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Dyonic giant magnons in CP^3: Strings and curves at finite J

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This paper studies giant magnons in $AdS_4 \times CP^3$ using both the string sigma model and the algebraic curve. We complete the dictionary of solutions by finding the dyonic generalization of the CP^1 string solution, which matches the "small" giant magnon in the algebraic curve, and by pointing out that the solution recently constructed by the dressing method is the "big" giant magnon. We then use the curve to compute finite-J corrections to all cases, which for the nondyonic cases always match the Arutyunov-Frolov-Zamaklar result. For the dyonic RP^3 magnon we recover the S^5 answer, but for the small and big giant magnons we obtain new corrections.

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I. INTRODUCTION

Classical and semiclassical strings allow us to explore some sectors of the $\mathcal{N} = 6$ ABJM/$AdS_4 \times CP^3$ duality [1], and these are much richer than their well-known counterparts in the $\mathcal{N} = 4$ SYM/$AdS_5 \times S^5$ duality [2]. It was known very early on that there are at least two kinds of giant magnons, created by placing the HM giant magnon [3] into various S^2-like subspaces, namely, CP^1 and RP^2 [4,5]. It is equally easy to place Dorey's S^3 dyonic giant magnon [6,7] into RP^3, giving a two-spin generalization of the RP^2 magnon [8].

Solutions of the string sigma model should be in exact correspondence to algebraic curves [9]. Here too several giant magnon solutions were known (compared to one in the S^5 case [10]) named "small" and "big" [11]. However these could not be the same two solutions as those known in the sigma model. In the S^5 case, and for the small magnon, one naturally obtains a dyonic (two-parameter, two-spin) solution. But the big magnon is something not seen in the S^5 case, a two-parameter solution with only one nonzero angular momentum, and thus cannot be the RP^3 magnon. There are also two distinct small giant magnons, and it has been observed that a pair of small magnons has all the properties we expect the RP^3 magnon to have [12].

The situation has improved with the recent publication of a new string solution, found using the dressing method, which like the big giant magnon is a two-parameter one-angular-momentum solution [13–15]. They have exactly the same dispersion relation, and in both cases we can take a nondyonic limit and recover the RP^2/pair of small magnons.

In this paper we complete the puzzle by finding a dyonic generalization of the CP^1 magnon. This is a solution which does not exist in S^5, exploring the four-dimensional subspace CP^2, and it has a dispersion relation matching that of the small giant magnon. It exists in two orientations, and like the small magnon has a third angular momentum which is $J_3 = \pm Q$ in these two cases. We have as yet only been able to find the $p = \pi$ case of this solution, but this is sufficient to see these properties.

Finite-J corrections are of increasing importance in the study of gauge and string integrability. They can sometimes be computed directly on the string side by finding solutions with $J < \infty$, and all existing finite-J giant magnons are embeddings of well-known S^5 results of this type [16–19]. Other methods that have been used to calculate finite-size corrections include the construction of corresponding algebraic curves [12,20–22] and the Lüscher formulas [12,19,23,24].

In this paper we extend the algebraic curve calculations of [12], by calculating finite-J corrections not only for a pair of small giant magnons, but also for a single small magnon and the big magnon. In the nondyonic case, all of these give the result AFZ [16] found for a magnon in S^2. Likewise the dyonic pair of giant magnons matches the S^3 result: this too is a simple embedding of that string solution. But for the dyonic small and "dyonic" big magnons, which correspond to string solutions not found in S^5, we find new formulas for these energy corrections.[1]

[1]A word about terminology. We use dyonic to mean two-parameter two-charge solutions (like Dorey's) but sometimes write"dyonic" (with scare quotes) for the two-parameter one-charge solution to specify that we mean the case $r \neq 1$. When we speak of the nondyonic limit, we always mean that we take the second parameter $r \to 1$, and this always takes us to some embedding of the simplest HM magnon.

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TABLE I. Summary of giant magnons in the string sigma model. The dressed solution of Hollowood and Miramontes [13], Kalousios, Spradlin, and Volovich [14], and Suzuki [15] also lives in CP^2. [The RP^2 solution has often been called $SU(2)\times SU(2)$, "big" and $S^2\times S^2$ in the literature.] To match the curves we want $p'=p/2$.

	$\mathcal{E}=\Delta-\frac{J}{2}$	$\delta\mathcal{E}$ (finite J)	Q	J_3
Vacuum	0			
CP^1 giant magnon	$\sqrt{2\lambda}\sin(\frac{p}{2})$	$-4\mathcal{E}\sin^2(\frac{p}{2})e^{-2\Delta/\mathcal{E}}$	0	0
Dyonic version in CP^2	$\sqrt{\frac{Q^2}{4}+2\lambda}$ when $p=\pi$	[Use "small" curve, (26)]	Q	$\pm Q$
RP^2 giant magnon	$2\sqrt{2\lambda}\sin(\frac{p'}{2})$	$-4\mathcal{E}\sin^2(\frac{p'}{2})e^{-2\Delta/\mathcal{E}}$	0	0
Dyonic version in RP^3	$\sqrt{\frac{Q^2}{4}+8\lambda\sin^2(\frac{p'}{2})}$	Like S^5 result, (9)	Q	0
HM/KSV/S dressed solution	$\sqrt{Q_f^2+8\lambda\sin^2(\frac{p'}{2})}$	[Use "big" curve, (29)]	0	0

TABLE II. Summary of giant magnons in the algebraic curve. In each case we list dyonic (or "dyonic") solutions, meaning $Q\sim\sqrt{\lambda}$, below the nondyonic case. We write these using λ rather than g for comparison with the string sigma-model results from Table I; the relation is $\sqrt{2\lambda}=4g$.

M_u, M_v, M_r	$[p_1, q, p_2]$	$\mathcal{E}=\Delta-\frac{J}{2}$	$\delta\mathcal{E}$ (finite J)	Q	J_3
Vacuum					
0, 0, 0	$[L,0,L]$	0	$\cdots$	0	0
Small giant magnon					
1, 0, 0	$[L-2,1,L]$	$\sqrt{2\lambda}\sin(\frac{p}{2})$	$-4\mathcal{E}\sin^2(\frac{p}{2})e^{-2\Delta/\mathcal{E}}$	1	1
Q, 0, 0	$[L-2Q,Q,L]$	$\sqrt{\frac{Q^2}{4}+2\lambda\sin^2(\frac{p}{2})}$	$\propto Q/\mathcal{E}\sqrt{S}$, see (26)	Q	Q
... and similar to $u\leftrightarrow v$:					
0, Q, 0	$[L,Q,L-2Q]$	(Same)	(Same)	Q	$-Q$
Big giant magnon					
1, 1, 1	$[L-1,0,L-1]$	$2\sqrt{2\lambda}\sin(\frac{p}{4})$	$-4\mathcal{E}\sin^2(\frac{p}{4})e^{-2\Delta/\mathcal{E}}$	0	0
Q_u, Q_u, Q_u	$[L-Q_u,0,L-Q_u]$	$\sqrt{Q_u^2+8\lambda\sin^2(\frac{p}{4})}$	$\propto S/\mathcal{E}Q_u^2$, see (29)	0	0
Pair of small giant magnons					
1, 1, 0	$[L-2,2,L-2]$	$2\sqrt{2\lambda}\sin(\frac{p}{4})$	$-4\mathcal{E}\sin^2(\frac{p}{4})e^{-2\Delta/\mathcal{E}}$	2	0
$\frac{Q}{2},\frac{Q}{2}$, 0	$[L-Q,Q,L-Q]$	$\sqrt{\frac{Q^2}{4}+8\lambda\sin^2(\frac{p}{4})}$	Like S^5 case, see (28)	Q	0

In Sec. II we set up the string sigma model, and discuss embeddings of S^5 giant magnons, including their finite-J corrections. We also discuss the recently published dressing method solution. Then in Sec. III we construct the dyonic generalization of the CP^1 magnon, which lives in CP^2.

We then turn to the algebraic curve, and in Sec. IV set this up, and discuss the known giant magnons in this formalism. In Sec. V we calculate finite-J corrections to all of these solutions. These corrections match the string results wherever they are known and are new results in the dyonic small and big cases.

Most of our results are summarized in Tables I and II.

II. THE STRING SIGMA MODEL FOR $AdS_4\times CP^3$

The string dual of ABJM theory in the 't Hooft limit is type IIA superstrings in $AdS_4\times CP^3$. At strong coupling in the gauge theory, leading to classical strings, these have large radii $R/2$ and R. The metric is then

$$\begin{aligned} ds^2 &= \frac{R^2}{4}ds^2_{\mathrm{AdS}}+R^2ds^2_{CP} \\ &= R^2\left(\frac{dy_\mu dy^\mu}{-4\mathbf{y}^2}+\frac{dz_i d\bar{z}_i}{|\mathbf{z}|^2}-\frac{|z_i d\bar{z}_i|^2}{|\mathbf{z}|^4}\right), \end{aligned}$$

where we have embedded $AdS_4\subset\mathbb{R}^{2,4}$ and $CP^3\subset\mathbb{C}^4$, parametrized by $\mathbf{y}$ and $\mathbf{z}$, respectively. To study strings in this space, we constrain the lengths of these embedding coordinate vectors: $\mathbf{y}^2=y_\mu y^\mu=-(y^{-1})^2-(y^0)^2+(y^1)^2+(y^2)^2+(y^3)^2=-1$ and $|\mathbf{z}|^2=z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3=+1$. In addition to these constraints, points in $\mathbb{C}^4$ differing by an overall phase are identified in CP^3. This can be dealt with by introducing a gauge field: write the conformal gauge Lagrangian as

$$\begin{aligned} 2\mathcal{L} &= \tfrac{1}{4}\partial_a\mathbf{y}\cdot\partial^a\mathbf{y}-\Lambda(\mathbf{y}^2+1)+\overline{D_a\mathbf{z}}\cdot D^a\mathbf{z} \\ &\quad -\Lambda'(\bar{\mathbf{z}}\cdot\mathbf{z}-1), \end{aligned}$$

where the covariant derivative is $D_a=\partial_a-A_a$. The equation of motion for the gauge field fixes $A_a=\bar{\mathbf{z}}\cdot\partial_a\mathbf{z}$. We can write the equations of motion for $\mathbf{y}$ and $\mathbf{z}$ as[2]

$$\partial_a\partial^a\mathbf{y} + (\partial_a\mathbf{y}\cdot\partial^a\mathbf{y})\mathbf{y} = 0,$$
$$D_aD^a\mathbf{z} + (\overline{D_a\mathbf{z}}\cdot D^a\mathbf{z})\mathbf{z} = 0.$$

The anti–de Sitter (AdS) and CP components are coupled by the Virasoro constraints, which read

$$\tfrac{1}{4}\partial_t\mathbf{y}\cdot\partial_t\mathbf{y} + \overline{D_t\mathbf{z}}\cdot D_t\mathbf{z} + \tfrac{1}{4}\partial_x\mathbf{y}\cdot\partial_x\mathbf{y} + \overline{D_x\mathbf{z}}\cdot D_x\mathbf{z} = 0,$$
$$\tfrac{1}{4}\partial_t\mathbf{y}\cdot\partial_x\mathbf{y} + \mathrm{Re}(\overline{D_t\mathbf{z}}\cdot D_x\mathbf{z}) = 0.$$

We now restrict to solutions in $\mathbb{R}\times CP^3$, with $y^{-1} + iy^0 = e^{2it}$ and $y^1 = y^2 = y^3 = 0$. We will always work in a gauge in which this t is world-sheet time (timelike, or static, conformal gauge).[3] The metric then reduces to

$$ds^2_{\mathbb{R}\times CP^3} = -dt^2 + |d\mathbf{z}|^2 - |\bar{\mathbf{z}}\cdot d\mathbf{z}|^2.$$

In writing the Lagrangian, and this metric, we have pulled out the large radius factor $R^2 = 2^{5/2}\pi\sqrt{\lambda}$ to give a prefactor to the action:

$$S = \int\frac{dxdt}{2\pi}R^2\mathcal{L} = 2\sqrt{2\lambda}\int dxdt\mathcal{L}.$$

This same factor appears when calculating conserved charges. The one from time translation (which we define with respect to AdS time, $\tan\tau_{\mathrm{AdS}} = y^0/y^{-1}$) is simply

$$\Delta = 2\sqrt{2\lambda}\int dx\frac{\partial\mathcal{L}}{\partial\partial_t\tau_{\mathrm{AdS}}} = \sqrt{2\lambda}\int dx 1,$$

where at the end we use the fact that $\tau_{\mathrm{AdS}} = 2t$ for the solutions we are studying. The charges from rotations of CP^3's embedding coordinate planes are

$$J(z_i) = 2\sqrt{2\lambda}\int dx\frac{\partial\mathcal{L}}{\partial\partial_t(\arg Z_i)}$$
$$= 2\sqrt{2\lambda}\int dx\Big[\mathrm{Im}(\bar{z}_i\partial_t z_i) - |z_i|^2\sum_j\mathrm{Im}(\bar{z}_j\partial_t z_j)\Big]$$
$$= 2\sqrt{2\lambda}\int dx\mathrm{Im}(\bar{z}_iD_tz_i),$$

where there is no summation over i. Only three of the four $J(z_i)$ are independent, since $\sum_{i=1}^4 J(z_i) = 0$. These three are the charges from the Cartan generators of $\mathfrak{su}(4)$, and the charges from all of the generators can be obtained using their Lie-algebra matrices $T^a = (T^a)_{ij}$:

$$J[T^a] = 2\sqrt{2\lambda}\int dx\mathrm{Im}(\bar{\mathbf{z}}\cdot T^aD_t\mathbf{z}).$$

The matrices T^a are Hermitian and traceless, and the charges $J(z_i)$ are those generated by diagonal T^a. The charges we will need for the giant magnons are

$$J = J(z_1) - J(z_4) = J[\mathrm{diag}(1,0,0,-1)],$$
$$Q = J(z_2) - J(z_3) = J[\mathrm{diag}(0,1,-1,0)], \quad (1)$$
$$J_3 = J\Big[\mathrm{diag}\Big(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\Big)\Big].$$

Instead of using four complex numbers as coordinates for CP^3, we can use six real angles. One set of these is defined by

$$\mathbf{z} = \begin{pmatrix}\sin\xi\cos(\vartheta_2/2)e^{-i\eta/2}e^{i\varphi_2/2}\\ \cos\xi\cos(\vartheta_1/2)e^{i\eta/2}e^{i\varphi_1/2}\\ \cos\xi\sin(\vartheta_1/2)e^{i\eta/2}e^{-i\varphi_1/2}\\ \sin\xi\sin(\vartheta_2/2)e^{-i\eta/2}e^{-i\varphi_2/2}\end{pmatrix} \quad (2)$$

and in terms of these angles, the metric is

$$ds^2_{CP^3} = d\xi^2 + \tfrac{1}{4}\sin^2 2\xi(d\eta + \tfrac{1}{2}\cos\vartheta_1 d\varphi_1 - \tfrac{1}{2}\cos\vartheta_2 d\varphi_2)^2$$
$$+ \tfrac{1}{4}\cos^2\xi(d\vartheta_1^2 + \sin^2\vartheta_1 d\varphi_1^2)$$
$$+ \tfrac{1}{4}\sin^2\xi(d\vartheta_2^2 + \sin^2\vartheta_2 d\varphi_2^2). \quad (3)$$

The charges of interest can be written using these angles: writing $J_{\varphi_2} = 2\sqrt{2\lambda}\int dx\frac{\partial\mathcal{L}}{\partial\partial_t\varphi_2}$ etc., we have $J = 2J_{\varphi_2}$, $Q = 2J_{\varphi_1}$, and $J_3 = J_\eta$.

A. Recycled giant magnons in CP^3

The Hofman-Maldacena giant magnon [3] is a rigidly rotating classical string solution in $\mathbb{R}\times S^2$. Writing $S^2\subset\mathbb{C}^2$, the solution is

$$\mathbf{w} = \begin{pmatrix}e^{it}[\cos\frac{p}{2} + i\sin\frac{p}{2}\tanh u]\\ \sin\frac{p}{2}\,\mathrm{sech}u\end{pmatrix}$$
$$= \begin{pmatrix}e^{i\phi_{\mathrm{mag}}(x,t)}\sin\theta_{\mathrm{mag}}(x,t)\\ \cos\theta_{\mathrm{mag}}(x,t)\end{pmatrix}, \quad (4)$$

where $u = \gamma(x - vt)$. The only parameter is the world-sheet velocity $v = \cos(p/2)$. There are two ways to embed this solution into CP^3:

(i) The first class of giant magnons in CP^3 is obtained by placing this solution into the subspace $CP^1 = S^2$ defined by $z_2 = z_3 = 0$, or $\xi = \frac{\pi}{2}$ [4]. With our conventions this is a sphere of radius $\frac{1}{2}$, and so to maintain timelike conformal gauge the solution should have angles on this sphere of $\vartheta_2 = \theta_{\mathrm{mag}}(2x,2t)$ and $\varphi_2 = \phi_{\mathrm{mag}}(2x,2t)$. In terms of $\mathbf{z}$ the solution is then

[2] Note that the CP^3 equation here reduces to that derived in [25], where instead of treating the total phase as a gauge symmetry it was fixed to a constant using another Lagrange multiplier.

[3] This implies that the length of the world sheet cannot be held fixed to 2π. Instead it is proportional to the energy Δ, and thus infinite for the giant magnon. Taking this to be finite makes Δ and J finite too, thus we use "finite J" and "finite size" interchangeably.

$$\mathbf{z}(x,t) = \begin{pmatrix} e^{(i/2)\phi_{\text{mag}}(2x,2t)} \sin(\frac{1}{2}\theta_{\text{mag}}(2x,2t)) \\ 0 \\ 0 \\ e^{-(i/2)\phi_{\text{mag}}(2x,2t)} \cos(\frac{1}{2}\theta_{\text{mag}}(2x,2t)) \end{pmatrix}. \tag{5}$$

These magnons have dispersion relation

$$\mathcal{E}(p) = \Delta - \frac{J}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right).$$

(ii) The second class of giant magnons lives in RP^2 [4], and can be written $\xi = \theta_{\text{mag}}(x,t)$, $\varphi_2 = 2\phi_{\text{mag}}(x,t)$, $\vartheta_1 = \vartheta_2 = \frac{\pi}{2}$, or [5]

$$\mathbf{z}(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_{\text{mag}}(x,t)}\sin\theta_{\text{mag}}(x,t) \\ \cos\theta_{\text{mag}}(x,t) \\ \cos\theta_{\text{mag}}(x,t) \\ e^{-i\phi_{\text{mag}}(x,t)}\sin\theta_{\text{mag}}(x,t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} w_1 \\ w_2 \\ \bar{w}_2 \\ \bar{w}_1 \end{pmatrix}. \tag{6}$$

The dispersion relation for this class of magnons is

$$\mathcal{E} = \Delta - \frac{J}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p'}{2}\right),$$

where we call the velocity $v = \cos(p'/2)$ in this case because it turns out that the momenta are related $p = 2p'$. The dyonic generalization of the RP^2 magnon is an embedding of Dorey's original S^3 dyonic magnon and carries a second momentum $Q \neq 0$. Parametrizing the S^3 by $\mathbf{w} \in \mathbb{C}^2$, the embedding needed is exactly the formula (6). The resulting solution lives in RP^3 and has a dispersion relation

$$\mathcal{E} = \Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 8\lambda\sin\left(\frac{p'}{2}\right)}.$$

The third angular momentum J_3 is still zero.

All giant magnons are pieces of a closed string, and for the CP^1 magnon, like the original S^2 solution, the condition that a set of magnons form a closed string is $\sum_i p_i = 0 \bmod 2\pi$, since p is the opening angle along the equator. For the RP^2 magnon, however, the $p' = \pi$ magnon is also a closed string, thanks to the $\mathbb{Z}_2$ identification in this space, and thus $\sum_i p'_i = 0 \bmod \pi$ instead.

For more detailed discussion of these subspaces, and of others such as $S^2 \times S^2$, see [25].

B. Finite-size corrections

For the two embeddings of S^2 giant magnons discussed above, we can obtain finite-J corrections by simply embedding the S^2 results [16,18] into CP^3. The explicit calculation of these corrections was done by [26] for the RP^2 case:

$$\Delta - \frac{J}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p}{2}\right)\left[1 - 4\sin^2\left(\frac{p'}{2}\right)e^{-2\Delta/2\sqrt{2\lambda}\sin(p'/2)} + \cdots\right], \tag{7}$$

and by [27] for the CP^1 case:

$$\Delta - \frac{J}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right) \times\left[1 - 4\sin^2\left(\frac{p}{2}\right)e^{-2\Delta/\sqrt{2\lambda}\sin(p/2)} + \cdots\right]. \tag{8}$$

For the RP^3 dyonic giant magnon, we can similarly embed the results from S^3. These were originally computed by [19] (from the all-J solutions of [17]) and were studied in CP^3 by [8,28]. The result is

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 8\lambda\sin^2\left(\frac{p'}{2}\right)} - 32\lambda\cos(2\phi)\frac{1}{\mathcal{E}}\sin^4\left(\frac{p'}{2}\right)e^{-\Delta\mathcal{E}/2S}, \tag{9}$$

where we define[4]

$$S = \frac{Q^2}{16\sin^2(\frac{p'}{2})} + 2\lambda\sin^2\left(\frac{p'}{2}\right). \tag{10}$$

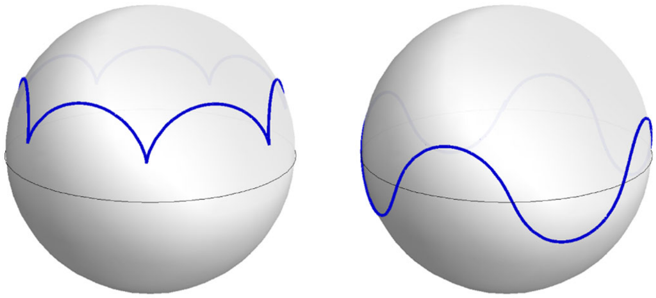
It remains to discuss the factor $\cos(2\phi)$. In the paper [19], this is set to be $+1$ for "type (i)" helical strings and -1 for "type (ii)" strings. These are two kinds of finite-J solutions, which in the nondyonic case in S^2 give a set of magnons in which adjacent magnons either have the same or opposite orientation. Type (i) strings thus have a cusp not touching the equator, while type (ii) strings cross the equator at less than a right angle—see Fig. 1. It seems very likely that we should interpret 2ϕ as the angle between the two magnons' orientation vectors. [The same factor could be included in the nondyonic cases (7) and (8)].

C. Dressing method solution

There is also another kind of giant magnon, which does not exist in S^5, recently constructed by several groups using the dressing method [13–15].

This method is a way of generating multisoliton solutions above a given vacuum in the principal chiral model, closely related to the Bäcklund transformation (see [29,30]). The "dressed" solution Ψ is obtained from the "bare" vacuum Ψ_0 by $\Psi = \chi(0)\Psi_0$, where the dressing matrix $\chi(\lambda)$ is a function of a spectral parameter. Each

[4]Note that $S(\frac{p'}{2}) \to \frac{1}{4}\mathcal{E}^2$ as $Q \to 0$. When comparing to [19], note that $\cosh(\theta/2) = \mathcal{E}/2\sqrt{2\lambda}\sin(\frac{p'}{2})$. In terms of our later notation, this θ is defined as $r = e^{\theta/2}$.



Type (i): $2\phi = 0$, $\delta\mathcal{E} < 0$ Type (ii): $2\phi = \pi$, $\delta\mathcal{E} > 0$

FIG. 1 (color online). The two classes of finite-J magnons found by [17].

independent pole[5] of $\chi(\lambda)$ results in one soliton, and in the cases of interest here, its position contains the solution's parameters.

The dressing method was first used to generate giant magnons in S^3 by [31], where the string sigma model was mapped to an $SU(2)$ principal chiral model and an $SO(3)$ vector model. In the $SU(2)$ case, a pole at $\lambda_1 = re^{ip/2}$ produces a dyonic giant magnon with charges

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\frac{1+r^2}{2r}\sin\left(\frac{p}{2}\right),$$
$$J_2 = \frac{\sqrt{\lambda}}{\pi}\frac{1-r^2}{2r}\sin\left(\frac{p}{2}\right),$$

which can then be combined to give the usual dispersion relation. (One then regards J_2 as the second parameter, instead of r.) In the $SO(3)$ case, only the nondyonic giant magnon can be obtained.

The recently constructed CP^3 solution uses the map to an $SU(4)/U(3)$ model. The position of the dressing pole $\lambda_1 = re^{ip'/2}$ again provides two parameters, but unlike the S^3 case, there is only one nonzero angular momentum

$$J = 2\Delta - 4\sqrt{2\lambda}\frac{1+r^2}{2r}\sin\left(\frac{p'}{2}\right).$$

(As in RP^2, we will call this momentum p'.) It is convenient, however, to use, instead of r, a new parameter defined as $Q_f = 2\sqrt{2\lambda}\frac{1-r^2}{2r}\sin(\frac{p'}{2})$. In terms of this, the dispersion relation becomes

$$\Delta - \frac{J}{2} = \sqrt{Q_f^2 + 8\lambda\sin^2\left(\frac{p'}{2}\right)}.$$

We have chosen the factor in front of Q_f to make this line up with the dispersion relation for the big giant magnon in the algebraic curve (after setting $p = 2p'$). Like this solution, the big giant magnon is a two-parameter single-momentum solution. We discuss it in Sec. IV C.

In the basis used by [13,14],[6] and writing only the case $p' = \pi$, the solution is[7]

$$\mathbf{z}' = N\begin{pmatrix} (1+r^2)\cos(t) + \cos(\frac{1-3r^2}{1+r^2}t) + r^2\cos(\frac{3-r^2}{1+r^2}t) + i(1-r^2)\sin(t)\sinh(\frac{4r}{1+r^2}x) \\ -(1+r^2)\sin(t) + \sin(\frac{1-3r^2}{1+r^2}t) - r^2\sin(\frac{3-r^2}{1+r^2}t) - i(1-r^2)\cos(t)\sinh(\frac{4r}{1+r^2}x) \\ 2(1-r^2)[\sin(\frac{1-r^2}{1+r^2}t)\sinh(\frac{2r}{1+r^2}x) - i\cos(\frac{1-r^2}{1+r^2}t)\cosh(\frac{2r}{1+r^2}x)] \\ 0 \end{pmatrix},$$

where N is a normalization factor ensuring $|\mathbf{z}|^2 = 1$. The vacuum used to derive this solution is $\mathbf{z}'_{\text{vac}} = (\cos(t), \sin(t), 0, 0)$, which carries a large charge under $J' = J[\sigma_2 \oplus 1]$. The value of this J' is altered by the presence of this magnon, but all other $J[T^a]$ charges vanish. Rotating the space to bring this vacuum $\mathbf{z}'_{\text{vac}}$ to match our $\mathbf{z}_{\text{vac}} = \frac{1}{\sqrt{2}}(e^{it}, 0, 0, e^{-it})$ will also rotate the charge J' into J, as used for the other magnons.

In the limit $r \to 1$, or $Q_f \to 0$, the dispersion relation becomes that of the RP^2 giant magnon. And the solution (in this basis) becomes the embedding of the ordinary magnon (4) given by $\mathbf{z}' = (\text{Re}w_1, \text{Im}w_1, \text{Re}w_2, 0) \in \mathbb{R}^4$.

Finite-J corrections to this solution are not known in the string sigma model (except trivially at $Q_f = 0$, where it is the RP^2 magnon) but we compute them using the algebraic curve in Sec. V.

The paper [14] also finds a second solution by dressing, equation (4.14).[8] This solution has the same angular momentum J as the big giant magnon, but lives in CP^1. It is in fact just an embedding of the bound state solution (5.14) of [31], which in that paper is written using parameter q instead of $r = e^{q/2}$. This bound state is an analytic continuation of a scattering state of two HM magnons, or in this case, of two CP^1 magnons.

[5]Sometimes there is also an image pole at $1/\lambda_1$.

[6]The paper [15] obtains this solution in the same basis as we use.

[7]Like the RP^2 and RP^3 magnons, this forms a closed string at $p' = \pi$.

[8]We thank Chrysostomos Kalousios for drawing our attention to this solution.

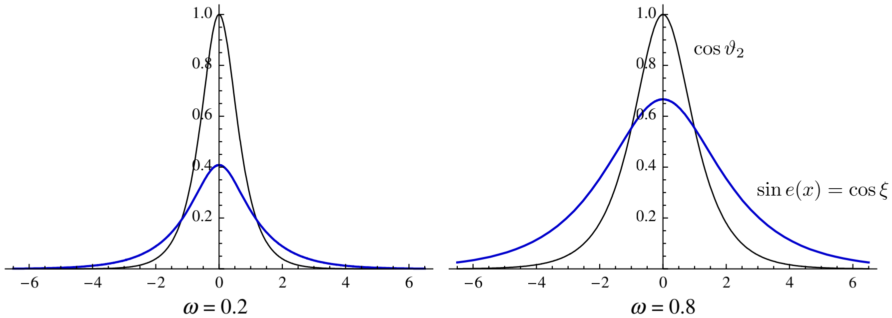


FIG. 2 (color online). Profiles of the CP^2 solution. $\cos\vartheta_2$ is the same as for Dorey's S^3 dyonic giant magnon, but the solution also spreads away from $\xi = \frac{\pi}{2}$ as we increase ω. This is shown for $\omega = 0.2$ and 0.8.

III. DYONIC GENERALIZATION OF THE CP^1 MAGNON

Consider the subspace CP^2 obtained by fixing $z_3 = 0$, or in terms of the angles, $\theta_1 = 0$ and $\eta = 0$, leaving

$$\mathbf{z} = \begin{pmatrix} \sin\xi\cos(\vartheta_2/2)e^{i\varphi_2/2} \\ \cos\xi e^{i\varphi_1/2} \\ 0 \\ \sin\xi\sin(\vartheta_2/2)e^{-i\varphi_2/2} \end{pmatrix}. \tag{11}$$

The metric for this subspace can be written

$$ds^2 = \tfrac{1}{4}\sin^2\xi[d\vartheta_2^2 + \sin^2\vartheta_2 d\varphi_2^2 + \cos^2\xi(d\varphi_1 - \cos\vartheta_2 d\varphi_2)^2] + d\xi^2.$$

At $\xi = \frac{\pi}{2}$ the space is CP^1, described by ϑ_2 and φ_2 only, but away from this value there is a second isometry direction $d\varphi_1$. It was proposed in [25] that the dyonic generalization of the CP^1 magnon might have momentum along this direction, but to do so, it must in addition have $\xi \neq \frac{\pi}{2}$ except at the end points of the string, at $x = \pm\infty$, where it must touch the same equator as the CP^1 solution.

We have not yet been able to find the full solution, but can find a GKP-like dyonic solution (i.e. a $p = \pi$ magnon[9]) using the ansatz:

$$\varphi_2 = 2t, \qquad \varphi_1 = -2\omega t,$$
$$\cos\vartheta_2 = \mathrm{sech}(\sqrt{1-\omega^2}2x), \qquad \xi = \frac{\pi}{2} - e(x).$$

This amounts to assuming that the backreaction on the original solution in ϑ_2, φ_2 created by giving it new momentum along φ_1 is exactly as for the S^3 dyonic solution, but unlike the S^3 case, there is one extra function $e(x)$ (Fig. 2).

With this ansatz the equations of motion for φ_1 and φ_2, and the second Virasoro constraint, are already solved. The equation of motion for ϑ_2 can be written

$$\partial_x(\cos^2 e(x)\mathrm{sech}X) = -2\frac{\cos^2 e(x)}{\sqrt{1-\omega^2}}\tanh X\{\mathrm{sech}X\cos^2 e(x) + \omega\sin^2 e(x)\},$$

where $X = \sqrt{1-\omega^2}2x$. Using a change of variables $y(x) = \ln(\cos^2 e(x))$, this equation can be written as

$$y'(x) = e^{y(x)}f(x) + g(x), \tag{12}$$

where

$$f(x) = \frac{2}{\sqrt{1-\omega^2}}(\omega - \mathrm{sech}X)\sinh X,$$
$$g(x) = -f(x) - \frac{2\omega^2}{\sqrt{1-\omega^2}}\tanh X.$$

Equation (12) has solutions of the form $y(x) = -\ln(-F(x)) + G(x)$, where $G'(x) = g(x)$ and $F'(x) = f(x)e^{G(x)}$. After some algebra, we find the following form for the solution:

$$\cos^2(e(x)) = \sin^2\xi = \frac{1}{1+\omega\cos\vartheta_2} = \frac{1}{1+\omega\,\mathrm{sech}(\sqrt{1-\omega^2}2x)}, \tag{13}$$

where $\omega \geq 0$.

Calculating charges J and Q for this solution, we find

$$\Delta - \frac{J}{2} = \sqrt{2\lambda}\frac{1}{\sqrt{1-\omega^2}}, \qquad \frac{Q}{2} = -\sqrt{2\lambda}\frac{\omega}{\sqrt{1-\omega^2}}$$

and therefore the dispersion relation is

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 2\lambda}.$$

Comparison with the solutions known in the algebraic curve formalism (see next section) leads us to conjecture that for the general case (allowing $p \neq \pi$) the dispersion relation is

[9] Gubser, Klebanov, and Polyakov [32] studied rotating folded strings, which at $J = \infty$ are the $p = \pi$ case of the HM magnon [3]. Two-spin folded string solutions were studied by Frolov and Tseytlin, [33] and are the $p = \pi$ case of Dorey's dyonic giant magnon [6,7].

$$\mathcal{E} = \Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 2\lambda \sin\left(\frac{p}{2}\right)}$$

matching the one for the "small giant magon" in the algebraic curve.

Unlike the RP^3 dyonic magnon, this one is charged not only under Q but also under J_3, with $J_3 = Q$. There is a second CP^2 solution, in the subspace with $z_2 = 0$ instead of $z_3 = 0$, which has $J_3 = -Q$ but is otherwise similar. In the limit $\omega \to 0$ both kinds become the same CP^1 solution. All of these properties match those of the two kinds of small giant magnons in the algebraic curve perfectly.

We summarize all the properties of the various string solutions in Table I.

IV. THE ALGEBRAIC CURVE FOR $AdS_4 \times CP^3$

The string equations of motion can also be studied using the formalism of algebraic curves. This has been a fruitful approach in $AdS_5 \times S^5$ [34–39]. The $AdS_4 \times CP^3$ case at hand was originally studied by Gromov and Vieira [9]. We start this section by a brief review of the construction of the algebraic curve and its most important properties.

A. From target space to quasimomenta

Begin by defining the connection $j = j_{\mathrm{AdS}} \oplus j_{CP}$, where [40,41]

$$(j_{\mathrm{AdS}})_{ij,\mu} = 2(y_i \partial_\mu y_j - (\partial_\mu y_i) y_j),$$
$$(j_{CP})_{ij,\mu} = 2(z_i D_\mu z_j - (D_\mu z_i) z_j).$$

By construction, the connection j is flat

$$dj + j \wedge j = 0.$$

The sigma-model action can be written in terms of j as

$$S = -\frac{g}{8} \int d\sigma d\tau \mathrm{STr} j^2,$$

leading to the equations of motion

$$d * j = 0. \tag{14}$$

The Lax connection is now given by

$$J(x) = \frac{1}{1 - x^2} j + \frac{x}{1 - x^2} * j,$$

where the new complex variable x is the spectral parameter.[10] $J(x)$ is a flat connection, for any x, provided j is flat and satisfies (14).

Using $J(x)$ we define the monodromy matrix

$$\Omega(x) = P e^{\int d\sigma J_\sigma(x)}.$$

Since the connection is flat, the eigenvalues of Ω are independent of τ. We write these eigenvalues as $e^{i\tilde{p}_1}$, $e^{i\tilde{p}_2}$, $e^{i\tilde{p}_3}$, and $e^{i\tilde{p}_4}$ for the CP^3 part, and $e^{i\hat{p}_1}$, $e^{i\hat{p}_2}$, $e^{i\hat{p}_3}$, and $e^{i\hat{p}_4}$ for the AdS part, and refer to the functions $\tilde{p}_i$ and $\hat{p}_i$ as "quasimomenta." The continuity in the complex plane of the function $\mathrm{eig}(\Omega(x))$ demands that when a branch cut C_{ij} connects sheets i and j, we must have $p_i^+ - p_j^- = 2\pi n$ when $x \in C_{ij}$.

At large x the monodromy matrix is

$$\Omega(x) = 1 + \frac{1}{x} \int d\sigma j_\tau + \cdots,$$

and thus the asymptotic behavior of the quasimomenta contains information about the charges. [The exact relation is (15).]

The quasimomenta $\tilde{p}_i$ and $\hat{p}_i$ describe the bosonic sector of type IIA string theory on $AdS_4 \times CP^3$. While this is all we will need in this paper, it will be convenient to work in a formalism with explicit $OSp(2,2|6)$ symmetry. To do this we define ten new quasimomenta q_i as [9]

$$\{q_1, q_2, q_3, q_4, q_5\} = \tfrac{1}{2}\{\hat{p}_1 + \hat{p}_2, \hat{p}_1 - \hat{p}_2, \tilde{p}_1 + \tilde{p}_2, -\tilde{p}_2 - \tilde{p}_4, \tilde{p}_1 + \tilde{p}_4\}$$

and

$$\{q_6, q_7, q_8, q_9, q_{10}\} = \{-q_5, -q_4, -q_3, -q_2, -q_1\}.$$

The functions q_i now define a ten-sheeted Riemann surface.

B. Relations and charges

The ten quasimomenta q_i must obey the following relations [9]:

(1) Only five are independent: $\{q_6, q_7, q_8, q_9, q_{10}\} = \{-q_5, -q_4, -q_3, -q_2, -q_1\}$.

(2) Square-root branch cut condition: $q_i^+(x) - q_j^-(x) = 2\pi n_{ij}$, $x \in C_{ij}$.

(3) Synchronized poles: the residues at $x = \pm 1$ are the same $\alpha_\pm/2$ for q_1, q_2, q_3, and q_4, while q_5 does not have a pole there.

(4) Inversion symmetry: $q_1(\frac{1}{x}) = -q_2(x)$, $q_3(\frac{1}{x}) = 2\pi m - q_4(x)$, and $q_5(\frac{1}{x}) = q_5(x)$. This $m \in \mathbb{Z}$ gives the momentum $p = 2\pi m$.

(5) Asymptotic behavior as $x \to \infty$:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \frac{1}{2gx} \begin{pmatrix} \Delta + S \\ \Delta - S \\ L - M_r \\ L + M_r - M_u - M_v \\ M_v - M_u \end{pmatrix} + o\left(\frac{1}{x^2}\right)$$
$$= \frac{1}{2gx} \begin{pmatrix} \Delta + S \\ \Delta - S \\ J_1 \\ J_2 \\ J_3 \end{pmatrix} + \cdots, \tag{15}$$

where $\lambda = 8g^2$ (i.e. $4g = \sqrt{2\lambda}$).

[10] In this section we use x to be the spectral parameter, and σ, τ to be the world-sheet coordinates which we called x, t before.

The ansatz used by [12], for solutions mostly in CP^3, is the following:

$$q_1(x) = \frac{\alpha x}{x^2-1}, \qquad q_2(x) = \frac{\alpha x}{x^2-1},$$
$$q_3(x) = \frac{\alpha x}{x^2-1} + G_u(0) - G_u\Big(\frac{1}{x}\Big) + G_v(0) - G_v\Big(\frac{1}{x}\Big) + G_r(x) - G_r(0) + G_r\Big(\frac{1}{x}\Big),$$
$$q_4(x) = \frac{\alpha x}{x^2-1} + G_u(x) + G_v(x) - G_r(x) + G_r(0) - G_r\Big(\frac{1}{x}\Big),$$
$$q_5(x) = G_u(x) - G_u(0) + G_u\Big(\frac{1}{x}\Big) - G_v(x) + G_v(0) - G_v\Big(\frac{1}{x}\Big). \tag{16}$$

From q_1 and q_2 we can read off $S = 0$ (zero AdS angular momentum) and $\alpha = \Delta/2g$.

The functions G_u, G_v, and G_r control the CP^3 part of the curve q_3, q_4, and q_5 and so, asymptotically, the $SU(4)$ excitation numbers M_u, M_v, and M_r. Their values at $x = 0$ control the momentum[11]

$$p = 2\pi m = q_3\Big(\frac{1}{x}\Big) + q_4(x). \tag{17}$$

The Dynkin labels of $SU(4)$ are related to the excitation numbers by

$$\begin{bmatrix} p_1 \\ q \\ p_2 \end{bmatrix} = \begin{bmatrix} L - 2M_u + M_r \\ M_u + M_v - 2M_r \\ L - 2M_v + M_r \end{bmatrix} \in \mathbb{Z}^3_{\geq 0}.$$

These can be combined into the $SO(6)$ charges:

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} q + (p_1+p_2)/2 \\ (p_1+p_2)/2 \\ (p_2-p_1)/2 \end{pmatrix} = \begin{pmatrix} L - M_r \\ L + M_r - M_u - M_v \\ M_u - M_v \end{pmatrix},$$

which are in turn combined into the magnons' major and minor charges:

$$J = J_1 + J_2 = 2L - M_u - M_v = p_1 + q + p_2,$$
$$Q = J_1 - J_2 = M_u + M_v - 2M_r = q.$$

C. Giant magnons in the curve

Giant magnons were first studied using the algebraic curve in [10], where it was shown that they correspond to logarithmic cuts (see also [42]). For the case of CP^3, two different kinds of giant magnons were given by [11], who named them small and big. These can be constructed by setting some of the resolvents in the above ansatz to

$$G_{\rm mag}(x) = -i\log\Big(\frac{x - X^+}{x - X^-}\Big), \tag{18}$$

where X^- is the complex conjugate of X^+.

(i) The first kind is the small giant magnon with

$$G_v(x) = G_{\rm mag}(x), \qquad G_u = G_r = 0.$$

The charges read off from this curve are

$$p = -i\log\frac{X^+}{X^-},$$
$$Q = -i2g\Big(X^+ - X^- + \frac{1}{X^+} - \frac{1}{X^-}\Big),$$
$$J = 2\Delta + i2g\Big(X^+ - X^- - \frac{1}{X^+} + \frac{1}{X^-}\Big), \tag{19}$$
$$J_3 = Q.$$

These can be put together to give the dispersion relation

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 16g^2\sin^2\Big(\frac{p}{2}\Big)}.$$

(ii) We can make another kind of small magnon with G_u instead of G_v. The only change is in the sign of $J_3 = -Q$.

(iii) Then there is the "big giant magnon," which has

$$G_u(x) = G_v(x) = G_r(x) = G_{\rm mag}(x)$$

from which we obtain the charges

$$p = -2i\log\frac{X^+}{X^-}, \qquad Q = 0,$$
$$J = 2\Delta + i4g\Big(X^+ - X^- - \frac{1}{X^+} + \frac{1}{X^-}\Big), \tag{20}$$
$$J_3 = 0.$$

This is the curve used by [11], and to get the dispersion relation, we must use not the total Q but rather Q_u, the contribution from just the u part (which is canceled by the v part in the full solution). This is the same function of $X^\pm$ as for the small giant magnon (19) above. The result is

$$\Delta - \frac{J}{2} = \sqrt{Q_u^2 + 64g^2\sin^2\Big(\frac{p}{4}\Big)}.$$

For this solution, $\mathcal{E} = \Delta - J/2$ is a function of two parameters, Q_u and p, but Q_u is not an asymptotic charge of the full solution. Unlike the ordinary dyonic giant magnons, which are two-parameter two-momentum solutions, here there is only one angular momentum.

[11] For a closed string $m \in \mathbb{Z}$, however, we want to consider a single giant magnon which in general is not a closed string. Hence we will relax this condition and consider general p. To get a physical state this momentum condition should be imposed. This can be done by considering multimagnon states [12,20].

Finally, we can also put one small magnon into each sector, $G_v(x) = G_u(x) = G_{\rm mag}(x)$ but with $G_r = 0$. For each of the charges (including both Δ and p) we obtain the sum of those of each of the constituent small giant magnons, and write $Q = Q_u + Q_v$, etc. Thus we get the dispersion relation

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q_u^2}{4} + 16g^2\sin^2\left(\frac{p_u}{2}\right)} + \sqrt{\frac{Q_v^2}{4} + 16g^2\sin^2\left(\frac{p_v}{2}\right)} = \sqrt{\frac{Q^2}{4} + 64g^2\sin^2\left(\frac{p}{4}\right)}.$$

If we were to write this in terms of the momentum p_u of one constituent magnon, rather than the total p, then we would have $\sin^2(p_u/2)$, as in [12]. Note that this solution has total $J_3 = 0$ (like the big magnon).

We summarize all of these properties, and more, in Table II.

D. Coalescence of nondyonic solutions

Notice that in the nondyonic limit $Q \ll g$, and $Q_u \ll g$, the dispersion relations for the pair of small magnons and the big magnon agree. This is not limited to just the dispersion relation: in this limit, $X^\pm = e^{\pm ip/4}$ (in both cases) and thus we have

$$G_{\rm mag}(x) - G_{\rm mag}(0) + G_{\rm mag}\left(\frac{1}{x}\right) = 0. \tag{21}$$

Looking at the ansatz (16), this is equivalent to setting $G_r = 0$. Thus the big giant magnon becomes the same algebraic curve as the pair of small magnons, in this limit.

For the small giant magnon, the same identity implies that $q_5 = 0$ in the nondyonic limit. This removes the difference between curves for the u and v small giant magnons.

V. FINITE-SIZE CORRECTIONS IN THE CURVE

These corrections were studied by Łukowski and Ohlsson Sax [12], where the basic technique is to replace $G_{\rm mag}(x)$ with the resolvent

$$G_{\rm finite}(x) = -2i\log\left(\frac{\sqrt{x - X^+} + \sqrt{x - Y^+}}{\sqrt{x - X^-} + \sqrt{x - Y^-}}\right), \tag{22}$$

where $Y^\pm$ are points shifted by some small amount $\delta \ll 1$ away from $X^\pm$[12]:

$$Y^\pm = X^\pm(1 \pm i\delta e^{\pm i\phi}). \tag{23}$$

When $\delta = 0$ this new $G_{\rm finite}(x)$ clearly reduces to the infinite-size magnon resolvent (18). This form of resolvent was found based on work [20], and finite-J corrections to the S^2 magnon were computed using this in [22].

[12]Note that this is a different choice of ϕ to that used in [12,20,22]. It is chosen to separate ϕ, which gives the orientation factor $\cos(2\phi)$ in $\delta\mathcal{E}$, from the phase of $X^\pm$, which is sometimes $p/2$ and sometimes $p/4$.

A. Finite-size small giant magnon

The first example we study is the magnon created by setting $G_u(x) = G_{\rm finite}(x)$ in the general ansatz (16), with $G_v = G_r = 0$. This one we discuss in the most detail, as subsequent examples are similar. We write

$$X^\pm = re^{ip_0/2}$$

in terms of which $p = p_0 + \delta p_{(1)} + \delta^2 p_{(2)} + o(\delta^3)$ and

$$\mathcal{E} = \Delta - \frac{J}{2} = 4g\frac{r^2+1}{2r}\sin\left(\frac{p}{2}\right) - \frac{\delta}{2}J_{(1)} - \frac{\delta^2}{2}J_{(2)} + o(\delta^3),$$
$$Q = 8g\frac{r^2-1}{2r}\sin\left(\frac{p}{2}\right) + \delta Q_{(1)} + \delta^2 Q_{(2)} + o(\delta^3). \tag{24}$$

We give formulas for these expansions in the Appendix. From the full asymptotic charges, we can calculate the energy correction in terms of δ. The first nonzero contribution is at order δ^2:

$$\delta\mathcal{E} = \left(\Delta - \frac{J}{2}\right) - \sqrt{\frac{Q^2}{4} + 16g^2\sin^2\left(\frac{p}{2}\right)} = -\delta^2\frac{g}{4}\cos(2\phi)\frac{2r}{1+r^2}\sin\left(\frac{p}{2}\right) + o(\delta^3). \tag{25}$$

The function $G_{\rm finite}(x)$ has a square-root branch cut from X^+ to Y^+, which in the curve (16) we choose to make connect sheets q_4 and $q_6 = -q_5$. We can then fix δ using the branch cut condition:

$$2\pi n = q_4(x^+) - q_6(x^-) = \frac{2\alpha x}{x^2-1} + G^+_{\rm finite}(X^+) + G^-_{\rm finite}(X^+) - G_{\rm finite}(0) + G_{\rm finite}\left(\frac{1}{X^+}\right).$$

The superscript G^- is to indicate that this term is evaluated on the other side of the cut from the others (and thus has the opposite sign between the terms of the numerator inside G). After taking this account we may take both evaluation points $x^\pm$ to be at $x = X^+$. Figure 3 shows the cuts and the points used. The result is

$$\delta = \frac{8ie^{-ip/4}e^{i\pi n}e^{-i\phi}\sqrt{r^2-1}\sin(\frac{p}{2})}{\sqrt{e^{-ip/2} - r^2e^{ip/2}}}\exp\left(\frac{i\Delta r/4g}{e^{-ip/2} - r^2e^{ip/2}}\right) = e^{i\psi}|\delta|.$$

In order to have a real energy correction, we demand that δ be real. We then find the correction to be

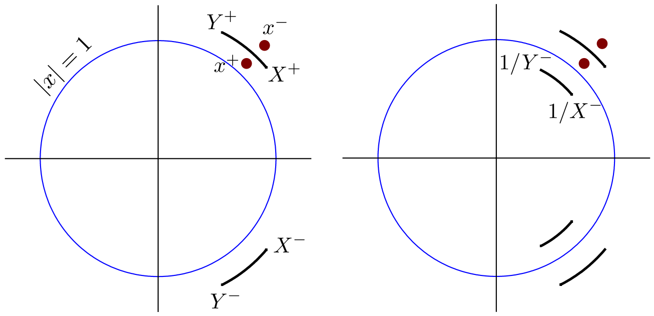


FIG. 3 (color online). Branch cuts and evaluation points. On the left is the situation for the pair of small magnons, where only the cuts for $G_{\text{finite}}(x)$ appear. The evaluation points $x^\pm$ straddle the cut from X^+ to Y^+, which is at radius $|x| = r$. On the right, the cuts in $G_{\text{finite}}(1/x)$ are drawn too, which is the situation encountered in the "small" and "big" magnons. The evaluation points are on the same side of the cut from $1/X^+$ to $1/Y^+$, and remain so even when we take the nondyonic limit $r \to 1$. (These are drawn for $\phi = 0$.)

$$\delta\mathcal{E} = -32g\cos(2\phi)\frac{r^2-1}{r^2+1} \times \frac{\sin^3(\frac{p}{2})}{\sqrt{r^2+\frac{1}{r^2}-2\cos(p)}}e^{-\Delta\mathcal{E}/S(p/2)} + o(\delta^3)$$
$$= -32g^2\cos(2\phi)\frac{Q}{\mathcal{E}\sqrt{S(\frac{p}{2})}}\sin^3\left(\frac{p}{2}\right)e^{-\Delta\mathcal{E}/S(p/2)} + o(\delta^3), \tag{26}$$

where we define

$$S\left(\frac{p}{2}\right) = 4g^2\frac{(r^2-1)^2}{r^2} + 16g^2\sin^2\left(\frac{p}{2}\right)$$

and note that, in the present small case,

$$S\left(\frac{p}{2}\right) = \frac{Q^2}{4\sin^2(\frac{p}{2})} + 16g^2\sin^2\left(\frac{p}{2}\right) \to \mathcal{E}^2$$

when $r \to 1$.

Three comments:

(i) Our result (26) is for the dyonic case $Q/g \sim 1$. As written it appears that $\delta\mathcal{E} \to 0$ in the nondyonic limit $r \to 1$, but this is not correct. We have implicitly assumed, when expanding in δ, that $\delta \ll r - 1 \sim \sqrt{Q/g}$, and this forbids taking $r \to 1$. However, we can derive the correction for the nondyonic case by writing $r = 1 + k\delta$ before assuming that δ is small, then expanding in δ, fixing δ using the branch cut, and only then taking the limit $k \to 0$. The result is the AFZ form:

$$\delta\mathcal{E}_{r=1} = -16g\cos(2\phi)\sin^3\left(\frac{p}{2}\right)e^{-2\Delta/\mathcal{E}} + o(\delta^3). \tag{27}$$

(ii) The condition that δ be real is that its phase ψ must be 0 or π:

$$\psi = n\pi - \frac{p}{4} - \phi - \frac{\Delta Q\cot(\frac{p}{2})}{4S(\frac{p}{2})} - \frac{1}{2}\arctan\left(\frac{2\mathcal{E}}{Q}\tan\left(\frac{p}{2}\right)\right) = 0 \quad \text{or} \quad \pi.$$

Like the energy correction (26), this expression is for the dyonic case; the phase of δ in the nondyonic limit $r \to 1$ is instead

$$\psi_{r=1} = 2\pi n - \frac{p}{2} = 0 \quad \text{or} \quad \pi,$$

where we have assumed $\cos(2\phi) = \pm 1$. This implies $p = 0 \bmod 2\pi$, which is exactly the usual condition for a closed string.

(iii) Finally, notice that the same factor $\cos(2\phi)$ appears in these results as in the sigma-model results (9), which we interpreted there as a geometric angle between adjacent magnons. Here we can observe that for the identity (21) to hold at the evaluation point $x = X^+$ (in the limit $\delta \to 0$, as well as $r \to 1$), we must set $\cos(2\phi) = \pm 1$.

B. Finite-size pair of small magnons

The nondyonic $r = 1$ case for the pair of small magnons was studied by Łukowski and Ohlsson Sax in [12], who obtained

$$\delta\mathcal{E}_{r=1} = -32g\cos(2\phi)\sin^3\left(\frac{p}{4}\right)e^{-2\Delta/\mathcal{E}} + \cdots.$$

This result can also be obtained by adding together all the charges of two small magnons, giving twice the correction (27). However the dyonic case cannot be obtained by adding together two dyonic finite-J small magnons: they interact with each other. For this case we must perform a similar analysis to that for the small giant magnon above.

The curve is $G_u = G_v = G_{\text{finite}}$ and $G_r = 0$, and we now set

$$X^\pm = re^{\pm ip_0/4}$$

giving $p = p_0 + \cdots$, $\mathcal{E} = 8g\frac{r^2+1}{2r}\sin(\frac{p}{4}) + \cdots$, and $Q = 16g\frac{r^2-1}{2r}\sin(\frac{p}{4}) + \cdots$.[13] The energy correction in terms of δ reads

$$\delta\mathcal{E} = -\delta^2\frac{g}{2}\cos(2\phi)\frac{2r}{r^2+1}\sin\left(\frac{p}{4}\right) + \cdots.$$

We then fix δ using the branch cut condition connecting sheets[14] q_4 and $q_7 = -q_4$

[13] As before, we give these expansions in δ in the Appendix.

[14] In [12] a condition for the G_v component to connect sheets q_4 and q_5 is used instead. (This involves separating the two cuts slightly, so that $q_5 \neq 0$. The G_u component has instead a cut connecting q_4 and q_6, which gives the same equation.) The resulting condition is the same as that given here except n is replaced by $2n$.

$$2\pi n = q_4(x^+) - q_7(x^-) = \frac{2\alpha x}{x^2-1} + 2G^+_{\rm finite}(X^+) + 2G^-_{\rm finite}(X^+)$$

and find the final energy correction

$$\delta\mathcal{E} = -256 g^2\cos(2\phi)\frac{1}{\mathcal{E}}\sin^4\left(\frac{p}{4}\right)e^{-\Delta\mathcal{E}/2S(p/4)} + \cdots. \tag{28}$$

This has the same form as the S^5 string result and exactly matches the RP^3 magnon's correction (9).

In this case there is no difficulty about the $r\to 1$ limit, where it reduces to the RP^2 correction (7). [Note that $S(\frac{p}{4})\to\frac{1}{4}\mathcal{E}^2$ in this limit, rather than $\mathcal{E}^2$ as in the small case.]

The phase of δ is, in this case,

$$\psi = \frac{n\pi}{2} - \frac{p}{4} - \phi - \frac{\Delta Q\cot(\frac{p}{4})}{8S(\frac{p}{4})} = 0 \quad \text{or} \quad \pi.$$

When $Q=0$ and $\phi=0$, this means $p/2 = p' = n'\pi$, $n'\in\mathbb{Z}$, exactly matching the condition for the RP^2 magnon to be a closed string.

C. Finite-size big magnon

The curve here is $G_u = G_v = G_r = G_{\rm finite}$. We write $X^\pm = re^{ip_0/4}$, and consider the dyonic case in the sense that $r>1$, even though $Q=0$. Define Q_u to be the Q from the small magnon, which, thanks to this choice of p_0, now reads $Q_u = 8g\frac{r^2-1}{2r}\sin(\frac{p}{4})+\cdots$, and we have $\mathcal{E} = 8g\frac{1+r^2}{2r}\sin(\frac{p}{4})+\cdots$. Calculating the expansions of the asymptotic charges in δ, we get (as for the pair case)

$$\delta\mathcal{E} = -\delta^2\frac{g}{2}\cos(2\phi)\frac{2r}{r^2+1}\sin\left(\frac{p}{4}\right)+\cdots.$$

Now for the branch cut condition. We connect sheets q_3 and $q_7 = -q_4$, at points $x^\pm$ on either side of the cut from X^+ to Y^+, but on the same side of the cut from $1/X^+$ to $1/Y^+$, obtaining the matching condition

$$2\pi n = q_3(x^+) - q_7(x^-) = 2\frac{\alpha x}{x^2-1} + G^+_{\rm finite}(x) + G^-_{\rm finite}(x) + 2G_{\rm finite}(0) - 2G_{\rm finite}\left(\frac{1}{x}\right).$$

This equation fixes δ, and after demanding that it be real, we obtain the correction:

$$\delta\mathcal{E} = -64g\cos(2\phi)\frac{r^2+\frac{1}{r^2}-2\cos(p/2)}{(r^2+1)(r^2-1)^2}r^3\sin^3\left(\frac{p}{4}\right)\times e^{-\Delta\mathcal{E}/S(p/4)}+\cdots = -1024g^2\cos(2\phi)\frac{S(\frac{p}{4})}{\mathcal{E}Q_u^2}\sin^6\left(\frac{p}{4}\right)e^{-\Delta\mathcal{E}/S(p/4)}+\cdots. \tag{29}$$

Similar to our small magnon result, this expression is valid only in the dyonic case. The nondyonic limit $r\to 1$ can be approached in the same way as for that case, by setting $r = 1+k\delta$ before expanding in δ. The limit $k\to 0$ then gives the result

$$\delta\mathcal{E}_{r=1} = -32g\cos(2\phi)\sin^3\left(\frac{p}{4}\right)e^{-2\Delta/\mathcal{E}}+\cdots$$

matching the $r=1$ limit of the pair of small magnons, (28) above, and thus the RP^2 string result (7).

The phase of δ in this case is

$$\psi = n\pi - \frac{p}{2} - \phi - \frac{\Delta Q_u\cot(\frac{p}{4})}{2S(\frac{p}{4})} + \arctan\left(\frac{\mathcal{E}}{Q_u}\tan\left(\frac{p}{4}\right)\right).$$

As for the small case, this expression is not valid in the nondyonic case, where instead we get [in the case $\cos(2\phi) = \pm 1$]

$$\psi_{r=1} = \frac{n\pi}{2} - \frac{p}{4} = 0 \quad \text{or} \quad \pi,$$

i.e. $p = 0 \bmod 2\pi$, thus $p' = 0 \bmod \pi$, which is the condition for a closed string in RP^2, and matches the "pair" case above.

VI. CONCLUSIONS

Let us summarize the dictionary of string and curve solutions which we have found:

(i) The small giant magnon in the curve matches the CP^1 giant magnon and its dyonic generalization in CP^2.

(ii) The RP^2 magnon is to be identified with the pair of small magnons. In the dyonic case this becomes the RP^3 magnon solution.

(iii) The dressed solution is identified with the big giant magnon. Both are two-parameter one-charge solutions, and when the additional parameter (Q_f or Q_u) is sent to zero, they become the RP^2 solution/pair of small magnons, respectively.

Note that the nondyonic RP^2 and CP^1 string solutions seem to have multiple descriptions in the algebraic curves: the big and pair of small magnons differ in their excitation numbers M_u, M_v, and M_r, as do the two kinds of small magnons. However these numbers are all of order $1 \ll 4g = \sqrt{2\lambda}$ and thus, like Q, invisible in the sigma model. In the limit $Q\to 0$ the curves forget these distinctions too.

Finite-size corrections to these magnons can be summarized as follows:

(i) In the nondyonic cases, the corrections are always of the AFZ form. These can be calculated in both the string and curve pictures.

(ii) For the RP^3/pair magnon, the corrections are the same as those for S^5 dyonic giant magnons and can again be calculated in both pictures.

(iii) For the dressed/big magnon, and also for the CP^2/small magnon, we have calculated corrections in the algebraic curve. These do not have the same form as in S^5.

Our result for the finite-J corrections to the small giant magnon differs from that of the algebraic curve calculation in [12]. This difference can be understood to arise due to an order-of-limits problem. As noted in Sec. V, we need to be careful in the nondyonic case with how we take limits $Q \to 0$ and $\delta \to 0$. However, the result of [12] is confirmed by the Lüscher calculations in [12,24]. It would be instructive to see if these results can be explained in a similar manner.

While the overall picture is now clear, there are various details which it would be nice to see explicitly in the string sigma model. First, our CP^2 solution should certainly exist at $p \neq \pi$, but so far we have not been able to find such solutions. Second, it would be interesting to understand exactly how the two different CP^2 solutions join (and interact) to form one RP^3 magnon. Finally, finite-J versions of both this CP^2 solution and the dressed solution should exist and would provide confirmation of the energy corrections calculated here.

We conclude by noting that our results fit well into the context of the integrable alternating spin chain for operators in the ABJM gauge theory [4,43–47]. The two small giant magnons correspond to simple magnons in either the fundamental or antifundamental part of the spin chain. The big magnon, on the other hand, carries the same charges as the heavy scalar excitation first discussed in [4]. In a recent paper, Zarembo [48] showed that in the Berenstein-Maldacena-Nastase [49] limit these heavy modes disappear from the spectrum as soon as quantum corrections are taken into account. It would be very interesting to understand to what extent these arguments carry over to the giant magnon regime.

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APPENDIX A: EXPANSIONS OF CHARGES

When working out finite-J corrections to the various algebraic curve magnons, we expanded the asymptotic charges in δ, defined by (23) $Y^\pm = X^\pm(1 \pm i\delta e^{\pm i\phi})$. We used these expansions to work out the correction $\delta\mathcal{E}$, for example, (25). Here we give the expansions of these charges explicitly.

We write all three cases at once, by setting $m = 1$ for the small magon and $m = 2$ for the pair and big. Thus we always have $X^\pm = re^{\pm ip/2m}$.

First, the momentum is $p = p_0 + \delta p_{(1)} + \delta^2 p_{(2)} + o(\delta^3)$, where

$$p_{(1)} = m\cos(\phi) \qquad p_{(2)} = \frac{3m}{8}\sin(2\phi).$$

Next, the angular momentum is $J = J_{(0)} + \delta J_{(1)} + \delta^2 J_{(2)} + o(\delta^3)$, with

$$J_{(0)} = 2\Delta - 4gm\frac{r^2+1}{r}\sin\left(\frac{p_0}{2m}\right),$$

$$J_{(1)} = -\frac{2gm}{r}\left[r^2\cos\left(\frac{p_0}{2m}+\phi\right)+\cos\left(\frac{p_0}{2m}-\phi\right)\right],$$

$$J_{(2)} = \frac{3gm}{2r}\sin\left(\frac{p_0}{2m}-2\phi\right).$$

Finally, the second angular momentum is $Q = Q_{(0)} + \delta Q_{(1)} + \delta^2 Q_{(2)} + o(\delta^3)$, where for the small and pair cases we have

$$Q_{(0)} = 4gm\frac{r^2-1}{r}\sin\left(\frac{p_0}{2m}\right),$$

$$Q_{(1)} = \frac{2gm}{r}\left[r^2\cos\left(\frac{p_0}{2m}+\phi\right)-\cos\left(\frac{p_0}{2m}-\phi\right)\right],$$

$$Q_{(2)} = \frac{3gm}{2r}\sin\left(\frac{p_0}{2m}-2\phi\right).$$

For the big magnon, $Q = 0$ to this order in δ. We used in the dispersion relation instead Q_u which (as a function of $X^\pm$) is the Q from the small magnon. For the purpose of these expansions (functions of r and p) it is easier to think of this as $Q_u = \frac{1}{2}Q_{\text{pair}}$ since the big and pair cases both have $m = 2$.

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