

**$Q$ -balls in flat potentials**

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We study the classical and absolute stability of  $Q$ -balls in scalar field theories with flat potentials arising in both gravity-mediated and gauge-mediated models. We show that the associated  $Q$ -matter formed in gravity-mediated potentials can be stable against decay into their own free particles as long as the coupling constant of the nonrenormalizable term is small, and that all of the possible three-dimensional  $Q$ -ball configurations are classically stable against linear fluctuations. Three-dimensional gauge-mediated  $Q$ -balls can be absolutely stable in the thin-wall limit, but are completely unstable in the thick-wall limit.

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**I. INTRODUCTION**

$Q$ -balls have recently attracted much attentions in cosmology [1] and astrophysics [2–4]. A  $Q$ -ball [5] is a nontopological soliton [6] whose stability is ensured by the existence of a continuous global charge  $Q$  (for a review see [7–11] and references therein), and a number of scalar field theory models have been proposed to support the existence of nontopological solitons. They include polynomial models [5], Sine-Gordon models [12], parabolic-type models [13], confinement models [14–17], two-field models [6,18], and flat models [1].

From a phenomenological point of view, the most interesting examples are probably the supersymmetric  $Q$ -balls arising within the framework of the minimal supersymmetric standard model (MSSM), which naturally contains a number of gauge invariant flat directions. Many of the flat directions can carry baryon (B) or/and lepton (L) number, which is/are essential for Affleck-Dine (AD) baryogenesis [19]. Following the AD mechanism, a complex scalar (AD) field acquires a large field value after a period of cosmic inflation and tends to form a homogeneous condensate, the AD condensate. In the presence of a negative pressure [20,21], the condensate is unstable against spatial fluctuations so that it develops into nonlinear inhomogeneous lumps, namely  $Q$ -balls. The stationary properties and cosmological consequences of the  $Q$ -balls depend on how the supersymmetry (SUSY) is broken in the hidden sector, transmitting to the observable sector through so-called messengers. In the gravity-mediated [22] or gauge-mediated scenarios [1], the messengers correspond, respectively, either to supergravity fields or to some heavy particles charged under the gauge group of the standard model.

$Q$ -balls can exist in scalar field potentials where SUSY is broken through effects in the supergravity hidden sector [23]. These type of  $Q$ -balls can be unstable to decay into baryons and the lightest supersymmetric particle dark matter, such as neutralinos [24], gravitinos [25,26], and axinos

[27]. Recently, McDonald has argued that enhanced  $Q$ -ball decay in AD baryogenesis models can explain the observed positron and electron excesses detected by PAMELA, ATIC, and PPB-BETS [28]. By imposing an upper bound on the reheating temperature of the Universe after inflation, this mode of decay through  $Q$ -balls has been used to explain why the observed baryonic ( $\Omega_b$ ) and dark matter ( $\Omega_{\text{DM}}$ ) energy densities are so similar [29,30], i.e.  $\Omega_{\text{DM}}/\Omega_b = 5.65 \pm 0.58$  [31].

Scalar field potentials arising through gauge-mediated SUSY breaking [22] tend to be extremely flat. Using one of the MSSM flat directions, namely, the  $QdL$  direction (where  $Q$  and  $d$  correspond to squark fields and  $L$  to a slepton field), which has a nonzero value of  $B - L$  and therefore does not spoil AD baryogenesis via the sphaleron processes that violate  $B + L$  [30], Shoemaker and Kusenko recently explored the minimum energy configuration for baryo-leptonic  $Q$ -balls, whose scalar field consists of both squarks and sleptons [32]. It had been assumed to that point that the lowest energy state of the scalar field corresponds to being exactly the flat direction; however, in [32], the authors showed that the lowest energy state lies slightly away from the flat directions, and that the relic  $Q$ -balls, which are stable against decay into both protons/neutrons (baryons) and neutrinos/electrons (leptons) [33], may end up contributing to the energy density of dark matter [29,34]; thus, the  $Q$ -balls can provide the baryon-to-photon ratio [34], i.e.  $n_b/n_\gamma \simeq (4.7\text{--}6.5) \times 10^{-10}$  [35], where  $n_b$  and  $n_\gamma$  are, respectively, the baryon and photon densities in the Universe.

In this paper we examine analytically and numerically the classical and absolute stability of  $Q$ -balls using flat potentials in the two specific models mentioned above. In order to study the possible existence of lower-dimensional  $Q$ -balls embedded in  $3 + 1$  dimensions, we will work in arbitrary spatial dimensions  $D$ ; although of course the  $D = 3$  case is of more phenomenological interest. Previous work [21,30,36] on the gravity-mediated potential has used either a steplike or Gaussian ansatz to study the analytical properties of the thin- and thick-wall  $Q$ -balls. Introducing more physically motivated ansätze, we will

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show that the thin-wall  $Q$ -balls can be quantum mechanically stable against decay into their own free particle quanta, that both thin- and thick-wall  $Q$ -ball solutions obtained are classically stable against linear fluctuations, and confirm that a Gaussian ansatz is a physically reasonable one for the thick-wall  $Q$ -ball. The one-dimensional  $Q$ -balls in the thin-wall limit are excluded from our analytical framework. The literature on  $Q$ -balls with gauge-mediated potentials has tended to use a test profile in approximately flat potentials. We will present an exact profile for a generalized gauge-mediated flat potential, and show that we naturally recover results previously published in [22,30,34].

The rest of this paper is organized as follows: In Sec. II, we briefly review the important  $Q$ -ball properties that were established in [37]. Section III provides a detailed analyses for gravity-mediated potentials, and in Sec. IV, we investigate the case of a generalized gauge-mediated potential. We confirm the validity of our analytical approximations with complete numerical  $Q$ -ball solutions in Sec. V before summarizing in Sec. VI. Two appendices are included. In Appendix A, we obtain an exact solution for the case of a logarithmic potential, and in Appendix B, we confirm that the adoption of a Gaussian ansatz is appropriate for the thick-wall  $Q$ -ball found in the gravity-mediated potentials.

## II. THE BASICS

Here, we review the basic properties of  $Q$ -balls as described in [37] and introduce a powerful technique that enables us to find the charge  $Q$  and energy  $E_Q$  of the  $Q$ -ball as well as the condition for its stability, and characteristic slope  $\gamma(\omega) \equiv E_Q/\omega Q$ , where  $\omega$  is defined through the  $Q$ -ball ansatz, which is given by decomposing a complex scalar field  $\phi$  into  $\phi = \sigma(r)e^{i\omega t}$ .  $\sigma$  is a real scalar field,  $r$  is a radial coordinate, and therefore  $\omega$  is a rotational frequency in the U(1) internal space. By scaling the radius  $r$  of the  $Q$ -ball ansatz, which minimizes  $E_Q$ , we can find the characteristic slopes in terms of the ratio between the surface energy  $\mathcal{S}$  and the potential energy  $\mathcal{U}$  of the  $Q$ -ball. When the characteristic slope  $\gamma$  is independent of  $\omega$ , we obtain the relation:  $E_Q \propto Q^{1/\gamma}$ . In general the charge, energy, and Euclidean action  $S_\omega$  are given by

$$Q = \omega \int_{V_D} \sigma^2; \quad S_\omega = \int_{V_D} \left( \frac{1}{2} \sigma'^2 + U_\omega \right); \quad (1)$$

$$E_Q = \omega Q + S_\omega,$$

where our metric is  $ds^2 = -dt^2 + h_{ij}dx^i dx^j$ , the determinant of the spherically symmetric spatial metric  $h_{ij}$  is defined by  $h \equiv \det(h_{ij})$ , and we have used the following notation:  $\int_{V_D} \equiv \int d^D x \sqrt{h} = \Omega_{D-1} \int_0^\infty dr r^{D-1}$ ,  $\Omega_{D-1} \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)}$ ,  $\sigma' \equiv \frac{d\sigma}{dr}$ , and  $D$  is the number of spatial dimensions. Without loss of generality, we can take positive values of  $\omega$  and  $Q$ . By defining the effective potential  $U_\omega$  of a potential

$U(\sigma)$

$$U_\omega \equiv U - \frac{1}{2}\omega^2\sigma^2, \quad (2)$$

the  $Q$ -ball equation is

$$\sigma'' + \frac{D-1}{r}\sigma' = \frac{dU_\omega}{d\sigma}, \quad (3)$$

where  $\sigma(r)$  is a monotonically decreasing function in terms of  $r$ . Given a potential  $U(\sigma)$ , which has a global minimum at  $\sigma = 0$ , it is possible to show that  $Q$ -balls exist within the restricted range of  $\omega$  [5]:

$$\omega_- \leq \omega < \omega_+, \quad (4)$$

where we have defined the lower limit  $\omega_-^2 \equiv \frac{2U}{\sigma^2}|_{\sigma_+(\omega_-)} \geq 0$ ,  $\sigma_+(\omega)$  is the nonzero value of  $\sigma$ , where  $U_\omega(\sigma_+(\omega))$  is minimized (see Fig. 1), and the upper limit  $\omega_+^2 \equiv \frac{d^2U}{d\sigma^2}|_{\sigma=0}$ . The existence condition Eq. (4) restricts the allowed form of the potential  $U$ , which implies that the potential should grow less quickly than the quadratic term (i.e. mass term) for small values of  $\sigma$ . The case  $\omega_- = 0$  corresponds to degenerate vacua potentials (DVPs), while  $\omega_- \neq 0$  has nondegenerate vacua (NDVPs). In [37] we examined the case of polynomial potentials and restricted ourselves to the case of  $\omega_+^2 = m^2$ , where  $m$  is a bare mass in the potentials. In this paper we extend our analysis allowing us to investigate the case  $\omega_+^2 \gg m^2$ , needed since the potentials include one-loop corrections to the bare mass  $m$ . Here, the potential, which we will consider in the gravity-mediated models, is  $U = U_{\text{grav}} + U_{\text{NR}}$ , where  $U_{\text{NR}}$  is a nonrenormalization term (to be discussed below), and

$$U_{\text{grav}} \equiv \frac{1}{2}m^2\sigma^2 \left( 1 + K \ln\left(\frac{\sigma^2}{M^2}\right) \right). \quad (5)$$

Here,  $K$  is a constant factor arising from the one-loop correction, and  $M$  is the renormalization scale. To proceed with analytical arguments, we consider the two limiting values of  $\omega$  or  $\sigma_0 \equiv \sigma(r=0)$ , which describe

- thin-wall  $Q$ -balls when  $\omega \simeq \omega_-$  or equivalently  $\sigma_0 \sim \sigma_+(\omega)$ ,
- thick-wall  $Q$ -balls when  $\omega \simeq \omega_+$  or equivalently  $\sigma_0 \simeq \sigma_-(\omega)$ .

Note, this limit does not imply that a thick-wall  $Q$ -ball has to have a large thickness that is comparable to the size of the core size. For the extreme thin-wall limit  $\omega = \omega_-$ , thin-wall  $Q$ -balls satisfy  $\frac{E_Q}{Q} = \gamma(\omega_-)\omega_-$ . In particular, Coleman demonstrated that a steplike profile for  $Q$ -balls, which generally exist for  $\omega_- \neq 0$ , satisfies  $\gamma = 1$ , which implies that the charge  $Q$  and energy  $E_Q$  are proportional to the volume, and he called this  $Q$ -matter [5]. For absolutely stable  $Q$ -balls, the energy per unit charge is smaller than the rest mass  $m$  for the field  $\phi$ ,

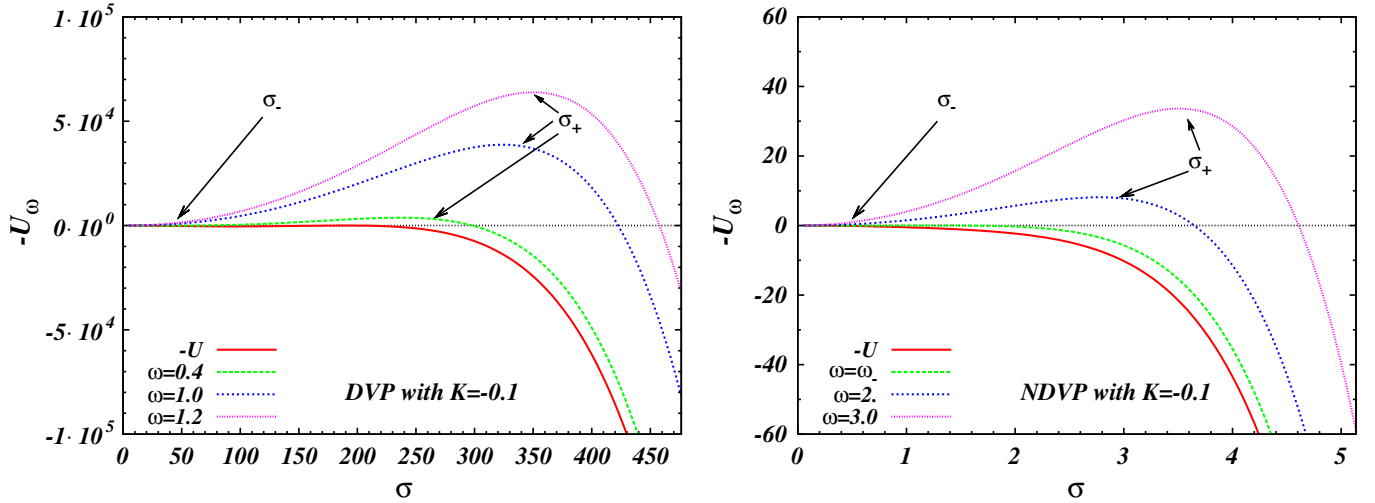


FIG. 1 (color online). Parameters  $\sigma_{\pm}(\omega)$  for a potential of the form  $U(\sigma) = 1/2\sigma^2(1 - |K| \ln \sigma^2) + \beta^2\sigma^6$  (effective potential  $U_{\omega} = U - 1/2\omega^2\sigma^2$ ) with  $|K| = 0.1$ . The left-hand figure corresponds to the case of a DVP with  $\beta^2 = \frac{|K|e^{-1}}{4} \exp(-\frac{2}{|K|}) \sim 1.90 \times 10^{-11}$ , while the right-hand side is the NDVP with  $\beta^2 = \frac{|K|e^{-1}}{4} \sim 9.20 \times 10^{-3}$ , see Eq. (19). The colored lines in each plot correspond to different values of  $\omega$ . The variable  $\sigma_{+}(\omega)$  is defined as the maximum of the inverse effective potential  $-U_{\omega}$ , whereas  $\sigma_{-}(\omega)$  corresponds to  $-U_{\omega}(\sigma_{-}(\omega)) = 0$  for  $\sigma_{-}(\omega) \neq 0$ . Recalling  $\omega_{-} = 0$  in DVP, the DVP has degenerate vacua at  $\sigma_{+}(0) = e^{1/4} \exp(\frac{1}{2|K|}) \sim 1.91 \times 10^2$  (red-solid line), while the NDVP does not. The inverse effective potential  $-U_{\omega}$  with  $\omega_{-} = 1$  in NDVP (green-dashed line), however, has degenerate vacua at  $\sigma_{+}(\omega_{-}) = e^{1/4} \sim 1.28$ , see the first relation in Eq. (18). For the lower limit  $\omega \simeq \omega_{-}$  (green-dashed lines), we could see  $\sigma_{+} = e^{1/4}$ , while the purple dotted-dashed lines show  $\sigma_{-}(\omega) \rightarrow 0$  near the thick wall-like limit  $\omega = 3.0 \sim \omega_{+}$  where  $\omega_{+} \gg 1$ .

$$\frac{E_Q}{Q} < m. \quad (7)$$

Thus, the  $Q$ -ball satisfying Eq. (7) is stable against free particle decays because the  $Q$ -ball energy  $E_Q$  is less than a collection of  $Q$  free particles of total energy  $E_{\text{free}} = mQ$ . If the  $Q$ -ball has decay channels into other fundamental scalar particles that have the lowest mass  $m_{\text{min}}$ , we need to replace  $m$  by  $m_{\text{min}}$  in the absolute stability condition Eq. (7). In the opposite limit  $\omega \simeq \omega_{+}$ , the  $Q$ -ball energy approaches the free particle energy  $E_Q \rightarrow mQ$ . For later convenience, we define two positive definite quantities,  $\epsilon_{\omega}$  and  $m_{\omega}$  by

$$\begin{aligned} \epsilon_{\omega} &\equiv -U_{\omega}(\sigma_{+}(\omega)) = \frac{1}{2}\omega^2\sigma_{+}^2(\omega) - U(\sigma_{+}(\omega)), \\ &\simeq \frac{1}{2}(\omega^2 - \omega_{-}^2)\sigma_{+}^2, \end{aligned} \quad (8)$$

$$m_{\omega}^2 \equiv m^2 - \omega^2, \quad (9)$$

which can be infinitesimally small for either thin- or thick-wall limits. By assuming  $\sigma_{+}(\omega) \simeq \sigma_{+}(\omega_{-}) \equiv \sigma_{+}$  in the thin-wall limit, we immediately obtain the second line in Eq. (8). Notice that this assumption was implicitly imposed in our previous thin-wall analysis [37]. While this is fine for the gravity-mediated case, with gauge-mediated potentials, which are extremely flat, this implicit assumption cannot hold because  $\sigma_{+}(\omega)$  does not exist. Therefore, we will not use the variable  $\epsilon_{\omega}$  for the case of the gauge-mediated potentials. Notice that the variable  $m_{\omega}^2$  cannot be

infinitesimally small when we consider the gravity-mediated case:  $\omega_{+}^2 \sim m^2$ . A powerful tool we can make use of when calculating some of the physical  $Q$ -ball parameters is the Legendre relation [37,38]. For example the energy follows from

$$S_{\omega} \rightarrow Q = -\left. \frac{dS_{\omega}}{d\omega} \right|_{E_Q} \rightarrow E_Q = \omega Q + S_{\omega}. \quad (10)$$

Assuming that  $\gamma$  is not a function of  $\omega$ , we can compute the advertised characteristic slope,

$$\frac{E_Q}{\omega Q} = \gamma \rightarrow E_Q \propto Q^{1/\gamma}, \quad (11)$$

where we have used another Legendre relation  $\omega = \left. \frac{dE_Q}{dQ} \right|_{S_{\omega}}$  in which we have fixed  $S_{\omega}$ . If a  $Q$ -ball is classically stable, it satisfies

$$\frac{\omega}{Q} \frac{dQ}{d\omega} \leq 0 \Leftrightarrow \frac{d}{d\omega} \left( \frac{E_Q}{Q} \right) = -\frac{S_{\omega}}{Q^2} \frac{dQ}{d\omega} \geq 0. \quad (12)$$

These classical stability conditions are equivalent to the fission condition, i.e.  $\frac{d\omega}{dQ} \leq 0$  in [37] so that the charge  $Q$  for classically stable  $Q$ -balls is a decreasing function in terms of  $\omega$ . By scaling a  $Q$ -ball solution with respect to the radius  $r$ , we also obtain the virial relation  $D\mathcal{U} = -(D-2)S + D\omega Q/2$  and the characteristic slope  $\gamma(\omega)$ ,

$$\gamma(\omega) = 1 + \left( D - 2 + D \frac{\mathcal{U}}{S} \right)^{-1} \quad (13)$$

once the ratio  $\mathcal{S}/\mathcal{U}$  is given where  $\mathcal{S} \equiv \int_{V_D} \frac{1}{2} \sigma^2$  and  $\mathcal{U} \equiv \int_{V_D} U$  are the surface and potential energies, respectively. For  $D \geq 2$ , we can see  $\gamma(\omega) \geq 1$  because  $\mathcal{S}$ ,  $\mathcal{U} \geq 0$ , which implies that  $S_\omega$  is positive definite for  $D \geq 2$ , see Eq. (1), while  $S_\omega$  is positive for  $D = 1$  only when  $\mathcal{U} \geq \mathcal{S}$ . It implies that we have to be careful to use the second relation of Eq. (12) for  $D = 1$  to evaluate the classical stability condition as we saw in the case of using the Gaussian ansatz, which is valid for  $D = 1$  for polynomial potentials [37]. Our key results for  $D \geq 2$  are

$$\gamma \simeq \begin{cases} 1 & \text{for } \mathcal{S} \ll \mathcal{U}, \\ (2D - 1)/2(D - 1) & \text{for } \mathcal{S} \sim \mathcal{U}, \\ (D - 1)/(D - 2) & \text{for } \mathcal{S} \gg \mathcal{U}. \end{cases} \quad (14)$$

The first case in Eq. (14) corresponds to the extreme case of thin- and thick-wall  $Q$ -balls. Furthermore, in [37], we saw that for the extreme thin-wall  $Q$ -balls in DVPs, then there was a virialization between  $\mathcal{S}$  and  $\mathcal{U}$ , which corresponds to the second case in Eq. (14). At present it is not known what kind of  $Q$ -ball potentials correspond to the third case; therefore, we will not be considering that case in the rest of our paper. Notice that in the case  $\mathcal{S} \gg \mathcal{U}$  for  $D = 2$ , we obtain the characteristic slope  $\gamma \gg 1$  from Eq. (13). Similarly, for  $D = 1$ , the characteristic slopes are obtained, i.e.  $\gamma \simeq 1, \gg 1, \simeq 0$ , respectively, for  $\mathcal{S} \ll \mathcal{U}$ ,  $\mathcal{S} \sim \mathcal{U}$ ,  $\mathcal{S} \gg \mathcal{U}$ . We will use these 1D analytic results to interpret numerical results of one-dimensional  $Q$ -balls in the thin-wall limit.

To end this section we note a nice duality that appears in Eqs. (13) and (14) between the two cases  $\mathcal{S} \sim \mathcal{U}$  and  $\mathcal{S} \gg \mathcal{U}$ . In particular, for  $\mathcal{S} \sim \mathcal{U}$  in  $D$  dimensions, the same result for  $\gamma$  is obtained (to leading order) in  $2 \times D$  dimensions when  $\mathcal{S} \gg \mathcal{U}$ .

### III. GRAVITY-MEDIATED POTENTIALS

The MSSM consists of a number of flat directions where SUSY is not broken. Those flat directions are, however, lifted by gauge, gravity, and/or nonrenormalizable interactions. In what follows the gravity interaction is included perturbatively via the one-loop corrections for the bare mass  $m$  in Eq. (5) and the nonrenormalizable interactions ( $U_{\text{NR}}$ ), which are suppressed by high energy scales such as the grand unified theory scale  $M_U \sim 10^{16}$  GeV or Planck scale  $m_{\text{pl}} \sim 10^{18}$  GeV. Here,  $m$  is of order of SUSY breaking scale, which could be the gravitino mass  $\sim m_{3/2}$ , evaluated at the renormalization scale  $M$  [23]. We note that, following the majority of work in this field, we will ignore A-term contributions [U(1) violation terms], thermal effects [39,40], which come from the interactions between the AD field and the decay products of the inflaton, and the Hubble-induced terms which gives a negative mass-squared contribution during inflation. It is possible that their inclusion could well change the results of the following analysis.

The scalar potential we are considering at present is [21,23]

$$U = U_{\text{grav}} + U_{\text{NR}} = \frac{1}{2} m^2 \sigma^2 \left( 1 + K \ln \left( \frac{\sigma^2}{M^2} \right) \right) + \frac{|\lambda|^2}{m_{\text{pl}}^{n-4}} \sigma^n, \quad (15)$$

where we used Eq. (5),  $K$  is a factor for the gaugino correction, which depends on the flat directions, and  $M$  is the renormalization scale. Also,  $\lambda$  is a dimensionless coupling constant, and  $U_{\text{NR}} \equiv \frac{|\lambda|^2}{m_{\text{pl}}^{n-4}} \sigma^n$ , where  $n > 2$ . If the MSSM flat directions include a large top quark,  $K$  can be positive and then  $Q$ -balls do not exist. For flat directions that do not have a large top quark component, we typically find  $K \simeq -[0.01-0.1]$  [21,41]. The power  $n$  of the nonrenormalizable term depends on the flat directions we are choosing along which we maintain R parity. As examples of the directions involving squarks, the  $u^c d^c d^c$  direction has  $n = 6$ , while the  $u^c u^c d^c e^c$  direction requires  $n = 10$ . A complete list of the MSSM flat directions can be found in Table 1 of [42]. Since the potential in Eq. (15) for  $K < 0$  satisfies the  $Q$ -ball existence condition in Eq. (4), where  $\omega_+ \gg m$ ,  $Q$ -balls naturally exist.

In the rest of this paper, we will focus on potentials of the form of Eq. (15) for general  $D (\geq 1)$  and  $\omega$  and  $n (> 2)$  so that  $M$  and  $m_{\text{pl}}$  have the same mass dimension,  $(D - 1)/2$ , as  $\sigma$ . It means that the parameters  $M$  and  $m_{\text{pl}}$  are only physical for  $D = 3$ . For several cases of  $n$  and  $D$ , the term  $U_{\text{NR}}$  can be renormalizable, but we will generally call it the nonrenormalizable term for the future convenience. The readers should note that the potential Eq. (15) has been derived only with  $\mathcal{N} = 1$  supergravity in  $D = 3$ , therefore the potential form could well be changed in other dimensions. Furthermore, the logarithmic correction breaks down for small  $\sigma$  and the curvature of Eq. (15) at  $\sigma = 0$  is finite due to the gaugino mass, which affects our thick-wall analysis and their dynamics. However, we concentrate our analysis on this potential form for arbitrary  $D$ ,  $n$  and any values of  $\sigma$  for two main reasons. The first is that it contains a number of general semiclassical features expected of all the potentials, and the second is that it offers the opportunity to consider the lower-dimensional  $Q$ -balls embedded in  $D = 3$ .

In Appendix A, we obtain the exact solution of Eq. (3) with the potential  $U = U_{\text{grav}}$ ; however, exact solutions of the general potential  $U$  in Eq. (15) are fully nonlinear and can be obtained only numerically. Therefore, we will analytically examine the approximate solutions in both the thin- and thick-wall limits. Before doing so, we shall begin by imposing a restriction on  $\lambda$  in Eq. (15) in order to obtain stable  $Q$ -matter in NDVPs. With the further restrictions on  $\lambda$  and  $|K|$ , we can proceed with our analytical arguments, and we will finally obtain the asymptotic  $Q$ -ball profile for large  $r$ , which will be used in the numerical section, Sec. V.



### A. The existence of absolutely stable $Q$ -matter

As we have seen, the first restriction on the gravity-mediated potential Eq. (15), which will allow for the existence of a  $Q$ -ball solution Eq. (4) is  $K < 0$ . However, given values for  $m$ ,  $m_{\text{pl}}$ ,  $M$ ,  $n$ , and  $K$  in Eq. (15), we need to restrict the allowed values of the parameter,  $\lambda$  in the potential in order to ensure we obtain absolutely stable  $Q$ -matter. Notice that  $Q$ -matter exists in NDVPs, while the extreme thin-wall  $Q$ -balls in DVPs, which will not be  $Q$ -matter as it will turn out, may exist with the lowest limit of  $\lambda$ .

By using the definitions of  $\omega_-$  and  $\sigma_+$ , namely,  $\omega_-^2 \equiv \frac{2U}{\sigma^2}|_{\sigma_+}$  and  $\frac{dU_{\omega_-}}{d\sigma}|_{\sigma_+} = 0$ , we shall find the range of values of  $\lambda$  for which absolutely stable  $Q$ -matter solutions exist. Moreover, we will obtain the curvature  $\mu$ , which is proportional to  $|K|$ , of the effective potential  $U_{\omega}$  at  $\sigma_+$ .

The effective potential for Eq. (15) can be rewritten in terms of new dimensionless variables  $\tilde{\sigma} = \sigma/M$ ,  $\tilde{\omega} = \omega/m$ , and

$$\beta^2 = \frac{|\lambda|^2 M^{n-2}}{m_{\text{pl}}^{n-4} m^2} > 0, \quad (16)$$

as

$$U_{\tilde{\omega}} = \frac{1}{2} M^2 m^2 \tilde{\sigma}^2 (1 - \tilde{\omega}^2 - 2|K| \ln \tilde{\sigma}) + M^2 m^2 \beta^2 \tilde{\sigma}^n. \quad (17)$$

After some simple algebra and introducing  $\tilde{\omega}_-^2 \equiv \frac{2U}{\sigma^2}|_{\sigma_+}$  and  $\frac{dU_{\tilde{\omega}_-}}{d\tilde{\sigma}}|_{\tilde{\sigma}_+} = 0$ , we obtain

$$\tilde{\sigma}_+ = \left( \frac{|K|}{(n-2)\beta^2} \right)^{1/(n-2)}, \quad (18)$$

$$\tilde{\omega}_-^2 = \frac{1}{n-2} \left[ n-2 + 2|K| - 2|K| \ln \left( \frac{|K|}{(n-2)\beta^2} \right) \right].$$

Notice that  $\tilde{\omega}_-^2 = 0$  corresponds to DVPs where  $Q$ -matter solutions do not exist [37], while the extreme thin-wall  $Q$ -balls do exist and are absolutely stable as we will see. In NDVPs,  $Q$ -matter solutions exist and are absolutely stable when  $0 < \tilde{\omega}_-^2 < 1$ , see Eq. (7). Combining these facts and using the second relation in Eq. (18), we have the constraint on  $\lambda$  for stable  $Q$ -matter solutions to exist, namely,

$$\frac{|K|e^{-1}}{n-2} \exp\left(-\frac{n-2}{2|K|}\right) < \beta^2 < \frac{|K|e^{-1}}{n-2}, \quad (19)$$

$$\begin{aligned} \Leftrightarrow \frac{|K|e^{-1}}{n-2} \frac{m_{\text{pl}}^{n-4} m^2}{M^{n-2}} \exp\left(-\frac{n-2}{2|K|}\right) < |\lambda|^2 \\ < \frac{|K|e^{-1}}{n-2} \frac{m_{\text{pl}}^{n-4} m^2}{M^{n-2}}, \end{aligned} \quad (20)$$

where we have used Eq. (16) to go from Eq. (19) to Eq. (20). Here, the lower limit of  $|\lambda|^2$  corresponds to  $\tilde{\omega}_-^2 = 0$ , while the upper limit corresponds to  $\tilde{\omega}_-^2 = 1$ . The

inequality in Eq. (20) implies that if the coupling constant  $\lambda$  of the non-renormalizable term in Eq. (15) is too small, then it does not support the existence of  $Q$ -balls, whereas a large  $\lambda$  coupling leads to unstable  $Q$ -matter. With the following parameter set,  $m = M = 1$ ,  $|K| = 0.1$ ,  $n = 6$  and the lower/upper limits of  $\beta^2$  in Eq. (19), Fig. 1 shows the inverse potentials in Eq. (17) and their inverse effective potentials  $-U_{\omega}$  with various values of  $\omega$ . The lower limit,  $\beta^2 = \frac{|K|e^{-1}}{4} \exp(-\frac{2}{|K|})$ , corresponds to DVPs case with  $\omega_- = 0$ , while in the upper limit,  $\beta^2 = \frac{|K|e^{-1}}{4}$ , the potentials do not have degenerate vacua with  $\omega_- = 1$ , hence are called NDVPs. By substituting the values of  $\beta^2$  into Eq. (18), we obtain the values of  $\sigma_+$  indicated in Fig. 1. Finally, we can obtain the curvature,  $\mu^2(\omega) \equiv \frac{d^2 U_{\omega}}{d\sigma^2}|_{\sigma_+(\omega)}$ , evaluated at  $\omega_-$ , i.e.

$$\mu^2 \equiv \mu^2(\omega_-) = m^2 |K| (n-2) \propto |K|, \quad (21)$$

which implies that a small logarithmic correction  $|K| \ll \mathcal{O}(1)$  in Eq. (15) gives an ‘‘extremely’’ flat effective potential  $U_{\omega}$  compared to the quadratic term  $m^2$  around  $\sigma = \sigma_+$  for a given  $n \sim \mathcal{O}(10^{0-1})$ .

### B. Thin-wall $Q$ -ball for $\sigma_0 \simeq \sigma_+$ , $R_Q \gg \delta$ , $1/\mu$ , $D \geq 2$

For the extreme limit  $\omega = \omega_-$ , Coleman demonstrated that the steplike ansatz [5] is applicable to the case of NDVPs because the surface effects of the thin-wall  $Q$ -ball in this limit are not significant. There are situations though where we would like to explore the region around  $\omega = \omega_-$ , corresponding to  $\sigma_0 \simeq \sigma_+(\omega)$ , and to do this we need to include surface effects. In [37] we explained how to do this under the assumptions:  $R_Q/\delta$ ,  $\mu R_Q \gg 1$ ,  $\sigma(R_Q) < \sigma_-(\omega)$ ,  $\sigma_+(\omega) \simeq \sigma_+(\omega_-) \equiv \sigma_+$ , and that the surface tension  $\tau \simeq \int_0^{\sigma_+} d\sigma \sqrt{2U_{\omega_-}}$  does not depend ‘‘sensitively’’ on  $\omega$ . Here,  $R_Q$ ,  $\delta$  are, respectively, the  $Q$ -ball core size and the shell thickness. We note that in [43], Coleman assumed  $U_{\omega} \simeq U_{\omega_-}$  in the shell region, and this is equivalent to saying  $\sigma(R_Q) < \sigma_-(\omega)$ . In what follows we will be making use of Coleman’s approach. By requiring this or  $\sigma(R_Q) < \sigma_-(\omega)$ , we can guarantee real values of shell thickness  $\delta$  and surface tension  $\tau$ . The assumption, in which  $\tau$  does not depend on  $\omega$ , is related to the assumptions  $\sigma_+(\omega) \simeq \sigma_+$  and  $U_{\omega} \simeq U_{\omega_-}$  is negligible in the shell region.

Under these assumptions and for  $D \geq 2$ , we now apply the previous thin-wall analysis in [37] to the present potential Eq. (15). The ansatz is given by

$$\sigma(r) = \begin{cases} \sigma_+ - s(r) & \text{for } 0 \leq r < R_Q, \\ \tilde{\sigma}(r) & \text{for } R_Q \leq r \leq R_Q + \delta, \\ 0 & \text{for } R_Q + \delta < r, \end{cases} \quad (22)$$

where  $R_Q$ ,  $\delta$ , the core profile  $s(r)$ , and the shell profile  $\tilde{\sigma}(r)$  will be obtained in terms of the underlying potential by extremizing  $S_{\omega}$  with respect to  $R_Q$ . Each of the profile

functions satisfies

$$s'' + \frac{D-1}{r} s' - \mu s = 0, \quad (23)$$

$$\bar{\sigma}'' - \frac{dU_\omega}{d\sigma} \Big|_{\bar{\sigma}} = 0. \quad (24)$$

By recalling Eq. (8), we have previously found that [37]

$$R_Q \simeq (D-1) \frac{\tau}{\epsilon_\omega}; \quad S_\omega \simeq \frac{\tau}{D} \partial V_D > 0; \quad (25)$$

$$Q \simeq \omega \sigma_+^2 V_D,$$

$$\frac{E_Q}{\omega Q} \simeq \frac{2D-1}{2(D-1)} - \frac{\omega_-^2}{2(D-1)\omega^2}, \quad (26)$$

$$\frac{\omega}{Q} \frac{dQ}{d\omega} \simeq 1 - \frac{2D\omega^2}{\omega^2 - \omega_-^2} < 0, \quad (27)$$

where we have taken the thin-wall limit  $\omega \simeq \omega_-$  in the last inequality. Notice that our analytical work cannot apply for the 1D thin-wall  $Q$ -ball, see Eq. (25).

*NDVPs:* This type of potential supports the existence of  $Q$ -matter that corresponds to the regime  $\mathcal{U} \gg \mathcal{S}$ . The  $Q$ -matter can be absolutely as well as classically stable for the extreme limit  $\omega \simeq \omega_-$ , when the coupling constant  $\lambda$  for the nonrenormalizable term in Eq. (15) satisfies Eq. (20). The characteristic slope is given by the first case of Eq. (14), and the charge and energy are proportional to the volume  $V_D$ .

*DVPs:* With the presence of degenerate minima in Eq. (15), in [37] we obtained the ratio  $\mathcal{U}/\mathcal{S} \sim 1$ , which corresponds to the second case of Eq. (14). The charge and energy are not proportional to the volume  $V_D$  itself in this case; hence, we cannot see the existence of  $Q$ -matter in the extreme limit  $\omega = \omega_- = 0$ . Instead, we can find the proportional relation simply from Eq. (11) and (26), namely,  $E_Q \propto Q^{2(D-1)/(2D-1)}$ .

Our main approximations are based on the assumptions  $\sigma_0 \simeq \sigma_+$ ,  $R_Q \gg \delta$ ,  $1/\mu$ , and  $U_\omega \simeq U_{\omega_-}$  in the shell region. In what follows we will see through numerical simulations that our analytic results agree well with the corresponding numerical results even in a ‘‘flat’’ potential choice  $|K| = 0.1$ ,  $m = M = 1$ ,  $n = 6$ , which implies that  $1/\mu \sim 1.58$ , see Eq. (21).

### C. Thick-wall $Q$ -balls for $\beta^2 \simeq |K| \simeq \mathcal{O}(1)$

In [37] we studied thick-wall  $Q$ -balls in general polynomial potentials, and extracted out the explicit  $\omega$  dependence from the integral in  $S_\omega$  by reparameterizing terms in the Euclidean action  $S_\omega$  in terms of dimensionless quantities and by neglecting higher order terms. We then made use of the technique Eq. (10) and obtained consistent classical and absolute stability conditions, Eqs. (7) and (12). For our present potential, Eq. (17), which satisfies

the condition,  $\beta^2 \simeq |K| \simeq \mathcal{O}(1)$ , we will be able to ignore the nonrenormalizable term by introducing  $\tilde{\sigma} = \sigma/M$  and  $\beta^2$  in Eq. (16). We can then obtain the stability conditions using the same technique as before. Indeed for the limit  $\omega \gtrsim \mathcal{O}(m)$ , we will see  $\tilde{\sigma}(r) \sim \mathcal{O}(\epsilon) < \mathcal{O}(1)$  where  $\epsilon$  is a small dimensionless constant [not  $\epsilon_\omega$  in Eq. (8)], and see  $\tilde{\sigma}_0 \equiv \tilde{\sigma}(0) \geq \tilde{\sigma}(r)$  for any  $r$  because  $\tilde{\sigma}(r)$  is a monotonically decreasing function in terms of  $r$ . Since the leading order of the logarithmic term,  $\tilde{\sigma}^2 \ln \tilde{\sigma}$ , in Eq. (17) is of  $\mathcal{O}(\epsilon^2)$  using the L'Hôpital's rules, we can ignore the nonrenormalizable term in Eq. (17) at the beginning of our analysis. To confirm this, in Appendix B we will keep all terms in Eq. (17) by introducing a Gaussian ansatz and show that the results below [Eqs. (33) and (34)] can also be recovered under the same assumption  $\beta^2 \simeq |K| \simeq \mathcal{O}(1)$ . By adapting the techniques introduced in [37], in this subsection we will show how to obtain the thick-wall solutions without involving Gaussian ansatz.

First of all we introduce two characteristic limits: the ‘‘moderate limit’’  $\omega \gtrsim \mathcal{O}(m)$  and the ‘‘extreme’’ limit  $\omega \gg m$ . We will see  $\tilde{\sigma}_0 \simeq \tilde{\sigma}_-(\omega) \rightarrow 0^+$ , which leads to  $\tilde{\sigma}_-(\omega) \ll \mathcal{O}(1)$  in the ‘‘extreme limit,’’ and then even in the moderate limit we will see that the contributions from the nonrenormalizable term are negligible and that  $\tilde{\sigma}_-(\omega)$  is a monotonically decreasing function in terms of  $\omega$ . Under the conditions  $\beta^2 \simeq |K| \simeq \mathcal{O}(1)$  in Eq. (17), we obtain

$$\left(\frac{\omega}{m}\right)^2 = 1 - 2|K| \ln \tilde{\sigma}_-(\omega) + 2\beta^2 \tilde{\sigma}_-^{n-2}(\omega) \sim 1 - 2|K| \log \tilde{\sigma}_-(\omega), \quad (28)$$

$$\frac{|K|m^2}{2\omega \tilde{\sigma}_-(\omega)} \frac{d\tilde{\sigma}_-(\omega)}{d\omega} = \left[ -1 + 2(n-2) \frac{\beta^2 \tilde{\sigma}_-^{n-2}(\omega)}{|K|} \right]^{-1} \sim -1 < 0, \quad (29)$$

$$\Leftrightarrow \omega \gtrsim \mathcal{O}(m), \quad \tilde{\sigma}_-(\omega) \sim \exp\left[\frac{m_\omega^2}{2|K|m^2}\right] \rightarrow 0, \quad (30)$$

where we used  $U_\omega(\tilde{\sigma}_-(\omega)) = 0$  to obtain Eq. (28). It follows that  $\tilde{\sigma}_-(\omega) \ll \mathcal{O}(1)$  for the thick-wall limit  $\omega \gg m$ , and we can ignore the nonrenormalizable term. Since Eq. (29) implies that  $\frac{d\tilde{\sigma}_-(\omega)}{d\omega} < 0$  in the limit  $\tilde{\sigma}_-(\omega) < \mathcal{O}(1)$ ,  $\tilde{\sigma}_-(\omega)$  is a monotonically decreasing function. Therefore, we can ignore the contributions from the nonrenormalizable term up to  $\omega \gtrsim \mathcal{O}(m)$ , which we call the moderate limit with the notion ‘‘ $\sim$ ’’ as seen in the second relations of Eqs. (28) and (29), instead of the extreme limit  $\omega \gg m$  with the notion ‘‘ $\rightarrow$ .’’ Thus, we obtain the desired results of the second relation in Eq. (30). From Eq. (28), the logarithmic term may be of  $\leq \mathcal{O}(1)$  for  $|K| < \mathcal{O}(1)$ ,  $\beta^2 \ll \mathcal{O}(1)$  in the moderate limit, which implies that the moderate limit is valid even when  $\omega \sim \mathcal{O}(m)$ .

Let us define  $\alpha(r)$  and  $\tilde{r}$  through  $\tilde{\sigma}(r) = a\alpha(r)$  and  $r = b\tilde{r}$  where  $a$  and  $b$  will be obtained in terms of the underlying parameters. By substituting these reparameterized parameters  $\alpha$ ,  $\tilde{r}$  and neglecting the nonrenormalizable term in Eq. (1) due to the L'Hôpital's rules, we obtain

$$S_\omega \sim \Omega_{D-1} \int d\tilde{r} \tilde{r}^{D-1} b^D \left[ \frac{1}{2} \left( \frac{aM}{b} \right)^2 \left( \frac{d\alpha}{d\tilde{r}} \right)^2 - \frac{1}{2} m^2 a^2 M^2 \left( 1 - \left( \frac{\omega}{m} \right)^2 - 2|K| \ln a \right) \alpha^2 + \frac{1}{2} m^2 |K| a^2 M^2 \alpha^2 \ln \alpha^2 \right], \quad (31)$$

$$= a^2 M^2 b^{D-2} \tilde{S}(\alpha), \quad (32)$$

where  $\tilde{S}(\alpha(\tilde{r}/b)) \equiv \Omega_{D-1} \int d\tilde{r} \tilde{r}^{D-1} \{ 1/2 (\frac{d\alpha}{d\tilde{r}})^2 - 1/2 \alpha^2 (1 - \ln \alpha^2) \}$ , which is independent of  $\omega$ . In going from Eq. (31) to Eq. (32) we have set the coefficients of the three terms in the brackets of Eq. (31) to be unity in order to explicitly remove the  $\omega$  dependence from the integral in  $S_\omega$ . In other words, we have set  $a = e^{-1/2} \exp[\frac{m_\omega^2}{2|K|m^2}] \sim e^{-1/2} \tilde{\sigma}_-(\omega)$ ,  $b = \frac{1}{m\sqrt{|K|}}$ . From Eq. (10), we can differentiate Eq. (32) with respect to  $\omega$  to obtain  $Q$  and then use the Legendre transformation to obtain  $E_Q$ . Coupled with Eqs. (7) and (12) we obtain both the classical and absolute stability conditions. This is straightforward and yields

$$Q \sim \frac{2\omega}{m^2|K|} S_\omega, \quad \frac{E_Q}{\omega Q} \sim 1 + \frac{m^2|K|}{2\omega^2} \rightarrow 1, \quad (33)$$

$$\frac{d}{d\omega} \left( \frac{E_Q}{Q} \right) \sim 1 - \frac{m^2|K|}{2\omega^2} \rightarrow 1 > 0, \quad (34)$$

$$\frac{\omega}{Q} \frac{dQ}{d\omega} \sim 1 - \frac{2\omega^2}{m^2|K|} \rightarrow -\frac{2\omega^2}{m^2|K|} < 0,$$

where we have taken the extreme limit  $\omega \gg m$  as indicated by  $\rightarrow$ . Equation (33) implies that the characteristic slope for the thick-wall  $Q$ -balls are tending toward the case  $S \ll U$  in Eqs. (14) and (34) shows that the  $Q$ -balls are classically stable. These results are independent of  $D$ . In Appendix B we will generalize the results of Eqs. (33) and (34) by adopting an explicit Gaussian ansatz without neglecting the nonrenormalizable term.

Before finishing this subsection, let us comment on possibilities to have absolutely stable thick-wall  $Q$ -balls in the case,  $|K| < \mathcal{O}(1)$ ,  $\beta^2 \ll \mathcal{O}(1)$ . The results present above still hold even in the moderate limit  $\omega \sim \mathcal{O}(m)$  for the present case. Thus, the thick-wall  $Q$ -balls, if they exist, can be absolutely stable when the following conditions from Eqs. (7) and (33) are met:

$$\omega_- < m, \quad \frac{\omega}{m} < \frac{1 + \sqrt{1 - 2|K|}}{2}, \quad (35)$$

$$|K| < \frac{1}{2}, \quad \beta^2 \ll \mathcal{O}(1).$$

It follows that for  $|K| \geq 1/2$ , the thick-wall  $Q$ -balls are always absolutely unstable. If  $\omega_- \geq m$ , we know  $\omega > \omega_-$  in both the moderate and extreme limits, hence the thick-wall  $Q$ -ball is always absolutely unstable again, see Eq. (7). Notice that the condition  $\beta^2 \leq |K|$  implies  $\omega_- \leq \mathcal{O}(m)$ , see Eq. (19), so the first condition in Eq. (35) can be satisfied. This then leaves only a small window of the parameter space for absolutely stable thick-wall  $Q$ -balls. In the numerical section, Sec. V, we will confirm that the thick-wall  $Q$ -ball can be absolutely stable against decay into their own quanta by choosing suitable parameters, i.e.  $\omega_- = 0$ ,  $\beta^2 \sim 1.90 \times 10^{-11}$ , and  $|K| = 0.1$ .

#### D. Asymptotic profile for large $r$ and $\beta^2 \leq |K| \leq \mathcal{O}(1)$

In order to obtain the full numerical profiles over all values of  $\omega$ , we should analytically determine the asymptotic profile for large  $r$  in the potential Eq. (15), which satisfies  $\beta^2 \leq |K| \leq \mathcal{O}(1)$  as in the previous subsection. As long as the value of  $r$  satisfies  $r > R_\omega$ , where  $R_\omega$  is some large length scale and depends on  $\omega$ , we can assume that the friction term in Eq. (3) and the nonrenormalizable term in Eq. (15) are negligible for large  $r$ . Hence, the  $Q$ -ball equation Eq. (3) reduces to the one-dimensional and integrable form

$$\sigma'' = \frac{dU_\omega}{d\sigma}, \quad (36)$$

where  $U_\omega \simeq 1/2 m^2 \sigma^2 (1 - (\frac{\omega}{m})^2 - |K| \log(\frac{\sigma^2}{M^2}))$ . Equation (36) implies that the profile has a symmetry under the variation of  $r$  because Eq. (36) does not depend on  $r$  explicitly. Multiplying both sides of Eq. (36) by  $\frac{d\sigma}{dr}$  leads to

$$\int_{\sigma(R_\omega)}^{\sigma(r)} \frac{d\sigma}{\sqrt{2U_\omega}} = R_\omega - r, \quad (37)$$

where we have used the boundary conditions  $\sigma'(\infty) \rightarrow 0$ ,  $U_\omega(\sigma(\infty) \rightarrow 0) \rightarrow 0$ , and  $\sigma'(r) < 0$ . After some elementary algebra, the final asymptotic profile becomes

$$\sigma(r) = M e^{M m_\omega^2 / 2 m^2} \exp\left(-\frac{m^2 |K| M}{2} (r - r_\omega)^2\right), \quad (38)$$

$$\frac{d}{dr} \left( -\frac{\sigma'}{\sigma} \right) = m^2 |K| M, \quad (39)$$

where  $r_\omega \equiv R_\omega - \sqrt{\frac{m_\omega^2}{m^2} - \frac{2|K|}{M} \log(\frac{\sigma(R_\omega)}{M})} / (|K|m)$ . Equation (38) is a consequence of the symmetry in Eq. (36) under the translation  $r \rightarrow r - r_\omega$  from a Gaussian profile as seen in Eq. (A1) of Appendix A. Furthermore, Eq. (39) depends on the parameters  $m$ ,  $M$ ,  $|K|$  in Eq. (15). We will later use the relation Eq. (39) as a criterion that must be

satisfied in obtaining full numerical profiles for all values of  $\omega$ .

We finish this section by recapping the key results we have derived for the case of the gravity-mediated potential, Eq. (15), in both the thin- and thick-wall limits. In the thick-wall limit, we imposed the restrictions  $\beta^2 \lesssim |K| \lesssim \mathcal{O}(1)$  on the potential to ignore the nonrenormalizable term. In both limits, we have derived the characteristic slopes in Eqs. (26) and (33) and the classical stability conditions in Eqs. (27) and (34) and shown that the  $Q$ -balls are classically stable in both cases. The thin-wall  $Q$ -balls in DVPs are always absolutely stable, and  $Q$ -matter in NDVPs can be absolutely stable when the coupling constant for the nonrenormalizable term satisfies Eq. (20); while absolutely stable  $Q$ -balls in the thick-wall limit may exist only for Eq. (35). Finally, we obtained the general asymptotic profile, Eq. (38), for large  $r$ .

#### IV. GAUGE-MEDIATED POTENTIAL

The gauge-mediated scalar potential can be written in quadratic form in the low energy regime for scales up to the messenger scale  $M_S$ , and carries a logarithmically (extremely) flat piece in the high energy regime [1,22]. This extreme flatness means that the thin-wall  $Q$ -ball we used in Eq. (22) cannot be applied to this situation, and so we now turn our attention to  $Q$ -balls in extreme flat potentials. We will generalize the results of [1] to an arbitrary number of spatial dimensions and show that the known  $Q$ -ball profiles in [1,34] are naturally recovered by our more general ansatz. Moreover, we will investigate both the classical and absolute stability of these  $Q$ -balls. The gauge-mediated potential, which we will use in this section, is approximated by [44,45]

$$U(\sigma) = \begin{cases} \frac{1}{2}m^2\sigma^2 & \text{for } \sigma(r) \leq \sigma(R) \\ U_0 = \text{const} & \text{for } \sigma(R) < \sigma(r) \end{cases}, \quad (40)$$

where  $U_0$  and  $R$  are free parameters that will be determined by imposing a condition that leads to a smooth matching of the profiles at  $\sigma(R)$ ,  $U_0 = 1/2m^2\sigma^2(R)$ . Notice that  $Q$ -balls exist within  $0 < \omega < m$  in Eq. (40), and the potential does not have degenerate vacua although  $\omega_- \simeq 0$ . Since Eq. (40) is not differentiable at  $\sigma(R)$ , we can approximate Eq. (40) by

$$U_{\text{gauge}} = \frac{1}{2}m^2\Lambda^2(1 - e^{-\sigma^2/\Lambda^2}), \quad (41)$$

which we will use in the numerical section. Note that  $\Lambda = \sigma(R)$  corresponds to the scale below which SUSY is broken, so that  $U_0 = 1/2m^2\Lambda^2$  in Eq. (40). The potential Eq. (41) differs from the one used in [46], but is similar to the potential used in [47]. Figure 2 shows the inverse potential Eq. (41) and the inverse effective potentials for various values of  $\omega$  with  $m = 1$ ,  $\Lambda^2 = 2$ , which implies  $U_0 = 1$ . The red-solid line shows the inverse potential of Eq. (41) [ $-U_{\text{gauge}}$ ], and the sky-blue dotted-dashed line

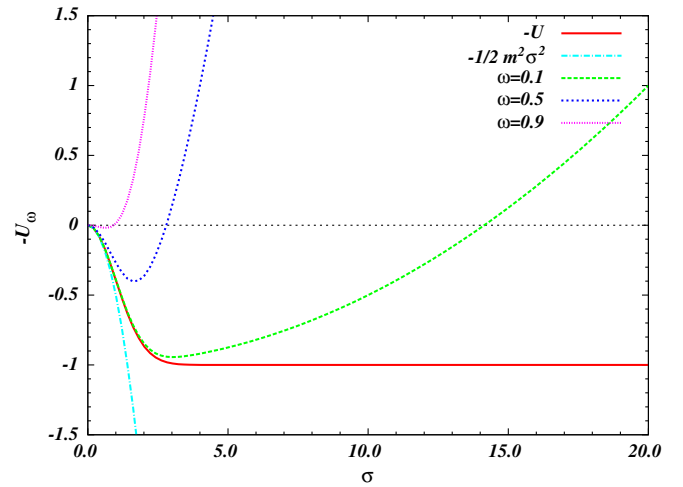


FIG. 2 (color online). The inverse potential  $-U_{\text{gauge}}$  in Eq. (41) (red-solid line) with  $m = 1$ ,  $\Lambda^2 = 2$ , which implies  $U_0 = 1$  and the inverse effective potentials  $-U_\omega$  for different values of  $\omega$ . In order to compare between Eqs. (40) and (41), we plot the inverse quadratic potential with the skyblue dotted-dashed line. The two potentials are asymptotically similar, but they are different around the intermediate region of  $\sigma$ , where  $1 \lesssim \sigma \lesssim 3$ .

corresponds to the inverse quadratic potential of Eq. (40). For sufficiently large and small  $\sigma$ , the two potentials in Eqs. (40) and (41) have similar behavior, but we can see the difference in the intermediate region of  $\sigma$  where  $1 \lesssim \sigma \lesssim 3$ . Hence, we can expect that profiles around the thick-wall limit are different between the potentials since the thick-wall profiles are constructed in the particular region,  $1 \lesssim \sigma \lesssim 3$ ; hence, it may lead to the different stationary properties and stability conditions.

Using Eq. (8), the  $Q$ -ball equation, Eq. (3), in the linearized potential Eq. (40) becomes

$$\sigma''_{\text{core}} + \frac{D-1}{r}\sigma'_{\text{core}} + \omega^2\sigma_{\text{core}} = 0, \quad \text{for } 0 \leq r < R, \quad (42)$$

$$\sigma''_{\text{shell}} + \frac{D-1}{r}\sigma'_{\text{shell}} - m_\omega^2\sigma_{\text{shell}} = 0, \quad \text{for } R \leq r, \quad (43)$$

where the profiles should be imposed to satisfy the boundary conditions,  $\sigma' < 0$ ,  $\sigma(0) \equiv \sigma_0 = \text{finite}$ ,  $\sigma(\infty) = \sigma'(\infty) = 0$ ,  $\sigma'(0) = 0$ . The solutions are

$$\begin{aligned} \sigma_{\text{core}}(r) &= Ar^{1-D/2}J_{D/2-1}(\omega r) \quad \text{for } 0 \leq r < R, \\ \sigma_{\text{shell}}(r) &= Br^{1-D/2}K_{D/2-1}(m_\omega r) \quad \text{for } R \leq r, \end{aligned} \quad (44)$$

where  $J$  and  $K$  are Bessel and modified Bessel functions, respectively, with constants  $A$  and  $B$ . By introducing  $\sigma_0$ , and expanding  $J_{D/2-1}(\omega r)$  for small  $\omega r$  in  $\sigma_{\text{core}}(r)$ , and by using the condition  $U_0 = 1/2m^2\sigma_{\text{shell}}^2(R)$  we obtain



$$A = \sigma_0 \Gamma(D/2) \left( \frac{2}{\omega} \right)^{D/2-1}, \quad (45)$$

$$U_0 = \frac{1}{2} m^2 B^2 R^{2-D} K_{D/2-1}^2(m_\omega R).$$

Since the energy density is smooth and finite everywhere, we have to impose a smooth continuity condition to the profiles  $\sigma_{\text{core}}(R) = \sigma_{\text{shell}}(R)$  and  $\sigma'_{\text{core}}(R) = \sigma'_{\text{shell}}(R)$ , which from Eq. (44) gives

$$\frac{A}{B} = \frac{K_{D/2-1}(m_\omega R)}{J_{D/2-1}(\omega R)} = \frac{m_\omega K_{D/2}(m_\omega R)}{\omega J_{D/2}(\omega R)}. \quad (46)$$

We will see that the particular value of  $\sigma_0$  does not change important features such as the stability condition and characteristic slope of the  $Q$ -ball solutions. Using Eq. (46) we obtain the following important identities, which we will make use of later [48]:

$$\omega \frac{J_{D/2}(\omega R)}{J_{D/2-1}(\omega R)} = m_\omega \frac{K_{D/2}(m_\omega R)}{K_{D/2-1}(m_\omega R)}, \quad (47)$$

$$\frac{J_{D/2}(\omega R) J_{D/2-2}(\omega R)}{J_{D/2-1}^2(\omega R)} = - \left( \frac{m_\omega}{\omega} \right)^2 \times \frac{K_{D/2}(m_\omega R) K_{D/2-2}(m_\omega R)}{K_{D/2-1}^2(m_\omega R)}, \quad (48)$$

where we used the recursion relations  $J_{\mu-1}(z) + J_{\mu+1}(z) = \frac{2\mu}{z} J_\mu(z)$ ,  $K_{\mu-1}(z) - K_{\mu+1}(z) = -\frac{2\mu}{z} K_\mu(z)$  for any real  $\mu$  and  $z$ . We can easily find  $S_\omega^{\text{core}} = U_0 V_D + (\frac{1}{2} \sigma_{\text{core}}(R) \sigma'_{\text{core}}(R)) \partial V_D$  and  $S_\omega^{\text{shell}} = -(1/2 \sigma_{\text{shell}}(R) \sigma'_{\text{shell}}(R)) \partial V_D$ , and then using  $S_\omega = S_\omega^{\text{core}} + S_\omega^{\text{shell}}$  it follows that

$$S_\omega = U_0 V_D, \quad (49)$$

where we have again used the continuity relations  $\sigma_{\text{core}}(R) = \sigma_{\text{shell}}(R)$  and  $\sigma'_{\text{core}}(R) = \sigma'_{\text{shell}}(R)$ . To find the charge  $Q$ , we do not make use of the Legendre relation  $Q = -\frac{dS_\omega}{d\omega}$  in Eq. (10), because  $R$  is a function of  $\omega$ , and is determined by Eq. (47). However, we can obtain  $Q$  by substituting Eq. (44) directly into Eq. (1):

$$Q = \frac{D U_0 V_D}{\omega} \left( \frac{K_{D/2}(m_\omega R) K_{D/2-2}(m_\omega R)}{K_{D/2-1}^2(m_\omega R)} \right), \quad (50)$$

where we have used Eqs. (45), (46), and (48), as well as the relation  $\int dy y Z_\mu^2(y) = [\frac{y^2}{2} (Z_\mu^2(y) - Z_{\mu-1}(y) Z_{\mu+1}(y))]$ , [45,48]. Here,  $\mu$  is real, and  $Z$  can be either the Bessel function  $J$  or the modified Bessel function  $K$ , and we have used the following recursion relations to obtain the indefinite integral:  $z \frac{dJ_\mu}{dz} \pm \mu J_\mu = \pm z J_{\mu\pm 1}$ ,  $J_{\mu-1} - J_{\mu+1} = 2 \frac{dJ_\mu}{dz}$ ,  $z \frac{dK_\mu}{dz} \pm \mu K_\mu = -z K_{\mu\pm 1}$ ,  $K_{\mu-1} + K_{\mu+1} = -2 \frac{dK_\mu}{dz}$ .

For future reference we obtain explicit expressions for  $R$  for case with an odd number of spatial dimensions. Equation (47) can be solved explicitly in terms of  $R$  to give

$$\omega R = \arctan\left(\frac{\omega}{m_\omega}\right), \quad \text{for } D = 1, \quad (51)$$

$$\omega R = \pi - \arctan\left(\frac{\omega}{m_\omega}\right), \quad \text{for } D = 3, \quad (52)$$

where we have used  $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} (\frac{\sin(x)}{x} - \cos(x))$ ,  $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$ ,  $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$ ,  $K_{3/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} (1 + \frac{1}{x})$ ,  $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} = K_{-1/2}(x)$ . We will discuss the classical stability for  $Q$ -balls in  $D = 1, 3$  in the numerical section, in which we will show stability plots arising from Eqs. (51) and (52).

### A. Thin-wall-like limit for $m_\omega R, \omega R \gg \mathcal{O}(1)$

We now discuss both the classical and absolute stability of gauge-mediated  $Q$ -balls in arbitrary dimensions  $D$ , in the limit  $m_\omega R, \omega R \gg 1$ , which implies that the ‘‘core’’ size  $R$  is large compared to  $1/m_\omega, 1/\omega$ . As we will see in the numerical section, Sec. V, the limit will turn out to be equivalent to the thin-wall limit  $\omega \simeq \omega_- \simeq 0$ . Recall that this potential does not have degenerate vacua. Using Eqs. (49) and (50),

$$S_\omega \simeq \frac{\omega Q}{D} \{1 + \mathcal{O}((m_\omega R)^{-1})\}, \quad (53)$$

where we have used  $\lim_{|z| \rightarrow \infty} K_\mu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} [1 + \frac{4\mu^2-1}{8z} + \mathcal{O}(z^{-2})]$ . The characteristic slope follows

$$\frac{E_Q}{\omega Q} \simeq \frac{D+1}{D} \quad (54)$$

from which we see immediately from Eq. (11) that we recover the published results of [1,44], namely,  $E \propto Q^{D/(D+1)}$ . From Eqs. (7) and (54), the thin-wall-like  $Q$ -ball is absolutely stable since the present limits will cover the thin-wall limit  $\omega \simeq \omega_- \simeq 0$  as we stated.

We can also obtain an explicit expression for  $R(\omega)$  and  $\frac{dR}{d\omega}$  in the limits  $m_\omega R \gg 1$  and  $\omega R \gg |\mu^2 - 1/4|$ , where  $\mu(\sim \mathcal{O}(1))$  is the argument of the Bessel function

$$\omega R = \left( \frac{D+1}{4} \right) \pi - \arctan\left(\frac{\omega}{m_\omega}\right), \quad (55)$$

$$\frac{dR}{d\omega} = -\frac{R}{\omega} \left( 1 - \frac{1}{m_\omega R} \right) \simeq -\frac{R}{\omega}. \quad (56)$$

Notice that Eq. (55) for  $D = 3$  reproduces the given profile in [1,34], and it coincides with the exact expression derived in Eq. (52). Using Eqs. (50), (55), and (56), we obtain

$$Q \simeq \frac{V_D U_0 D}{\omega}, \quad (57)$$

$$\frac{\omega}{Q} \frac{dQ}{d\omega} \simeq -D - 1 < 0, \quad (58)$$

which shows that the  $Q$ -ball in this limit is classically stable. One can also check both  $Q \simeq -\frac{dS_\omega}{d\omega} = DU_0 V_D / \omega$  from Eq. (56) and  $\frac{d}{d\omega} \left( \frac{E_Q}{Q} \right) \simeq \frac{D+1}{D} > 0$  from Eq. (54), which are, respectively, consistent with Eq. (58) and with the result in Eq. (12).

### B. Thick-wall limit for $D = 1, 3, \dots$

Having just discussed the thin-wall-like properties for arbitrary  $D$ , we turn our attention now to the other limit  $\omega \simeq \omega_+$ . This is much more difficult to analytically explore because Eq. (48) can only give a closed form expression for  $R$  for the case where  $D$  is an odd number of spatial dimensions. Therefore, we will concentrate here on the cases, e.g.  $D = 1, 3$ .

$D = 3$  case: From Eq. (52) and recalling that in the thick-wall limit,  $m_\omega \rightarrow 0$ ,  $\omega \simeq \omega_+ = m$ , we obtain  $R \simeq \frac{\pi}{2\omega}$ ,  $\frac{dR}{d\omega} \simeq -\frac{R}{\omega}$ , and by substituting these into Eq. (50) we find

$$\frac{\omega}{Q} \frac{dQ}{d\omega} \simeq -1 + \frac{\omega^2}{m_\omega^2} \rightarrow \frac{\omega^2}{m_\omega^2} > 0, \quad (59)$$

$$\frac{E_Q}{\omega Q} = 1 + \frac{\pi m_\omega}{6\omega} \rightarrow 1, \quad (60)$$

which shows that the three-dimensional thick-wall  $Q$ -ball is classically unstable. This fact is consistent with the relation that  $\frac{d}{d\omega} \left( \frac{E_Q}{Q} \right) = 1 - \frac{\pi\omega}{6m_\omega} \rightarrow -\frac{\pi\omega}{6m_\omega} < 0$ , where we have used Eq. (60). It also follows that the thick-wall  $Q$ -ball is not absolutely stable, and the solution will decay to free particles satisfying  $E_Q \rightarrow mQ$ , which is the first case of Eq. (14).

$D = 1$  case: As in the case  $D = 3$ , Eq. (51) implies  $R \rightarrow 0$ ,  $\frac{dR}{d\omega} \simeq -\frac{m^2}{m_\omega \omega^3}$  in the thick-wall limit. Using the above results, we obtain

$$\frac{\omega}{Q} \frac{dQ}{d\omega} \simeq -1 - \frac{m^2}{\omega^2} + \frac{\omega^2}{m_\omega^2} \rightarrow \frac{\omega^2}{m_\omega^2} > 0, \quad (61)$$

$$\frac{E_Q}{\omega Q} = 1 + \left( 1 + \frac{1}{m_\omega R} \right)^{-1} \rightarrow 1. \quad (62)$$

Note that the approximate value in Eq. (61) is the same as Eq. (59). Then the one-dimensional thick-wall  $Q$ -ball is also classically unstable. This fact is again consistent with the result that  $\frac{d}{d\omega} \left( \frac{E_Q}{Q} \right) \simeq 1 + m_\omega R - \frac{m^2}{\omega^2} - \frac{\omega^2 R}{m_\omega} \rightarrow -\frac{\omega^2 R}{m_\omega} < 0$ . As in the three-dimensional case, the thick-wall  $Q$ -ball is not absolutely stable, and the solution decays into its free particles.

### C. Asymptotic profile

The asymptotic profile for the large  $r$  regime in this model can be described by the contribution from the quadratic term in the potential Eq. (40), from which the profile then becomes

$$\sigma(r) \sim E \sqrt{\frac{\pi}{2m_\omega}} r^{-(D-1)/2} e^{-m_\omega r} \Leftrightarrow -\frac{\sigma'}{\sigma} \sim \frac{D-1}{2r} + m_\omega, \quad (63)$$

where  $E$  is a constant [37]. Note that we have used the fact that the modified Bessel function of the second kind has the relation  $K_\mu(r) \sim \sqrt{\frac{\pi}{2r}} e^{-r}$  for large  $r$  and any real  $\mu$ . We will use the criterion in the second relation of Eq. (63) in the following section.

Summarizing our most important results, the thin-wall-like  $Q$ -ball is classically stable for a general  $D$ , while it is absolutely stable as seen in Eqs. (54) and (58). On the other hand, for thick-wall  $Q$ -balls in  $D = 1, 3$ , the  $Q$ -balls are both classically and absolutely unstable, as can be seen from Eqs. (59)–(62). Finally, we obtained the general asymptotic profile Eq. (63) for large  $r$ .

## V. NUMERICAL RESULTS

In this section, we obtain exact numerical solutions for  $Q$ -balls for both the gravity-mediated potential in Eq. (17) and the gauge-mediated potential in Eq. (41) with dimensionless parameters by setting  $m = M = 1$  and  $\Lambda^2 = 2$ . We adopt the 4th-order Runge-Kutta algorithm and usual shooting methods to solve the second order differential equations Eq. (3) (for full details see the numerical techniques developed in [37]). The raw numerical data contains errors for large  $r$ , thus we introduce the previously obtained analytical asymptotic profiles to help control these uncertainties. In particular, we use Eq. (39) for the gravity-mediated potential and Eq. (63) for the gauge-mediated case. Using these techniques, the numerical profiles match smoothly and continuously onto the analytic ones. In order to check the previously obtained analytic results, we calculate  $Q$ -ball properties numerically over the whole parameter space  $\omega$  except around the extreme thin-wall limit  $\omega = \omega_-$ , because it is difficult to obtain reliable numerical results in that limit.

### A. Gravity-mediated potential

We shall investigate two choices of  $\lambda$  in Eq. (15) for  $|K| = 0.1$  and  $n = 6$ , which can be seen as the red lines in Fig. 1. The choice of the parameters,  $|K|$  and  $n$ , are simply from phenomenological reasons. The DVP on the left has  $\omega_- = 0$  ( $\beta^2 = \frac{|K|e^{-1}}{4} \exp(-\frac{2}{|K|}) \sim 1.90 \times 10^{-11} \ll \mathcal{O}(1)$ ), and the NDVP on the right has  $\omega_- = 1$  ( $\beta^2 = \frac{|K|e^{-1}}{4} \sim 9.20 \times 10^{-3} \ll \mathcal{O}(1)$ ), recalling Eq. (19). Figure 1 also shows plots of the inverse effective potentials  $-U_\omega$  for various values of  $\omega$ . Because of numerical com-

plications, we are unable to fully examine the properties in the extreme thin-wall limit; however, by solving close to this wall limit, our numerical results recover the expected analytical results we derived in Eqs. (26) and (27). With the above choice of parameters, the curvature  $\mu$  of  $U_\omega$  at  $\sigma_+(\omega_-) \equiv \sigma_+$  in Eq. (21) is  $\mu^2 \sim 0.4$ , which implies that  $1/\mu \sim 1.58$ . From the first relation in Eq. (18), we have found  $\sigma_+ \sim 1.28$  in NDVP and  $\sigma_+ \sim 1.91 \times 10^2$  in DVP. Since we have assumed  $R_Q \gg 1/\mu$ ,  $\sigma_0 \simeq \sigma_+$  in our thin-wall analysis for the gravity-mediated potential, we see that it breaks down when the core size  $R_Q$  becomes the same order as  $1/\mu$  and/or  $\sigma_0 \simeq \sigma_+$ . Although the full definition of the core size  $R_Q$  is presented in [37], it is very time-consuming to evaluate it properly in the simulations; hence, in this analysis we have used a more naive

approach, in which we have estimated the value of  $r = R_Q$  when the field profile drops quickly from its core value. For the thick-wall limit, we required the condition  $\beta^2 \leq |K| \leq \mathcal{O}(1)$ , which is satisfied with the above chosen parameter set; hence, the analysis is valid for  $\omega \gtrsim \mathcal{O}(1)$ . Because of the choice of  $|K| = 0.1 < \mathcal{O}(1)$  and  $\omega_- = 0$  in NDVP, we will see our analysis holds even for  $\omega \sim \mathcal{O}(1)$ .

*Hybrid profile:* The numerical profiles have errors for large  $r$ , which correspond to either undershooting or overshooting cases; thus, to minimize the errors in the region of large  $r$  we replace the numerical data by the predicted asymptotic analytical profile using the criterion Eq. (39) to obtain the solution for the whole range of  $r$ . We then have the hybrid profile which can be written as

$$\sigma(r) = \begin{cases} \sigma_{\text{num}}(r), & \text{for } r < R_{\text{num}}, \\ \sigma_{\text{num}}(R_{\text{num}}) \exp\left(-\frac{|K|}{2} R_{\text{num}}^2 - \frac{\sigma'_{\text{num}}(R_{\text{num}})}{\sigma_{\text{num}}(R_{\text{num}})} R_{\text{num}}\right) \exp\left(-\frac{|K|r^2}{2} + (R_{\text{num}}|K| + \frac{\sigma'_{\text{num}}(R_{\text{num}})}{\sigma_{\text{num}}(R_{\text{num}})})r\right) & \text{for } R_{\text{num}} \leq r \leq R_{\text{max}}, \end{cases} \quad (64)$$

where  $\sigma_{\text{num}}$  is the numerical raw data,  $R_{\text{num}}$  is determined by  $|(-\sigma'_{\text{num}}/\sigma_{\text{num}})' - 1|_{r=R_{\text{num}}} < 0.001$ , and we have set  $R_{\text{max}} = 60$  throughout our numerical simulations in this subsection. We have calculated the following numerical properties using the above hybrid profile, Eq. (64), for  $D = 1, 2, 3$ :

*Profile:* In the top two panels of Fig. 3 (DVP on the left and NDVP on the right), the red-solid and blue-dotted lines show the numerical slopes  $-\sigma'/\sigma$  for two typical values of  $\omega$  in  $D = 3$ . We smoothly continue them to the corresponding analytic profiles by the methods just described in the numerical techniques, see green-dashed and purple-dotted-dashed lines. The linear lines correspond to the Gaussian tails in Eq. (38) and for the cases of  $\omega = 0.14$  (DVP) and  $\omega = 1.01$  (NDVP) corresponding to the thin-wall solution we see that it is shifted from the origin to  $r \simeq 21$ . The middle panels show the obtained hybrid profiles of Eq. (64) for the various values of  $\omega$  and  $D$ . The higher the spatial dimension, the larger the core size  $Q$ -balls can have. The energy density configurations  $\rho_E(r)$  can be seen in the bottom panels of Fig. 3. Outside of the cores of the DVP profiles for  $\omega \sim \omega_-$ , we can see the same features as we saw in the polynomial potentials we investigated in [37], namely, highly concentrated energy density spikes. In NDVP, however, the spikes cannot be seen. The presence of the spike contributes to the increase in the surface energy  $\mathcal{S}$ , which in turn leads to the different virialization ratio for  $\mathcal{S}/\mathcal{U}$ , where  $\mathcal{U}$  is the potential energy, as can be seen in Eq. (14).

*Criterion for the existence of a thin-wall Q-ball:* Figure 4 shows the numerical results for  $\sigma_0(\omega)$  against  $\omega$  for both types of potentials—DVP (left) and NDVP (right). Our main analytical approximation relies on  $\sigma_0(\omega) \simeq \sigma_+(\omega) \sim \sigma_+ \equiv \sigma_+(\omega_-)$ , where we have found  $\sigma_+ \sim$

$1.28 \sim \mathcal{O}(1)$  in NDVP and  $\sigma_+ \sim 1.91 \times 10^2 \gg \mathcal{O}(1)$  in DVP. The 3D thin-wall  $Q$ -ball (green-crossed dots) appears for a wider range of  $\omega$  than the 2D  $Q$ -ball (red-plus signs) in DVP as well as NDVP. For each case, the approximation can be valid, respectively, up to  $\omega \sim 0.24$  or  $\omega \sim 1.04$  with about 10% errors for the 3D case. Near the thick-wall limit  $\omega \simeq \omega_+$  for both potentials, we see  $\sigma_0 \simeq \sigma_- \rightarrow 0$ . The one-dimensional values (skyblue-circled dots) always lie on  $\sigma_-$ . Note that in the 3D region  $\omega \gtrsim 0.53$  for DVP, we can see  $\sigma_0(\omega) \leq \mathcal{O}(10^2)$ , which implies that the contribution from the nonrenormalizable term in Eqs. (28) and (B9), i.e.  $\beta^2 \tilde{\sigma}^4 \leq \mathcal{O}(10^{-3}) \ll \mathcal{O}(1), \mathcal{O}(|K|)$ , is negligible compared to other terms in Eqs. (28) and (B9). Hence, our analytic solution still holds in the limit  $\omega \sim \mathcal{O}(1)$  as discussed in Sec. III C.

*Virialization and characteristic slope:* Figure 5 shows the  $Q$ -ball properties plotted against the ratio of  $\mathcal{S}/\mathcal{U}$ , where  $\mathcal{S}$  and  $\mathcal{U}$  are the surface and potential energies (top panels), and the characteristic slope  $E_Q/\omega Q$  (bottom panels). For the DVP case where the thin-wall  $Q$ -ball satisfies  $\sigma_0 \sim \sigma_+$  it appears to be heading toward  $\mathcal{S}/\mathcal{U} \sim 1$  as  $\omega \rightarrow \omega_- = 0$  [see Eq. (14)], in all three cases. Also we predict that the thin-wall  $Q$ -ball in NDVP has  $\mathcal{S}/\mathcal{U} \sim 0$  [see Eq. (14)] and that it is consistent with what can be seen in the top-right panel around  $\omega = \omega_- = 1$ . The bottom panels show analytically and numerically the characteristic slopes  $E_Q/\omega Q$  in both the thin- and thick-wall limits. The analytic thin-wall lines (purple-dotted line for 2D and blue-dotted line for 3D) based on Eq. (26) are well fitted for the NDVP case with the corresponding numerical dots (red plus-dots for 2D and green crossed-dots for 3D) as long as  $\sigma_0 \simeq \sigma_+$ , see the criteria in Fig. 4. For the DVP case, our numerical data is seen to be heading in the right direction. The numerical solutions for both cases in the

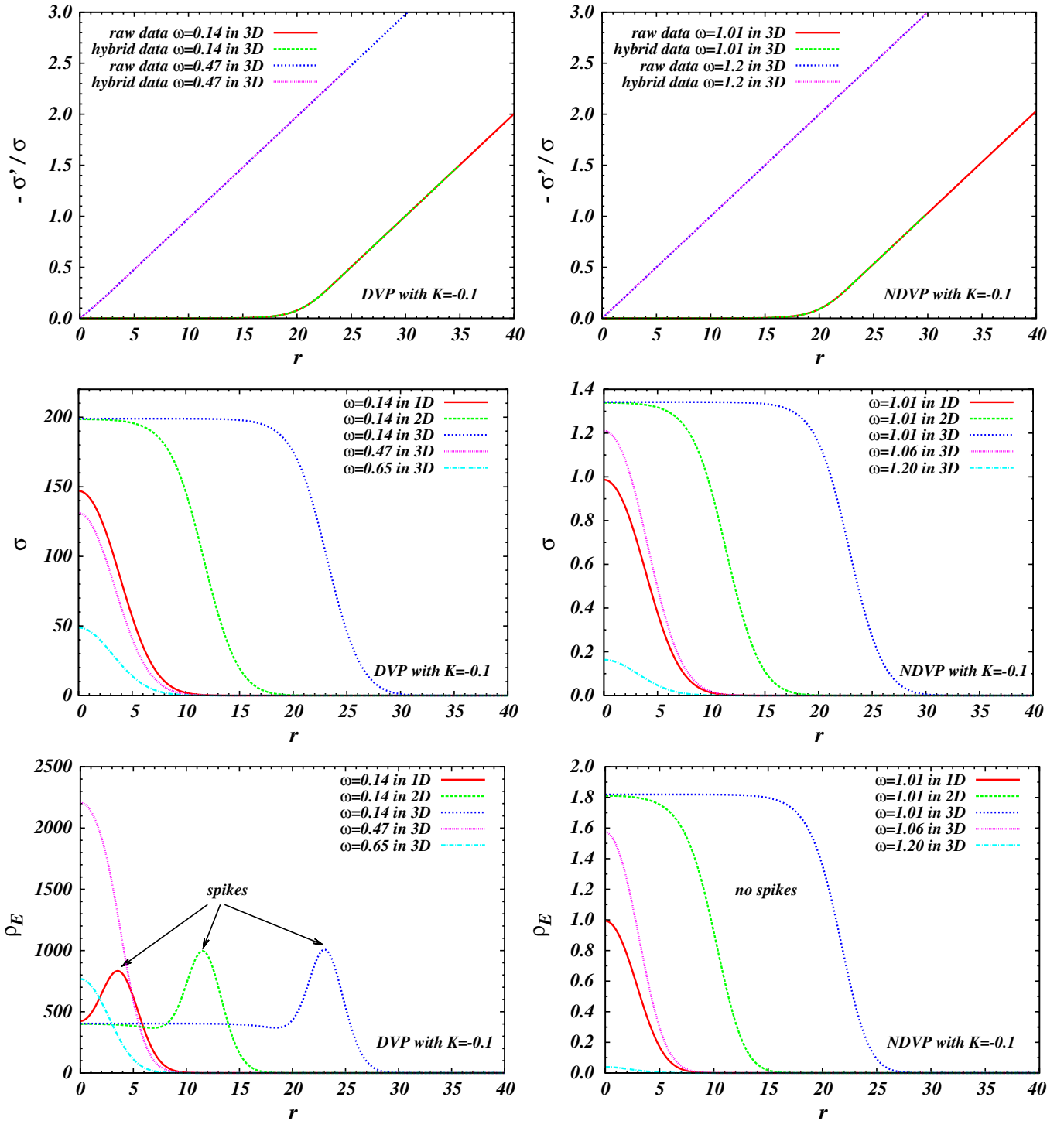


FIG. 3 (color online). The top two panels show the three-dimensional numerical slopes  $-\sigma'/\sigma$  for two typical values of  $\omega$  for both DVP (left) and NDVP (right). The raw numerical data (red-solid and blue-dotted lines) matches continuously onto the analytical asymptotic profiles for large  $r$  (green-dashed and purple-dotted-dashed lines). The linear lines correspond to the Gaussian tails in Eq. (38), where we can see the large shifts in the thin-wall limits of  $\omega$ . The middle and bottom panels show, respectively, the hybrid profiles Eq. (64) and the energy density configurations for the various values of  $\omega$  and  $D$ . The spikes of the energy density configurations exist in the DVP case but not in the NDVP case.

thick-wall region are well fitted by the analytic solution in general  $D$  given by the orange-dotted-dashed lines, in the second relation of Eq. (33) or Eq. (B13). From the virial

relation Eq. (13) for  $D = 1$ , we can only predict the extreme values of the 1D characteristic slope,  $\gamma$ , in either the DVP or NDVP case once we know what  $S/\mathcal{U}$  is. To



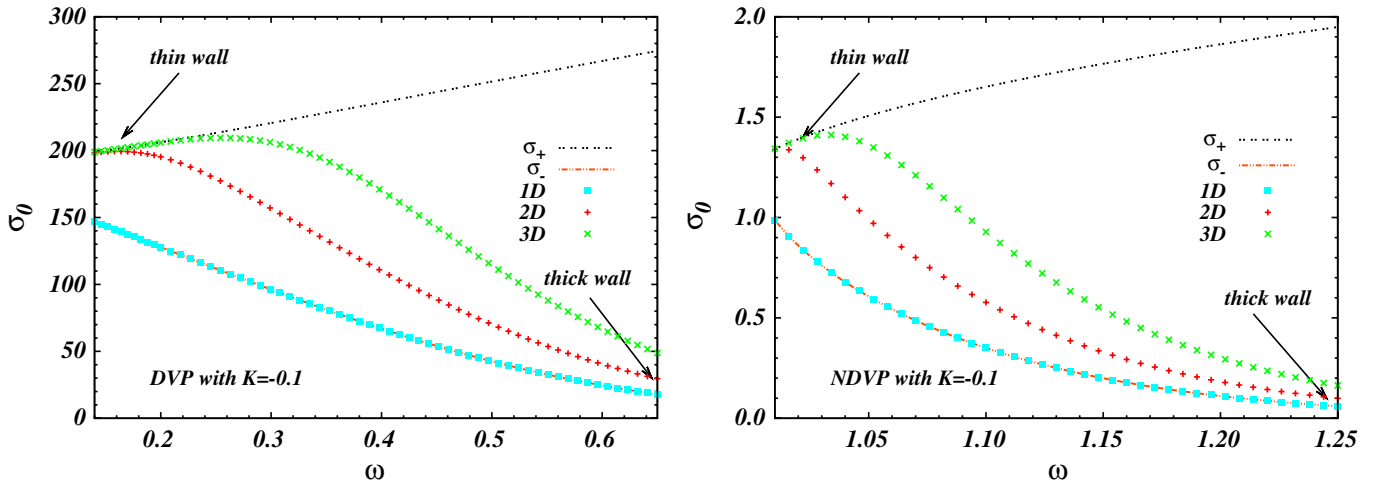


FIG. 4 (color online). The initial value  $\sigma_0(\omega) \equiv \sigma(0)$  is plotted against  $\omega$ . In the two panels the black-dashed and orange dotted-dashed lines show  $\sigma_{\pm}(\omega)$ , and these lines become closer for  $\omega = \omega_-$  for both types of the potentials DVP (left,  $\omega_- = 0$ ) and NDVP (right,  $\omega_- = 1$ ). Since  $\sigma_0 \simeq \sigma_+$  for  $D = 2, 3$  in the region  $\omega \sim \omega_-$  where  $\sigma_+ \sim 1.28$  in NDVP and  $\sigma_+ \sim 1.91 \times 10^2$  in DVP, our analytical results in Sec. III B, are valid in this region.

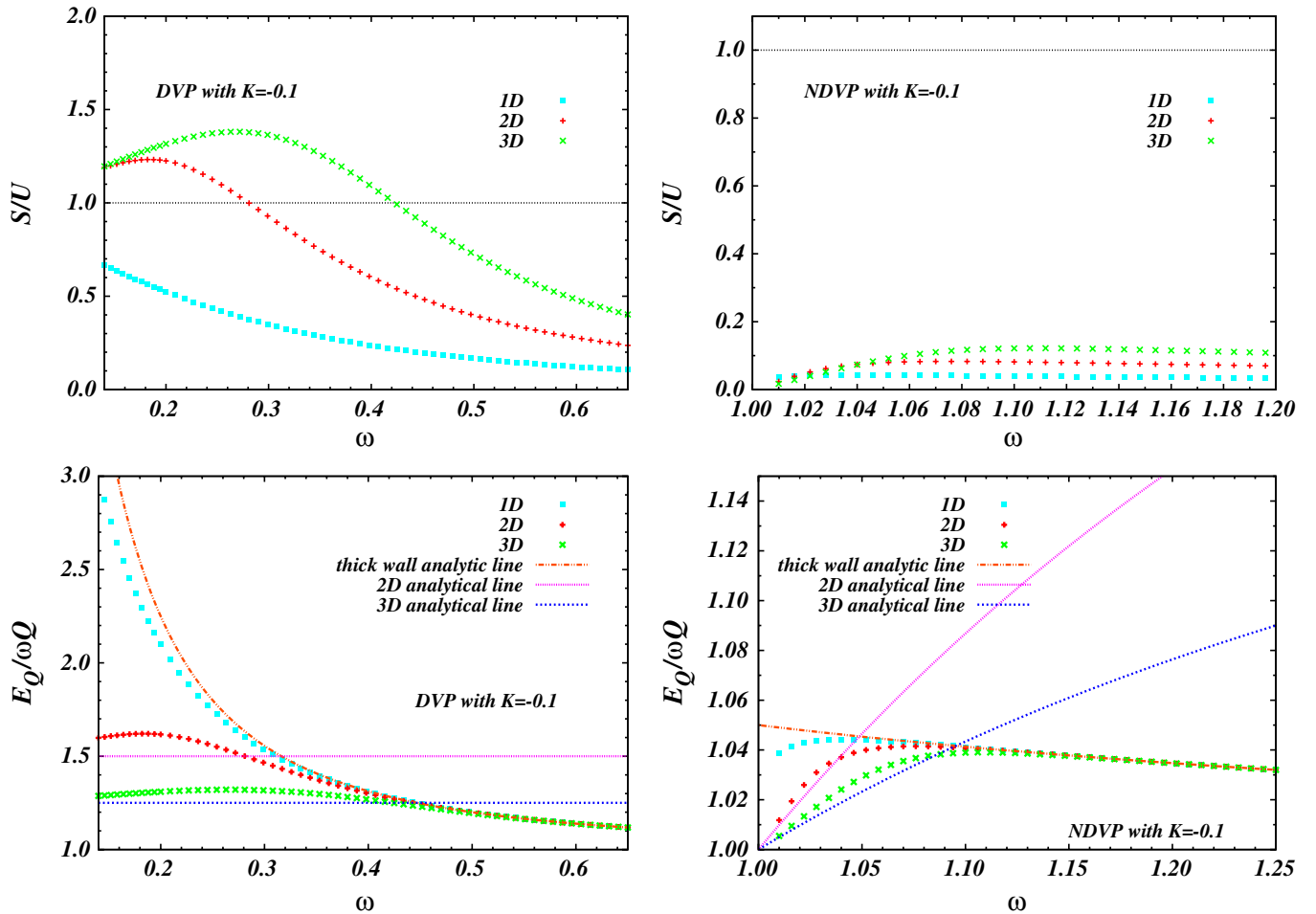


FIG. 5 (color online). The top panels show the ratio  $S/\mathcal{U}$  where  $S$  and  $\mathcal{U}$  are the surface and potential energies, and the bottom panels show the numerically obtained characteristic slope  $E_Q/\omega Q$ , in 1D (skyblue circled-dots), 2D (red plus-dots) and 3D (green crossed-dots). For comparison, in the bottom panels, the thin-wall analytic lines obtained using Eq. (26) are also shown (purple-dotted line for 2D and blue-dotted line for 3D) as are the thick-wall analytic lines obtained from Eqs. (33) and (B13) (orange-dotted-dashed for all  $D$ ). The analytic lines match well with the numeric data in the appropriate limits, especially for the NDVP case.

obtain that we rely on the numerical simulations and from the top two panels in Fig. 5, we see that for the DVP case with  $D = 1$ ,  $S/U$  appears to be heading toward unity, implying  $\gamma \gg 1$  in Eq. (13), whereas for the NDVP case  $S/U \ll 1$ , implying  $\gamma \rightarrow 1$  in Eq. (13). Comparing these with the bottom two panels we see the behavior for  $\gamma$  appears to follow these predictions.

*Q-ball stability:* Figure 6 shows plots for both the classical (top panels) and absolute stability (bottom panels) with the stability threshold lines (black-dashed) for the cases of DVP (left) and NDVP (right). Let us consider the classical stability case first. For the thin-wall regime in DVP, notice that the numerical data of  $\frac{\omega}{Q} \frac{dQ}{d\omega}$  (red-dot-circles for 2D and green-dot-crosses for 3D) are heading toward the analytic lines of Eq. (27). For the thick-wall case, on the other hand, the analytical lines of Eq. (34) (orange-dotted-dashed) fit excellently with the numerical data in all dimensions, because Eq. (34) is independent of  $D$ .

Furthermore, the  $Q$ -ball is classically stable over all values of  $\omega$  except for the 1D thin-wall case where our analytical work cannot be applied. We saw this feature of unstable 1D thin-wall  $Q$ -balls for the case of polynomial models in [37]. For the absolute stability in the bottom panels, the analytical lines using Eqs. (26), (33), and (B13) are matched with the numerical dots for both the thin- and thick-wall limits. Here, we note how well the three-dimensional  $Q$ -ball (and also the higher dimensional ones as predicted in [37]) can be described simply by our thin- and thick-wall  $Q$ -balls. As our parameter set satisfies Eq. (35), we can see that absolutely stable  $Q$ -balls exist in DVP near the thick-wall limit. Because of the choice of  $\omega_- = 1$ , the  $Q$ -ball in the NDVP case, however, is always absolutely unstable and most of the features are similar in terms of  $D$ . The analytical lines (top-right panel) in NDVP agree with the corresponding numerical data qualitatively better than the lines for DVP.

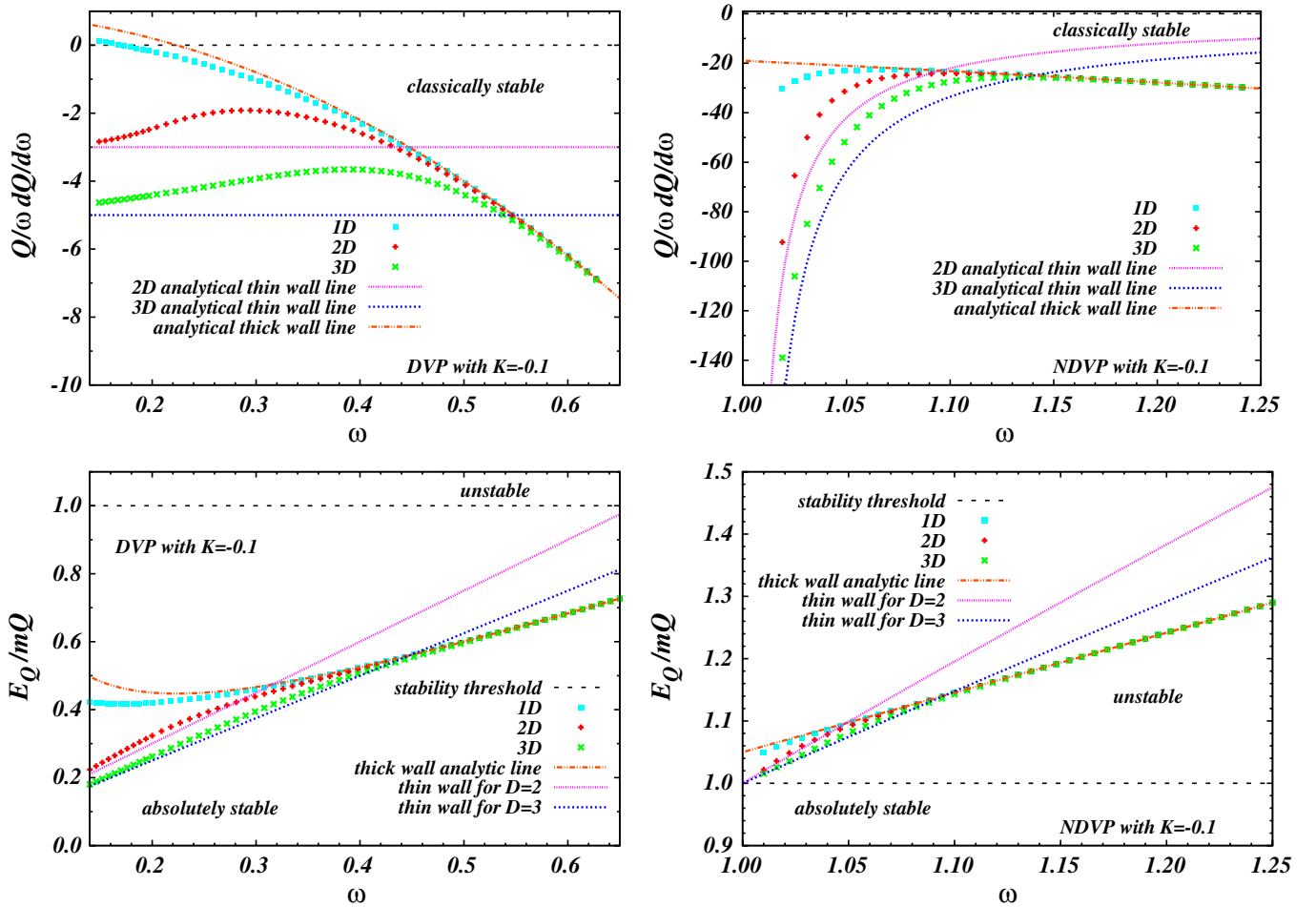


FIG. 6 (color online). Classical stability for the top panels and absolute stability for the bottom panels for both DVP (left) and NDVP (right). The black-dashed lines indicate the stability thresholds for both classical and absolute stability in all panels.  $Q$ -balls found below the lines are stable either (both) classically or (and) absolutely. In the top panels, the analytical lines using Eqs. (27) and (34) agree well quantitatively with the corresponding numerical data for thick-wall regimes, but not well in the thin-wall regimes. However, the numerical plots look qualitatively similar to the analytical lines in the thin-wall limit as seen in the polynomial models [37]. In addition, the analytical lines for  $E_Q/mQ$  using Eqs. (26) and (B13) match the numerical lines for both the thin and thick-wall limits.

To sum up our discussion of the gravity-mediated model, our analytical estimates of the characteristic slope and other properties of the  $Q$ -balls are well checked against the corresponding numerical results, even though we set a “flatter” potential with  $|K| = 0.1 < \mathcal{O}(1)$ .

### B. Gauge-mediated potential

This subsection presents numerical results showing the properties of gauge-mediated  $Q$ -balls with  $m = 1, \Lambda^2 = 2$  in Eq. (41). Although we have obtained analytical results for the potential, Eq. (40), the potential is neither analytic

nor smooth for all  $\sigma$ . Therefore, we shall use the approximate potential, Eq. (41), see Fig. 2 and we expect that Eq. (41) is a suitable approximation especially for the thin-wall limit  $\omega$  and large  $D$ . We will also see and explain the expected discrepancies that exist between the numerical and analytic results.

*Hybrid profile:* As we saw in earlier examples the numerical profiles we have obtained have errors for large  $r$ , which correspond to either undershooting or overshooting; thus, we replace the numerical data in that regime by the exact asymptotic analytic solutions we obtained using the

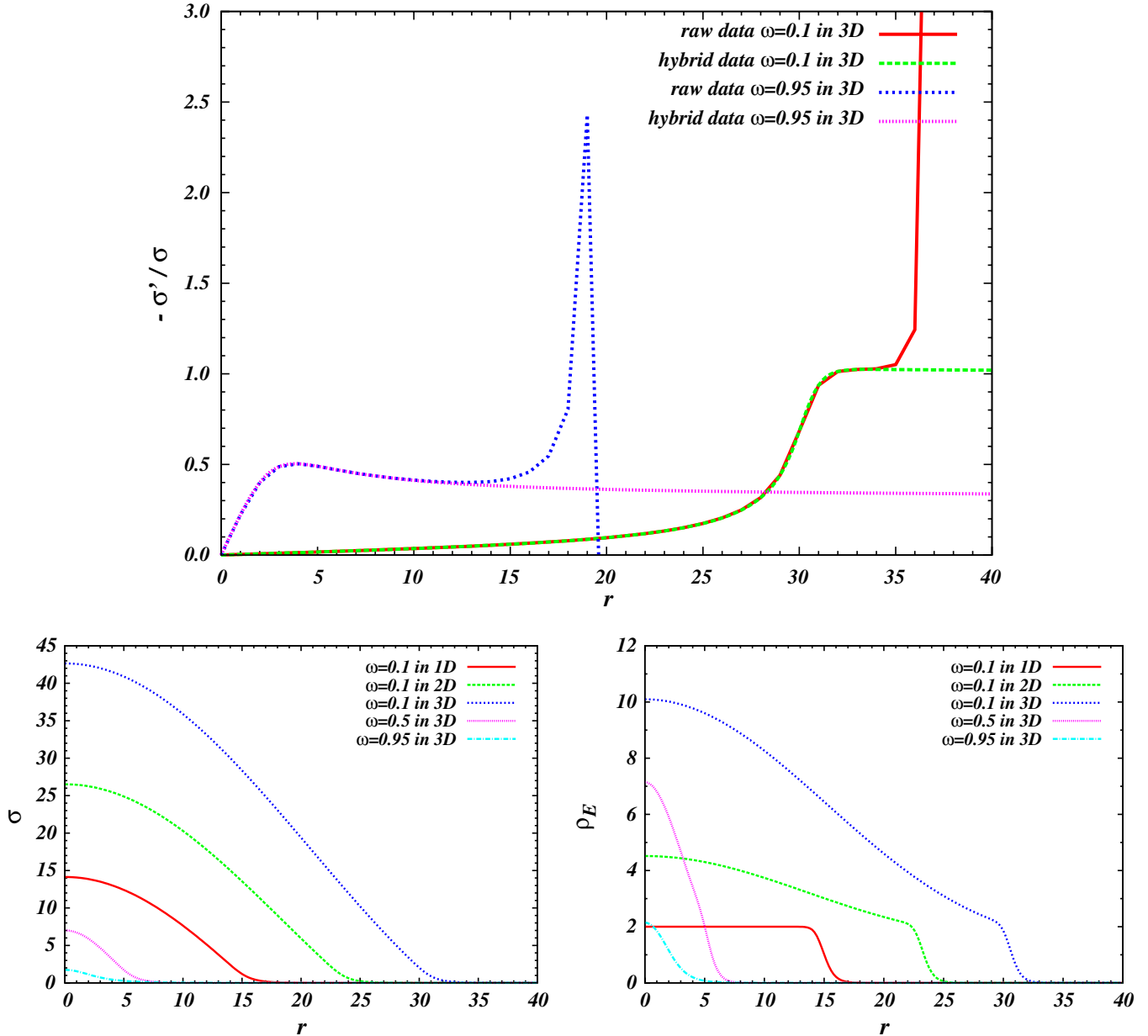


FIG. 7 (color online). The top panel shows the three-dimensional numerical slopes  $-\sigma'/\sigma$  for two values of  $\omega$ . The raw numerical data (red-solid and blue-dotted lines) matches smoothly to the corresponding analytical asymptotic profiles for large  $r$  (green-dotted and purple-dashed lines). Both the left and right-bottom panels show, respectively, the hybrid profiles Eq. (65) and the energy density configurations for the various values of  $\omega$  and  $D$ . The spikes of energy density configurations do not exist even in the thin-wall limits.

second relation of Eq. (63) to smoothly continue the numerical solutions to the corresponding analytical ones. The hybrid profile in this model is

$$\sigma(r) = \begin{cases} \sigma_{\text{num}}(r) & \text{for } r < R_{\text{num}}, \\ \sigma_{\text{num}}(R_{\text{num}}) \left(\frac{R_{\text{num}}}{r}\right)^{(D-1)/2} e^{-m_\omega(r-R_{\text{num}})} & \text{for } R_{\text{num}} \leq r \leq R_{\text{max}}, \end{cases} \quad (65)$$

where  $\sigma_{\text{num}}$  is the numerical raw data,  $R_{\text{num}}$  is determined by  $|\frac{D-1}{2r} + m_\omega + (\sigma'_{\text{num}}/\sigma_{\text{num}})| < 0.001$ , and we have again set  $R_{\text{max}} = 60$ . We have calculated the following numerical properties using the above hybrid profile, Eq. (65), up to  $D = 3$ .

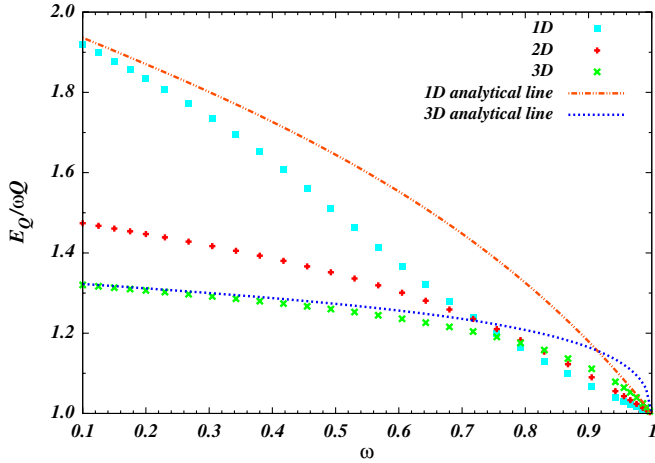


FIG. 8 (color online). The numeric characteristic slopes  $E_Q/\omega Q$  and the analytic lines (orange-dashed line for 1D and blue-dotted line for 3D), which are calculated using Eqs. (51) and (52) in the whole range of  $\omega$ . The 3D analytic line agrees with the numeric data well except near the thick-wall limit. Similarly the 1D analytic line agree well only in the extreme thin-wall limit.

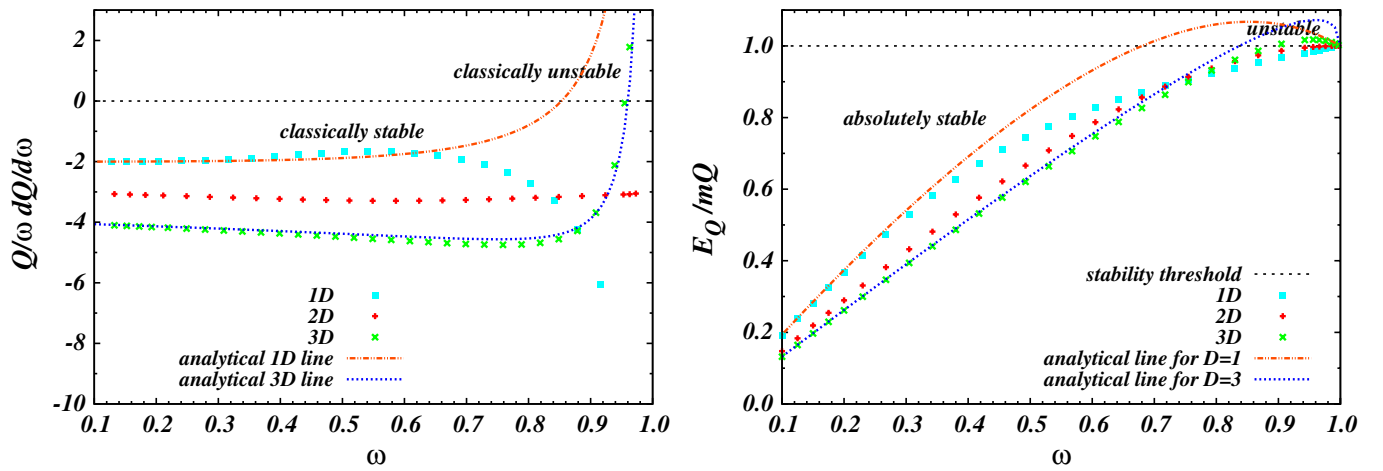


FIG. 9 (color online). The stability of  $Q$ -balls—Classical (left panel) and absolute (right panel). The black-dashed lines in the two panels indicate the stability thresholds for both classical and absolute stability where  $Q$ -balls under the lines are classically/absolutely stable. The analytic lines for  $D = 1, 3$  are calculated by substituting Eqs. (51) and (52) into Eq. (50) and differentiating it with respect to  $\omega$ .

*Profile and energy density configuration:* Figure 7 shows the three-dimensional numerical slopes  $-\sigma'/\sigma$  for two values of  $\omega$  (top), hybrid profiles (left-bottom) as in Eq. (65), and the configurations for energy density (right-bottom). In the top panel, the raw numerical data (red-solid and blue-dotted lines) is matched smoothly onto the continuous asymptotic profiles Eq. (65) for large  $r$  (green-dotted and purple-dashed lines). By fixing the numerical raw data using the technique Eq. (65), we show the profiles for various values of  $\omega$  and  $D$ , see the left-bottom panel. Also the peaks of the energy density cannot be observed in the whole range of  $\omega$ , see the right-bottom panel.

*Characteristic slope:* In Fig. 8, we plot both the numeric and analytic characteristic slopes  $E_Q/\omega Q$  (orange-dashed line for 1D and blue-dotted line for 3D). By substituting Eqs. (51) and (52) into Eqs. (1) and (50), we have obtained the analytic slopes covering the whole range of  $\omega$ . The 3D analytic line agrees well with the numerical data except near the thick-wall limit. Similarly, the 1D analytic line agrees well only in the thin-wall limit. The origin of the discrepancies in the analytic versus numerical fits are the differences between the potentials themselves [Eq. (40) in the analytical section (Sec. IV) and Eq. (41)]. These differences are largest between  $1 \leq \sigma \leq 3$ , which in turn affects the region around  $0.9 \leq \omega < 1.0$ , see Figs. 2 and 8.

*Q-ball stability:* Figure 9 illustrates the stability of  $Q$ -balls: classical stability in the left panel and absolute stability in the right panel. The black-dashed lines in both panels indicate their respective stability thresholds where



$Q$ -balls under the lines are stable. We calculate the analytic lines for  $D = 1, 3$  by substituting Eqs. (51) and (52) into Eq. (50) and differentiating it with respect to  $\omega$ . The 3D numerical data can be matched with the analytic lines in both the thin- and thick-wall limits. As in Eq. (59), the three-dimensional  $Q$ -ball in the thick-wall limit is classically unstable. The numerical thick-wall  $Q$ -ball in 1D is classically stable, which differs from the prediction in Eq. (61). In the right panel, the analytic line for  $D = 3$  agrees with the numerical data except in the thick-wall limit where the analytical lines for both 1D and 3D do not match the corresponding numerical data. Furthermore, the thick-wall  $Q$ -ball in 1D is absolutely unstable as predicted analytically in Eq. (62), but this fact cannot be observed numerically. The reasons for this discrepancy are as before a problem with our choice of potentials. We can see that the thin-wall  $Q$ -balls for any  $D$  are both classically and absolutely stable.

To recap, our numerical results in the gauge-mediated case are generally well fitted by our analytical estimations. Observed discrepancies between the analytical predictions and numerical data arise from the artifact of our approximated smooth potential Eq. (41) for the generalized gauge-mediated potential Eq. (40). We have confirmed that the thin-wall  $Q$ -balls for any  $D$  are both absolutely and classically stable.

## VI. CONCLUSION AND DISCUSSION

We have explored stationary properties of  $Q$ -balls in two kinds of flat potentials, which are the gravity-mediated potential, Eq. (15), and the generalized gauge-mediated potential, Eq. (40). Generally, the gauge-mediated potential is extremely flat compared to the gravity one; therefore, we cannot apply our thin-wall ansatz Eq. (22) to the gauge-mediated case. By linearizing the gauge-mediated potential, we obtained the analytical properties instead. For both potential types, we both analytically and numerically examined characteristic slopes as well as the stability of the  $Q$ -balls in the thin- and thick-wall limits. Our main analytical results are summarized in Fig. 1.

This present paper is of course related to our previous work [37]. The key differences are that in the present work on thin-wall  $Q$ -balls we are assuming the value of  $\sigma_+(\omega)$  for the thin-wall limit  $\omega \simeq \omega_-$  depends weakly on  $\omega$  and we have replaced the assumption  $\sigma(R_Q) < \sigma_-(\omega)$  by the equivalent assumption (made by Coleman)  $U_\omega \simeq U_{\omega_-}$  in the  $Q$ -ball shell region [43]. These in turn are related to the previous requirement that the surface tension  $\tau$  depends weakly on  $\omega$ , which can be translated into the main assumptions:  $R_Q \gg \delta$ ,  $1/\mu$ ,  $\sigma_0 \simeq \sigma_+$ , and  $U_\omega \simeq U_{\omega_-}$  in the shell region. Furthermore, our analytic work agrees well with the numerical results for small curvature  $\mu$  with  $|K| = 0.1$ ; however, it is not clear that our analytic framework still holds even in the case of  $|K| \ll \mathcal{O}(1)$ , which

corresponds to a case where the potential is extremely flat, see Eq. (21).

*Q-balls in gravity-mediated potentials:* It is possible to obtain absolutely stable  $Q$ -matter with a small coupling constant, Eq. (20), for the nonrenormalizable term in Eq. (15). For  $|K| \ll \mathcal{O}(1)$ , a gravity-mediated potential can not be really considered as flat, which allows us to apply our previous results, Eqs. (26) and (27), in [37] to describe the thin-wall  $Q$ -ball where  $\sigma_0 \simeq \sigma_+$ . In the thick-wall limit by reparametrizing parameters in  $S_\omega$  and neglecting the nonrenormalizable term under the conditions  $\beta^2 \lesssim |K| \lesssim \mathcal{O}(1)$ , we have obtained the stationary properties of the  $Q$ -ball. We showed that the thick-wall  $Q$ -ball is classically stable, and demonstrated that under certain conditions Eq. (35) it can be absolutely stable. Although this analysis is much simpler than the analysis associated with imposing a Gaussian ansatz developed in Appendix B, the former analysis assumed that the nonrenormalizable term is negligible at the beginning of the analysis. In the latter analysis, we have kept all terms in Eq. (17) and shown that the nonrenormalizable term is indeed negligible in the limit  $\omega \gtrsim \mathcal{O}(m)$ . Our results, Eqs. (34) and (B13), for the thick-wall  $Q$ -ball have recovered the previous results obtained in [30,49] without any contradictions for classical stability conditions as opposed to the case of using a Gaussian ansatz in a general polynomial potential in which we showed that the ansatz led to a contradiction and corrected it by introducing a physically motivated ansatz [37]. This is because the Gaussian ansatz, Eq. (B1), becomes the exact solution, Eq. (A1), in the gravity-mediated potential in the limit  $\omega \gtrsim \mathcal{O}(m)$ , where the nonrenormalizable term is negligible. In Figs. 5 and 6 the analytical lines agree well with the corresponding numerical plots in both the thin-wall and thick-wall limits. Under our numerical parameter sets, the  $Q$ -balls in DVP are both classically and absolutely stable up to  $\omega \lesssim m$ , while all of the  $Q$ -balls in NDVP are absolutely unstable because of our choice,  $\omega_- = m$ . We believe that an absolutely stable  $Q$ -matter exists in NDVP when we take  $\omega_- < m$ . Since the  $Q$ -balls in both potential types are always classically stable, as can be seen in the top two panels of Fig. 6 except for the case of 1D  $Q$ -balls in the thin-wall limit to which our analytical work cannot be applied since it holds only for  $D \geq 2$ . We have also found the asymptotic profile Eq. (38) for all possible values of  $\omega$ , see the top two panels in Fig. 3.

Our analytical estimations on the value of  $\frac{\omega}{Q} \frac{dQ}{d\omega}$  do not agree well with the numerical results, because  $\sigma_0 \simeq \sigma_+$ . Nevertheless, the other analytical properties are well fitted especially in NDVP, see bottom panels in Figs. 5 and 6. The DVP in Eq. (15) for small  $|K|$  is extremely flat as the gauge-mediated potential in Eq. (40), where both of the potentials have  $\omega_- \simeq 0$ . Notice that the asymptotic profile for the former case has a Gaussian tail, while the latter profile is determined by the usual quadratic mass term, see

Eqs. (38) and (63). By assuming that the shell effects are much smaller than the core effects in the thin-wall limit, the difference of the tails can be negligible. Indeed, we can see the thin-wall numerical lines for both the classical stability and the characteristic slope look qualitatively and quantitatively similar to each other, as can be seen in both the top/bottom left panels of Fig. 6 and the panels of Fig. 9. Notice that the spikes of energy density in the gauge-mediated potential cannot be seen even though  $\omega_- \simeq 0$ , see Fig. 7.

Furthermore, we know that the potential  $U_{\text{grav}}$  can be approximated by  $1/2m^2M^{2|K|}\sigma^{2-2|K|}$  for small  $|K| \ll \mathcal{O}(1)$ , then the potential in Eq. (15) looks similar to the confinement model in [15,16]. By neglecting the nonrenormalizable terms in the thick-wall limit, we can easily obtain the characteristic slope,  $\gamma = \frac{2+|K|(D-1)}{2+|K|(D-2)} \simeq 1$ , [9] by following the same technique as in Eq. (10), which does not depend on  $\omega$  but does depend on  $D$  and  $|K|$ . It follows that  $E_Q \propto Q^{1/\gamma}$  from Eq. (11). This result is obviously worse than our main results in Eqs. (33) and (B13), see bottom two panels in Fig. 5, because we know that the Gaussian ansatz Eq. (B1) can be the exact solution Eq. (A1) for  $U = U_{\text{grav}}$ ; thus, it is not so powerful to approximate  $U_{\text{grav}}$  by  $1/2m^2M^{2|K|}\sigma^{2-2|K|}$  for small  $|K|$ .

*Q-balls in gauge-mediated potentials:* For the gauge-mediated potential in Eq. (40), we obtained the full analytical results in  $D = 1, 3$  over the whole range of  $\omega$  using Eqs. (51) and (52), see Figs. 8 and 9. In the thin-wall limit for  $m_\omega R, \omega R \gg \mathcal{O}(1)$ , we reproduced the previously obtained results, Eq. (54), in [1,44,45] and showed that they are both classically and absolutely stable in Eqs. (54) and (58). The one- and three- dimensional thick-wall  $Q$ -balls, on the other hand, are neither classically nor absolutely stable, see Eqs. (61) and (62) or Eqs. (59) and (60), respectively. Since the potential, Eq. (40), is not differentiable everywhere, we have used the approximate potential, Eq. (41), instead in the numerical section. Figures 8 and 9 show that the numerical results agree with the analytical results in the thin-wall limit. The numerical data near the thick-wall limit and/or in the 1D case differ from the analytic lines since the profiles are computed in the region where the two potentials between Eq. (40) and (41) are different, see Fig. 2. This differences come from the artifact of our approximated smooth potential Eq. (41) against the generalized gauge-mediated potential Eq. (40).

*The 3DQ-balls:* Although we have shown  $Q$ -ball results for an arbitrary number of spatial dimensions  $D$ , only three-dimensional cases are phenomenologically interesting.  $Q$ -balls in flat potentials give the proportional relation  $E_Q \propto Q^{1/\gamma}$ , where  $\gamma$  generally depends on  $D$ . The actual values of  $1/\gamma$  for three-dimensional thin-wall  $Q$ -balls are 4/5, 1, and 3/4 in DVP, NDVP of gravity-mediated potentials and in gauge-mediated potentials, respectively. It implies that the gauge-mediated  $Q$ -balls would be formed

in the most energetically compact state for a large charge  $Q$ , so it is likely that such formed  $Q$ -balls would have survived any possible decay processes and thermal evaporation until the present day, and possibly become a dark matter candidate [29].

*Dynamics and cosmological applications:* The dynamics of a pair of one-dimensional  $Q$ -balls has been recently analyzed using momentum flux [50]. For a large separation between the  $Q$ -balls, the profiles develop the usual exponential tail,  $e^{-m_\omega r}$ , in general polynomial potentials and in [50] the authors showed that there was a solitonic force between them. Profiles in the gravity-mediated models and other confinement models, however, have different asymptotic tails, which may affect the detailed dynamics and the  $Q$ -ball formation [51–54].

In a cosmological setting (thermal background), SUSY  $Q$ -balls are generally unstable via evaporation, diffusion, dissociation, and/or decay into today's baryons and lowest mass supersymmetric particles, if the AD field couples to the thermal plasma, which are decay products from inflaton, and/or if the field possesses a lepton number for the MSSM flat directions [21,30]. Following our detailed analytical and numerical analyses of both gravity-mediated and gauge-mediated  $Q$ -balls, it is clear that this whole area of dynamics and cosmological implications of these  $Q$ -balls deserves further analyses.

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## APPENDIX A: AN EXACT SOLUTION

In this appendix we will show that a Gaussian profile is an exact solution of the  $Q$ -ball equation Eq. (3) with  $U_\omega = U_{\text{grav}} - 1/2\omega^2\sigma^2$  in Eq. (15). Notice that the potential  $U_{\text{grav}}$  becomes negative for  $e^{1/2|K|M} < \sigma$ ; hence, the system is not bounded from below. The additional contribution from the nonrenormalizable term  $U_{\text{NR}}$  compensates the negative term and supports the existence of  $Q$ -balls in the system. Although the Gaussian exact solution is no longer a solution for the full potential  $U_{\text{grav}} + U_{\text{NR}}$  in Eq. (15), the solution we will obtain here provides hints in suggesting a reasonable ansatz for the thick-wall  $Q$ -ball as we will see later.

Let us consider the following Gaussian profile:

$$\sigma_{\text{sol}}(r) = \rho_\omega \exp\left(-\frac{|K|m^2r^2}{2}\right), \quad (\text{A1})$$

where we will see that  $m$ ,  $M$ , and  $|K|$  are the same parameters as in Eq. (15) and  $\rho_\omega$  will be shortly determined in terms of the underlying parameters. By substituting

TABLE I. Key analytical results. Recall that the  $\omega$ -independent characteristic slope  $\gamma \equiv E_Q/\omega Q$  leads to the proportionality relation  $E_Q \propto Q^{1/\gamma}$ . The symbols,  $\circ$ ,  $\times$ ,  $\triangle$ , indicate that  $Q$ -balls are stable, unstable, or can be stable with conditions, respectively. The symbol,  $\blacktriangle$ , means that we may need the condition  $|K| \ll \mathcal{O}(1)$ .

Model	Gravity mediated			Gauge mediated	
	Thin wall	Thick wall	Thin wall	Thick wall	Thick wall
Q-ball type	$\blacktriangle$	$\beta^2 \lesssim  K  \lesssim \mathcal{O}(1)$	None	$D = 1, 3, \dots$	
Conditions	$R_Q \gg \delta, 1/\mu; \sigma_0 \simeq \sigma_+$ and $U_\omega \simeq U_{\omega_-}$ in shell		$R \gg 1/m_\omega, 1/\omega$		
Assumptions	DVPs	NDVPs	Both	NDVPs	
Potential type					
$1/\gamma$	$\frac{2D-1}{2(D-1)}$	1	1	$\frac{D}{D+1}$	1
Absolute stability	$\circ$	$\triangle$	$\triangle$	$\circ$	$\times$
Classical stability	$\circ$	$\circ$	$\triangle$	$\circ$	$\times$

Eq. (A1) into the left-hand side of Eq. (3) it leads to

$$U_{\text{grav}} = \frac{m^2}{2} \sigma^2 \left( 1 - |K| \ln \left( \frac{\sigma}{M} \right)^2 \right) \quad (\text{A2})$$

and

$$\rho_\omega = M \exp \left( \frac{D-1}{2} + \frac{m_\omega^2}{2|K|m^2} \right), \quad (\text{A3})$$

where we set an integration constant as zero. Recall  $m_\omega^2 \equiv m^2 - \omega^2$ . Note that the constant  $M$  has the same mass dimension,  $(D-1)/2$ , as  $\sigma$  so that the only physical case is  $D=3$ . The profile Eq. (A1) is an exact solution for  $U_{\text{grav}}$  with the core radius  $R_Q = \sqrt{2/m^2|K|}$  [30], which is very large compared with  $m^{-1}$  for small  $|K| \ll \mathcal{O}(1)$ , and satisfies the boundary conditions for  $Q$ -balls, namely,  $\sigma'(0) = 0 = \sigma(\infty) = \sigma'(\infty)$  [37]. In the extreme limit  $\omega \gg m$  we obtain  $\rho_\omega \rightarrow 0$  for  $|K| \lesssim \mathcal{O}(1)$ , which implies  $\sigma_0 \equiv \sigma(0) \rightarrow 0$ . For large  $\sigma$ , the potential becomes asymptotically flat, tending toward an infinite negative value. By adding the nonrenormalizable term  $U_{\text{NR}}$ , the potential  $U_{\text{grav}}$  is lifted for large  $\sigma$  in Eq. (15), then the full potential  $U_{\text{grav}} + U_{\text{NR}}$  is bounded from below, see Sec. III A. We can see the ansatz given in [30] corresponds to the case where  $\rho_\omega \simeq M$ , which is valid only for  $|K| \ll \mathcal{O}(1)$  and  $\omega \simeq m$ , see in Eq. (A3).

### APPENDIX B: THICK-WALL Q-BALL WITH A GAUSSIAN ANSATZ

In this appendix, we will investigate the thick-wall  $Q$ -ball in gravity-mediated models by introducing a Gaussian ansatz and keeping all terms in Eq. (17) as opposed to the analysis in Sec. III C. By using this profile we can perform the Gaussian integrations, and will obtain the generalized results of Eqs. (33) and (34) in Sec. III C. The test profile for the case,  $\omega \gtrsim \mathcal{O}(m)$ , coincides with the solution  $\sigma_{\text{sol}}$  in Eq. (A1), which implies that the nonrenormalizable term  $U_{\text{NR}}$  in Eq. (15) is negligible.

To recap, the notation we have adopted in Eq. (17) is  $\tilde{\sigma} = \sigma/M$ ,  $\tilde{\omega} = \omega/m$ ,  $\beta^2$  is defined in Eq. (16) and we are considering the case of  $n > 2$ . To begin with we introduce a

Gaussian ansatz inspired by Eq. (A1) for the potential Eq. (17)

$$\tilde{\sigma}(r) = \lambda_\omega \exp(-\kappa_\omega^2 r^2/2), \quad (\text{B1})$$

where  $\tilde{\sigma}_0 \equiv \tilde{\sigma}(0) = \lambda_\omega = \text{finite}$ , and  $\lambda_\omega, \kappa_\omega$  will be functions of  $\omega$  implicitly.  $\lambda_\omega$  should not be confused with the coupling constant  $\lambda$  in Eq. (15). Both  $\lambda_\omega$  and  $\kappa_\omega$  can be determined by extremizing the Euclidean action  $S_\omega$ ; hence, the actual free parameter here will be only  $\omega$ . It is crucial to note that  $\lambda_\omega$  cannot be infinite in the thick-wall limit since we know that  $\lambda_\omega$  is finite and tending to 0. If the nonrenormalizable term  $U_{\text{NR}}$  is negligible, we can expect  $\lambda_\omega \sim \rho_\omega/M \sim \tilde{\sigma}_-(\omega)$  and  $\kappa_\omega^2 \sim |K|m^2$  due to Eq. (A1), which implies that the core radius  $R_Q$  of the thick-wall  $Q$ -ball is  $R_Q \sim \sqrt{2/m^2|K|}$ . For the extreme thick-wall limit  $\omega \gg m$ , we shall also confirm  $\lambda_\omega \rightarrow 0$ , which means  $\tilde{\sigma}_0 \rightarrow 0$ .

By substituting Eq. (B1) into Eq. (1) with the potential Eq. (17), we obtain  $Q$  and  $S_\omega$  using the following Gaussian integrations:  $\Omega_{D-1} \int_0^\infty dr r^{D-1} e^{-kr^2} = (\frac{\pi}{k})^{D/2}$  for real  $k$  where  $\Omega_{D-1} \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)}$ . Thus,

$$Q = M^2 \pi^{D/2} \omega \lambda_\omega^2 \kappa_\omega^{-D}, \quad (\text{B2})$$

$$S_\omega = M^2 \pi^{D/2} \kappa_\omega^{-D} [A(\kappa_\omega, \lambda_\omega) + B(\omega, \lambda_\omega) + C(\lambda_\omega)], \quad (\text{B3})$$

where

$$A(\kappa_\omega, \lambda_\omega) \equiv \frac{D\lambda_\omega^2}{4} (\kappa_\omega^2 + |K|m^2), \quad (\text{B4})$$

$$B(\omega, \lambda_\omega) \equiv \frac{m^2 \lambda_\omega^2}{2} \left( 1 - \frac{\omega^2}{m^2} - 2|K| \ln \lambda_\omega \right), \quad (\text{B5})$$

$$C(\lambda_\omega) = m^2 \beta^2 \lambda_\omega^n \left( \frac{2}{n} \right)^{D/2}.$$

Notice that  $A(\kappa_\omega, \lambda_\omega)$  comes from the gradient term and the logarithmic term in  $S_\omega$  and depends on both  $\kappa_\omega$  and  $\lambda_\omega$ . Similarly,  $B(\omega, \lambda_\omega)$  is given by the quadratic term in the potential Eq. (17) and depends both on  $\lambda_\omega$  and explicitly on  $\omega$ , where as  $C(\lambda_\omega)$  arises simply from the nonrenormaliz-

able term in the potential. An alternative (but in this case more complicated) approach to obtain  $Q$  would be the use of Legendre transformations in Eq. (10).

By extremizing  $S_\omega$  in terms of the two free parameters  $\kappa_\omega, \lambda_\omega$ :

$$\frac{\partial S_\omega}{\partial \kappa_\omega} = 0, \quad \frac{\partial S_\omega}{\partial \lambda_\omega} = 0, \quad (\text{B6})$$

we obtain

$$A + B + C = \frac{\lambda_\omega^2 \kappa_\omega^2}{2}, \quad A + B + \frac{nC}{2} = \frac{m^2 \lambda_\omega^2 |K|}{2}, \quad (\text{B7})$$

which implies that

$$\frac{\kappa_\omega^2}{m^2} = |K| - (n-2)\beta^2 \lambda_\omega^{n-2} \left(\frac{2}{n}\right)^{D/2} \geq 0, \quad (\text{B8})$$

where we have eliminated the  $A + B$  terms in the two expressions of Eq. (B7). Using Eq. (B8) and the second expression of Eq. (B7), we obtain the relations between  $\omega$  and  $\lambda_\omega$

$$\frac{\omega^2}{m^2} = 1 + |K|(D-1-2\ln\lambda_\omega) + \frac{2(n+D)-nD}{2} \beta^2 \lambda_\omega^{n-2} \left(\frac{2}{n}\right)^{D/2}, \quad (\text{B9})$$

$$\sim \begin{cases} 1 + |K|(D-1-2\ln\lambda_\omega) & \text{for } |K| \sim \mathcal{O}(1) \\ 1 - 2|K|\ln\lambda_\omega & \text{for } |K| < \mathcal{O}(1) \end{cases}, \quad (\text{B10})$$

$$\frac{d\lambda_\omega}{d\omega} = -\frac{\lambda_\omega \omega}{|K|m^2 F} \sim -\frac{\lambda_\omega \omega}{|K|m^2} < 0, \quad (\text{B11})$$

where we have differentiated Eq. (B9) with respect to  $\omega$  to obtain Eq. (B1) and have defined  $F \equiv 1 - (n-2)\frac{2(n+D)-nD}{4}\frac{\beta^2}{|K|}\lambda_\omega^{n-2}\left(\frac{2}{n}\right)^{D/2} = 1 + \frac{2(n+D)-nD}{4|K|m^2}(\kappa_\omega^2 - m^2|K|)$ . Equations (B8) and (B9) imply that both  $\kappa_\omega$  and  $\lambda_\omega$  are functions of  $\omega$ ; however, these are not solvable in closed forms unless the particular limits, which were introduced in Sec. III C, are taken, as we will now show. Comparing Eqs. (B9) and (B11) with Eqs. (28) and (29), we can see an extra contribution of  $\mathcal{O}(|K|)$  in Eq. (B9), which is not present in Eq. (28). This difference of  $(D-1)|K|$  arises because in calculating Eq. (B9) we have used  $\lambda_\omega$  whereas we have used  $\tilde{\sigma}_-(\omega)$  in obtaining Eq. (28), and although related they are not precisely the same. In the extreme thick-wall limit  $\omega \gg m$ , and from Eq. (B9) this implies  $\lambda_\omega \rightarrow 0^+$  [recall from Eq. (B1) that  $\lambda_\omega$  has to remain finite]. Considering the nonrenormalizable term in Eq. (B9), the fact that  $\beta^2 \lesssim |K| \lesssim \mathcal{O}(1)$  and  $\lambda_\omega \rightarrow 0^+$  with  $n > 2$ , implies that this term is subdominant and can be ignored. As long as  $\lambda_\omega < \mathcal{O}(1)$ , then  $F \sim 1$  and the second relation of Eq. (B11) follows, which implies that  $\lambda_\omega$  is a monotonically decreasing function in terms of  $\omega$ . The limit  $\lambda_\omega \sim \mathcal{O}(1)$  corresponds to  $\omega \gtrsim \mathcal{O}(m)$ , see Eq. (B9).

We will call this the moderate limit and represent it by “ $\sim$ ”. The other case,  $\omega \gg m$  (or equivalently  $\lambda_\omega \ll \mathcal{O}(1)$ ), we shall call the extreme limit and represent it by  $\rightarrow$ . Depending on the logarithmic strength of  $|K|$ , we can obtain Eq. (B10), which leads to the approximated expressions for  $\lambda_\omega$  and can also obtain  $\kappa_\omega$  from Eq. (B8)

$$\lambda_\omega \sim \begin{cases} \rho_\omega/M & \text{for } |K| \sim \mathcal{O}(1) \\ \tilde{\sigma}_-(\omega) & \text{for } |K| < \mathcal{O}(1) \end{cases} \rightarrow 0; \quad (\text{B12})$$

$$\frac{\kappa_\omega^2}{m^2} \sim |K| \rightarrow |K| \quad \text{for } |K| \lesssim \mathcal{O}(1),$$

where  $\kappa_\omega$  is independent of  $\omega$  in both the moderate and extreme limits.

Using Eqs. (B2), (B3), and (B7), we obtain the characteristic slope in both the moderate and extreme limits,

$$\frac{E_Q}{\omega Q} = 1 + \frac{\kappa_\omega^2}{2\omega^2} \sim 1 + \frac{m^2|K|}{2\omega^2} \rightarrow 1. \quad (\text{B13})$$

In order to show their classical stability, we shall differentiate  $Q$  with respect to  $\omega$  using Eqs. (B8), (B9), and (B11):

$$\frac{\omega}{Q} \frac{dQ}{d\omega} = 1 - \frac{2\omega^2}{m^2|K|F} \left[ 1 - \frac{D(n-2)}{4\kappa_\omega^2} (\kappa_\omega^2 - m^2|K|) \right] \sim 1 - \frac{2\omega^2}{m^2|K|} \rightarrow -\frac{2\omega^2}{m^2|K|} < 0, \quad (\text{B14})$$

$$\frac{d}{d\omega} \left( \frac{E_Q}{Q} \right) = 1 - \frac{1}{2\omega^2} \left[ \kappa_\omega^2 + \frac{(n-2)\omega^2}{m^2|K|F} (\kappa_\omega^2 - m^2|K|) \right] \sim 1 - \frac{m^2|K|}{2\omega^2} \rightarrow 1 > 0, \quad (\text{B15})$$

where we have taken the moderate limit and extreme limit and used  $\kappa_\omega^2 \sim m^2|K|$ ,  $F = 1 + \frac{2(n+D)-nD}{4|K|m^2} \times (\kappa_\omega^2 - m^2|K|) \sim 1$ . The classical stability condition Eq. (B14) is consistent with Eq. (B15), and is consistent with Eq. (12). This is different from the result we obtained for the polynomial potentials [37] (see Eq. (74) in [37]), because in that case the Gaussian ansatz does not give the exact solution unlike here in Eq. (B1) where it does become the exact solution Eq. (A1) in both limits. The results Eqs. (B13)–(B15) in both the moderate and extreme limits recover the key results, Eqs. (33) and (34), and are independent of  $D$ ; hence, the thick-wall  $Q$ -balls for all  $D$  have similar properties. We can also see the small additional effects arising from the nonrenormalizable term in Eqs. (B14) and (B15).

Let us summarize the important results we found in this appendix. By introducing a Gaussian test profile Eq. (B1) inspired by the exact solution Eq. (A1) for  $U_{\text{grav}}$ , we computed the Euclidean action  $S_\omega$  and the charge  $Q$  using Gaussian integrations. Then we extremized  $S_\omega$  in terms of  $\lambda_\omega$  and  $\kappa_\omega$  in Eq. (B6), which gave the relations of both  $\lambda_\omega$  and  $\kappa_\omega$  as a function of  $\omega$ . By introducing two limits called



moderate limit and extreme limit, we confirmed that the ansatz, Eq. (B1), approaches Eq. (A1) in the moderate limit. We established that the results Eqs. (B13)–(B15) recovered the previous results in Eqs. (33) and (34), which are obtained simply by reparametrizing in  $S_\omega$  and extracting the explicit  $\omega$  dependence from the integral in  $S_\omega$  with  $U = U_{\text{grav}}$  where the nonrenormalizable term was neglected at the beginning of the analysis by applying L'Hôpital rules.

In addition, we would like to emphasize the main differences between our work and other earlier analyses in the literature [30,49]. The analytical framework adopted in [49] is valid only for  $|K| = 1$ ,  $D = 3$ ,  $n = 4$ . Our work has shown that this can be generalized to arbitrary integer values of  $D$  and  $n(>2)$  under the conditions  $\beta^2 \lesssim |K| \lesssim \mathcal{O}(1)$ , and that the thick-wall  $Q$ -ball can be classically stable. In Sec. III C, we also found that the thick-wall

$Q$ -ball may be absolutely stable under certain additional conditions, Eq. (35). Furthermore, Enqvist and McDonald in [30] analytically obtained the same core size of thick-wall  $Q$ -balls, although they obtained a slightly different value for  $E_Q/Q$  [see their Eq. (112)]. The reason for this is because their ansatz assumed  $\lambda_\omega \simeq 1$  in Eq. (B1) by simply neglecting the nonrenormalizable term, which implies that the third term of  $B(\omega, \lambda_\omega)$  and term  $C(\lambda_\omega)$  in Eq. (B5) are absent. Hence, their analysis is valid for  $|K| \ll \mathcal{O}(1)$  and  $\omega \simeq m$ , see Eq. (A3). We, however, have kept all the terms in Eq. (17) and used a more general ansatz, which can be applied for  $|K| \lesssim \mathcal{O}(1)$  and  $\omega \gtrsim \mathcal{O}(m)$  with the restricted coupling constant of the nonrenormalizable term  $\beta^2 \lesssim |K|$ . In summary, in this appendix we have extensively investigated analytically both the absolute and classical stability of  $Q$ -balls in Eqs. (B13) and (B14).

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