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Sequential flavor symmetry breaking

Thorsten Feldmann,[1] Martin Jung,[2,3] and Thomas Mannel[2]

[1]*Physik Department T31, Technische Universität München, D-85747 Garching, Germany*

[2]*Theoretische Physik 1, Fachbereich Physik, Universität Siegen, D-57068 Siegen, Germany*

[3]*Instituto de Física Corpuscular, CSIC-Universitat de València, Apartado de Correos 22085, E-46071 Valencia, Spain*

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The gauge sector of the standard model exhibits a flavor symmetry that allows for independent unitary transformations of the fermion multiplets. In the standard model the flavor symmetry is broken by the Yukawa couplings to the Higgs boson, and the resulting fermion masses and mixing angles show a pronounced hierarchy. In this work we connect the observed hierarchy to a sequence of intermediate effective theories, where the flavor symmetries are broken in a stepwise fashion by vacuum expectation values of suitably constructed spurion fields. We identify the possible scenarios in the quark sector and discuss some implications of this approach.

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I. INTRODUCTION

The origin of flavor remains one of the main mysteries in modern particle physics, and many attempts have been made to understand the phenomenon of flavor by postulating certain (discrete) flavor symmetries (see e.g. [1–6] and references therein), or by localizing fermions in extra dimensions (see e.g. [7–14] and references therein), to name two popular ideas. While such scenarios can successfully explain some of the issues related to the hierarchies observed in fermion masses and mixings, the origin of the proposed new mechanisms (e.g. from the embedding into a grand unified theory or even string theory, respectively) still remains an open issue.

Alternatively, we may start from a bottom-up approach in which the phenomenon of flavor is just parametrized as in the standard model (SM). In fact, the SM has an approximate global flavor symmetry G_F (see below), which is broken by the Yukawa couplings, inducing the fermion masses and mixings. Such an explicit symmetry breaking is usually parametrized by introducing spurion fields with a definite behavior under the symmetry to be broken. In the case at hand, focusing on the quark sector, the Yukawa matrices Y_U and Y_D are considered as complex spurion fields [15], transforming nontrivially under G_F.

A special role is played by the top quark, which has a Yukawa coupling of order one, breaking the original flavor symmetry group G_F to a smaller subgroup G_F' (see below), which is still a good symmetry as long as the remaining Yukawa couplings are negligible. In a recent paper [16], two of us have shown that in such a case it is convenient to consider a nonlinear representation of G_F in which the subgroup G_F' is linearly realized. In this context, it turned out to be useful to assign a canonical mass dimension to the Yukawa spurion fields, since in this way the top Yukawa coupling could be understood as originating from a dimension-four operator, while the remaining Yukawa terms are dimension five, thereby reflecting the hierarchy between the top mass and the lighter quark masses.[1] If we take this approach seriously, two immediate implications arise:

(i) The spontaneous breaking $G_F \to G_F'$ induces Goldstone modes, which call for a dynamical interpretation. One possibility is to consider *local* flavor symmetries, where Goldstone modes become the longitudinal modes for massive gauge bosons. Another alternative is to keep the Goldstone modes as physical axionlike degrees of freedom.[2] These issues will be discussed in somewhat more detail in a separate publication [19].

(ii) The breaking $G_F \to G_F'$ induced by the top-quark Yukawa coupling can be considered the first step in a sequence of flavor symmetry breaking steps taking place at different physical scales $\Lambda \gg \Lambda' \gg \Lambda'' \gg \ldots$ Through the vacuum expectation values (VEVs) of the spurion fields, the hierarchy of scales should be directly related to the observed hierarchy for quark masses and mixings.

In the following, we shall identify the flavor subgroups for each of the intermediate effective theories in the construction above and identify the corresponding representations for quark fields, spurions, and Goldstone modes. We will also briefly discuss the requirements for the spurion potential necessary for such a scenario.

II. SUCCESSIVE FLAVOR SYMMETRY BREAKING

In this section we identify the sequence of intermediate (residual) flavor symmetries which arise when the original flavor symmetry of the SM gauge sector is broken in a

[1]A similar construction can be performed in the lepton sector, when the SM is minimally extended by a dimension-five operator in order to describe nonvanishing neutrino masses [17,18].

[2]An option to avoid Goldstone modes altogether is to restrict oneself to *discrete* flavor symmetries.

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stepwise fashion at different scales, set by the VEVs of the relevant spurion fields and linked to the observed hierarchies in the quark masses and Cabibbo-Kobayashi-Maskawa (CKM) angles. Considering the Yukawa sector for the quarks,

$$-\mathcal{L}_Y = Y_U \bar{Q}_L \tilde{H} U_R + Y_D \bar{Q}_L H D_R + \text{H.c.}, \tag{2.1}$$

we may consider independent phase transformations for the three quark multiplets (Q_L, U_R, D_R) and the Higgs field (H). Among these four phases, two are identified as baryon number $U(1)_B$ and weak hypercharge $U(1)_Y$, which are not broken by the Yukawa matrices, whereas the 2 remaining $U(1)$ symmetries are broken by $\langle Y_U\rangle \neq 0$ or $\langle Y_D\rangle \neq 0$. We thus define the flavor group in the quark sector as[3]

$$\begin{aligned} G_F &= SU(3)^3 \times U(1)^4/(U(1)_B \times U(1)_Y) \\ &= SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R} \times U(1)_{U_R} \times U(1)_{D_R}. \end{aligned} \tag{2.2}$$

For the Yukawa sector to be formally invariant under G_F, we assign the following transformation properties to the spurion fields:

$$Y_U \sim (3, \bar{3}, 1)_{-1,0}, \qquad Y_D \sim (3, 1, \bar{3})_{0,-1}, \tag{2.3}$$

where the terms in brackets refer to the three $SU(3)$ factors, and the subscripts to the two $U(1)$ factors, respectively. Counting parameters, we have $2 \times 18 = 36$ entries for the spurions $Y_{U,D}$ and $3 \times 8 + 2 = 26$ symmetry generators, leaving $36 - 26 = 10$ physical parameters in the quark Yukawa sector, which can be identified with the six quark masses, the three CKM angles, and the CP-violating CKM phase (see also [20]).[4]

[3]Our discussion differs from the one in [15] where the independent phase rotations for the Higgs fields have been overlooked.

[4]Similarly, considering the $U(1)$ phases in the lepton sector, we obtain the SM flavor group

$$G_F^{\text{lepton}} = U(3)^2/(U(1)_e \times U(1)_\mu \times U(1)_\tau)$$

for massless neutrinos, and

$$\tilde{G}_F^{\text{lepton}} = U(3)^2$$

for massive neutrinos which are generated by a lepton-number violating dimension-five term in the Lagrangian

$$-\mathcal{L}_{\text{Maj}} = \frac{1}{\Lambda_{\text{L}}} g_\nu (H\ell_L)^T (H\ell_L).$$

In the first case, we have 18 parameters in the spurion Y_E and $2 \times 8 - 1 = 15$ symmetry generators, leaving three physical parameters to be identified with the three charged lepton masses. In the second case, we have $18 + 12 = 30$ parameters from the spurions Y_E and g_ν, from which we subtract $2 \times 9 = 18$ symmetry generators, to obtain 12 physical parameters, which are the 6 lepton masses, the three Pontecorvo-Maki-Nakagawa-Sakata (PMNS) angles, one Dirac phase, and the two Majorana phases.

In order to specify the sequence of flavor symmetry breaking, we have to identify a hierarchy between the Yukawa entries $(Y_U)_{ij}$ and $(Y_D)_{ij}$. However, before the flavor symmetry is actually broken, the Yukawa matrices can be freely rotated by transformation matrices in G_F, and therefore the *a priori* ranking of individual entries in the Yukawa matrices seems to be somewhat ambiguous. On the other hand, the right-handed rotations and a *common* left-handed rotation for up- and down-quarks are not observable in the SM, anyway, leaving the quark masses and CKM angles as the only relevant parameters. We therefore find it sufficient to choose a basis where the right-handed rotations are unity, while for the left rotations we restrict ourselves to matrices V_{u_L} and V_{d_L}, which scale in the same manner as the CKM matrix. This leaves us with the generic power counting[5]

$$\begin{aligned} \langle Y_U\rangle_{ij} &\sim (V_{u_L})_{ij}(y_u)_j \sim \begin{pmatrix} \lambda^{n_u} & \lambda^{1+n_c} & \lambda^3 \\ \lambda^{1+n_u} & \lambda^{n_c} & \lambda^2 \\ \lambda^{3+n_u} & \lambda^{2+n_c} & 1 \end{pmatrix}, \\ \langle Y_D\rangle_{ij} &\sim (V_{d_L})_{ij}(y_d)_j \sim \begin{pmatrix} \lambda^{n_d} & \lambda^{1+n_s} & \lambda^{3+n_b} \\ \lambda^{1+n_d} & \lambda^{n_s} & \lambda^{2+n_b} \\ \lambda^{3+n_d} & \lambda^{2+n_s} & \lambda^{n_b} \end{pmatrix}, \end{aligned} \tag{2.4}$$

where we introduced the scaling for quark Yukawa couplings with the Wolfenstein parameter ($\lambda \sim 0.2 \ll 1$) as $y_i \sim \lambda^{n_i}$ (with $n_t = 0$), and inserted the standard power counting for CKM elements,

$$V_{u_L} \sim V_{d_L} \sim V_{\text{CKM}} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}. \tag{2.5}$$

The scaling of the quark masses can be constrained from the phenomenological information in Table I, where we assume in the following that renormalization-group effects (in the sequence of effective theories to be constructed) do not change the hierarchies observed at low scales a lot. More precisely, to keep the discussion simple, we restrict ourselves to

(i) $n_d > n_s > n_b > 0$ and $n_u > n_c > n_t \equiv 0$,
(ii) $n_c \geq n_b$ and $n_s > n_c$.

The remaining degree of freedom in choosing values for the n_i leads to several options, among which are also cases where one or two spurions receive their VEV at the same scale simultaneously. To be concrete, we focus on three cases with more or less natural and distinct scale separation, (a1) $n_c < n_b + 2 < n_b + 3 < n_s$, (a2) $n_c < n_b + 2 < n_s < n_b + 3$, and (b) $n_b + 2 < n_c < n_s < n_c + 1$, which are summarized in Table II. A detailed derivation

[5]During the sequence of flavor symmetry breaking, some of the entries can actually be set to zero by exploiting the freedom to rotate the VEVs of certain spurion fields with respect to the corresponding residual flavor group.

TABLE I. SM values for the quark masses [21], and approximate scaling with the Wolfenstein parameter $\lambda \sim 0.2$. Light-quark masses (u, d, s) are given in the $\overline{\mathrm{MS}}$ scheme at $\mu = 2$ GeV, charm and bottom masses as $\bar{m}_c(\bar{m}_c)$ and $\bar{m}_b(\bar{m}_b)$, and the top mass is evolved down to the scale m_b. The evolution between the scales m_b and m_c is negligible for our considerations.

	u	d	s
m_q	[1.5–4.5] MeV	[5.0–8.5] MeV	[80–155] MeV
$n_q = \log_\lambda(m_q/m_t)$	6–9	6–8	4–6
	c	b	t
m_q	[1.0–1.4] GeV	[4.0–4.5] GeV	[250–300] GeV
$n_q = \log_\lambda(m_q/m_t)$	3–4	2–3	0

TABLE II. Three alternative sequences of flavor symmetry breaking, and associated parameter counting for the Yukawa matrices. Notice that the following equalities always hold: #Spurions + #VEVs − #Symmetries = 10, #Goldstones + #Spurions + #VEVs = 36, where 10 refers to the 6 quark masses + three CKM rotations + one CKM phase, and 36 refers to the original 2×18 real parameters in the Yukawa matrices Y_U and Y_D.

Flavor symmetry		GBs	Spur.	VEVs	Symm.	Scale
$SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R} \times U(1)^2$		0	36	0	26	
$SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(3)_{D_R} \times U(1)^3$		9	26	1	17	$\Lambda \sim y_t \Lambda$
$SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(2)_{D_R} \times U(1)^3$		14	20	2	12	$\Lambda' \sim y_b \Lambda$
(a1)	$SU(2)_{D_R} \times U(1)^4$	19	14	3	7	$\Lambda^{(2a)} \sim y_c \Lambda$
	$SU(2)_{D_R} \times U(1)^3$	20	12	4	6	$\Lambda^{(3a1)} \sim y_b \lambda^2 \Lambda$
	$SU(2)_{D_R} \times U(1)^2$	21	10	5	5	$\Lambda^{(4a1)} \sim y_b \lambda^3 \Lambda$
	$U(1)^2$	24	6	6	2	$\Lambda^{(5a1)} \sim y_s \Lambda$
(a2)	$SU(2)_{D_R} \times U(1)^3$	20	12	4	6	$\Lambda^{(3a1)} \sim y_b \lambda^2 \Lambda$
	$U(1)^3$	23	8	5	3	$\Lambda^{(4a2)} \sim y_s \Lambda$
	$U(1)^2$	24	6	6	2	$\Lambda^{(5a2)} \sim y_b \lambda^3 \Lambda$
(b)	$SU(2)_{U_R} \times SU(2)_{D_R} \times U(1)^3$	17	16	3	9	$\Lambda^{(2b)} \sim y_b \lambda^2 \Lambda$
	$SU(2)_{D_R} \times U(1)^3$	20	12	4	6	$\Lambda^{(3b)} \sim y_c \Lambda$
	$U(1)^3$	23	8	5	3	$\Lambda^{(4b)} \sim y_s \Lambda$
	$U(1)^2$	24	6	6	2	$\Lambda^{(5b)} \sim y_c \lambda \Lambda$
$U(1)^2$	($\not{C}\not{P}$)	24	4	7 + 1	2	$\Lambda^{(6)} \sim y_s \lambda \Lambda$
...	($\not{C}\not{P}$)	26	0	9 + 1	0	$\Lambda^{(7)} \sim y_{u,d} \Lambda$

of the various steps in the flavor symmetry breaking can be found in the appendix.

Let us discuss some common and distinct features of the different scenarios:

(i) Common to all scenarios is the second step of symmetry breaking, which (at least in our setup with only one electroweak Higgs doublet) is unambiguously induced by the VEV for the $(Y_D)_{33}$ element, which gives rise to the bottom-quark mass. Below the scale $\Lambda' \sim y_b \Lambda$, the residual flavor symmetry is

$$G_F'' = U(2)_{Q_L} \times U(2)_{U_R} \times U(2)_{D_R}. \tag{2.6}$$

At first glance, it appears as just the two-family analogue of the original flavor group G_F. However, there are two important differences: First, it appears one additional $U(1)$ factor compared to G_F. Second, it still contains an off-diagonal spurion field χ_s, which is a doublet of $SU(2)_{Q_L}$ and the only spurion which is charged under the additional $U(1)$. Only if this spurion field (and the associated breaking of the extra $U(1)$ symmetry) were absent, we would recover an effective two-family model where, as is well known, one would have no CP violation in the quark Yukawa sector.

(ii) From an aesthetic point of view, the alternative labeled (a1) in Table II is somewhat favored. It can be realized with a rather natural hierarchy of scales. For instance, taking

$$n_b = 2, \qquad n_c = 3, \qquad n_s = 6, \qquad n_{u,d} = 8,$$

which fits well to the phenomenological mass spectrum, one obtains an equal separation of scales,[6]

$$\Lambda^{(n)} = \lambda^{(n+1)}\Lambda.$$

Moreover, the smallest non-Abelian subgroup for this case is given by

$$SU(2)_{D_R} \times U(1)^2.$$

This residual flavor symmetry may thus be taken as the simplest nontrivial example to study the dynamics of flavor spurions and its consequences for flavor physics, including the construction of higher-dimensional operators for flavor transitions with minimal flavor violation (or beyond [22]), the dynamics of Goldstone modes, and the construction of realistic scalar potentials.

(iii) In all cases, the symmetry is eventually broken down to

$$U(1)^2 = U(1)_{u_R} \times U(1)_{d_R}.$$

The corresponding effective theory now still contains three complex spurion fields, among which one spurion is uncharged under either of the two $U(1)$ groups. Consequently, when the latter acquires its VEV, its phase cannot be rotated away by symmetry transformation.[7] At this very step, we therefore generically encounter a CP-violating phase, which in our case is associated with the $(Y_D)_{12}$ element.

(iv) Finally, the two $U(1)$ symmetries will be broken by the $(Y_U)_{11}$ and $(Y_D)_{11}$ elements associated with the up- and down-quark mass. Notice that these symmetries are chiral, and the corresponding $U(1)$ anomalies contribute to the effective θ parameter in QCD. The related spurion fields may serve as a solution to the strong CP problem as in the general Peccei-Quinn setup [24–26]. This will be discussed in more detail in [19].

III. INVARIANTS AND POTENTIALS FOR SCALAR SPURION FIELDS

In this section we consider how the sequential symmetry breaking, described in the last section, could be achieved spontaneously. The question of how an appropriate potential could look like is discussed in many different contexts (see e.g. [27–29]), but no general recipe for constructing a potential that leads to a specific symmetry breaking has been found.

In any case, a potential for the spurion fields can only depend on invariants under the flavor symmetry group G_F. Because of the form of the potential these invariants should take the appropriate VEVs, which finally specify the ten physical parameters (six quark masses and four CKM parameters). Of course we are unable to derive a potential that achieves this complicated symmetry breaking, but we may at least identify ten independent invariants in terms of which we may express the physical quantities. These invariants can be constructed from monomials of the basic scalar spurion fields $Y_U(x)$ and $Y_D(x)$, and may thus be classified by their canonical dimension.

Before considering the three-family case, it is instructive to look at the simpler example of two families with the flavor symmetry $g_F = SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(2)_{D_R} \times U(1)^2$, first. It exhibits 11 symmetry generators, which leaves 5 physical parameters (4 masses and the Cabibbo angle) from the 16 parameters in the Yukawa matrices. Classifying the invariants by increasing canonical dimension, we find

$$\begin{aligned}
i_1^{(2)} &= \mathrm{tr}(U), \qquad v_1^{(2)}/\Lambda^2 = y_u^2 + y_c^2,\\
i_2^{(2)} &= \mathrm{tr}(D), \qquad v_2^{(2)}/\Lambda^2 = y_d^2 + y_s^2,\\
i_1^{(4)} &= \mathrm{tr}(U^2) - (i_1^{(2)})^2, \qquad v_1^{(4)}/\Lambda^4 = -2y_u^2 y_c^2,\\
i_2^{(4)} &= \mathrm{tr}(UD) - i_1^{(2)} i_2^{(2)},\\
v_2^{(4)}/\Lambda^4 &= \sin^2\theta(y_c^2 - y_u^2)(y_d^2 - y_s^2) - y_c^2 y_d^2 - y_u^2 y_s^2,\\
i_3^{(4)} &= \mathrm{tr}(D^2) - (i_2^{(2)})^2, \qquad v_3^{(4)}/\Lambda^4 = -2y_d^2 y_s^2,
\end{aligned} \tag{3.1}$$

where we introduced the combinations

$$U = Y_U Y_U^\dagger, \qquad D = Y_D Y_D^\dagger, \tag{3.2}$$

which transform homogeneously under $SU(2)_{Q_L}$, and where we denote with $v_\alpha^{(k)} = \langle i_\alpha^{(k)} \rangle$ the VEVs of the 5 invariants. The potential $V = V(i_\alpha^{(m)})$ may now be expanded around its minimal value in the form

$$V = \sum_{k,m} \sum_{\alpha,\beta} \frac{1}{\Lambda^{m+k-4}} (i_\alpha^{(m)} - v_\alpha^{(m)}) M_{\alpha,\beta}^{(m,k)} (i_\beta^{(k)} - v_\beta^{(k)}), \tag{3.3}$$

where Λ is a UV scale, which renders the positive semi-definite matrix $M_{\alpha,\beta}^{(m,k)}$ dimensionless. Notice that higher-dimensional operators appear unavoidably if we assign canonical mass dimension to the (scalar) spurion fields $Y_{U,D}$. As already mentioned, the mechanism showing how such an effective potential could be generated by integrating out some new degrees of freedom in an underlying theory, remains an open issue.

In principle we may also invert the relations to obtain the Cabibbo angle and the masses as functions of the $v_i^{(k)}$;

[6]For comparison, scenario (a2) can be realized, for instance, by $n_b = 2.5$, $n_c = 3.5$, $n_s = 5$, $n_{u,d} = 7$, leading to the tower of scales $(\lambda^{2.5}, \lambda^{3.5}, \lambda^{4.5}, \lambda^5, \lambda^{5.5}, \lambda^6, \lambda^7)\Lambda$. Similarly, case (b) could be realized by $n_b = 2$, $n_c = 4.5$, $n_s = 5$, $n_{u,d} = 7$, with $(\lambda^2, \lambda^4, \lambda^{4.5}, \lambda^5, \lambda^{5.5}, \lambda^6, \lambda^7)\Lambda$.

[7]Alternatively, in a previous step of the construction, one could have identified two spurion fields with the same quantum numbers, whose VEVs in general cannot be made real simultaneously. This mechanism thus gives a particular realization of spontaneous CP violation [23].

however, the above invariants are not yet very suitable for the further discussion:

(i) As we have seen in the previous section, the order of the different symmetry-breaking steps depends on the relative size of the Yukawa entries, which in the two-family case are characterized by the exponents $\{n_u, n_c, n_d, n_s, (1+n_s)\}$ (in the hierarchical limit). It is therefore desirable to consider invariants that feature the very same exponents.

(ii) To put the invariants on a similar footing, they should have the same canonical dimension (i.e. we have to introduce rational functions of the above invariants).

(iii) Instead of $i_2^{(4)}$ it would be desirable to have an invariant that vanishes in the no-mixing case ($\theta = 0$). Such invariants can be constructed from the commutator $[U, D]$,

$$\begin{aligned} i_1^{(8)} &= \det([U, D]), \\ v_1^{(8)}/\Lambda^8 &= \tfrac{1}{4}(y_c^2 - y_u^2)^2(y_s^2 - y_d^2)^2\sin^2 2\theta. \end{aligned} \tag{3.4}$$

We therefore modify the above definitions as follows:

$$\begin{aligned} I_1 &= \mathrm{tr}(U), \qquad V_1/\Lambda^2 = y_u^2 + y_c^2, \\ I_2 &= \mathrm{tr}(D), \qquad V_2/\Lambda^2 = y_d^2 + y_s^2, \\ I_3 &= \frac{1}{2}(I_1 - \mathrm{tr}(U^2)/I_1), \qquad V_3/\Lambda^2 = \frac{y_u^2 y_c^2}{y_u^2 + y_c^2}, \\ I_4 &= \frac{1}{2}(I_2 - \mathrm{tr}(D^2)/I_2), \qquad V_4/\Lambda^2 = \frac{y_d^2 y_s^2}{y_s^2 + y_d^2}, \\ I_5 &= 4\frac{\det([U, D])}{I_1 I_2 (I_1 + I_2)}, \\ V_5/\Lambda^2 &= \frac{(y_c^2 - y_u^2)^2 (y_s^2 - y_d^2)^2 \sin^2 2\theta}{(y_u^2 + y_c^2)(y_d^2 + y_s^2)(y_u^2 + y_c^2 + y_d^2 + y_s^2)}. \end{aligned} \tag{3.5}$$

The invariants $I_{1\text{–}5}$ now take their VEVs according to the power counting for masses and mixing angles. For instance, with our standard case, $n_c < n_s < 1 + n_s < n_u \sim n_d$, we have

$$V_1 \sim \lambda^{2n_c} \gg V_2 \sim \lambda^{2n_s} \gg V_5 \sim \lambda^{2+2n_s} \gg V_{3,4} \sim \lambda^{2n_{u,d}},$$

which defines the sequence of symmetry breaking. We may then solve (3.5) for masses and mixing angle to obtain

$$\begin{aligned} y_{c,u}^2 &= \frac{V_1 \pm \sqrt{V_1(V_1 - 4V_3)}}{2\Lambda^2} \simeq \begin{cases} V_1/\Lambda^2 \\ V_3/\Lambda^2 \end{cases}, \\ y_{s,d}^2 &= \frac{V_2 \pm \sqrt{V_2(V_2 - 4V_4)}}{2\Lambda^2} \simeq \begin{cases} V_2/\Lambda^2 \\ V_4/\Lambda^2 \end{cases}, \\ \sin^2 2\theta &= \frac{(V_1 + V_2)V_5}{(V_1 - 4V_3)(V_2 - 4V_4)} \simeq \frac{V_5}{V_2}, \end{aligned} \tag{3.6}$$

where the approximate relations refer to the SM hierarchies.

We note in passing that models based on texture zeros, which imply relations between the masses and the mixing angles [30], may be mapped onto relations between invariants. In turn, a relation between invariants always characterizes a class of Yukawa matrices that may or may not feature texture zeros in a particular flavor basis. This may be explicitly demonstrated by considering a simple two-family model with one texture zero. We use the basis in which Y_U is diagonal and

$$Y_D = \begin{pmatrix} 0 & a \\ a & 2b \end{pmatrix} \tag{3.7}$$

is given in terms of two parameters a and b. This model implies the relation

$$V_5 = \frac{4(V_1 - 4V_3)(\sqrt{V_2 V_4} - 2V_4)}{V_1 + V_2} \simeq 4\sqrt{V_2 V_4}, \tag{3.8}$$

which translates into a relation between the Cabibbo angle and the down-type masses,

$$\tan\theta \simeq \sqrt{\frac{m_d}{m_s}}, \tag{3.9}$$

which is phenomenologically reasonable.

We now turn to the three-family case, which can be studied along the same lines. We have to identify in total ten independent invariants. The two quadratic and the three quartic invariants are again given by

$$i_1^{(2)} = \mathrm{tr}(U), \qquad i_2^{(2)} = \mathrm{tr}(D), \tag{3.10}$$

and

$$\begin{aligned} i_1^{(4)} &= \mathrm{tr}(U^2) - (i_1^{(2)})^2, \qquad i_2^{(4)} = \mathrm{tr}(UD) - i_1^{(2)} i_2^{(2)}, \\ i_3^{(4)} &= \mathrm{tr}(D^2) - (i_2^{(2)})^2. \end{aligned} \tag{3.11}$$

The remaining five invariants, which are necessary to specify the physical quark flavor parameters, thus have to be built from even higher-dimensional invariants. For the dimension-six terms, we choose

$$\begin{aligned} i_1^{(6)} &= \mathrm{tr}(U^3) - \tfrac{3}{2} i_1^{(4)} i_1^{(2)} - (i_1^{(2)})^3 \equiv 3\det(U), \\ i_2^{(6)} &= \mathrm{tr}(U^2 D) - \tfrac{1}{2} i_1^{(4)} i_2^{(2)} - i_2^{(4)} i_1^{(2)} - i_2^{(2)} (i_1^{(2)})^2, \\ i_3^{(6)} &= \mathrm{tr}(U D^2) - \tfrac{1}{2} i_3^{(4)} i_1^{(2)} - i_2^{(4)} i_2^{(2)} - i_1^{(2)} (i_2^{(2)})^2, \\ i_4^{(6)} &= \mathrm{tr}(D^3) - \tfrac{3}{2} i_3^{(4)} i_2^{(2)} - (i_2^{(2)})^3 \equiv 3\det(D). \end{aligned} \tag{3.12}$$

Finally, among the dimension-eight invariants only one is independent, and we choose

$$i_1^{(8)} = \mathrm{tr}(U[U, D]D), \tag{3.13}$$

which completes the list of invariants for the 3×3 case.

The potential $V = V(i_\alpha^{(m)})$ can again be expanded as in (3.3). The sequential breaking of G_F as proposed in the last section can emerge only through a hierarchy of VEVs for the various invariants. This hierarchy has to be put in by hand in (3.3) and may perhaps find its explanation in an underlying theory above the scale Λ.[8] In fact, our choice of VEVs is such that the first breaking of $G_F \to G_F'$ is obtained, if the potential V generates a (sizeable) VEV for the $i_1^{(2)}$ invariant, only,

$$v_1^{(2)} \simeq y_t^2\Lambda^2, \qquad v_k^{(m)} \simeq 0 \qquad \text{otherwise}, \tag{3.14}$$

in which case we obtain a nonvanishing top-quark Yukawa coupling, while all other parameters (which give rise to the lighter quark masses and CKM parameters) still (approximately) vanish.

The next step is the breaking of $G_F' \to G_F''$. Clearly, the relevant potential V' can only depend on the invariants of G_F', which we denote as $j_k^{(m)}$. As before, we introduce quadratic terms which transform under $SU(2)_L \times U(1)_T$, namely, two triplets,

$$U' = Y_U^{(2)} Y_U^{(2)\dagger}, \qquad D' = \tilde{Y}_D^{(2)} \tilde{Y}_D^{(2)\dagger}, \tag{3.15}$$

one charged doublet

$$X' = \tilde{Y}_D^{(2)} \xi_b, \tag{3.16}$$

and one singlet

$$\Xi' = \xi_b^\dagger \xi_b. \tag{3.17}$$

In terms of these, the invariants of dimension-two can be written as

$$j_1^{(2)} = \mathrm{tr}(U'), \qquad j_2^{(2)} = \mathrm{tr}(D'), \qquad j_3^{(2)} = \Xi', \tag{3.18}$$

while the fourth-order invariants are

$$\begin{aligned} j_1^{(4)} &= \mathrm{tr}((U')^2) - (j_1^{(2)})^2, & j_2^{(4)} &= \mathrm{tr}(U'D') - j_1^{(2)} j_2^{(2)}, \\ j_3^{(4)} &= \mathrm{tr}((D')^2) - (j_2^{(2)})^2, & j_4^{(4)} &= X'^\dagger X' - j_2^{(2)} j_3^{(2)}. \end{aligned} \tag{3.19}$$

Finally, there are two independent invariants of dimension six,

$$j_1^{(6)} = X'^\dagger U' X', \qquad j_2^{(6)} = X'^\dagger D' X'. \tag{3.20}$$

At tree level, the potential V' simply follows from the original potential V by expressing the invariants $i_k^{(m)}$ by the invariants $j_\alpha^{(m)}$ and the VEV for the top Yukawa coupling, see Appendix C. Including radiative corrections in the effective theory below the scale Λ (or more precisely, below the mass scale of the scalar degree of freedom related to the VEV $y_t\Lambda$), the parameters of the effective potential might change accordingly. The general form is thus again given by

$$V' = \sum_{k,m} \sum_{\alpha,\beta} \frac{1}{(\Lambda')^{m+k-4}} (j_\alpha^{(m)} - w_\alpha^{(m)}) N_{\alpha,\beta}^{(m,k)} (j_\beta^{(k)} - w_\beta^{(k)}). \tag{3.21}$$

The next step in the symmetry breaking $G_F' \to G_F''$ will then be achieved by $w_3^{(2)} \simeq y_b^2(\Lambda')^2$. This scheme can be repeated until the complete flavor symmetry is broken.

Note, that the invariants $i_i^{(m)}$ and $j_i^{(m)}$ introduced above are all real. Therefore, the parameters $M_{i,j}^{(m,k)}$ and $N_{i,j}^{(m,k)}$ have to be real as well to yield a Hermitian potential. As described above, the CKM phase, corresponding to the SM mechanism for CP violation, appears when one of the spurion fields receives a complex VEV. The potential allows for spontaneous CP violation, as soon as an invariant of one of the residual flavor symmetries becomes complex. In the scenarios discussed above, this is the case for

$$\begin{aligned} G_F^{3a1}\colon L_1^{(4)} &= \mathrm{Re}(\chi_{13}^* \xi_d^\dagger \xi_s \chi_{23}), \\ L_2^{(4)} &= \mathrm{Im}(\chi_{13}^* \xi_d^\dagger \xi_s \chi_{23}), \end{aligned} \tag{3.22}$$

$$\text{and} \quad \begin{aligned} G_F^{(3b)}\colon L_1'^{(4)} &= \mathrm{Re}(\xi_u^\dagger \xi_c \xi_s^\dagger \xi_d), \\ L_2'^{(4)} &= \mathrm{Im}(\xi_u^\dagger \xi_c \xi_s^\dagger \xi_d), \end{aligned} \tag{3.23}$$

where $L_2^{(\prime)(4)}$ is odd under CP.

As in the two-family example, we again introduce rational functions of the invariants that are convenient for the discussion of power counting or parameter relations in models with texture zeros. The modified set of invariants for the three-family case reads

[8] We note, however, that restricting ourselves to the most general set of dimension-four operators, where

$$V = \sum_i m_i^2 i_i^{(2)} + \sum_{i,j} 2\rho_{ij} i_i^{(2)} i_j^{(2)} + \sum_i \lambda_i i_i^{(4)},$$

only part of the flavor symmetry will be broken by the minimum of the potential, including the case $G_F \to G_F'$ for a particular subset of parameter space.

$$
\begin{aligned}
I_1 &= \mathrm{tr}(U), \qquad V_1/\Lambda^2 = y_u^2 + y_c^2 + y_t^2 \sim \lambda^0,\\
I_2 &= \mathrm{tr}(D), \qquad V_2/\Lambda^2 = y_d^2 + y_s^2 + y_b^2 \sim \lambda^{2n_b},\\
I_3 &= \frac{1}{2}(I_1 - \mathrm{tr}(U^2)/I_1),\\
V_3/\Lambda^2 &= \frac{y_u^2 y_c^2 + y_u^2 y_t^2 + y_c^2 y_t^2}{y_u^2 + y_c^2 + y_t^2} \sim \lambda^{2n_c},\\
I_4 &= \frac{1}{2}(I_2 - \mathrm{tr}(D^2)/I_2),\\
V_4/\Lambda^2 &= \frac{y_d^2 y_s^2 + y_d^2 y_b^2 + y_s^2 y_b^2}{y_s^2 + y_d^2 + y_b^2} \sim \lambda^{2n_s},\\
I_5 &= \det(U)/I_1/I_3,\\
V_5/\Lambda^2 &= \frac{y_u^2 y_c^2 y_t^2}{y_u^2 y_c^2 + y_u^2 y_t^2 + y_c^2 y_t^2} \sim \lambda^{2n_u},\\
I_6 &= \det(D)/I_2/I_4,\\
V_6/\Lambda^2 &= \frac{y_d^2 y_s^2 y_b^2}{y_d^2 y_s^2 + y_d^2 y_b^2 + y_s^2 y_b^2} \sim \lambda^{2n_d},
\end{aligned}
\tag{3.24}
$$

which determines the 6 Yukawa couplings corresponding to the quark masses, and

$$
\begin{aligned}
I_7 &= \frac{\mathrm{tr}(U[U,D]D)}{I_1 I_2 (I_1 + I_2)}, \qquad V_7/\Lambda^2 \simeq y_b^2 \theta_{23}^2 \sim \lambda^{2(n_b+2)},\\
I_8 &= \frac{1}{2}\frac{\det([U,[U,D]])}{I_1^2 I_2 (I_1 + I_2)^2 I_3^2 I_7},\\
V_8/\Lambda^2 &\simeq y_b^2 \theta_{13}^2 + y_s^2 \frac{\theta_{12}\theta_{13}}{\theta_{23}} \cos\delta \sim \lambda^{2(n_b+3)} + \lambda^{2(n_s+1)},\\
I_9 &= \frac{1}{2}\frac{\det([[U,D],D])}{I_2^2 I_1 (I_1 + I_2)^2 I_4^2 I_7},\\
V_9/\Lambda^2 &\simeq y_b^2(\theta_{13}^2 + \theta_{12}^2\theta_{23}^2 - 2\theta_{12}\theta_{23}\theta_{13}\cos\delta) \sim \lambda^{2(n_b+3)},\\
I_{10} &= -\frac{i}{2}\frac{\det([U,D])}{I_1^2 I_2^2 (I_3 + I_4)},\\
V_{10}/\Lambda^2 &\simeq y_s^2 \theta_{12}\theta_{23}\theta_{13}\sin\delta \sim \lambda^4 \lambda^{2(n_s+1)},
\end{aligned}
\tag{3.25}
$$

which determines the angles and the CP-violating phase in the standard parametrization [31]. Again, the invariants I_{7-10} are defined in such a way that they vanish in the no-mixing case. Moreover, $I_{10} \neq 0$ signals CP violation.

We may again solve for the SM parameters to obtain the quark Yukawa couplings $y_{t,c,u}^2(V_{1,3,5})$ and $y_{b,s,d}^2(V_{2,4,6})$, as well as the (approximate) solutions for the mixing angles

$$
\begin{aligned}
&\theta_{23}^2 \simeq \frac{V_7}{V_2}, \qquad \theta_{13}^2 \simeq \frac{V_8}{V_2},\\
&\left(\theta_{12}\cos\delta - \frac{\theta_{13}}{\theta_{23}}\right)^2 \simeq \frac{V_9}{V_7} - \frac{V_2^2 V_{10}^2}{V_4^2 V_7 V_8},\\
&\theta_{12}^2 \sin^2\delta \simeq \frac{V_2^2 V_{10}^2}{V_4^2 V_7 V_8},
\end{aligned}
\tag{3.26}
$$

where we also neglected terms of order $\lambda^{-4} y_s^2/y_b^2$. Finally, we consider again a simple model with texture zeros in the 3×3 Yukawa matrices [32],

$$
Y_U = \begin{pmatrix} 0 & C_u & 0 \\ C_u^* & 0 & B_u \\ 0 & B_u^* & |A_u| \end{pmatrix}, \qquad Y_D = \begin{pmatrix} 0 & C_d & 0 \\ C_d^* & 0 & B_d \\ 0 & B_d^* & |A_d| \end{pmatrix},
\tag{3.27}
$$

which yields the following approximate relations between quark masses and mixing angles:

$$
\begin{aligned}
&\frac{|V_{ub}|^2}{|V_{cb}|^2} \simeq \frac{\theta_{13}^2}{\theta_{23}^2} \simeq \frac{m_u}{m_c},\\
&\frac{|V_{td}|^2}{|V_{ts}|^2} \simeq \frac{\theta_{13}^2 + \theta_{12}^2\theta_{23}^2 - 2\theta_{12}\theta_{23}\theta_{13}\cos\delta}{\theta_{23}^2} \simeq \frac{m_d}{m_s}.
\end{aligned}
\tag{3.28}
$$

As before, this can be formulated in a basis-independent way in terms of the following approximate relations between invariants:

$$
\frac{V_8}{V_7} \simeq \sqrt{\frac{V_5}{V_3}}, \qquad \frac{V_9}{V_7} \simeq \sqrt{\frac{V_6}{V_4}}.
\tag{3.29}
$$

IV. CONCLUSIONS

In this paper we have shown how the hierarchies in quark masses and mixings can be associated with a particular sequence of flavor symmetry breaking. The different scales at which the individual steps of partial flavor symmetry breaking occur are separated among each other by not more than 1–2 orders of magnitude. Depending on the assumed power counting for the quark masses, we have identified different scenarios that are compatible with phenomenology. We have also given some general arguments for the possible form of scalar potentials that may realize the sequence of flavor symmetry breaking and identified the invariants that may be used to expand the potential around its minimum or to classify ansätze for the Yukawa matrices involving texture zeros in a basis-independent way.

In all cases, the minimal *non-Abelian* flavor subgroup is given by $SU(2)_{D_R} \times U(1)^{2(3)}$. Its further breaking eventually leads to an effective theory with a residual $U(1)^2$ flavor symmetry, where one of the spurion fields is uncharged. When this spurion achieves a complex VEV, its phase cannot be rotated away and provides the one and only source for CP violation in the quark Yukawa sector. The CP-violating phase is thus generated at rather low scales (compared to, say, a grand unified theory scale).

A dynamical interpretation of the Goldstone modes, appearing at each step of the (global) flavor symmetry breaking, can be achieved by promoting the flavor symmetries to local ones, where the Goldstone modes become the longitudinal modes of the corresponding massive gauge bosons. One the other hand, the final chiral $U(1)^2$ symmetries are anomalous, and the associated Goldstone bosons couple to the QCD instantons. They may thus be used to resolve the strong *CP* problem as in the general Peccei-Quinn setup, with the corresponding Goldstone modes appearing as axion fields. Details will be presented in a separate publication [19].

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Note added in proof.—While completing this work, the paper [33] appeared, where a 2-Higgs-doublet scenario (2HDM) with a large ratio of VEVs ($\tan\beta \sim m_t/m_b \gg 1$) was considered. In this case, the original flavor symmetry is broken in one step as $G_F \to G_F'' = U(2)^3$, see Eq. (2.6) and the discussion in [16]. The possible enhancement with $\tan\beta$ allows for interesting observable deviations from the SM and from minimally flavor-violating scenarios with minimal Higgs sector. In [33] it has been shown that they can be identified in a very transparent way using the nonlinear representation of flavor symmetries suggested in [16]. It is evident, that the related change in the hierarchies of the Yukawa matrices for $\tan\beta \gg 1$ would also imply a different pattern for the sequence of flavor symmetry breaking, which could be worked out in an analogous way as presented in our work.

APPENDIX A: SEQUENCE OF FLAVOR-SYMMETRY BREAKING IN THE SM

In this appendix, we present the detailed derivation of the different scenarios for sequential flavor-symmetry breaking as discussed in the text.

1. Leading order

Neglecting all terms of $\mathcal{O}(\lambda)$ in Y_U and Y_D, only the top-quark Yukawa coupling in $(Y_U)_{33}$ survives, due to our general assumption $n_q > 0$. We thus obtain the breaking (which—apart from the additional $U(1)$ factors—coincides with the discussion in [16])

$$G_F \to G_F' = SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(3)_{D_R} \times U(1)_T \times U(1)_{U_R^{(2)}} \times U(1)_{D_R}, \tag{A1}$$

$$\sim SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(3)_{D_R} \times U(1)_{Q_L^{(2)}} \times U(1)_{U_R^{(2)}} \times U(1)_{D_R}, \tag{A2}$$

where the equivalence in the second line arises if we take into account the globally conserved baryon number, implying the relation

$$3B = T + Q_L^{(2)} + U_R^{(2)} + D_R$$

for the quark charges, where T counts the quark number for the third generation in Q_L and U_R, and $Q_L^{(2)}$ and $U_R^{(2)}$ for the first two generations (see also Appendix B). The decomposition of Y_U and Y_D in terms of irreducible representations of G_F' and the representation of the nine Goldstone modes ($\Pi_{L,U_R}^{a=4..8}$, $\Pi_L^8 = -\Pi_{U_R}^8$) remains as in [16], with

$$\begin{aligned} Y_U &= \mathcal{U}(\Pi_L)\begin{pmatrix} \begin{matrix} Y_U^{(2)} \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & y_t\Lambda \end{pmatrix}\mathcal{U}^\dagger(\Pi_{U_R}), \\ Y_D &= \mathcal{U}(\Pi_L)\begin{pmatrix} \tilde{Y}_D^{(2)} \\ \xi_b^\dagger \end{pmatrix}, \end{aligned} \tag{A3}$$

and $\mathcal{U}(\Pi) = \exp[i\Pi^a T^a/\Lambda]$.

2. Order Λ'/Λ

Let us first consider the transformation properties of the residual spurion fields with respect to G_F' [here the subscripts refer to the $U(1)$ factors defined in Eq. (A1)], and their scaling with λ,

$$\begin{aligned} Y_U^{(2)} &\sim (2, 2, 1)_{1,-1,0} \propto \begin{pmatrix} \lambda^{n_u} & \lambda^{1+n_c} \\ \lambda^{1+n_u} & \lambda^{n_c} \end{pmatrix}\Lambda, \\ \tilde{Y}_D^{(2)} &\sim (2, 1, \bar{3})_{1,0,-1} \propto \begin{pmatrix} \lambda^{n_d} & \lambda^{1+n_s} & \lambda^{3+n_b} \\ \lambda^{1+n_d} & \lambda^{n_s} & \lambda^{2+n_b} \end{pmatrix}\Lambda, \\ \xi_b^\dagger &\sim (1, 1, \bar{3})_{0,0,-1} \propto (\lambda^{3+n_d} \quad \lambda^{2+n_s} \quad \lambda^{n_b})\Lambda. \end{aligned} \tag{A4}$$

We now assume that at the scale $\Lambda' \ll \Lambda$, the next-highest entry in the residual spurion fields gets its VEV. For $n_c > n_b$, the spurion $\xi_b^\dagger$ will have the largest eigenvalue,[9]

$$\langle \xi_b^\dagger \rangle = (0, 0, \tilde{y}_b)\Lambda \equiv (0, 0, x_b)\Lambda', \tag{A5}$$

with $x_b = \mathcal{O}(1)$ such that $\tilde{y}_b \sim \Lambda'/\Lambda \sim m_b/m_t$. Similarly, as for the discussion of the 2HDM with large $\tan\beta$ in [16], this further breaks the flavor symmetry to

$$G_F' \to G_F'' = SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(2)_{D_R} \times U(1)_{Q_L^{(2)}} \times U(1)_{U_R^{(2)}} \times U(1)_{D_R^{(2)}}, \tag{A6}$$

$$\sim SU(2)_{Q_L} \times SU(2)_{U_R} \times SU(2)_{D_R} \times U(1)_{\mathrm{III}} \times U(1)_{U_R^{(2)}} \times U(1)_{D_R^{(2)}}, \tag{A7}$$

[9] If we allow for $n_c = n_b$, the spurion $Y_U^{(2)}$ also will get its VEV simultaneously, such that in the scenario (a) discussed below, the scales Λ' and Λ'' would coincide.

where now $U(1)_{\rm III}$ acts on all quarks in the third generation. The five additional Goldstone modes ($\Pi'_{D_R}{}^{a=4..8}$) are introduced as

$$\tilde{Y}_D^{(2)} = \begin{pmatrix} Y_D^{(2)} & \chi_s \end{pmatrix} \mathcal{U}^\dagger(\Pi'_{D_R}), \quad \xi_b^\dagger = \begin{pmatrix} 0 & 0 & x_b \Lambda' \end{pmatrix} \mathcal{U}^\dagger(\Pi'_{D_R}), \tag{A8}$$

with $\mathcal{U}(\Pi') = \exp[i\Pi'^a T^a/\Lambda']$.

3. Alternative (a1): $n_c < n_b + 2 < n_b + 3 < n_s$

At this stage, the further breaking of the flavor symmetry depends on the details about the assumed power counting for the quark masses. Let us first discuss the scenario (a1):

a. Order Λ''/Λ

In the case of $n_c < n_b + 2$, it is convenient to classify the residual spurion fields of G_F'' according to (A7),

$$\begin{aligned}
Y_U^{(2)} &\sim (2,2,1)_{0,-1,0} \propto \begin{pmatrix} \lambda^{1+n_u} & \lambda^{1+n_c} \\ \lambda^{1+n_u} & \lambda^{n_c} \end{pmatrix} \Lambda, \\
Y_D^{(2)} &\sim (2,1,2)_{0,0,-1} \propto \begin{pmatrix} \lambda^{n_d} & \lambda^{1+n_s} \\ \lambda^{1+n_d} & \lambda^{n_s} \end{pmatrix} \Lambda, \\
\chi_s &\sim (2,1,1)_{-1,0,0} \propto \begin{pmatrix} \lambda^{3+n_b} \\ \lambda^{2+n_b} \end{pmatrix} \Lambda,
\end{aligned} \tag{A9}$$

such that the next spurion getting a VEV is

$$\langle Y_U^{(2)} \rangle = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{y}_c \end{pmatrix} \Lambda \equiv \begin{pmatrix} 0 & 0 \\ 0 & x_c \end{pmatrix} \Lambda'', \tag{A10}$$

with $x_c = \mathcal{O}(1)$, implying $\tilde{y}_c \sim \Lambda''/\Lambda \sim m_c/m_t$. The VEV of $Y_U^{(2)}$ thus further breaks the flavor symmetry as

$$G_F'' \to G_F^{(3a1)} = SU(2)_{D_R} \times U(1)_C \times U(1)_{\rm III} \times U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(1)}}, \tag{A11}$$

$$\sim SU(2)_{D_R} \times U(1)_C \times U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(2)}}, \tag{A12}$$

where $U(1)_C$ refers to the second-generation quarks in Q_L and U_R. This implies five additional Goldstone bosons ($\Pi''_{L,U_R}{}^{a=1..3}$, $\Pi_L''^3 = -\Pi''_{U_R}{}^3$), appearing via

$$Y_U^{(2)} = \mathcal{U}(\Pi_L'') \begin{pmatrix} Y_U^{(1)} & 0 \\ 0 & x_c \Lambda'' \end{pmatrix} \mathcal{U}^\dagger(\Pi''_{U_R}), \tag{A13}$$

$$Y_D^{(2)} = \mathcal{U}(\Pi_L'') \begin{pmatrix} \xi_d^\dagger \\ \xi_s^\dagger \end{pmatrix}, \qquad \chi_s = \mathcal{U}(\Pi_L'') \begin{pmatrix} \chi_{13} \\ \chi_{23} \end{pmatrix}. \tag{A14}$$

b. Order $\Lambda^{(3)}/\Lambda$

The residual spurions of $G_F^{(3a1)}$ now scale/transform as

$$\begin{aligned}
Y_U^{(1)} &\sim (1)_{0,1,-1,0} \propto \lambda^{n_u}, \\
\xi_d^\dagger &\sim (2)_{0,1,0,-1} \propto \begin{pmatrix} \lambda^{n_d} & \lambda^{1+n_s} \end{pmatrix}, \\
\chi_{13} &\sim (1)_{0,1,0,0} \propto \lambda^{3+n_b}, \\
\xi_s^\dagger &\sim (2)_{1,0,0,-1} \propto \begin{pmatrix} \lambda^{1+n_d} & \lambda^{n_s} \end{pmatrix}, \\
\chi_{23} &\sim (1)_{1,0,0,0} \propto \lambda^{2+n_b},
\end{aligned} \tag{A15}$$

where the subscripts refer to the $U(1)$ charges in (A12). In this case, assuming $n_s > n_b + 2$, the next spurion to receive a VEV is χ_{23}, which breaks the $U(1)_C$ symmetry,

$$G_F^{(3a1)} \to G_F^{(4a1)} = SU(2)_{D_R} \times U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(2)}}. \tag{A16}$$

The associated Goldstone boson ϕ''' appears as a simple phase,

$$\chi_{23} = x_{sb} e^{i\phi'''/\Lambda'''} \Lambda''' \quad \text{and} \quad \xi_s^\dagger \to \xi_s^\dagger e^{i\phi'''/\Lambda'''}, \tag{A17}$$

with $\Lambda'''/\Lambda \sim y_b \lambda^2$.

c. Order $\Lambda^{(4)}/\Lambda$

The residual spurions for $G_F^{(4a1)}$ read

$$\begin{aligned}
Y_U^{(1)} &\sim (1)_{1,-1,0} \propto \lambda^{n_u}, \\
\xi_d^\dagger &\sim (2)_{1,0,-1} \propto \begin{pmatrix} \lambda^{n_d} & \lambda^{1+n_s} \end{pmatrix}, \\
\xi_s^\dagger &\sim (2)_{0,0,-1} \propto \begin{pmatrix} \lambda^{1+n_d} & \lambda^{n_s} \end{pmatrix}, \\
\chi_{13} &\sim (1)_{1,0,0} \propto \lambda^{3+n_b}.
\end{aligned} \tag{A18}$$

For $n_s > n_b + 3$, the next spurion to get a VEV is χ_{13}, which breaks another $U(1)$ symmetry,

$$G_F^{(4a1)} \to G_F^{(5a1)} = SU(2)_{D_R} \times U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(2)}}. \tag{A19}$$

The corresponding Goldstone mode $\phi^{(iv)}$ appears again as a phase factor used to redefine the spurions $Y_U^{(1)}$ and ξ_d according to their $U(1)$ charge. It should be noted that after the $U(1)_{Q_L^{(1)}}$ is broken, the residual spurions ξ_d and ξ_s have the same quantum numbers with respect to $G_F^{(5a1)}$, and therefore the relative phase of their VEVs will provide the source for spontaneous CP violation.

d. Order $\Lambda^{(5)}/\Lambda$

Taking $\langle \xi_s \rangle \neq 0$, we next break

$$G_F^{(5a1)} \to G_F^{(6a1)} = U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(1)}}, \tag{A20}$$

and the remaining spurion fields are given by

$$Y_U^{(1)} \sim (1)_{-1,0} \propto \lambda^{n_u}, \qquad Y_D^{(1)} \sim (1)_{0,-1} \propto \lambda^{n_d}, \\ \chi_{12} \sim (1)_{0,0} \propto \lambda^{1+n_s}. \tag{A21}$$

Here, we have decomposed the $SU(2)_{D_R}$ doublet ξ_d into two complex singlets $Y_D^{(1)}$ and χ_{12},

$$\xi_s^\dagger = (0, x_{ss}\Lambda^{(5a1)})\mathcal{U}^\dagger(\Pi_R^{(v)}), \\ \xi_d^\dagger = (Y_D^{(1)}, \chi_{12})\mathcal{U}^\dagger(\Pi_R^{(v)}). \tag{A22}$$

4. Alternative (a2): $n_c < n_b + 2 < n_s < n_b + 3$

a. Order Λ''/Λ and order $\Lambda^{(3)}/\Lambda$

These steps are the same as for the alternative (a1) above.

b. Order $\Lambda^{(4)}/\Lambda$

In this case, i.e. for $n_s < n_b + 3$, the next spurion to receive a VEV is $\langle \xi_s \rangle \sim y_s\Lambda$, which breaks

$$G_F^{(4a1)} \to G_F^{(5a2)} = U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(1)}}, \tag{A23}$$

leaving us with four singlet spurion fields

$$Y_U^{(1)} \sim (1)_{1,-1,0} \propto \lambda^{n_u}, \qquad Y_D^{(1)} \sim (1)_{1,0,-1} \propto \lambda^{n_d}, \\ \chi_{12} \sim (1)_{1,0,0} \propto \lambda^{n_s+1}, \qquad \chi_{13} \sim (1)_{1,0,0} \propto \lambda^{n_b+3}. \tag{A24}$$

c. Order $\Lambda^{(5)}/\Lambda$

Taking now $\langle \chi_{13} \rangle \neq 0$, we break

$$G_F^{(5a2)} \to G_F^{(6a2)} = U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(1)}}, \tag{A25}$$

with the remaining spurion fields as for case (a1).

5. Alternative (b): $n_b + 2 < n_c < n_s < n_c + 1$

The case $n_c > n_b + 2$ may be considered as somewhat less likely, because in order to have $y_c/y_b \lesssim \lambda^2$ at some high scale we would have to require sizeable renormalization effects in order to recover $m_c/m_b \sim 0.3$ at low scales.

a. Order Λ''/Λ

In that case, the next spurion to get a VEV would be

$$\langle \chi_s \rangle = \begin{pmatrix} 0 \\ y_{23} \end{pmatrix}\Lambda = \begin{pmatrix} 0 \\ x_{23} \end{pmatrix}\Lambda'', \tag{A26}$$

with $x_{23} = \mathcal{O}(1)$ and thus $\Lambda''/\Lambda = \lambda^2 m_b/m_t$. This leads to the breaking

$$G_F'' \to G_F^{(3b)} = SU(2)_{U_R} \times SU(2)_{D_R} \times U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(2)}} \\ \times U(1)_{D_R^{(2)}}. \tag{A27}$$

Introducing three new Goldstone bosons $((\Pi_L'')^{a=1,2,3})$, we parametrize

$$Y_U^{(2)} = \mathcal{U}(\Pi_L'')\begin{pmatrix} \xi_u^\dagger \\ \xi_c^\dagger \end{pmatrix}, \qquad Y_D^{(2)} = \mathcal{U}(\Pi_L'')\begin{pmatrix} \xi_d^\dagger \\ \xi_s^\dagger \end{pmatrix}, \\ \chi_s = \mathcal{U}(\Pi_L'')\langle \chi_s \rangle. \tag{A28}$$

b. Order $\Lambda^{(3)}/\Lambda$

The residual spurions of $G_F^{(3b)}$ scale/transform as

$$\xi_u^\dagger \sim (2,1)_{1,-1,0} \propto (\lambda^{n_u} \quad \lambda^{1+n_c}), \\ \xi_d^\dagger \sim (1,2)_{1,0,-1} \propto (\lambda^{n_d} \quad \lambda^{1+n_s}), \\ \xi_c^\dagger \sim (2,1)_{0,-1,0} \propto (\lambda^{1+n_u} \quad \lambda^{n_c}), \\ \xi_s^\dagger \sim (1,2)_{0,0,-1} \propto (\lambda^{1+n_d} \quad \lambda^{n_s}). \tag{A29}$$

In this case, the next spurion to receive a VEV is $\xi_c^\dagger$, which breaks

$$G_F^{(3b)} \to G_F^{(4b)} = SU(2)_{D_R} \times U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(2)}}, \tag{A30}$$

introducing three new Goldstone bosons at the scale $\Lambda^{(3)} \sim y_c\Lambda$.

c. Order $\Lambda^{(4)}/\Lambda$

Decomposing the doublet $\xi_u^\dagger$ into two singlets $Y_U^{(1)}$ and φ_{12}, the remaining spurions of $G_F^{(4b)}$ are

$$Y_U^{(1)} \sim (1)_{1,-1,0} \propto \lambda^{n_u}, \quad \text{and} \quad \varphi_{12} \sim (1)_{1,0,0} \propto \lambda^{1+n_c}, \\ \xi_d^\dagger \sim (2)_{1,0,-1} \propto (\lambda^{n_d} \quad \lambda^{1+n_s}), \\ \xi_s^\dagger \sim (2)_{0,0,-1} \propto (\lambda^{1+n_d} \quad \lambda^{n_s}). \tag{A31}$$

Notice that the flavor group and the representations of the spurion fields are the same as for $G_F^{(4a1,4a2)}$, only that the role of χ_{13} is now played by φ_{12}.

As in the case of scenario (a2), we assume that the next spurion to get a VEV is ξ_s, breaking the flavor symmetry at $\Lambda^{(4b)} \sim y_s\Lambda$,

$$G_F^{(4b)} \to G_F^{(5b)} = U(1)_{Q_L^{(1)}} \times U(1)_{U_R^{(1)}} \times U(1)_{D_R^{(1)}}. \tag{A32}$$

The remaining steps in the flavor symmetry breaking follow scenario (a2), except for $\Lambda^{(5b)} \sim \langle \varphi_{12} \rangle \sim \lambda y_c\Lambda$.

6. Order $\Lambda^{(6)}/\Lambda$ and order $\Lambda^{(7)}/\Lambda$

Since in all scenarios the residual spurion field χ_{12} is uncharged under $G_F^{(6)}$, its VEV will in general be a complex number whose phase cannot be rotated away by flavor transformations. The CP symmetry in the Yukawa sector will thus be broken spontaneously by $\langle \chi_{12} \rangle/\Lambda \sim \lambda y_s \sim \Lambda^{(5)}/\Lambda$, if the potential singles out a nonvanishing imaginary part.

Finally, the VEVs for $Y_U^{(1)}$ and $Y_D^{(1)}$ break the remaining flavor symmetry

$$G_F^{(6a1)} \rightarrow \text{nothing} \tag{A33}$$

and give masses to the up- and down-quark, where the order of symmetry breaking is not really important.

APPENDIX B: VARIOUS $U(1)$ CHARGES

For convenience, we collect in Table III the various $U(1)$ charges appearing in the construction of the flavor symmetry breaking. Notice that some $U(1)$ charges are linear dependent,

$$3B = T + Q_L^{(2)} + U_R^{(2)} + D_R, \tag{B1}$$

$$= \mathrm{III} + Q_L^{(2)} + U_R^{(2)} + D_R^{(2)}, \tag{B2}$$

$$= \mathrm{III} + C + Q_L^{(1)} + U_R^{(1)} + D_R^{(2)}. \tag{B3}$$

TABLE III. Various $U(1)$ charges appearing in the discussion of the sequential flavor symmetry breaking.

	$3B$	U_R	D_R	T	III	$Q_L^{(2)}$	$U_R^{(2)}$	$D_R^{(2)}$	C	$Q_L^{(1)}$	$U_R^{(1)}$	$D_R^{(1)}$
$(u,d)_L$	1	0	0	0	0	1	0	0	0	1	0	0
$(c,s)_L$	1	0	0	0	0	1	0	0	1	0	0	0
$(t,b)_L$	1	0	0	1	1	0	0	0	0	0	0	0
u_R	1	1	0	0	0	0	1	0	0	0	1	0
c_R	1	1	0	0	0	0	1	0	1	0	0	0
t_R	1	1	0	1	1	0	0	0	0	0	0	0
d_R	1	0	1	0	0	0	0	1	0	0	0	1
s_R	1	0	1	0	0	0	0	1	0	0	0	0
b_R	1	0	1	0	1	0	0	0	0	0	0	0

APPENDIX C: EXPRESSING THE G_F INVARIANTS THROUGH G'_F INVARIANTS

The explicit relations between the ten invariants $i_\alpha^{(m)}$ of the full flavor group G_F and the nine invariants $j_\alpha^{(m)}$ of the residual flavor group G'_F read

$$i_1^{(2)} = j_1^{(2)} + y_t^2\Lambda^2, \qquad i_2^{(2)} = j_2^{(2)} + j_3^{(2)}, \tag{C1}$$

for the dimension-two invariants, and

$$\begin{aligned} i_1^{(4)} &= j_1^{(4)} - 2y_t^2\Lambda^2 j_1^{(2)}, \\ i_2^{(4)} &= j_2^{(4)} - j_1^{(2)} j_3^{(2)} - y_t^2\Lambda^2 j_2^{(2)}, \\ i_3^{(4)} &= j_3^{(4)} + 2j_4^{(4)}, \end{aligned} \tag{C2}$$

for the dimension-four terms, together with

$$\begin{aligned} i_1^{(6)} &= -\tfrac{3}{2}y_t^2\Lambda^2 j_1^{(4)}, \\ i_2^{(6)} &= -y_t^2\Lambda^2 j_2^{(4)} - \tfrac{1}{2}j_3^{(2)} j_1^{(4)} \\ i_3^{(6)} &= j_1^{(6)} - \tfrac{1}{2}y_t^2\Lambda^2 j_3^{(4)} - j_1^{(2)}(j_4^{(4)} + j_2^{(2)} J_3^{(2)}) - j_3^{(2)} j_2^{(4)}, \\ i_4^{(6)} &= 3j_2^{(6)} - 3j_2^{(2)}(j_4^{(4)} + j_2^{(2)} j_3^{(2)}) - \tfrac{3}{2}j_3^{(2)} j_3^{(4)}, \end{aligned} \tag{C3}$$

and

$$\begin{aligned} i_1^{(8)} = {} & (j_1^{(2)} - 2y_t^2\Lambda^2) j_1^{(6)} + (y_t^4\Lambda^4 + \tfrac{1}{2}j_1^{(4)}) \\ & \times (j_4^{(4)} + j_2^{(2)} J_3^{(2)}) + j_1^{(4)} j_3^{(4)} - j_2^{(4)}(j_2^{(4)} + j_1^{(2)} j_2^{(2)}) \\ & + \tfrac{1}{2}(j_1^{(2)})^2 j_3^{(4)} + \tfrac{1}{2}(j_2^{(2)})^2 j_1^{(4)}. \end{aligned} \tag{C4}$$

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