PHYSICAL REVIEW D **80,** 034028 (2009)

Chiral magnetic conductivity

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(Received 30 July 2009; published 26 August 2009)

Gluon field configurations with nonzero topological charge generate chirality, inducing $\mathcal{P}$- and $\mathcal{CP}$-odd effects. When a magnetic field is applied to a system with nonzero chirality, an electromagnetic current is generated along the direction of the magnetic field. The induced current is equal to the chiral magnetic conductivity times the magnetic field. In this article we will compute the chiral magnetic conductivity of a high-temperature plasma for nonzero frequencies. This allows us to discuss the effects of time-dependent magnetic fields, such as produced in heavy ion collisions, on chirally asymmetric systems.

DOI: 10.1103/PhysRevD.80.034028 PACS numbers: 12.38.−t, 12.38.Mh

I. INTRODUCTION

Quantum chromodynamics (QCD) contains gauge field configurations which carry topological charge [1]. These configurations interpolate between the different vacua of the gluonic sector of QCD [2] and induce interesting $\mathcal{P}$- and $\mathcal{CP}$-odd effects [3]. Neglecting the masses of quarks (as appropriate at high energies), it holds for each individual flavor that [4]

$$\Delta N_5 \equiv \Delta(N_R - N_L) = -2Q, \tag{1}$$

where ΔN_5 denotes the change in chirality (N_5) which is the difference between the number of modes with right- and left-handed chirality. In the limit of zero quark mass N_5 is also equal to the total number of particles plus antiparticles with right-handed helicity minus the total number of particles plus antiparticles with left-handed helicity. Right-handed helicity means that spin and momentum are parallel, whereas left-handed helicity implies they are opposite.

There exist different mechanisms to generate topological charge during heavy ion collisions. One possible way is by longitudinal fields created just after the collision [5–7], another due to QCD sphaleron transitions in the quark-gluon-plasma [8–11], and also plasma instabilities can lead to generation of topological charge [12]. Furthermore, it has been argued that instanton "ladders" may describe a significant fraction of multiparticle production at high energies [13–15]. Finally, metastable $\mathcal{P}$- and $\mathcal{CP}$- odd domains may exist in the quark-gluon plasma close to the critical temperature [16]. No matter what the precise mechanism behind the generation of topological charge is, this leads to generation of chirality as can be seen from Eq. (1).

Assuming that the θ angle vanishes, there is no explicit $\mathcal{P}$ and $\mathcal{CP}$ breaking in QCD. Hence positive and negative topological charges are being generated with equal probability. But because of fluctuations, a finite amount of topological charge can be generated in each event within a given region of phase space determined by the experimental acceptance. Only if one averages over many events the net produced topological charge should vanish. Therefore in each event a difference between the total number of right- and left-handed particles can be expected.

When two heavy ions collide with a nonzero impact parameter, a (electromagnetic) magnetic field of enormous magnitude is created in the direction of angular momentum of the collision; it has been evaluated in Refs. [17,18] (see, also, [19] for a proposal to utilize electromagnetic fields in the search for disoriented chiral condensates). If a nonzero chirality is present in such a situation, an electromagnetic current will be induced in the direction of the magnetic field. This is the so-called chiral magnetic effect [17,20–22]; see the recent Ref. [23] for the first study of this effect in lattice QCD. To understand this effect qualitatively, let us imagine a situation in which the $\mathcal{P}$-and $\mathcal{CP}$-odd processes made N_5 positive, so that we have an excess of quarks plus antiquarks with right-handed helicity. In a background magnetic field, the quarks will align their magnetic moments along the magnetic field. And assuming the quarks can be treated as massless, the momenta of the quarks will be aligned along the field as well. Consequently a quark with left-handed helicity tends to move exactly in the opposite direction to a quark with right-handed helicity. Since the magnetic moment is proportional to the charge, an antiquark with right-handed helicity will move exactly opposite to a quark with right-handed helicity. Accordingly, in this case of positive N_5, an excess of positive charge will move parallel to the magnetic field and an excess of negative charge will go in the opposite direction. Thus an electromagnetic current is generated along the magnetic field.

In a heavy ion collision this current leads to an excess of positive charge on one side of the reaction plane (the plane in which the beam axis and the impact parameter lies) and negative charge on the other; the resulting charge asymmetry is also modulated by the radial flow and the transport properties of the medium. This charge asymmetry can be

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investigated experimentally using the observables proposed in Ref. [24]. Preliminary data of the STAR collaboration has been presented in Refs. [25,26]. Implications of the chiral magnetic effect on astrophysical phenomena have recently been discussed in Ref. [27]; another astrophysical implication can be found in [28].

A system of massless fermions with nonzero chirality can be described by a chiral chemical potential μ_5 which couples to the zero component of the axial vector current in the Lagrangian. The induced current in such a situation can be written as $\boldsymbol{j} = \sigma_\chi \boldsymbol{B}$, where σ_χ is the chiral magnetic conductivity. For constant and homogeneous magnetic fields its value is determined by the electromagnetic axial anomaly and for one flavor and one color equal to [22,29,30] (see, also, [31])

$$\sigma_\chi(\omega = 0, \boldsymbol{p} = 0) \equiv \sigma_0 = \frac{e^2}{2\pi^2}\mu_5, \tag{2}$$

where ω and $\boldsymbol{p}$ denote frequency and momentum, respectively, and e equals the unit charge. For a finite number of colors N_c and flavors f, one has to multiply this result by $N_c \sum_f q_f^2$ where q_f denotes the charge of a quark in units of e. The generation of currents due to the anomaly in background fields or rotating systems is also discussed in related contexts in Refs. [31–35].

For constant magnetic fields which are inhomogeneous in the plane transverse to the field, one finds that the total current J along B equals [22],

$$J = e\left\lfloor \frac{e\Phi}{2\pi} \right\rfloor \frac{L_z \mu_5}{\pi}, \tag{3}$$

where L_z is the length of the system in the z direction and the flux Φ is equal to the integral of the magnetic field over the transverse plane,

$$\Phi = \int d^2 x B(x, y). \tag{4}$$

The floor function $\lfloor x \rfloor$ is the largest integer smaller than x. The quantity $\lfloor e\Phi/(2\pi) \rfloor$ in Eq. (3) is equal to the number of zero modes in the magnetic field [36].

To compute the current generated by a configuration of specific topological charge, one should express μ_5 in terms of the chirality N_5. By using the anomaly relation, one can then relate N_5 to the topological charge. This is discussed in detail in Ref. [22].

The aim of this paper is to study how a system with constant nonzero chirality responds to a time-dependent magnetic field. This is interesting for phenomenology since the magnetic field produced with heavy ion collisions depends strongly on time. To obtain the induced current in a time-dependent magnetic field, we will compute the chiral magnetic conductivity for nonzero frequencies and nonzero momenta using linear response theory. We will compute the leading order conductivity and leave the inclusion of corrections due to photon and or gluon exchange for future work. In leading order, the chiral magnetic conductivity for an electromagnetic plasma and quark gluon plasma are equal (up to a trivial factor of $N_c \sum_f q_f^2$). Since we do not take into account higher order corrections, some of our results for QCD will only be a good approximation in the limit of very high temperatures where the strong coupling constant α_s is sufficiently small.

We will take the metric $g^{\mu\nu} = \mathrm{diag}(+, -, -, -)$. The gamma matrices in the complete article satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. We will use the notation p for both the four-vector $p^\mu = (p_0, \boldsymbol{p})$ and the length of a three-vector $p = |\boldsymbol{p}|$.

II. KUBO FORMULA FOR CHIRAL MAGNETIC CONDUCTIVITY

For small magnetic fields, the induced vector current can be found using the Kubo formula. This formula tells us that to first order in the time-dependent perturbation, the induced vector current is equal to a retarded correlator of the vector current with the perturbation evaluated in equilibrium. More explicitly, one finds that

$$\langle j^\mu(x) \rangle = \int d^4 x' \Pi_{\mathrm{R}}^{\mu\nu}(x, x') A_\nu(x'), \tag{5}$$

where $j^\mu(x) = e\bar{\psi}(x)\gamma^\mu \psi(x)$ and the retarded response function $\Pi_{\mathrm{R}}^{\mu\nu}$ is given by

$$\Pi_{\mathrm{R}}^{\mu\nu}(x, x') = i\langle [j^\mu(x), j^\nu(x')] \rangle \theta(t - t'). \tag{6}$$

The equilibrium Hamiltonian is invariant under translations in time and space, therefore we can use that $\Pi_{\mathrm{R}}^{\mu\nu}(x, x') = \Pi_{\mathrm{R}}^{\mu\nu}(x - x')$. Let us take a vector field of the following specific form $A_\nu(x) = \tilde{A}_\nu(p)e^{-ipx}$. The Kubo formula now becomes

$$\langle j^\mu(x) \rangle = \tilde{\Pi}_{\mathrm{R}}^{\mu\nu}(p)\tilde{A}_\nu(p)e^{-ipx}, \tag{7}$$

where

$$\tilde{\Pi}_{\mathrm{R}}^{\mu\nu}(p) = \int d^4 x e^{ipx} \Pi_{\mathrm{R}}^{\mu\nu}(x). \tag{8}$$

In order to compute the chiral magnetic conductivity, we will take a time-dependent magnetic field pointing in the z direction. Because of Faraday's law ($\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$), such a time-dependent magnetic field comes always together with a perpendicular electric field. Let us choose a gauge such that the only component of the vector field that is nonvanishing is A_y. Then $B_z(x) = \partial_x A_y(x)$ so that $\tilde{B}_z(p) = ip^1 \tilde{A}^2(p)$. Using Eq. (7) we find that the induced vector current in the direction of the magnetic field can now be written as

$$\langle j_z(x) \rangle = \sigma_\chi(p)\tilde{B}_z(p)e^{-ipx}, \tag{9}$$

where the chiral magnetic conductivity equals

$$\sigma_\chi(p) = \frac{1}{ip^1}\tilde{\Pi}_{\mathrm{R}}^{23}(p) = \frac{1}{2ip^i}\tilde{\Pi}_{\mathrm{R}}^{jk}(p)\epsilon^{ijk}. \tag{10}$$

The right-hand side of the last equation is a result of rotational and gauge invariance. We can now write the chiral magnetic conductivity in the following way:

$$\sigma_\chi(p) = \frac{1}{ip^i} G^i_{\mathrm{R}}(p), \tag{11}$$

with

$$G^i_{\mathrm{R}}(p) = \tfrac{1}{2}\epsilon^{ijk}\tilde{\Pi}^{jk}_{\mathrm{R}}(p). \tag{12}$$

The last two equations show that the chiral magnetic conductivity follows directly from evaluating $G^i(p)$, which is the spatially antisymmetric part of the off diagonal retarded current-current correlator, or equivalently the photon polarization tensor.

Another related quantity that follows from the off diagonal part of the photon polarization tensor is the Hall conductivity. The Hall current is generated in the presence of a magnetic field that is perpendicular to an electric field. Unlike the chiral magnetic current, the Hall current is in a direction perpendicular to the magnetic field. In order to obtain the Hall conductivity one usually computes the photon polarization tensor in the presence of a homogeneous background field. The electric field is then treated as a perturbation (see, e.g., Ref. [37] for a recent calculation using a holographic model of QCD).

In general, the chiral magnetic conductivity will be complex. Let us therefore write

$$\sigma_\chi(p) = \sigma'_\chi(p) + i\sigma''_\chi(p), \tag{13}$$

where $\sigma'_\chi(p)$ and $\sigma''_\chi(p)$ are real functions given by

$$\sigma'_\chi(p) = \frac{1}{p^i}\,\mathrm{Im}G^i_{\mathrm{R}}(p), \tag{14}$$

$$\sigma''_\chi(p) = -\frac{1}{p^i}\,\mathrm{Re}G^i_{\mathrm{R}}(p). \tag{15}$$

For convenience we will write the zero momentum limit of the chiral magnetic conductivity as follows:

$$\sigma_\chi(\omega) \equiv \lim_{\boldsymbol{p}\to 0}\sigma_\chi(p_0 = \omega, \boldsymbol{p}). \tag{16}$$

The real part of the conductivity is an even function of ω while the imaginary part is odd, i.e., $\sigma'_\chi(\omega) = \sigma'_\chi(-\omega)$ and $\sigma''_\chi(\omega) = -\sigma''_\chi(-\omega)$. If we apply a homogeneous magnetic field $\boldsymbol{B}(t) = B_\omega\cos(\omega t)\hat{\boldsymbol{z}}$ the response $j(t) = \langle j_z(t)\rangle$ will be

$$j(t) = [\sigma'_\chi(\omega)\cos(\omega t) + \sigma''_\chi(\omega)\sin(\omega t)]B_\omega. \tag{17}$$

Hence the real part and imaginary part correspond to the in- and out-phase response, respectively. To find the response to a general time-dependent magnetic field $\boldsymbol{B} = B(t)\hat{\boldsymbol{z}}$ one first has to compute the Fourier transform of the magnetic field

$$\tilde{B}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}B(t). \tag{18}$$

The response will then be

$$j(t) = \int_0^\infty \frac{d\omega}{\pi}[\sigma'(\omega)\cos(\omega t) + \sigma''(\omega)\sin(\omega t)]\tilde{B}(\omega). \tag{19}$$

The real and imaginary parts of the chiral magnetic conductivity related to each other by the Kramers-Kroning relation

$$\sigma'_\chi(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} dq_0\frac{\sigma''_\chi(q_0)}{q_0 - \omega}, \tag{20}$$

$$\sigma''_\chi(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} dq_0\frac{\sigma'_\chi(q_0)}{q_0 - \omega}. \tag{21}$$

In the next section we will study the fermion propagator in the presence of a chiral chemical potential. Then using this propagator we will compute the retarded current-current correlator from which we can derive the chiral magnetic conductivity.

III. FERMION PROPAGATOR

The bare fermion propagator as a function of Euclidean momentum Q in the presence of a chiral chemical potential equals

$$S(Q) = \frac{1}{i\gamma^0(\tilde{\omega}_m - i\mu - i\mu_5\gamma^5) - \boldsymbol{\gamma}\cdot\boldsymbol{q}}, \tag{22}$$

where $\tilde{\omega}_m = (2m+1)\pi T$ with $m \in \mathbb{Z}$ is a fermionic Matsubara mode. By computing the inverse, this propagator can be written as

$$S(Q) = \mathcal{P}_+\frac{\not{Q}_+}{Q_+^2} + \mathcal{P}_-\frac{\not{Q}_-}{Q_-^2}, \tag{23}$$

where we have defined $Q^\mu_\pm = (i\tilde{\omega}_\pm, \boldsymbol{q})$ with $\tilde{\omega}_\pm = \tilde{\omega}_m - i\mu_\pm$ and $\mu_\pm = \mu \pm \mu_5$. Accordingly, $Q_{\pm\mu} = (i\tilde{\omega}_\pm, -\boldsymbol{q})$ and $Q^2_\pm = -(\tilde{\omega}^2_\pm + \boldsymbol{q}^2)$. Furthermore $\not{Q}_\pm = Q^\mu_\pm\gamma_\mu$. The right- (+) and left-handed (−) chirality projection operators are given by $\mathcal{P}_\pm = \frac{1}{2}(1 \pm \gamma_5)$. They satisfy $\mathcal{P}^2_\pm = \mathcal{P}_\pm$ and $\mathcal{P}_+\mathcal{P}_- = 0$. Let us now define

$$\Delta_\pm(q_0, \boldsymbol{q}) = \frac{1}{q_0 \mp E_q}, \tag{24}$$

where $E_q = |\boldsymbol{q}|$. Furthermore we introduce $\hat{\boldsymbol{q}} = \boldsymbol{q}/|\boldsymbol{q}|$ and $\hat{q}^\mu_\pm \equiv (1, \pm\hat{\boldsymbol{q}})$. With these definitions, the fermion propagator can be rewritten as

$$S(Q) = \frac{1}{2}\sum_{s,t=\pm}\Delta_t(i\tilde{\omega}_s, \boldsymbol{q})\mathcal{P}_s\gamma_\mu\hat{q}^\mu_t. \tag{25}$$

As can be seen from Eq. (25), the propagator consists of two parts describing the modes with right- ($s = +$) and left-handed chirality ($s = -$). Because we took $m = 0$, the opposite chiralities do not mix. The different values of t correspond to particles (modes with positive energy, $t = +$) and antiparticles (modes with negative energy, $t = -$). In the following section we will use this expression for the propagator to compute the retarded current-current correlator.

IV. COMPUTATION OF RETARDED CORRELATOR

We will compute the retarded correlator $G^i_{\rm R}(p)$, defined in Eq. (12), using the imaginary time formalism of thermal field theory. The retarded correlator can be obtained from the Euclidean correlator by analytic continuation in the following way:

$$G^i_{\rm R}(p_0, \boldsymbol{p}) = G^i_{\rm E}(\omega_n, \boldsymbol{p})|_{i\omega_n \to p_0 + i\epsilon}, \tag{26}$$

where $\epsilon = 0^+$.

At very high temperatures, the gluons and quark masses can be neglected to first approximation and the current-current correlator is a convolution of two bare massless fermion propagators $S(Q)$ (see, e.g., Ref. [38] for the correct expression). Using Eq. (12), we find

$$G^i_{\rm E}(P) = \frac{e^2}{2\beta}\sum_{\tilde{\omega}_m}\int \frac{d^3q}{(2\pi)^3}\epsilon^{ijk}\,{\rm tr}[\gamma^k S(Q)\gamma^j S(P+Q)]. \tag{27}$$

With use of Eq. (25) and the properties of the chirality projection operators, the integrand of Eq. (27) can now be written as

$$\frac{1}{4}\sum_{s,t,u=\pm}\epsilon^{ijk}\,{\rm tr}[\gamma_\nu\gamma^k\gamma_\mu\gamma^j\mathcal{P}_s]\Delta_t(i\tilde{\omega}_s, \boldsymbol{q}) \times \Delta_u(i\tilde{\omega}_s + i\omega_n, \boldsymbol{p}+\boldsymbol{q})\hat{q}^\mu_t(\widehat{p+q})^\nu_u. \tag{28}$$

It can be seen from the last equation that the opposite chiralities ($s = \pm$) do not mix. As long as $m = 0$, the chiral magnetic conductivity is a sum of a contribution from purely right-handed modes and purely left-handed modes.

We can now use that ${\rm tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma}$ where $\epsilon^{\mu\nu\rho\sigma}$ is the complete antisymmetric tensor with $\epsilon^{0123} = 1$. Then it follows that

$$\epsilon^{ijk}\,{\rm tr}[\gamma_\nu\gamma^k\gamma_\mu\gamma^j\gamma^5]a^\mu b^\nu = 8i(a^ib^0 - a^0b^i). \tag{29}$$

As a result, we obtain

$$G^i_{\rm E}(P) = \frac{ie^2}{2\beta}\sum_{\tilde{\omega}_m}\int\frac{d^3q}{(2\pi)^3}\sum_{s,t,u=\pm}s\left[t\frac{q^i}{E_q} - u\frac{p^i+q^i}{E_{p+q}}\right] \times \Delta_t(i\tilde{\omega}_s, \boldsymbol{q})\Delta_u(i\tilde{\omega}_s + i\omega_n, \boldsymbol{p}+\boldsymbol{q}). \tag{30}$$

We can now perform the sum over Matsubara frequencies. Using that $\omega_n = 2n\pi T$ is a bosonic Matsubara frequency one finds

$$\frac{1}{\beta}\sum_{\tilde{\omega}_m}\Delta_t(i\tilde{\omega}_s, \boldsymbol{q})\Delta_u(i\tilde{\omega}_s + i\omega_n, \boldsymbol{p}+\boldsymbol{q}) = \frac{t\tilde{n}(E_q - t\mu_s) - u\tilde{n}(E_{p+q} - u\mu_s) + \frac{1}{2}(u-t)}{i\omega_n + tE_q - uE_{p+q}}, \tag{31}$$

where $\tilde{n}(x) = [\exp(\beta x) + 1]^{-1}$ is the Fermi-Dirac distribution function. We can now perform the analytic continuation in order to obtain the retarded correlator $G^i_{\rm R}(p)$, which amounts to replacing $i\omega_n$ by $p_0 + i\epsilon$ in Eq. (31). We obtain

$$G^i_{\rm R}(p) = \frac{ie^2}{2}\int\frac{d^3q}{(2\pi)^3}\sum_{s,t,u=\pm}s\left[t\frac{q^i}{E_q} - u\frac{p^i+q^i}{E_{p+q}}\right] \times \frac{t\tilde{n}(E_q - t\mu_s) - u\tilde{n}(E_{p+q} - u\mu_s) + \frac{1}{2}(u-t)}{p_0 + i\epsilon + tE_q - uE_{p+q}}. \tag{32}$$

Before we compute the integral over q, let us try to interpret Eq. (32). The retarded correlator $G^i_{\rm R}(p)$ has a real part when $p_0 = uE_{p+q} - tE_q$. From Eq. (15) it can be seen that in that case the chiral magnetic conductivity acquires an imaginary part. When $p_0 = uE_{p+q} - tE_q$, the virtual particles in the loop of the photon polarization tensor go on the mass shell, which by the optical theorem corresponds to production or scattering of real particles in the electromagnetic field.

If $u = -t$ this implies that particle-antiparticle pairs are produced from the electromagnetic field that oscillates with frequency p_0. If $t = u$, particles ($u = t = 1$) or antiparticles ($u = t = -1$) scatter from the electromagnetic field and acquire or lose some momentum and energy. Let us take a closer look at the pair-production process.

If $p_0 > 0$, the produced particles and antiparticles have energies E_{p+q} ($u = 1$) and E_q ($t = -1$), respectively. If the system we consider consists mainly of particles, μ_s is positive. At zero temperature all particle states up to the Fermi energy μ_s are filled, so then it is impossible to produce particles with energy per particle less than μ_s due to Pauli blocking. This is reflected in the Fermi-Dirac distributions in the integrand of Eq. (32), for $u = t = -1$, $\mu_s > 0$, and $T = 0$, the integrand vanishes if $E_{p+q} < \mu_s$. If $E_{p+q} > \mu_s$ there is no Pauli blocking, in that case $E_q > \mu_s - p$. Hence at $T = 0$, $G^i_{\rm R}(p)$ develops a real part and particle-antiparticle pairs with chirality s can be produced if $p_0 > 2\mu_s - p$. If both E_{p+q} and E_q are larger than μ_s the produced pair does not feel the influence of the nonzero chirality, so then the imaginary part of the chiral magnetic conductivity should vanish as well. In that case, $E_q < \mu_s$ so that $E_{p+q} < \mu_s + p$ which gives $p_0 < 2\mu_s + p$. So concluding, we expect the chiral magnetic conductivity to have an imaginary part for $2\mu_s - p < p_0 < 2\mu_s + p$ due to pair production.

Now let us continue the calculation of $G^i_{\rm R}(p)$. The last term in the integrand of Eq. (32) vanishes after summing over chiralities (s). It then follows that the function $G^i_{\rm R}(p)$ is ultraviolet finite because the high-momentum part of the integrand is exponentially suppressed by the Fermi-Dirac distribution functions. We can therefore shift and reflect the integration variable as follows: $\boldsymbol{q} \to -\boldsymbol{q} - \boldsymbol{p}$ in order to make both Fermi-Dirac distribution functions dependent on E_q. Then after interchanging t with u and inversions $t \to -t$ and $u \to -u$, we arrive at

$$G^i_{\rm R}(p) = \frac{ie^2}{2} \int \frac{d^3q}{(2\pi)^3} \sum_{s,t,u=\pm} s\left[t\frac{q^i}{E_q} + u\frac{p^i + q^i}{E_{p+q}} \right] \times \frac{\tilde{n}(E_q - \mu_s) - \tilde{n}(E_q + \mu_s)}{p_0 + i\epsilon + tE_q + uE_{p+q}}. \quad (33)$$

Here we used that since $t = \pm$,

$$t[\tilde{n}(E_q - t\mu_s) - \tilde{n}(E_q + t\mu_s)] = \tilde{n}(E_q - \mu_s) - \tilde{n}(E_q + \mu_s). \quad (34)$$

We can now perform the sum over u, which gives us

$$G^i_{\rm R}(p) = ie^2 \int \frac{d^3q}{(2\pi)^3} \sum_{s,t=\pm} s\left[t\frac{p_0}{E_q} q^i - p^i \right] \times \frac{\tilde{n}(E_q - \mu_s) - \tilde{n}(E_q + \mu_s)}{(p_0 + i\epsilon + tE_q)^2 - E^2_{p+q}}. \quad (35)$$

Now it is possible to perform the angular integral. We use the fact that the integral has to be proportional to p^i. Hence we can replace q^i by $(\boldsymbol{q} \cdot \boldsymbol{p})p^i/p^2$. We find after integrating over angles

$$G^i_{\rm R}(p) = \frac{ie^2}{16\pi^2} \frac{p^i}{p} \frac{p^2 - p_0^2}{p^2} \int_0^\infty dq f(q) \sum_{t=\pm} (2q + tp_0) \times \log\left[\frac{(p_0 + i\epsilon + tq)^2 - (q+p)^2}{(p_0 + i\epsilon + tq)^2 - (q-p)^2} \right], \quad (36)$$

where

$$f(q) \equiv \sum_{s=\pm} s[\tilde{n}(q - \mu_s) - \tilde{n}(q + \mu_s)]. \quad (37)$$

We now have all results in order to obtain the chiral magnetic conductivity which follows from combining Eq. (36) with Eq. (15). We will discuss the result in the next section.

V. COMPUTATION OF CHIRAL MAGNETIC CONDUCTIVITY

We will discuss the leading order contribution to the chiral magnetic conductivity as a function of frequency and momentum. The zero frequency and momentum value is constrained by the axial anomaly, so loop corrections by gluons and/or photons will not alter this result [22]. However, loop corrections will change the conductivity at nonzero frequencies.

In an electromagnetic (EM) plasma, the leading order result is a sensible approximation since loop effects are of order $\alpha_{\rm EM}$ which is small. The results for an electromagnetic plasma are only reliable if the temperature is larger than the mass of the electron, since we have assumed massless particles. In a quark gluon plasma loop corrections are only negligible at very high temperatures due to asymptotic freedom. Hence one should keep in mind that the leading order result is in QCD only a valid approximation at high temperatures.

A. Zero frequency limit

Let us first rederive the zero frequency, zero momentum limit of the chiral magnetic conductivity. Since

$$\lim_{p\to 0} \lim_{p_0 \to 0} \sum_{t=\pm} \log\left[\frac{(p_0 + i\epsilon + tq)^2 - (q+p)^2}{(p_0 + i\epsilon + tq)^2 - (q-p)^2} \right] = \frac{2p}{q}, \quad (38)$$

it immediately follows from Eq. (36) that

$$\lim_{\boldsymbol{p}\to 0} G^i_{\rm R}(p_0 = 0, \boldsymbol{p}) = \frac{ie^2\mu_5}{2\pi^2} p^i. \quad (39)$$

As a result we recover the known zero frequency result of the chiral magnetic conductivity for homogeneous magnetic fields ($\boldsymbol{p} \to 0$),

$$\sigma_0 \equiv \sigma_\chi(\omega = 0) = \frac{e^2}{2\pi^2}\mu_5. \quad (40)$$

The zero frequency limit is independent of μ and T. Since this value is constrained by the axial anomaly, loop corrections by gluons and/or photons will not alter this result [22].

B. Imaginary part

To obtain the imaginary part of the chiral magnetic conductivity we need to compute the imaginary part of the logarithm in Eq. (36). This is a sum of step functions times π. We find using that $q \geq 0$ and $p = |\boldsymbol{p}| \geq 0$,

$$\begin{aligned} &{\rm Im} \sum_{t=\pm} (2q + tp_0) \log\left[\frac{(p_0 + i\epsilon + tq)^2 - (q+p)^2}{(p_0 + i\epsilon + tq)^2 - (q-p)^2} \right] \\ &= \pi[2q - |p_0|\theta(p_0^2 - p^2)][\theta(q_+ - q) - \theta(q_- - q)] \\ &\quad + \pi p_0 \theta(p^2 - p_0^2)[\theta(q - q_+) + \theta(q - q_-)], \end{aligned} \quad (41)$$

where $q_\pm = \frac{1}{2}|p_0 \pm p|$.

To compute the imaginary part of the chiral magnetic conductivity we furthermore need the to perform the following two indefinite integrals:

$$\int dq f(q) = T \sum_{s,t=\pm} st \log[1 + e^{(q+t\mu_s)/T}],$$
$$\int dq q f(q) = \sum_{s,t=\pm} st\{qT \log[1 + e^{(q+t\mu_s)/T}] + T^2 \mathrm{Li}_2[-e^{(q+t\mu_s)/T}]\}, \tag{42}$$

where $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$ denotes a polylogarithm of second order. We now find for the imaginary part of the chiral magnetic conductivity at nonzero frequency and momentum

$$\sigma''_\chi(p) = \frac{e^2}{16\pi}\frac{p^2 - p_0^2}{p^3}\Big\{8\theta(p^2 - p_0^2)p_0\mu_5 + T\sum_{s,t,r=\pm} st\{2rT\mathrm{Li}_2[-e^{(q_r+t\mu_s)/T}] + p[\mathrm{sgn}(p_0)\theta(p_0^2 - p^2) + r\theta(p^2 - p_0^2)] \times \log[1 + e^{(q_r+t\mu_s)/T}]\}\Big\}. \tag{43}$$

In the limit of homogeneous magnetic fields ($\boldsymbol{p} \to 0$), the imaginary part of the chiral magnetic conductivity becomes

$$\sigma''_\chi(\omega) = \frac{e^2}{3\pi}\omega\delta(\omega)\mu_5 + \frac{e^2\omega|\omega|}{96\pi}\sum_{s,t=\pm} st\left[\frac{d}{dq}\tilde{n}(q + t\mu_s)\right]_{q=|\omega|/2}. \tag{44}$$

The derivative with respect to the Fermi-Dirac distribution shows that for small temperatures only states near the Fermi-surface contribute to the chiral magnetic conductivity.

At zero temperature, the imaginary part of the leading order chiral magnetic conductivity becomes

$$\sigma''_\chi(\omega) = \frac{e^2}{3\pi}\omega\delta(\omega)\mu_5 - \frac{e^2\omega^2}{96\pi}\sum_{s,t=\pm} st\delta(\omega/2 + t\mu_s). \tag{45}$$

The last equation shows that in the limit of $p \to 0$ the leading order contribution to the chiral magnetic conductivity develops resonances at $\omega = \pm 2\mu_s$. As argued in the previous section, these resonances can be attributed to particle-antiparticle pair production in the time-dependent electromagnetic field.

For large temperatures ($T > \mu_5$), we can approximate Eq. (44) by

$$\sigma''_\chi(\omega) = \frac{e^2}{3\pi}\omega\delta(\omega)\mu_5 + \frac{e^2\omega|\omega|}{24\pi T^2}\tilde{n}(|\omega|/2)^3 \times [e^{|\omega|/T} - e^{|\omega|/(2T)}]\mu_5. \tag{46}$$

The imaginary part at large temperatures has a maximum at $\omega/T \approx 5.406$ with the value $\sigma'' \approx 0.394\sigma_0$.

C. Real part

At zero temperature we can now recover the real part of the chiral magnetic conductivity by applying the Kramers-Kronig relation, Eq. (20). We find

$$\sigma'_\chi(\omega = 0) = \frac{e^2}{2\pi^2}\mu_5, \tag{47}$$

$$\sigma'_\chi(\omega \neq 0) = \frac{e^2}{3\pi^2}\sum_{s=\pm}\frac{s\mu_s}{4 - (\omega/\mu_s)^2}. \tag{48}$$

For finite temperatures we were unfortunately not able to obtain an analytic expression for the real part.

The delta function of the imaginary part at $\omega = 0$ only contributes to $2/3$ of the real part at $\omega = 0$. The other $1/3$ part comes from the pair-production processes. This conclusion holds for any temperature. This is because both the real [see, Eq. (40)] and imaginary part [Eq. (44)] at $\omega = 0$ are independent of temperature. From the Kramers-Kronig relation it follows that for any nonzero frequency the imaginary part at $\omega = 0$ does not contribute to the real part at $\omega \neq 0$. Therefore for any temperature the real part of the chiral magnetic conductivity drops from σ_0 at $\omega = 0$ to $\sigma_0/3$ just away from $\omega = 0$.

D. Discussion

We display the real and imaginary part for $T = 0$, $p = 0.1\mu_5$, and $\mu = 0$ in Fig. 1. As was argued at the end of the previous subsection, it can be seen in this figure that the real part of the chiral magnetic conductivity drops from σ_0

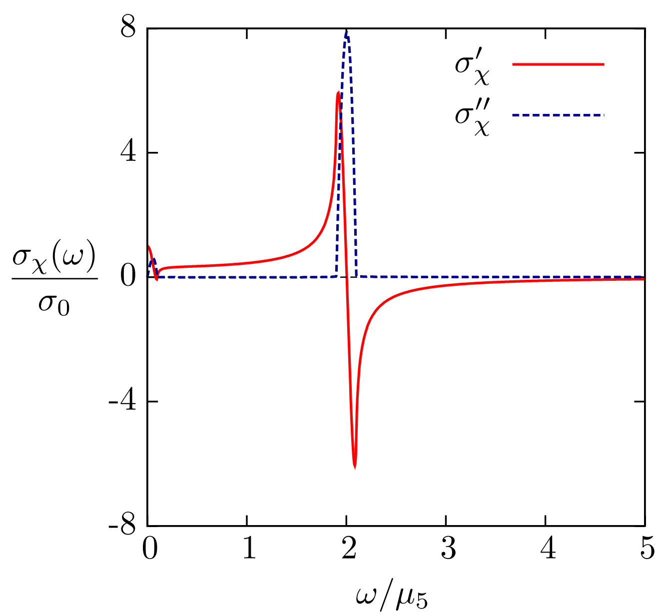

FIG. 1 (color online). Real (red, solid) and imaginary (blue, dashed) part of the leading order chiral magnetic conductivity as a function of frequency, at $T = 0$ and $\mu = 0$ for $p = 0.1\mu_5$. The result is normalized to the zero frequency conductivity $\sigma_0 = e^2\mu_5/(2\pi^2)$.

at $\omega = 0$ to $\sigma_0/3$ just away from $\omega = 0$. Also the resonance at $\omega = 2\mu_5$ is clearly visible. The width of the imaginary part at the resonance is equal to $2p$. The real part of the conductivity becomes negative above the resonance frequency. This is a typical resonance behavior and implies that when the imaginary part vanishes the response is 180 degrees out of phase with the applied magnetic field.

In Fig. 2 we display the real and imaginary part for $T = 0$, $p = 0.1\mu_5$, and $\mu = 1.5\mu_5$. In this case there are resonances at $\omega = 5\mu_5$ and $\omega = \mu_5$. Equation (45) shows that the imaginary part is proportional to ω^2, therefore the second resonance at $\omega = 5\mu_5$ is much stronger than the first one at $\omega = \mu_5$. Because the second resonance is due to the right-handed modes, and the first one due to left-handed modes, the contribution of the second resonance has opposite sign to the first resonance.

The real and imaginary part of the chiral magnetic conductivity at high temperatures ($T > \mu_5$) are displayed in Fig. 3. This figure is the most relevant for QCD at very high temperatures, since then loop corrections will be small. As argued in the previous subsection, it can be seen in the figure that the real part of the conductivity drops from σ_0 at $\omega = 0$ to $\sigma_0/3$ just away from $\omega = 0$.

Let us now study the induced current in a magnetic field of the form created during heavy ion collisions. For simplicity we approximate the two colliding nuclei by pointlike particles like in Ref. [19]. This gives a reasonable approximation to the more accurate methods discussed in Refs. [17,18] and is most reliable for large impact parameters. The magnetic field at the center of the collision can

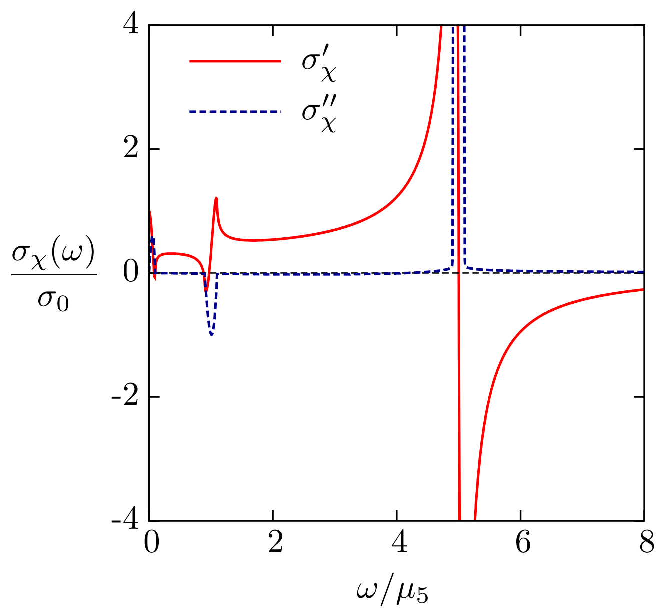

FIG. 2 (color online). Real (red, solid) and imaginary (blue, dashed) part of the leading order normalized chiral magnetic conductivity as a function of frequency, at $T = 0$, $\mu = 1.5\mu_5$, and $p = 0.1\mu_5$. The result is normalized to the zero frequency conductivity $\sigma_0 = e^2\mu_5/(2\pi^2)$.

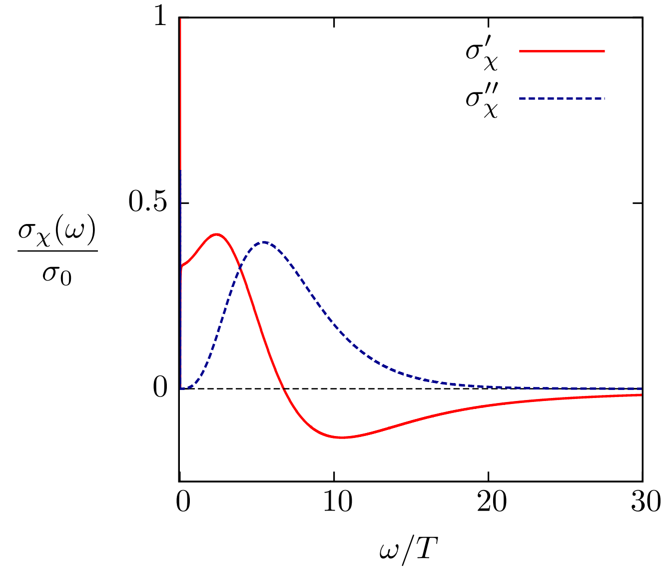

FIG. 3 (color online). Real (red, solid) and imaginary (blue, dashed) part of the leading order normalized chiral magnetic conductivity at high temperatures ($T > \mu_5$) for homogeneous magnetic fields ($p = 0$). At $\omega = 0$, the normalized conductivity is equal to 1.

then be written as

$$B(t) = \frac{1}{[1 + (t/\tau)^2]^{3/2}} B_0, \tag{49}$$

with $\tau = b/(2\sinh Y)$ and $eB_0 = 8Z\alpha_{\rm EM}\sinh Y/b^2$. Here b denotes the impact parameter, Z the charge of the nucleus, and Y the beam rapidity. For Gold-Gold ($Z = 79$) collisions at 100 GeV per nucleon, one has $Y = 5.36$. At typical large impact parameters (say $b = 10$ fm) one finds $eB_0 \sim 1.9 \times 10^5$ MeV^2 and $\tau = 0.05$ fm$/c$. For 31 GeV per nucleon ($Y = 4.19$) Gold-Gold collisions, one finds at $b = 10$ fm, $eB_0 \sim 5.9 \times 10^4$ MeV^2 and $\tau = 0.15$ fm$/c$. The Fourier transform of Eq. (49) equals

$$\tilde{B}(\omega) = 2\tau^2|\omega|K_1(\tau|\omega|)B_0, \tag{50}$$

where $K_1(z)$ denotes the first-order modified Bessel function of the second kind.

For illustration purposes, we will assume that our magnetic field is (unlike in heavy ion collisions) homogeneous. The induced current can be found by applying Eq. (19). We display the induced current in the magnetic field of Eq. (49) in a system with nonzero chirality at very high temperatures in Fig. 4. The induced current is plotted as a function of time for three different characteristic time scales τ of the magnetic field.

In any general decaying magnetic field, the only relevant frequencies are the ones which are smaller than the inverse lifetime of the magnetic field, $\omega \lesssim \mathcal{O}(1/\tau)$. In Fig. 3 it can be seen that even in the noninteracting case there still is sizable response at high temperatures as long as $\tau \gtrsim 1/(5T)$. Hence even for such fast changing fields there

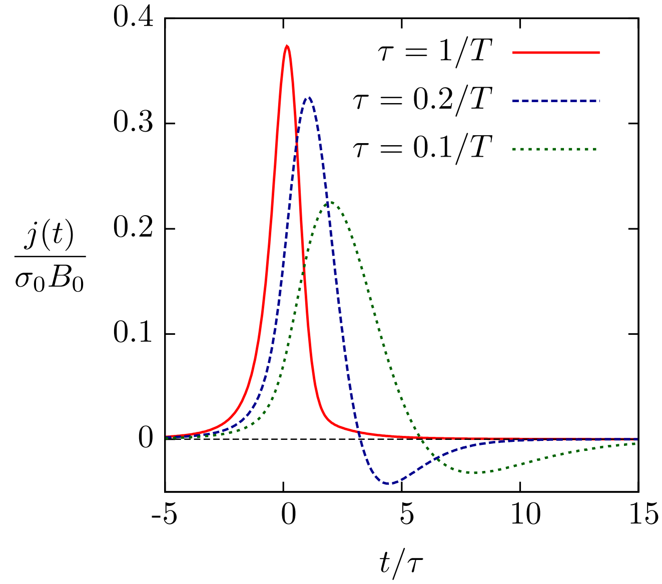

FIG. 4 (color online). Induced current in time-dependent magnetic field, Eq. (49), as a function of time, at very high temperature. The results are plotted for different values of the characteristic time scale τ of the magnetic field.

will really be an induced current, although it will be about $1/3$ of the adiabatic approximation which is $j(t) = \sigma_0 B(t)$. This can be more clearly seen in Fig. 4. The red solid line describes a relatively slowly varying field with $\tau = 1/T$, in that case indeed $j(t) \sim \sigma_0 B(t)/3$. The red solid line can therefore also be seen as a guide to eye of the time dependence of the magnetic field. Realizing this, in a fast changing magnetic field with $\tau = 0.1/T$ it clearly takes some time for the current to respond. The maximal current will be smaller, but it takes also more time for the current to diminish after the magnetic field has gone away. At late times, the current can even become negative. The situation in which $\tau = 0.1/T$ approximately corresponds to 100 GeV per nucleon Gold-Gold collisions at $b = 10$ fm.

Thermal fluctuations will increase in magnitude when the temperature is increased. These fluctuations can cause the spins of the particles to align along the magnetic field. Hence one would expect that when keeping τ fixed and increasing the temperature, the system will respond faster to the changing magnetic field. This can be seen in Fig. 4, by increasing the temperature at fixed τ one goes from the dotted to the solid line, which indicates faster response.

The discussion will alter when loop corrections due to interactions are taken into account. In that case the quasiparticles obtain a thermal width Γ which is of order αT. Because of the thermal broadening, we expect that the peak in the chiral magnetic conductivity (Fig. 3) at $\omega = 0$ will get a width of order Γ. At the same time the value at $\omega = 0$ will not change since it is constrained by the anomaly. As long as $\tau \gtrsim \mathcal{O}(1/\Gamma)$, the zero frequency result will therefore be a reasonable estimate and the induced current will be more or less equal to the adiabatic approximation $j(t) \sim \sigma_0 B(t)$. Hence the stronger the system interacts at fixed μ_5 and T, the stronger the response at small frequencies. Qualitatively one would expect such behavior, because the stronger a particle interacts the faster the particle aligns its spin and hence momentum along the field.

Let us finally discuss how much charge is separated by the current. Let us divide our infinite system into an upper half (u) and a lower half (l), and set the boundary to be the plane $z = 0$. Using the fact that the current is conserved $\partial_\mu j^\mu = 0$, one finds that the change in charge in the upper hemisphere per unit of time equals

$$\frac{dQ_u}{dt} = -\int_u d^3x \boldsymbol{\nabla} \cdot \boldsymbol{j} = \int d^2x j_z(z = 0, t). \tag{51}$$

By integrating this equation over time and using Eq. (19), we find that the change in charge in the upper hemisphere ΔQ_u equals

$$\Delta Q_u = A \int_{-\infty}^{\infty} dt j(t) = \sigma_0 A \tilde{B}(\omega = 0), \tag{52}$$

where A is the area of the plane $z = 0$. Because of global charge conservation, the charge change in the lower hemisphere is equal to $\Delta Q_l = -\Delta Q_u$. In Eq. (52) we find the surprising result that as long as linear response is valid and the chirality is constant, the total induced charge difference between upper and lower hemisphere is independent of detailed dynamics and is just determined by the zero frequency chiral magnetic conductivity σ_0. In the magnetic field of Eq. (49) we obtain for the induced charge difference between the upper and lower hemisphere

$$\Delta Q \equiv \Delta Q_u - \Delta Q_l = 4\sigma_0 A \tau B_0. \tag{53}$$

VI. CONCLUSIONS

In a quark gluon plasma it is possible to generate nonzero chirality by gluon configurations with nonzero topological charge. In the presence of a magnetic field, nonzero chirality leads to a current along the field. This is the chiral magnetic effect which can potentially give rise to observable effects in heavy ion collisions. Since the magnetic field in heavy ion collisions is rapidly decreasing as a function of time, it is desirable to study the chiral magnetic effect in a time-dependent magnetic field. In this article, we have shown for the first time that such study is possible in a systematic way using linear response theory.

To obtain the induced current in a time-dependent magnetic field we have derived a general Kubo formula for the chiral magnetic conductivity. We have shown that the chiral magnetic conductivity is proportional to the antisymmetric part of the off diagonal photon polarization tensor.

Since we have applied linear response theory, our results are only valid for small magnetic fields. This means that the magnetic fields should not alter the plasma dynamics much, implying that the magnetic length $1/\sqrt{eB}$ has to be larger than the (color) electric screening length $\sim 1/gT$. Hence for reliable results, the magnetic field should satisfy $eB \lesssim g^2T^2$.

We have computed the leading order chiral magnetic conductivity for constant chirality using perturbation theory. As such, our result is only applicable for QCD at high temperatures where loop corrections can be neglected. We have shown that pair production in the time-dependent electromagnetic field gives rise to nontrivial behavior of the chiral magnetic conductivity at nonzero frequencies. Our result can be systematically improved by including loop corrections, in a similar way to what has been done for the electrical conductivity in, for example, Ref. [39].

The general formula for the chiral magnetic conductivity we have obtained allows for the evaluation using other methods. For example, as suggested in Ref. [40], one could study the chiral magnetic effect in holographic models of QCD. Also since μ_5 does not give rise to a sign problem [22], one could in principle study the chiral magnetic conductivity using lattice QCD (see, also, Ref. [23] for another approach). However, because one has to perform an analytic continuation in order to obtain the retarded correlator, this is not completely straightforward to do.

The main message of this calculation is the finding that even in the leading order the chiral magnetic conductivity has sizable response at nonzero frequencies. This is a clear proof that even a rapidly decaying magnetic can give rise to a non-negligible current. By taking into account interactions, the quasiparticles will obtain a thermal width, which as we argued will increase the response at small frequencies but does not change the zero frequency result.

This calculation of the chiral magnetic conductivity is a small step in order to improve our understanding of the dynamics of the chiral magnetic effect. To apply our results to heavy ion phenomenology, one has to take into account also other dynamical effects like the time dependence of the chirality, the radial flow, and possible screening mechanisms. Also it would be interesting to study the effects of a time-dependent magnetic field on other physical quantities and effects, like, for example, the chiral condensate, the chiral phase transition, and dynamical chiral symmetry breaking. So far these have only been investigated in a constant magnetic field [41–47].

ACKNOWLEDGMENTS

We are grateful to Kenji Fukushima, Larry McLerran, Dirk Rischke, and Andreas Schmitt for discussions. This manuscript has been authored under Contract No. #DE-AC02-98CH10886 with the U.S. Department of Energy. The work of H. J. W. was supported partly by the Alexander von Humboldt Foundation and partly by the ExtreMe Matter Institute EMMI in the framework of the Helmholtz Alliance Program of the Helmholtz Association (HA216/EMMI).

[1] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
[2] C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. **63B**, 334 (1976).
[3] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).
[4] J. S. Schwinger, Phys. Rev. **82**, 664 (1951); S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[5] D. Kharzeev, A. Krasnitz, and R. Venugopalan, Phys. Lett. B **545**, 298 (2002).
[6] T. Lappi and L. McLerran, Nucl. Phys. **A772**, 200 (2006).
[7] D. Kharzeev, E. Levin, and K. Tuchin, Phys. Rev. C **75**, 044903 (2007).
[8] L. D. McLerran, E. Mottola, and M. E. Shaposhnikov, Phys. Rev. D **43**, 2027 (1991).
[9] G. D. Moore, Phys. Lett. B **412**, 359 (1997); G. D. Moore and K. Rummukainen, Phys. Rev. D **61**, 105008 (2000); D. Bödeker, G. D. Moore, and K. Rummukainen, Phys. Rev. D **61**, 056003 (2000); G. D. Moore, arXiv:hep-ph/0009161.
[10] D. T. Son and A. O. Starinets, J. High Energy Phys. 09 (2002) 042.
[11] E. Shuryak and I. Zahed, Phys. Rev. D **67**, 014006 (2003).
[12] P. Arnold and G. D. Moore, Phys. Rev. D **73**, 025006 (2006).
[13] D. E. Kharzeev, Y. V. Kovchegov, and E. Levin, Nucl. Phys. **A690**, 621 (2001); **A699**, 745 (2002).
[14] E. V. Shuryak and I. Zahed, Phys. Rev. D **62**, 085014 (2000).
[15] M. A. Nowak, E. V. Shuryak, and I. Zahed, Phys. Rev. D **64**, 034008 (2001).
[16] D. Kharzeev, R. D. Pisarski, and M. H. G. Tytgat, Phys. Rev. Lett. **81**, 512 (1998).
[17] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, Nucl. Phys. **A803**, 227 (2008).
[18] V. Skokov, A. Illarionov, and V. Toneev, arXiv:0907.1396.
[19] H. Minakata and B. Muller, Phys. Lett. B **377**, 135 (1996).
[20] D. Kharzeev, Phys. Lett. B **633**, 260 (2006).
[21] D. Kharzeev and A. Zhitnitsky, Nucl. Phys. **A797**, 67 (2007).
[22] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[23] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya, and M. I. Polikarpov, arXiv:0907.0494.
[24] S. A. Voloshin, Phys. Rev. C **70**, 057901 (2004).
[25] I. V. Selyuzhenkov (STAR Collaboration), Romanian Reports in Physics **58**, 049 (2006).

[26] S. A. Voloshin (STAR Collaboration), arXiv:0806.0029; arXiv:0907.2213.
[27] J. Charbonneau and A. Zhitnitsky, arXiv:0903.4450.
[28] A. A. Andrianov, D. Espriu, P. Giacconi, and R. Soldati, arXiv:0907.3709.
[29] H. B. Nielsen and M. Ninomiya, Phys. Lett. **130B**, 389 (1983).
[30] E. D'Hoker and J. Goldstone, Phys. Lett. **158B**, 429 (1985).
[31] M. A. Metlitski and A. R. Zhitnitsky, Phys. Rev. D **72**, 045011 (2005).
[32] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[33] C. G. Callan and J. A. Harvey, Nucl. Phys. **B250**, 427 (1985).
[34] E. V. Gorbar, V. A. Miransky, and I. A. Shovkovy, arXiv:0904.2164.
[35] D. T. Son and P. Surowka, arXiv:0906.5044.
[36] Y. Aharonov and A. Casher, Phys. Rev. A **19**, 2461 (1979).
[37] G. Lifschytz and M. Lippert, arXiv:0904.4772.
[38] J. I. Kapusta and C. Gale, *Finite-temperature Field Theory: Principles and Applications* (Cambridge University Press, Cambridge, England, 2006).
[39] G. Aarts and J. M. Martinez Resco, J. High Energy Phys. 11 (2002) 022.
[40] A. Rebhan, A. Schmitt, and S. A. Stricker, J. High Energy Phys. 05 (2009) 084.
[41] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. D **52**, 4747 (1995).
[42] I. A. Shushpanov and A. V. Smilga, Phys. Lett. B **402**, 351 (1997).
[43] T. D. Cohen, D. A. McGady and E. S. Werbos, Phys. Rev. C **76**, 055201 (2007).
[44] N. O. Agasian and S. M. Fedorov, Phys. Lett. B **663**, 445 (2008).
[45] N. Sadooghi and K. S. Anaraki, Phys. Rev. D **78**, 125019 (2008).
[46] E. S. Fraga and A. J. Mizher, Phys. Rev. D **78**, 025016 (2008); A. J. Mizher and E. S. Fraga, arXiv:0810.5162.
[47] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya, and M. I. Polikarpov, arXiv:0906.0488.