, $A_0 = (2\pi ikT/g)(\sqrt{2}t^{NN})z/L$. This is a linear ansatz, and is only valid for the tree level action. Loop corrections generate a potential for A_0, and turn the true solution into a domain wall, of width $\sim 1/(gT)$ [6]. At present, however, we need not trouble ourselves with such details. All that

matters is that this can be modified to represent a solution of the quantum equations of motion.

Consider the sum of the following two diagonal generators:

$$\lambda_3 = \sqrt{2}((N-1)t^{N-1,N-1} + t^{NN}) = \sqrt{2(N-1)(N-2)}\lambda_C^{N-1}, \quad \text{(A3)}$$

Eq. (11).

Consider a gauge transformation generated by this transformation, $\Omega(z) = \exp(2\pi i \lambda_3 z/L)\mathbf{1}_N$. This is a "U(1)" transformation, in that it is the same on both ends of the box, $\Omega(L) = \Omega(0) = \mathbf{1}_N$ [29]. This is evident from (A3), where we see that it is a combination of two Z(N) transformations: one of strength $N-1$, along the $t^{N-1,N-1}$ direction in group space, plus one of unit strength, along the t^{NN} direction. The total strength is then N, which for Z(N) is equivalent to the identity; i.e., to no winding.

In [29], U(1) interfaces were suggested as a way to perform the semiclassical matching between an effective and an original theory. For the pure gauge theory, the U(1) interfaces are not necessary: we can use the k possible Z(N) interfaces to perform the matching. We remark, rather trivially, that if the Z(N) interfaces do match, then so will the U(1) interfaces, since by the above, they are just a combination of two Z(N) interfaces in different directions in group space, one along $t^{N-1,N-1}$, and the other along t^{NN}.

The real use of a U(1) interface, however, is for the theory with dynamical quarks, where the Z(N) invariance is broken by the presence of quarks. Before considering this case, consider a simpler example: a Higgs model, where a charged scalar field, ϕ, acquires a vacuum expectation value, $\langle\phi\rangle$. Then in going from one end of the box, to the other, one can define a winding number by the number of times that the phase of $\langle\phi\rangle$ winds around 2π. The topology is elementary, just $\pi_1(\mathrm{U}(1)) \simeq Z$.

Now consider an Abelian gauge theory at nonzero temperature, such as QED, where the gauge group is unbroken. Instead of the expectation value of a Higgs field, we consider windings of the thermal Wilson line. The thermal Wilson line is $= +1$ at both ends of the box, but it can wind nontrivially as it goes along the box. The topology remains $\pi_1(\mathrm{U}(1)) \simeq Z$. Thus there are thermal interfaces in QED. This was first pointed out by Smilga [15], who computed their properties in the Schwinger model.

Thermal U(1) interfaces are of interest in the electroweak theory. They are unlike standard domain walls, in that the interface tension is strongly dependent upon the temperature. In the high temperature phase, where the Higgs field is unbroken, the domain walls lie in the U(1) for hypercharge. As the system cools through the electroweak phase transition, they then rotate into the U(1) for electromagnetism. Since the potential is generated by quantum effects, it vanishes exponentially at low temperatures, $\exp(-m_e/T)$, where m_e is the electron mass. Thus, while U(1) domain walls dominate the stress energy tensor while they exist, unlike standard domain walls, they naturally vanish at low temperature. Whether their presence, during the period in which they dominate the stress energy tensor, can lead to characteristic cosmological signals is an interesting question.

We return to a SU(N) gauge theory. For each of the N diagonal directions, we can define a U(1) winding number. For example, λ_3 represents winding once in the first $N-2$ directions, $-(N-2)$ times in the $(N-1)$th direction, and no winding in the last, Nth direction.

Semiclassically, one expands about the configuration along the λ_3 direction, in both the original and effective theories. The configuration is a local minimum of each action, and has no instabilities under small fluctuations. Thus, such a U(1) interface can be used to match the parameters of the effective theory to the original theory, as suggested in Ref. [29].

To define such a configuration nonperturbatively, such as on the lattice, it is necessary to fix the freedom to perform global gauge rotations. Only after fixing the freedom to change overall gauge rotations can one define U(1) winding numbers for each of the diagonal directions [29].

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