PHYSICAL REVIEW D **80,** 034007 (2009)

$U_A(1)$ breaking and phase transition in chiral random matrix model

T. Sano,[1,2] H. Fujii,[1] and M. Ohtani[3]

[1]*Institute of Physics, The University of Tokyo, Tokyo 153-8902, Japan*
[2]*Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan*
[3]*Physics Department, School of Medicine, Kyorin University, Tokyo 181-8611, Japan*
(Received 15 April 2009; published 6 August 2009)

We propose a chiral random matrix model which properly incorporates the flavor-number dependence of the phase transition owing to the $U_A(1)$ anomaly term. At finite temperature, the model shows the second-order phase transition with mean-field critical exponents for two massless flavors, while in the case of three massless flavors the transition turns out to be of the first order. The topological susceptibility satisfies the anomalous $U_A(1)$ Ward identity and decreases gradually with the temperature increased.

DOI: 10.1103/PhysRevD.80.034007 PACS numbers: 12.38.Lg, 11.30.Qc, 11.30.Rd, 25.75.Nq

I. INTRODUCTION

Chiral symmetry breaking manifests itself in the accumulation of the Dirac modes with zero eigenvalues through the Banks-Casher relation [1]. In the chiral random matrix (ChRM) theory, the Dirac operator is restricted to the space of the constant modes and replaced with a matrix of random entities, retaining the global symmetries of QCD. The ChRM theory has been successful for providing a universal framework to investigate the correlation properties of the low-lying Dirac eigenvalues in the so-called epsilon regime in QCD [2,3]. The ChRM theory can be also regarded as the simplest schematic model for qualitative study of the QCD-like phase diagram from the viewpoint of chiral symmetry [4]. Effects of the external fields such as the temperature T [5,6], the quark chemical potential μ [4,7–10], and the Polyakov loop [11] have been investigated within the ChRM models.

The conventional ChRM model predicts a second-order phase transition at finite temperature T [5,6], irrespective of the number of flavors N_f. This is a shortcoming as a model of QCD. Based on the universality argument [12], the chiral transition for $N_f = 2$ with the $U_A(1)$ anomaly is expected to be of the second order with the O(4) critical exponents, while the transition becomes of the first order for $N_f \geq 3$. Even in a mean-field description, the Ginzburg-Landau effective potential for $N_f = 3$ involves the $U_A(1)$ breaking determinant term [13,14], which gives rise to a first-order transition. However, it is unknown so far how to incorporate the $U_A(1)$-breaking determinant term into the ChRM models.

The explicit $U_A(1)$ breaking, or the anomaly, is included in the ChRM models by treating ν exact zero modes with definite chirality, which should be interpreted as the zero modes associated with the gauge field configurations of the topological charge ν. The fluctuation of ν is effectively modeled with the instanton ensemble. The ChRM model supplemented with a Gaussian distribution in ν has been shown to provide a screening of the topological charge fluctuation at zero temperature, and thus a resolution of the $U_A(1)$ problem [15]. At finite temperatures, however, this ChRM model results in an unphysical suppression of the topological susceptibility [16,17]. The resolution of the $U_A(1)$ problem in the ChRM model is then limited only at zero temperature. A modification of the model has been proposed in [16,17] to remove the unphysical suppression of the topological susceptibility at finite temperature.

The ground state is fixed exactly with the saddle point condition in the thermodynamic limit. Because the topological charge ν fluctuates around zero with the variance of $\mathcal{O}(V)$, as dictated by the central limit theorem, the nonzero ν configurations are irrelevant to the ground state condition in the ChRM model, and therefore the $U_A(1)$ breaking seemingly cannot alter the order of the phase transition at finite temperature.

In this paper we propose a ChRM model involving the $U_A(1)$ breaking effect, which describes a first-order phase transition for $N_f = 3$ and at the same time reproduces the physical behavior for the topological susceptibility. The proposed model consists of the ‘‘near-zero’’ modes and the ‘‘topological zero’’ modes following Refs. [18,19]. The latter are interpreted as the zero modes with the right and left chiralities, respectively, associated with instantons and anti-instantons. We introduce number distributions of the topological zero modes based on the instanton gas picture. Summing over the number of the topological zero modes in the partition function, we can derive the $U_A(1)$ breaking interaction. Furthermore, this model satisfies the anomalous $U_A(1)$ Ward identity [15,20] and predicts physical temperature dependence for the topological susceptibility [16,17].

Interplay between the topological zero modes and the near-zero modes is studied in Ref. [21] to derive the instanton-induced interactions in the coarse-grained instanton-liquid model. The topological zero modes are also introduced in Ref. [18] from the determinant interaction in the Nambu–Jona-Lasinio (NJL) model in 0 + 1 dimensions, but the potential there is unbound in the $N_f = 3$ case.

This paper is organized as follows. After reviewing the conventional ChRM model in the next section, we introduce an extended ChRM model which involves two kinds of zero modes, the near-zero modes and the topological zero modes, motivated by the instanton gas picture in subsection II A. Unlike in the conventional ChRM model, the total number of the topological zero modes is assumed to vary according to the instanton distribution. In subsection II B we propose the binomial distribution for the instanton numbers in a finite space-time volume, and show that after summing up the topological zero modes the resulting effective potential has the $U_A(1)$ breaking term and is bounded from below. General properties of the ground state and fluctuations of the model are presented for a certain set of parameters and equal quark masses in Sec. III. Section IV is devoted to discussions and summary.

II. CHIRAL RANDOM MATRIX MODEL

The QCD partition function with N_f quark flavors of mass m_f is written as

$$Z_\theta^{\rm QCD} = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu^{\rm QCD} = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} \left\langle \prod_{f=1}^{N_f} \det(D+m_f) \right\rangle_\nu, \tag{1}$$

where $\langle\cdots\rangle_\nu$ denotes the average over the gauge field configurations of fixed topological charge ν, and the θ parameter has been introduced.

The ChRM model is defined for fixed ν conventionally as [5]

$$Z_\nu = \int dW e^{-N\Sigma^2 {\rm tr} W^\dagger W} \prod_{f=1}^{N_f} \det(D+m_f), \tag{2}$$

where the Dirac operator has been replaced with an anti-Hermite matrix of constant modes

$$D = \begin{pmatrix} 0 & iW + it\mathbf{1}_{N-|\nu|/2} \\ iW^\dagger + it\mathbf{1}_{N-|\nu|/2} & 0 \end{pmatrix} \tag{3}$$

with an $(N+\nu/2)\times(N-\nu/2)$ random complex matrix W in the chiral representation $\gamma_5 = {\rm diag}(1_{N+\nu/2}, -1_{N-\nu/2})$. Here we have introduced the effective temperature t as a deterministic part in the Dirac operator. It may be interpreted as the lowest Matsubara frequency $t=\pi T$. It is readily shown that the matrix D has $|\nu|$ exact zero eigenvalues with definite chirality, which are interpreted as the exact zero modes associated with the topological number ν. The total number of modes $2N$ is finite and proportional to the space-time volume V; $N/V = \mathcal{O}(1)$.

After bosonization of this model we obtain the effective potential $\Omega_\nu(S)$ on the chiral manifold

$$\begin{aligned} Z_\nu &= \int dS e^{-N\Sigma^2 {\rm tr}(SS^\dagger)} \det((S+m_f) \\ &\quad \times (S^\dagger + m_f) + t^2)^{N-(|\nu|/2)} \\ &\quad \times \begin{cases} \det(S+m_f)^\nu & (\nu \geq 0) \\ \det(S^\dagger + m_f)^{-\nu} & (\nu < 0) \end{cases} \\ &\equiv \int dS e^{-2N\Omega_\nu(S)}, \end{aligned} \tag{4}$$

where S is an $N_f \times N_f$ complex matrix of the chiral order parameter (see next subsection). The complete partition function is obtained after summing over ν

$$Z_\theta = \sum_{\nu=-2N}^{2N} e^{-(\nu^2/2(2N)\tau)} e^{i\nu\theta} Z_\nu. \tag{5}$$

Here we need to supplement the distribution of ν characterized by the quenched topological susceptibility τ, which is determined by the pure gluonic dynamics. Note that the range of the topological charge ν is limited within $\pm 2N$.

In the thermodynamic limit $N \to \infty$, the ground state is exactly determined by the saddle point equation. Since the fluctuation of ν scales as $\nu^2 \sim N$ and nonzero ν contribution to the ground state becomes negligible, the ground state is given by the saddle point condition for $\Omega_\nu(S)$ with $\nu = 0$, which is symmetric under $U(N_f)\times U(N_f)$ for $m_f = 0$. Furthermore, the N_f dependence is factored out in the potential $\Omega_0(S)$ for $S \propto \mathbf{1}_{N_f}$ and $m_f = 0$,[1] yielding a second-order phase transition irrespective of N_f. Concerning the fluctuation properties, the model (5) at zero temperature gives a nonzero singlet pseudoscalar susceptibility $\chi_{\rm ps0}$ in the chiral limit, resolving the $U_A(1)$ problem [15]. At finite temperature, however, this leads to an unphysical suppression of the topological susceptibility as mentioned in the introduction.

A. Model with near- and topological zero modes

Let us consider a variation of the conventional ChRM model with the instanton gas picture in mind. An isolated instanton is a localized object accompanying a right-handed fermion zero mode. In a dilute system of N_+ instantons and N_- anti-instantons, we expect N_+ right-handed and N_- left-handed zero modes (even at finite temperature). In an effective theory at long distances, effects of the instantons should be integrated out, which will result in $U_A(1)$-breaking effective interactions [14]. The fundamental assumption in our modeling is the classification of the constant modes into the near-zero modes and topological zero modes [15]. We deal with the $2N$ near-zero modes appearing in the conventional models and include additionally the $N_+ + N_-$ topological zero modes which we regard as the modes accompanied by the instan-

[1] $S \propto \mathbf{1}_{N_f}$ can be derived automatically from the saddle point condition in the case of equal quark masses.

tons. Distributions of N_+ and N_- lead to fluctuations of the topological charge $\nu = N_+ - N_-$ as well as the total number $N_+ + N_-$. Eventually we shall sum over N_+ and N_- with the mean value of $\mathcal{O}(N)$ and take the thermodynamic limit $N \to \infty$.

For definite numbers of zero modes, we write a Gaussian ChRM model as [18]

$$Z^N_{N_+,N_-} = \int dAdBdXdY e^{-N\Sigma^2 \mathrm{tr}(AA^\dagger + BB^\dagger + XX^\dagger + YY^\dagger)} \times \prod_{f=1}^{N_f} \det(D + m_f) \tag{6}$$

with

$$D = \begin{pmatrix} 0 & iA + it\mathbf{1}_N & 0 & iX \\ iA^\dagger + it\mathbf{1}_N & 0 & iY & 0 \\ 0 & iY^\dagger & 0 & iB \\ iX^\dagger & 0 & iB^\dagger & 0 \end{pmatrix}. \tag{7}$$

Here we use the chiral representation in which $\gamma_5 = \mathrm{diag}(1_N, -1_N, 1_{N_+}, -1_{N_-})$. Then the matrix D, which satisfies $\{D, \gamma_5\} = 0$, has a block structure with complex matrices A, B, X, and Y as shown above. The $N \times N$ complex matrix A acts on the near-zero modes, while the rectangular complex matrix B of size $N_+ \times N_-$ acts on the topological zero modes. The matrices $X \in \mathbb{C}^{N\times N_-}$ and $Y \in \mathbb{C}^{N\times N_+}$ represent the interactions between the near-zero modes and the topological zero modes. In the following we take the quark mass m_f of flavor f to be diagonal $\propto \mathbf{1}_{2N+N_++N_-}$ in the space of the zero modes. Notice that the temperature term t is introduced only for the near-zero modes, while the topological zero modes are assumed to be insensitive to the temperature t. This discrimination is physically legitimate in the instanton gas picture where each topological zero mode is localized around an (anti-)instanton and its eigenvalue is not much affected by the antiperiodic boundary condition in the temporal direction unless the temporal size $1/T$ becomes the same order of the typical temporal extent of the instanton or the topological zero mode.[2]

The effective potential on the chiral manifold can be derived from Eq. (6) in a standard manner. First, we recast the partition function in the form of the integration over Grassmann variables of the near-zero modes ψ and of the topological zero modes χ

$$\det(D + m_f) = \int d\psi^\dagger d\psi d\chi^\dagger d\chi \exp\left[-(\psi_L^\dagger, \psi_R^\dagger, \chi_L^\dagger, \chi_R^\dagger)(D + m_f) \begin{pmatrix} \psi_R \\ \psi_L \\ \chi_R \\ \chi_L \end{pmatrix} \right], \tag{8}$$

where, for each flavor, $\psi^\dagger_{R,L}$ and $\psi_{R,L}$ have N components while $\chi^\dagger_{L(R)}$ and $\chi_{R(L)}$ have $N_{+(-)}$ components. Then integration over the random matrices A, B, X, and Y with the Gaussian distributions gives rise to the four-fermion interaction term. We unfold this term with the aid of the Hubbard-Stratonovich transformation as

$$\begin{aligned} &\exp\left[\frac{1}{N\Sigma^2}(\psi_L^{\dagger f}\psi_R^g + \chi_L^{\dagger f}\chi_R^g)(\psi_R^{\dagger g}\psi_L^f + \chi_R^{\dagger g}\chi_L^f)\right] \\ &= \int dS \exp[-(\psi_L^{\dagger f}\psi_R^g + \chi_L^{\dagger f}\chi_R^g)S_{fg} \\ &\quad - (\psi_R^{\dagger f}\psi_L^g + \chi_R^{\dagger f}\chi_L^g)S^\dagger_{fg}]\exp[-N\Sigma^2 \mathrm{tr}S^\dagger S] \end{aligned} \tag{9}$$

with an $N_f \times N_f$ complex matrix S. Here summation over the flavor indices f and $g = 1, \ldots, N_f$ should be understood, and the zero-mode sum is also implicit in (e.g.) $\psi_L^{\dagger f}\psi_R^g \equiv \sum_i \psi_{Li}^{\dagger f}\psi_{Ri}^g$. This transformation allows us to perform the Grassmann integration. We thus obtain the desired form for the partition function:

$$Z^N_{N_+,N_-} = \int dS e^{-N\Sigma^2 \mathrm{tr}S^\dagger S} \det{}^N[(S + \mathcal{M})(S^\dagger + \mathcal{M}^\dagger) + t^2] \times \det{}^{N_+}(S + \mathcal{M})\det{}^{N_-}(S^\dagger + \mathcal{M}^\dagger). \tag{10}$$

The mass matrix $\mathcal{M} = \mathrm{diag}(m_1, \ldots, m_{N_f})$ will be taken as a general $N_f \times N_f$ matrix of the source field later in Sec. III.

In the chiral limit $\mathcal{M} = 0$, the integrand of (10) is invariant under $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ transformation, $S \to USV^{-1}$ with $U, V \in \mathrm{SU}(N_f)$, in addition to $\mathrm{U_V}(1)$. On the other hand, under $\mathrm{U_A}(1)$ transformation, $S \to e^{2i\theta}S$, it acquires a phase factor $e^{2iN_f\theta\nu}$ due to the difference $\nu \equiv N_+ - N_-$ in the powers of the determinants. In other words, the nonzero ν breaks the $\mathrm{U_A}(1)$ symmetry explicitly down to $\mathbb{Z}_{2N_f}$.

B. Distributions of topological zero modes

The complete partition function is obtained after summing $Z^N_{N_+,N_-}$ over the "instanton" numbers N_+ and N_- with an appropriate weight reflecting the pure gluon dynamics.[3] Although the distributions of N_+ and N_- will be correlated in general, here we assume independent distri-

[2] If T is high enough to modify the nature of the topological zero modes, one should deal with the modification of the instanton configurations and distributions, too, which is beyond the scope of this study.

[3] We keep here the number of near-zero modes $2N$ fixed. In Ref. [19], the total number of the zero modes $2N + N_+ + N_-$ was fixed.

butions $P(N_\pm)$ for N_+ and N_-, i.e.,

$$Z_\theta = \sum_{N_+,N_-} e^{i\nu\theta} P(N_+)P(N_-)Z^N_{N_+,N_-} = \int dS e^{-2N\Omega(S;t,m,\theta)}. \tag{11}$$

First we consider $P(N_\pm)$ in a dilute instanton gas picture. For a one-instanton configuration, one may assign a weight κ compared with a no-instanton configuration, and multiply a factor $N \propto V$ taking into account the integration over the instanton location. For a configuration with N_+ instantons, we have a weight

$$P_{\rm Po}(N_+) = \frac{1}{N_+!}(\kappa N)^{N_+}, \tag{12}$$

where $N_+!$ is the symmetry factor [14]. The same distribution is assumed for N_- as well. This is nothing but the Poisson distribution up to a normalization. The summation with $P_{\rm Po}(N_\pm)$ in Eq. (11) results in the exponentiation of the determinant term [18]

$$\Omega_{\rm Po} = \frac{1}{2}\Sigma^2 {\rm tr} SS^\dagger - \frac{1}{2}\ln\det[(S+\mathcal{M})(S^\dagger+\mathcal{M}^\dagger)+t^2] - \frac{1}{2}\kappa[e^{i\theta}\det(S+\mathcal{M}) + e^{-i\theta}\det(S^\dagger+\mathcal{M}^\dagger)]. \tag{13}$$

The same determinant term as in $\Omega_{\rm Po}$ is commonly incorporated in other effective models as the "anomaly term" to break the $U_A(1)$ symmetry.[4] In the ChRM model, however, this potential is unbound for $N_f = 3$. Indeed, for large $S = \phi \mathbf{1}_{N_f}$ the term $\det(S+\mathcal{M}) \sim \phi^3$ dominates over the other terms in $\Omega_{\rm Po}$.

It should be noticed here that the fermion coupling distorts the $N_\pm$ distribution itself. With including the determinant term of the topological zero modes in Eq. (10), the effective distribution for N_+ reads

$$\tilde{P}_{\rm Po}(N_+) = \frac{1}{N_+!}(\kappa N d)^{N_+} \tag{14}$$

with $d = |\det(S+\mathcal{M})|$, and similarly for N_-. We find that the average value of $N_\pm$ increases indefinitely with increasing d as $\langle N_\pm\rangle = \kappa N d$. However, the possibility of infinitely many topological zero modes $N_\pm$ should be avoided in the ChRM model as a low-energy effective theory within a finite volume $V \propto N$.

Here we regularize the distribution by setting explicitly a maximum value of $\mathcal{O}(N)$ for $N_\pm$. We split the finite space-time volume into γN cells with γ being a constant of $\mathcal{O}(1)$, and assign a probability p for a cell to be occupied by a single (anti-)instanton and $(1-p)$ for a cell unoccupied. We exclude the possibility of the double occupation of a cell, which is justified by a repulsion between the instantons. Just like in the lattice gas model in statistical mechanics, this assumption results in the binomial distributions for $N_\pm$:

$$P(N_\pm) = \binom{\gamma N}{N_\pm} p^{N_\pm}(1-p)^{\gamma N - N_\pm}. \tag{15}$$

For a small p and a large γN, the binomial distribution $P(N_\pm)$ is accurately approximated with the Poisson distribution with the mean $\gamma N p$. But it cannot be a good approximation for a large p. The binomial distribution provides a stringent upper bound γN for the number of topological zero modes $N_\pm$, in contrast to the Poisson distribution. The corresponding effective potential for S is found to be

$$\begin{aligned}\Omega(S;t,m,\theta) = {}& \frac{1}{2}\Sigma^2 {\rm tr} SS^\dagger \\ & - \frac{1}{2}\ln\det[(S+\mathcal{M})(S^\dagger+\mathcal{M}^\dagger)+t^2] \\ & - \frac{1}{2}\gamma[\ln(e^{i\theta}\alpha\det(S+\mathcal{M})+1) \\ & + \ln(e^{-i\theta}\alpha\det(S^\dagger+\mathcal{M}^\dagger)+1)]\end{aligned} \tag{16}$$

with $\alpha = p/(1-p)$. This is the ChRM model that we propose and analyze in this paper.

The effective potential (16) is bounded from below by the ${\rm tr} SS^\dagger$ term in contrast to $\Omega_{\rm Po}$ in Eq. (13). The anomaly terms are accommodated under the logarithms in the square brackets. Moreover, for a small value of $|\alpha \det(S + m)|$, it reduces to the potential (13) with $\kappa = \gamma\alpha$. On the other hand, the Poisson approximation fails when $\langle N_\pm\rangle$ becomes $\mathcal{O}(\gamma N)$. We stress here again the fact that the distribution $P(N_+)$ is deformed in the presence of the coupling with the topological zero modes as

$$\tilde{P}(N_+) = \binom{\gamma N}{N_+}(pd)^{N_+}(1-p)^{\gamma N - N_+}, \tag{17}$$

which changes the probability p to an effective one $\tilde{p} \equiv pd/(pd + 1 - p)$. The same deformation applies to $P(N_-)$. Accordingly, the mean number of the zero modes is modified to $\langle \tilde{N}_\pm\rangle = \gamma N \tilde{p}$.

Two remarks are here in order. In the conventional models, the total number of the modes is kept fixed, while the fluctuation of ν is allowed for resolving the $U_A(1)$ problem. On the other hand, in the instanton gas picture, both N_+ and N_- are expected to fluctuate naturally. As stressed in this subsection, the mean number of the modes depends on the magnitude of S, which gives an $\mathcal{O}(N)$ effect. The anomaly term appearing in the effective potential in turn affects the saddle point value of S. This point is essential to the first-order transition for $N_f = 3$, which is overlooked in the earlier works.

Secondly, in Eq. (16) one needs to introduce a dimensionful scale to compensate the dimension of the determi-

[4]Although one assumes no mass term in the determinant interaction conventionally, there is an ambiguity [22]. Our choice of the mass dependence has been fixed in Eq. (6).

nant if the dimension of mass is assigned to $\mathcal{M}$ as well as S. Here we leave all the quantities dimensionless for demonstration of the general feature of the model.

C. θ dependence

The variance of the topological charge $\nu = N_+ - N_-$ for the binomial distribution is computed as $2N\tau = 2N\gamma p(1-p)$, where τ is the quenched topological susceptibility. In the presence of the fermion coupling, this susceptibility will be replaced with

$$\tilde{\tau} = \gamma\tilde{p}(1-\tilde{p}) = \gamma\frac{\alpha d}{(\alpha d + 1)^2}. \tag{18}$$

We can confirm this fact also by rewriting the anomaly term in Eq. (16) as

$$-\frac{1}{2}\gamma\ln\bigg[1 + |\alpha\det(S+\mathcal{M})|^2 + 2|\alpha\det(S+\mathcal{M})| \times\cos\bigg(\theta - \frac{i}{2}\ln\frac{\det(S+\mathcal{M})}{\det(S^\dagger+\mathcal{M}^\dagger)}\bigg)\bigg], \tag{19}$$

from which we find again the replaced topological susceptibility $\tilde{\tau}$ as a coefficient of θ^2. Such a series in θ of the anomaly term gives a connection to the general arguments on the η' in the $1/N_c$ expansion [23].

To make a clear connection with the conventional model, let us ignore for a moment the fluctuation of the total number $N_+ + N_-$ and apply the Gaussian approximation for the ν distribution. We then obtain the partition function as

$$\begin{aligned} Z_\theta &= \int_{-\infty}^{\infty} d\nu e^{-(\nu^2/2(2N)\tau)} e^{i\nu\theta} Z^N_{N_+,N_-} \\ &= \text{const}\int dS e^{-N\Sigma^2 \text{tr} S^\dagger S}\det{}^N[(S+\mathcal{M})(S^\dagger+\mathcal{M}^\dagger)+t^2] \\ &\quad\times\det{}^{(N_++N_-)/2}[(S+\mathcal{M})(S^\dagger+\mathcal{M}^\dagger)] \\ &\quad\times\exp\bigg[-N\tau\bigg(\theta-\frac{i}{2}\ln\frac{\det(S+\mathcal{M})}{\det(S^\dagger+\mathcal{M}^\dagger)}\bigg)^2\bigg]. \end{aligned}\tag{20}$$

This is almost identical with the model discussed in Refs. [16,17], which reproduces the screening of the (unquenched) topological susceptibility (see Eq. (38)) as measured on a lattice at finite temperatures. However, this model (20) fails to describe a first-order phase transition at finite temperature for $N_f = 3$. In fact, the anomaly term appears only as a phase in (20) in contrast to (19) and drops out when we determine the magnitude of $S = S^\dagger$ in the ground state (with $\mathcal{M} = \mathcal{M}^\dagger$). The variation of $N_+ + N_-$ is essential for the anomaly term to affect the saddle point condition, and thus the order of the phase transition.

III. GROUND STATE AND FLUCTUATIONS

In this section we shall study the ground state properties of the system with equal mass $\mathcal{M} = m\mathbf{1}_{N_f}$ for simplicity. Setting $S = \phi\mathbf{1}_{N_f}$ with real ϕ and with $\theta = 0$, we obtain a simple form of the grand potential,

$$\begin{aligned}\Omega(\phi; t, m) &= \frac{1}{2}N_f\Sigma^2\phi^2 - \frac{1}{2}N_f\ln[(\phi+m)^2+t^2] \\ &\quad - \gamma\ln|\alpha(\phi+m)^{N_f}+1|.\end{aligned}\tag{21}$$

The factor N_f cannot be factored out in the potential Ω because of the anomaly term here.

A. Chiral phase transition: Ground state

In the thermodynamic limit $N \to \infty$, the ground state can be analyzed with the solution of the saddle point equation

$$\Sigma^2\phi - \frac{\phi+m}{(\phi+m)^2+t^2} - \gamma\frac{\alpha(\phi+m)^{N_f-1}}{\alpha(\phi+m)^{N_f}+1} = 0. \tag{22}$$

The scalar quark condensate is related to the solution ϕ as

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{2NN_f}\frac{\partial}{\partial m}\ln Z(m) = -\Sigma^2\phi. \tag{23}$$

Without the anomaly term $\alpha = 0$, Eq. (22) recovers the flavor-independent gap equation of the conventional model, which has the solution $\phi_0^2 = \Sigma^{-2} - t^2$ in the chiral limit. One can estimate the effect of small m and α on ϕ^2 at the leading order as

$$\phi^2 \sim \phi_0^2 - \frac{m}{\phi_0}(\Sigma^{-2} - 2t^2) + \frac{\alpha\gamma}{\Sigma^4}\phi_0^{N_f-2}. \tag{24}$$

At lower temperatures $2t^2 < \Sigma^{-2}$ the value of the condensate is decreased by the quark mass term, on the contrary to our intuition. This is because the leading order term of m appears in a combination $-2m\phi/(\phi^2+t^2)$ in the potential, which simplifies to $-2m/\phi$ for $t = 0$, favoring smaller ϕ for the potential to be more stabilized.

There is no chiral symmetry for $N_f = 1$ because of the anomaly term. Let us discuss phenomenologically interesting cases, $N_f = 2$ and 3.

1. $N_f = 2$

Expanding the potential Ω around $\phi = 0$, we find

$$\Omega = c_0 + c_2\phi^2 + c_4\phi^4 - h\phi + \mathcal{O}(\phi^6, \phi^3 m, m^2) \tag{25}$$

with $c_0 = -\ln t^2$, $c_2 = \Sigma^2 - \alpha\gamma - t^{-2}$, $c_4 = (\gamma\alpha^2 + t^{-4})/2$, and $h = 2(\alpha\gamma + t^{-2})m$. This is the standard form of the Landau-Ginzburg potential for a second-order phase transition ($m = 0$) with the critical temperature t_c

$$t_c = \frac{1}{\sqrt{\Sigma^2 - \alpha\gamma}}. \tag{26}$$

Inclusion of the anomaly term increases the value of t_c. The behavior in the vicinity of the critical point is characterized by the mean-field exponents; we find $\beta = 1/2$ and $\delta = 3$, respectively, from the solutions $\phi^2 = \epsilon t_c^{-2}/c_4 \propto \epsilon^{2\beta}$ for

$\epsilon = (t_c - t)/t_c$ with $0 < \epsilon \ll 1$, and $\phi = [m\Sigma^2/(2c_4)]^{1/3} \propto m^{1/\delta}$ with $m \neq 0$ at $t = t_c$.

2. $N_f = 3$

Expanding the potential Ω around $\phi = 0$, we find

$$\Omega = c_0 + c_2\phi^2 + c_3\phi^3 + c_4\phi^4 - h\phi + \mathcal{O}(\phi^6, \phi^2 m, m^2) \tag{27}$$

with $c_0 = -(3/2)\ln t^2$, $c_2 = (3/2)(\Sigma^2 - t^{-2})$, $c_3 = -\alpha\gamma$, $c_4 = (3/4)t^{-4}$, and $h = 3t^{-2}m$. The $\mathrm{U_A}(1)$ symmetry is explicitly broken even for $m = 0$ due to the anomalous ϕ^3 term, which leads to a first-order phase transition.

In Fig. 1, we display the chiral condensate ϕ as a function of the temperature t and the quark mass m. For numerical demonstration, we have chosen the parameters as $\Sigma = 1$, $\gamma = 2$, and $\alpha = 0.3$ in this paper. We clearly see the second-order transition for $N_f = 2$, while the first-order transition for $N_f = 3$ in the chiral limit $m = 0$. In $N_f = 3$ case, as we increase the current quark mass m, we find a terminating point of the first-order line at $m_c = 0.0265$.

For $N_f \geq 4$, the anomaly term only affects the coefficients of ϕ^n $(n \geq N_f)$ in the series expansion of the potential.

B. Mesonic masses

Once the ground state acquires the nonzero expectation value $S = \phi\mathbf{1}_{N_f} \neq 0$, the symmetry is spontaneously broken down to $\mathrm{U_V}(N_f)$, $S \to USU^{-1}$. Degeneracy of the vacua in the chiral limit dictates the massless fluctuations $S = \phi e^{i\pi^a\lambda^a/(\sqrt{2}\phi)} \sim \phi + i\pi^a\lambda^a/\sqrt{2}$ corresponding to the generators of the broken symmetry.

To find the mesonic masses, we use the parametrization $S = \phi + \lambda^a(\sigma^a + i\pi^a)/\sqrt{2}$ with σ^a, $\pi^a \in \mathbb{R}$, and the $\mathrm{U}(N)$ generators λ^a normalized as $\mathrm{tr}[\lambda^a\lambda^b] = 2\delta^{ab}$ $(a = 0, \ldots, N_f^2 - 1)$. Using a formula for a matrix X with a small parameter ϵ, $\det(1 + \epsilon X) = 1 + \epsilon\,\mathrm{tr}X + \frac{1}{2}\epsilon^2[(\mathrm{tr}X)^2 - \mathrm{tr}X^2] + \mathcal{O}(\epsilon^3)$, we can easily expand Ω (16) around the saddle point solution to define the massses with $\Omega = \Omega_0 + \frac{1}{2}M_{\mathrm{s}a}^2\sigma^{a2} + \frac{1}{2}M_{\mathrm{ps}a}^2\pi^{a2} + \cdots$.

The flavor nonsinglet masses in the scalar and pseudoscalar channels, respectively, are found to be

$$M_{\mathrm{s}}^2 = \Sigma^2 - \frac{1}{(\phi + m)^2 + t^2} + \frac{2(\phi + m)^2}{[(\phi + m)^2 + t^2]^2} + \gamma\frac{\alpha(\phi + m)^{N_f - 2}}{\alpha(\phi + m)^{N_f} + 1}, \tag{28}$$

$$M_{\mathrm{ps}}^2 = \Sigma^2 - \frac{1}{(\phi + m)^2 + t^2} - \gamma\frac{\alpha(\phi + m)^{N_f - 2}}{\alpha(\phi + m)^{N_f} + 1}. \tag{29}$$

With the saddle point equation, we can reexpress the π mass as

$$M_{\mathrm{ps}}^2 = \frac{\Sigma^2 m}{\phi + m}. \tag{30}$$

In the chiral limit, we have the massless π. With the explicit breaking of m, we find a relation reminiscent of the Gell-Mann-Oakes-Renner relation,

$$(\phi + m)^2 M_{\mathrm{ps}}^2 = m\Sigma^2(\phi + m) \sim -m\langle\bar{\psi}\psi\rangle, \tag{31}$$

if we identify $\phi + m$ as the pion decay constant f_π. Interestingly, there is a mass hierarchy, $M_{\mathrm{s}}^2:\Sigma^2:M_{\mathrm{ps}}^2 = 2:1:0$, in the chiral limit at zero temperature $t = 0$.

On the other hand, the flavor-singlet masses for the scalar and pseudoscalar singlet channels have an additional contribution from the anomaly term as

$$M_{\mathrm{s}0}^2 = M_{\mathrm{s}}^2 - \Delta M_0^2, \tag{32}$$

$$M_{\mathrm{ps}0}^2 = M_{\mathrm{ps}}^2 + \Delta M_0^2 \tag{33}$$

with

$$\Delta M_0^2 \equiv N_f\gamma\frac{\alpha(\phi + m)^{N_f - 2}}{[\alpha(\phi + m)^{N_f} + 1]^2} = N_f\frac{\tilde{\tau}}{(\phi + m)^2}. \tag{34}$$

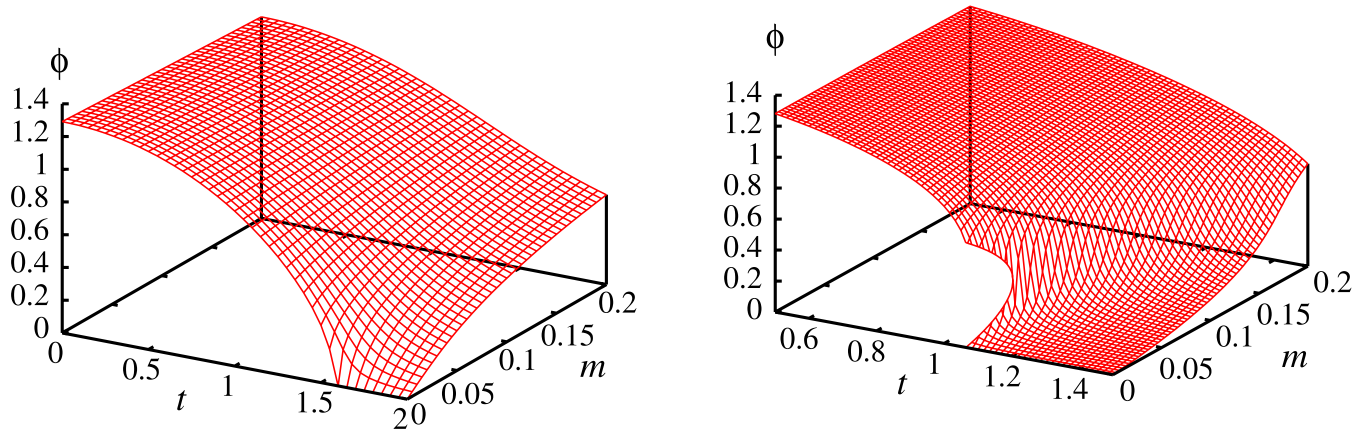

FIG. 1 (color online). Chiral condensate ϕ as a function of t and m for $N_f = 2$ (left) and 3 (right).

The would-be Nambu-Goldstone (NG) mode in the singlet channel becomes massive due to the coupling to the $U_A(1)$ interaction ΔM_0^2. This same effect appears as a reduction in the scalar singlet channel. This mass gap is related to the (replaced) quenched susceptibility $\tilde{\tau}$, similarly to the Witten-Veneziano formula [23]. It is interesting to note that for $N_f \geq 3$ the $U_A(1)$ breaking effect disappears $\Delta M_0^2 = 0$ from the meson masses when the condensate vanishes in the symmetric phase in the chiral limit [24]. In contrast, N_f-point functions are affected by the $U_A(1)$ breaking term.

In Fig. 2 we show the mesonic masses in the flavor-singlet scalar and pseudoscalar channels and in the flavor-nonsinglet scalar and pseudoscalar channels. Both for $N_f = 2$ and 3, the flavor-singlet pseudoscalar meson acquires the nonzero mass via (33).

C. Susceptibilities

The scalar and pseudoscalar susceptibilities are defined as the responses to the external fields $\mathcal{M} = (s^a + ip^a)\lambda^a/\sqrt{2}$ in Eq. (16):

$$\chi_s^{ab} = -\frac{\partial^2}{\partial s^a \partial s^b}\Omega(S(\mathcal{M}); \mathcal{M})\bigg|_{\mathcal{M}=m}, \quad \chi_{ps}^{ab} = -\frac{\partial^2}{\partial p^a \partial p^b}\Omega(S(\mathcal{M}); \mathcal{M})\bigg|_{\mathcal{M}=m}. \tag{35}$$

In our basis, those susceptibilities are found to be diagonal $\chi_{s,ps}^{ab} \propto \delta^{ab}$ with a simple form,

$$\chi = \chi^{(0)}\frac{1}{1 - \chi^{(0)}/\Sigma^2} = \chi^{(0)}\frac{\Sigma^2}{M^2} \tag{36}$$

with $\chi^{(0)} = \Sigma^2 - M^2$ for each channel. A susceptibility diverges when the corresponding M^2 vanishes (e.g.) at a critical point where the effective potential $\Omega(S)$ has a flat direction. Especially, in the broken phase, the susceptibilities of the nonsinglet NG modes are indefinite.

The topological susceptibility χ_{top} is defined as the response to the angle θ

$$\chi_{top} \equiv -\frac{1}{2N}\frac{\partial^2}{\partial\theta^2}\ln Z_\theta\bigg|_{\theta=0}. \tag{37}$$

Using the saddle point solution with small θ, parametrized as $S(\theta) = \phi + i\eta_0(\theta)\lambda^0/\sqrt{2}$, the susceptibility in the mean-field approximation is obtained by differentiating $\Omega(S(\theta); \theta)$ around $\theta = 0$ as

$$\begin{aligned}\chi_{top} &= \frac{\partial^2\Omega}{\partial\theta^2} - \left(\frac{\partial^2\Omega}{\partial\theta\partial\eta_0}\right)^2\left(\frac{\partial^2\Omega}{\partial\eta_0^2}\right)^{-1} \\ &= \tilde{\tau} - \frac{N_f\tilde{\tau}^2}{(\phi+m)^2}\frac{1}{M_{ps0}^2} \\ &= \left[\frac{1}{\tilde{\tau}} + \frac{N_f}{m\Sigma^2(\phi+m)}\right]^{-1},\end{aligned} \tag{38}$$

or

$$\frac{1}{\chi_{top}} = \frac{1}{\tilde{\tau}} + \frac{1}{\tau_m}. \tag{39}$$

Here $\tilde{\tau}$ is the modified susceptibility defined in Eq. (18) in the previous section. As is well known [25], the most prominent effect of the fermion coupling is the screening of the topological susceptibility via the contribution

$$\tau_m = \frac{\Sigma^2 m(\phi+m)}{N_f} = \frac{M_{ps}^2(\phi+m)^2}{N_f}. \tag{40}$$

In the quenched limit $N_f \to 0$, the χ_{top} recovers the quenched susceptibility $\tau = \gamma p(1-p)$, while in the massless limit the susceptibility χ_{top} is screened completely to zero.

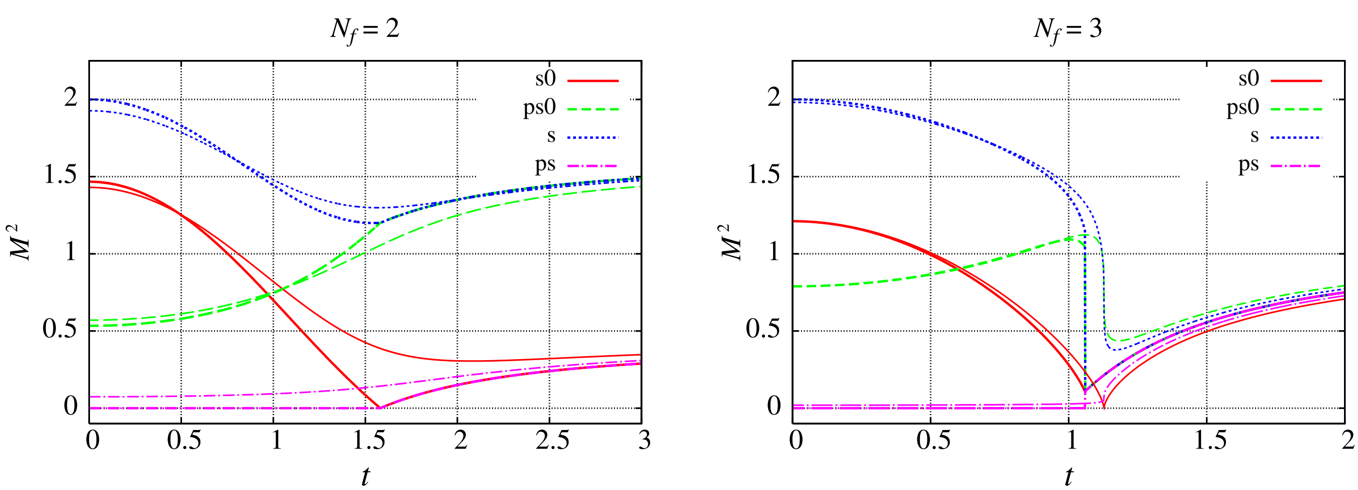

FIG. 2 (color online). Temperature dependence of the mesonic masses in the flavor-singlet scalar (s0) and pseudoscalar (ps0) channels and in the flavor-nonsinglet scalar (s) and pseudoscalar (ps) channels in the chiral limit ($m = 0$, thick lines) and with explicit breaking ($m \neq 0$, thin lines) for $N_f = 2$ (left) and $N_f = 3$ (right). As nonzero quark mass, we set $m = 0.1$ for $N_f = 2$ and $m = m_c = 0.0265$ for $N_f = 3$.

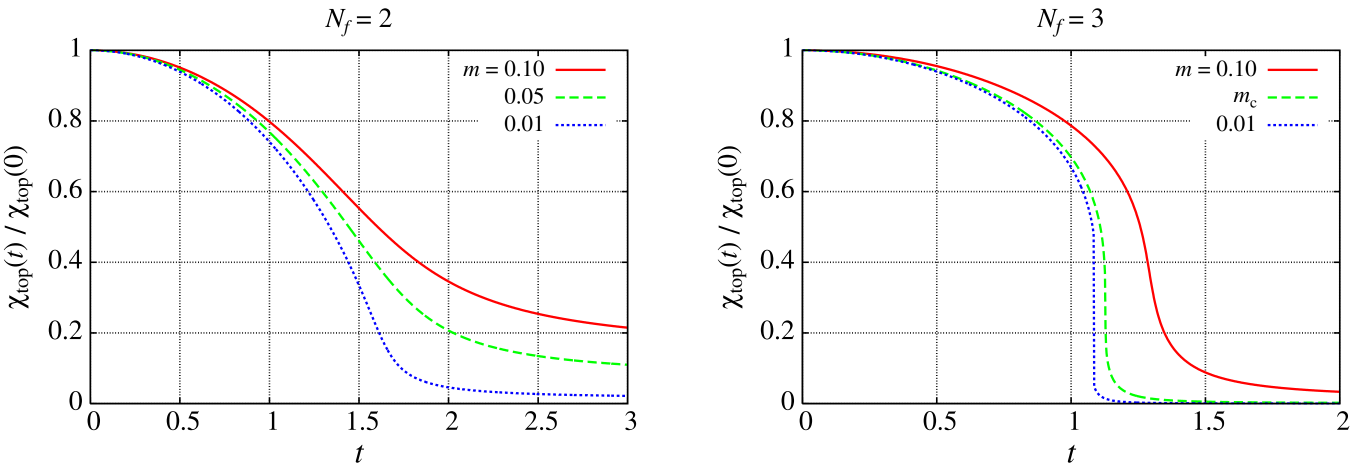

FIG. 3 (color online). Temperature dependence of topological susceptibilities for $N_f = 2$ (left) and 3 (right).

By changing the variable $S \to \tilde{S} = Se^{i\theta/N_f}$ under the integral (16) and then applying the saddle point approximation, we find $\Omega(t, m, \theta) = \Omega(t, me^{i\theta/N_f}, 0)$. Here the θ appears only in the combination with the mass m. In this latter form, the source term becomes a linear combination of the scalar and pseudoscalar sources in the flavor-singlet channel $\mathcal{M} = me^{i\theta/N_f} = (s_0 + ip_0)\lambda^0/\sqrt{2}$, and the $U_A(1)$ relation for the topological susceptibility [20] is immediately derived as

$$\chi_{top} = -\frac{1}{2N}\left(\frac{\partial^2 \ln Z_\theta}{\partial p_0^2}\left(\frac{\partial p_0}{\partial \theta}\right)^2 + \frac{\partial \ln Z_\theta}{\partial s_0}\frac{\partial^2 s_0}{\partial \theta^2}\right)\Bigg|_{\theta=0}$$
$$= -\frac{m^2}{N_f}\chi_{ps0} - \frac{m}{N_f}\langle\bar{\psi}\psi\rangle. \qquad (41)$$

We have seen that the pseudoscalar meson in the flavor-singlet channel has nonzero mass (33) because of the $U_A(1)$ breaking term, and accordingly the pseudoscalar singlet susceptibility remains finite in the broken phase in the chiral limit. Thus for the small but nonzero quark mass m, the topological susceptibility χ_{top} decreases following the chiral condensate $\langle\bar{\psi}\psi\rangle \sim \phi$ with increasing temperature t, as clearly observed in Fig. 3.

IV. DISCUSSIONS AND SUMMARY

In this work we have considered the ChRM model with the near-zero and topological zero modes. It was known before that summation over the topological zero modes with the Poisson distribution results in the determinant interaction, but unfortunately it gives a pathological unbound potential for the ChRM model with $N_f = 3$ [18]. On the other hand, the Gaussian distribution for topological charge ν with the total number of the modes fixed, leads to the log-determinant type interaction (20). This term in the potential resolves the $U_A(1)$ problem at zero and finite temperatures as suggested in Refs. [16,17], but still yields a second-order phase transition at finite temperature, irrespective of N_f.

We have proposed that (i) the numbers of the topological zero modes with right and left chiralities, respectively, have the distributions related to instanton dynamics in a finite volume, and thus (ii) these distributions have an upper bound of $\mathcal{O}(N)$. We have adopted the binomial distribution as such a distribution. This gives rise to the stable potential which describes the chiral phase transitions of the second and the first order depending on the number of flavors $N_f = 2$ and 3, respectively, and resolves the $U_A(1)$ problem as well. We have confirmed through numerical evaluations that the proposed model also reasonably reproduces temperature dependence of meson masses, (pseudo)scalar susceptibilities and topological susceptibility. Notably, the topological susceptibility is consistent with the universal quark mass dependence, satisfying the anomalous Ward identity.

In the conventional ChRM models, the near-zero modes themselves are customarily assumed to emerge from the instanton dynamics, in contrast to the explicit separation of the near-zero modes and the topological zero modes in our modeling. To assess the foundations and relations of these models, a rigorous analysis of these low-lying modes based on the microscopic dynamics would be needed, which is beyond the scope of our current study.

We have adopted the independent binomial distribution for $N_\pm$ as a simple improvement from the Poisson distribution in order to obtain a stable effective potential. However, the distributions of $N_\pm$ can be correlated nontrivially in general. It will be interesting to examine other possibilities for number distributions of the topological zero modes. Incidentally, we note here that the determinant interactions of the form (13) in the NJL models in 3 + 1 dimensions also lead to the unbound effective potentials for a large quark condensate beyond the cutoff although the local minimum is usually chosen as the physical ground state.

Concerning phenomenological applications, first we need to tune the model parameters Σ, α, and γ as well as the light and strange quark masses m_{ud} and m_s so as to reproduce the empirical properties in the vacuum. Extension to the case at finite baryochemical potential is straightforward [4,8], which allows us to study the phase diagram of the ChRM model with the flavor dependence in the space of the temperature, the chemical potential, and the quark masses. Especially the existence of the analog of the QCD critical point(s) [26] will be an important subject to be studied. Applications of this model at finite isospin and strangeness chemical potentials are also planned. Progress in this direction will be reported elsewhere.

ACKNOWLEDGMENTS

The authors are grateful to T. Wettig for fruitful discussions on this work. M. O. thanks T. Hatsuda, C. Lehner, and J. J. M. Verbaarschot for collaboration on a related subject. H. F. and T. S. acknowledge useful discussions in the workshop on ''Non-equilibrium Quantum Field Theories and Dynamic Critical Phenomena'' at Yukawa Institute of Theoretical Physics in March 2009. They are also very grateful to the members of Komaba theory group for their interests and encouragements. This work is supported in part by Grants-in-Aid No. 19540273 of MEXT, Japan.

APPENDIX: TRINOMIAL DISTRIBUTION

Here we deal with the trinomial distribution for N_+ and N_-, as a simplest example of the nonfactorizable distribution

$$P(N_+, N_-) = \frac{(\gamma N)!}{N_+!N_-!(\gamma N - N_+ - N_-)!} \times p_+^{N_+} p_-^{N_-} (1 - p_+ - p_-)^{\gamma N - N_+ - N_-}, \quad \text{(A1)}$$

where $p_{+(-)}$ is the probability for a single cell to be occupied by an (anti-)instanton. Note that the distribution is symmetric under the exchange of $+$ with $-$. Replacing $P(N_+)P(N_-)$ in Eq. (11) with $P(N_+, N_-)$ and setting $p_+ = p_- = p$, we obtain the effective potential

$$\Omega_{\mathrm{Tri}}(S; m, \theta, t) = \frac{1}{2}\Sigma^2 \mathrm{tr} S^\dagger S - \frac{1}{2} \ln\det[(S + \mathcal{M})(S^\dagger + \mathcal{M}^\dagger) + t^2] - \frac{1}{2}\gamma \ln[\alpha e^{i\theta} \det(S + \mathcal{M}) + \alpha e^{-i\theta} \det(S^\dagger + \mathcal{M}^\dagger) + 1] \quad \text{(A2)}$$

with $\alpha = p/(1 - 2p)$. This is quite similar to Eq. (16), and qualitative features of the model are unchanged. The quenched topological susceptibility, or the variance of $\nu = N_+ - N_-$, for Eq. (A1) is computed as

$$\tau = \gamma p. \quad \text{(A3)}$$

[1] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[2] J. J. M. Verbaarschot and T. Wettig, Annu. Rev. Nucl. Part. Sci. **50**, 343 (2000).
[3] E. V. Shuryak and J. J. M. Verbarrschot, Nucl. Phys. **A560**, 306 (1993); J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993).
[4] A. M. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov, and J. J. M. Verbaarschot, Phys. Rev. D **58**, 096007 (1998).
[5] A. D. Jackson and J. J. M. Verbaarschot, Phys. Rev. D **53**, 7223 (1996).
[6] T. Wettig, A. Schafer, and H. A. Weidenmuller, Phys. Lett. B **367**, 28 (1996); **374**, 362(E) (1996).
[7] A. M. Halasz, A. D. Jackson, and J. J. M. Verbaarschot, Phys. Rev. D **56**, 5140 (1997).
[8] M. A. Stephanov, Phys. Rev. Lett. **76**, 4472 (1996).
[9] M. A. Stephanov, Phys. Rev. D **73**, 094508 (2006).
[10] J. Han and M. A. Stephanov, Phys. Rev. D **78**, 054507 (2008).
[11] M. A. Stephanov, Phys. Lett. B **375**, 249 (1996).
[12] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
[13] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **44**, 1422 (1970); M. Kobayashi, H. Kondo, and T. Maskawa, Prog. Theor. Phys. **45**, 1955 (1971).
[14] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976); **18**, 2199(E) (1978).
[15] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Nucl. Phys. **B498**, 313 (1997).
[16] M. Ohtani, C. Lehner, T. Wettig, and T. Hatsuda, Mod. Phys. Lett. A **23**, 2465 (2008).
[17] C. Lehner, M. Ohtani, J. J. M. Verbaarschot, and T. Wettig, Phys. Rev. D **79**, 074016 (2009).
[18] R. A. Janik, M. A. Nowak, and I. Zahed, Phys. Lett. B **392**, 155 (1997).
[19] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, AIP Conf. Proc. **494**, 408 (1999).
[20] R. J. Crewther, Phys. Lett. **70B**, 349 (1977).
[21] M. A. Nowak and I. Zahed, Phys. Lett. B **230**, 108 (1989).
[22] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B163**, 46 (1980).
[23] E. Witten, Nucl. Phys. **B156**, 269 (1979); Ann. Phys. (N.Y.) **128**, 363 (1980); G. Veneziano, Nucl. Phys. **B159**, 213 (1979).
[24] S. H. Lee and T. Hatsuda, Phys. Rev. D **54**, R1871 (1996).
[25] H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607 (1992).
[26] For recent review, M. Stephanov, Proc. Sci., LAT2006 (2007) 024.