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Pushing the asymptotics of the 6j-symbol further

Maïté Dupuis[1,*] and Etera R. Livine[1,†]

[1] *Laboratoire de Physique, ENS Lyon, CNRS-UMR 5672, 46 Allée d'Italie, Lyon 69007, France*

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In the context of spin-foam models for quantum gravity, we investigate the asymptotical behavior of the $\{6j\}$-symbol at next-to-leading order. This gives the first quantum gravity correction to the (3d) Regge action. We compute it analytically and check our results against numerical calculations. The $\{6j\}$-symbol is the building block of the Ponzano-Regge amplitudes for 3d quantum gravity, and the present analysis is directly relevant to deriving the quantum corrections to gravitational correlations in the spin-foam formalism.

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I. INTRODUCTION: SPIN FOAMS AND THE $\{6j\}$-SYMBOL

The spin-foam formalism is an attempt to define rigorously a path integral for Quantum Gravity. Spin foams can be interpreted from several perspectives, as a covariant history formulation for Loop Quantum Gravity describing the evolution of the spin network states, as an improved and quantized version of the Regge calculus for general relativity, as a quantization of "almost topological" field theories, as a higher-dimensional generalization of the matrix models generating 2d surfaces. And they have been shown to be related to many other approaches to quantum gravity. The spin-foam model for 3d quantum gravity is the Ponzano-Regge model [1], which was the first spin-foam model ever written. It has been shown to provide a consistent quantization of general relativity in three space-time dimensions (for both Riemannian and Lorentzian signatures). The main spin-foam models for 4d quantum gravity are the Barrett-Crane model [2] and the more recent EPR-FK-LS family of spin-foam models [3–5]. They are related to the reformulation of general relativity as a constrained topological BF-theory and were mostly constructed as a discretization of the path integral over space-time geometries.

Such spin-foam models provide a description of the quantum geometry of space-time at the Planck scale. The main issue is then to extract semiclassical information from the formalism and to show its relation to the more standard perturbative approach to the quantization of general relativity (as a quantum field theory). Solving this question amounts to proving that we recover general relativity in a large scale (or low energy) regime of the spin-foam models and to showing how to compute the quantum corrections to the classical dynamics of the gravitational field. A proposal to address this problem is the "spin-foam graviton" framework proposed by Rovelli and collaborators [6]. It defines the propagators and correlation functions for geometric observables, mainly the area, from which we can extract information about the (effective) space-time metric and its (quantum) fluctuations. Most explicit calculations in this framework have been done at the leading order (in the scale parameter) and for the simplest space-time triangulation (a single tetrahedron in 3d and a single 4-simplex in 4d). In order to make the link with the standard QFT perturbative expansion, we need to be able to push these calculations further and calculate the correlations both at higher order ("loop corrections") and for more refined triangulations (smoother boundary state). In the present work, we focus on the first aspect: the leading order of the correlations gives the classical propagator of the graviton and we would like to compute the higher order (quantum) corrections. Following the lines of [7–9], this requires understanding the corrections to the asymptotical behavior of the spin-foam vertex amplitude, which is the amplitude associated to a single tetrahedron in 3d quantum gravity or to a single 4-simplex in 4d models. This is the basic building blocks of spin-foam models, which are then constructed by gluing these spin-foam vertices in some particular way in order to describe the whole space-time. In the Ponzano-Regge model, the spin-foam vertex is given by the $\{6j\}$-symbol from the recoupling theory of the representations of SU(2). The Barrett-Crane model is defined by the $\{10j\}$-symbol and the more recent models use the EPR or FK vertex amplitudes. The present paper focuses on the $\{6j\}$-symbol, relevant for 3d quantum gravity.

They are three basic ways to compute the leading order asymptotics of the $\{6j\}$-symbol and show its relation to the Regge action for 3d gravity:

(i) *Recursion relations* [10]: Using the invariance of the $\{6j\}$-symbol under Pachner moves (Biedenharn-Elliott identity) or directly its definition as a recoupling coefficient, one can derive a recursion relation for $\{6j\}$-symbol. This recursion formula is actually very useful for numerical computations, but it can also be approximated at large spins by a (second order) differential equation. One then derive the asymptotics from a WKB approximation.

(ii) *Integral formula* [11,12]: One can write the square of the $\{6j\}$-symbol as an integral over four copies of

*maite.dupuis@ens-lyon.fr
†etera.livine@ens-lyon.fr

SU(2). In the large spin regime, we can use saddle point techniques and one derives the right asymptotics after a careful analysis of nondegenerate and degenerate configurations for the saddle points. This is the technique used to derive the asymptotics of the Barrett-Crane and EPR-FK vertex amplitudes.

(iii) *Brute-force approximation* [13]: One can start from the explicit algebraic formula of the $\{6j\}$-symbol as a sum over some products of factorials. Using the Stirling formula and after lengthy calculations, we approximate the sum by an integral and use saddle point techniques again which lead to the same asymptotics.

We also point out the more sophisticated and rigorous proof of the asymptotics by Roberts [14] based on geometric quantization, but that also uses an integral formula and saddle point methods.

The present goal is to push these approaches one step further and derive the first correction to the leading order formula. As a first attempt, we focus on the third method and show how to extract the next-to-leading order corrections through a brutal approximation of the explicit algebraic expression of the $\{6j\}$-symbol. We compare our results with the previous calculations for the cases of the equilateral and isosceles tetrahedra [8] and check the general case against numerical calculations. Although the final explicit formula for this next-to-leading order in the general case is not particularly pretty, we prove that it is indeed possible to compute it analytically exactly and we show that we could extract all orders of the $\{6j\}$-symbol using the same procedure. This is a necessary step towards providing explicit formulas or procedures to compute all orders of the perturbative expansion (in term of the length scale) of the graviton correlations in spin-foam models. Moreover, we interpret this next-to-leading order of the $\{6j\}$-symbol as providing quantum gravity corrections to the standard 3d Regge action.

We could also use the more subtle approach of the recursion relation. This requires a careful analysis of the recursion relation and computing the corrections to the WKB approximation [15]. Or we could use the integral formula technique, then one should be particularly careful when dealing with the degenerate contributions to the $\{6j\}$-symbol.

II. THE $\{6j\}$-SYMBOL

The $\{6j\}$-symbol is the basic building block of the Ponzano-Regge model which is a state sum model for 3d Euclidean gravity formulated as a SU(2) gauge theory. The Ponzano-Regge model is defined over a triangulation of space-time: we build the 3d space-time manifold from tetrahedra glued together along their respective triangles and edges. We assign an irreducible representation (irreps) of SU(2) to each edge e of the triangulation. These irreps are labeled by a half-integer $j_e \in \mathbb{N}/2$, the spin, and the dimension of the corresponding representation space is given by $d_{j_e} = 2j_e + 1$. Each tetrahedron of the triangulation has six edges labeled by six spins $j_{e_1}, \ldots, j_{e_6}$ and we associate it with the corresponding $\{6j\}$-symbol, which is the unique (nontrivial) SU(2) invariant built from these six representations. It is giving by combining four normalized Clebsh-Gordan coefficients corresponding to the four triangles of the tetrahedron. Finally, the Ponzano-Regge amplitude for a given colored triangulation is simply given by the product of the $\{6j\}$-symbols associated to all its tetrahedra.

Looking more closely at a single tetrahedron, we label its four triangles by $I = 0, \ldots, 3$. Then each of its six edges is labeled by the couple of triangles to which it belongs, (IJ) with $0 \leq I < J \leq 3$. To each edge is attached a SU(2) irrep of spin j_{IJ}, which defines the length of that edge $j_{IJ} + \frac{1}{2} = \frac{d_{j_{IJ}}}{2}$ (see Fig. 1). There are several ways of expressing the $\{6j\}$-symbol. The basic formula is the Racah's single sum formula which expresses the $\{6j\}$-symbol as a sum over some products of factorials (see Appendix A). This is our starting point as in [13]:

$$\begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{Bmatrix} = \sqrt{\Delta(j_{01}, j_{02}, j_{03})\Delta(j_{23}, j_{02}, j_{12})\Delta(j_{23}, j_{13}, j_{03})\Delta(j_{01}, j_{13}, j_{12})} \sum_{\max v_I}^{\min p_J} \frac{(-1)^t (t+1)!}{\prod_I (t - v_I)! \prod_J (p_J - t)!}, \tag{1}$$

where the v_I and p_i are given by the following sums:

$$\forall\ K = 0 \ldots 3, \qquad v_K = \sum_{I \neq K} j_{IK}, \qquad \forall\ k = 1 \ldots 3, \qquad p_k = \sum_{i \neq 0, k} (j_{0i} + j_{ki}).$$

The factors $\Delta(j_{01}, j_{02}, j_{03})$ are weights associated to each triangle and are defined by:

$$\Delta(j_{01}, j_{02}, j_{03}) = \frac{(j_{01} + j_{02} - j_{03})!(j_{01} - j_{02} + j_{03})!(-j_{01} + j_{02} + j_{03})!}{(j_{01} + j_{02} + j_{03} + 1)!}.$$

From this point, in all sums and products throughout this paper, capital indices K will run from 0 to 3 and lower-case indices k will run from 1 to 3.

We are interested in the large spin expansion of the $\{6j\}$-symbol when scaling all the spins homogeneously. Actually we will scale the lengths $d_{j_{IJ}}/2$ instead of the spins j_{IJ} because the structure of the expansion will be simpler (we expect an alternation of cosines and sines without any mixing up at all orders as in [8]) and the geometrical interpretation (when possible) is expected to be simpler. Then we rescale all $d_{j_{IJ}}$ by $\lambda d_{j_{IJ}}$ in (1), which is equivalent to changing $j_{IJ} = \frac{d_{j_{IJ}}}{2} - \frac{1}{2}$ to $\lambda \frac{d_{j_{IJ}}}{2} - \frac{1}{2}$. This gives:

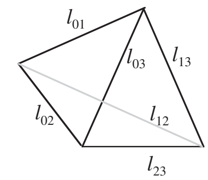

FIG. 1. A single tetrahedron: the edge lengths are given by $l_{IJ} = \frac{d_{j_{IJ}}}{2}$.

$$
\begin{aligned}
&\begin{Bmatrix} \lambda d_{j_{01}}/2 - 1/2 & \lambda d_{j_{02}}/2 - 1/2 & \lambda d_{j_{03}}/2 - 1/2 \\ \lambda d_{j_{23}}/2 - 1/2 & \lambda d_{j_{13}}/2 - 1/2 & \lambda d_{j_{12}}/2 - 1/2 \end{Bmatrix} \\
&= \sqrt{\Delta(\lambda d_{j_{01}}, \lambda d_{j_{02}}, \lambda d_{j_{03}})\Delta(\lambda d_{j_{23}}, \lambda d_{j_{02}}, \lambda d_{j_{12}})\Delta(\lambda d_{j_{23}}, \lambda d_{j_{13}}, \lambda d_{j_{03}})\Delta(\lambda d_{j_{01}}, \lambda d_{j_{13}}, \lambda d_{j_{12}})} \sum_{\lambda \max \tilde{v}_I - (3/2)}^{\lambda \min \tilde{p}_j - 2} (-1)^t \\
&\quad \times \frac{(t+1)!}{\prod_i (t - \lambda \tilde{v}_I + \frac{3}{2})! \prod_j (\lambda \tilde{p}_j - t - 2)!}
\end{aligned} \tag{2}
$$

with the new conventions:

$$
\tilde{v}_K = \sum_{I \neq K} \frac{d_{j_{IK}}}{2}, \qquad \tilde{p}_k = \sum_{i \neq 0,k} \frac{(d_{j_{0i}} + d_{j_{ki}})}{2},
$$

$$
\Delta(\lambda d_{j_{01}}, \lambda d_{j_{02}}, \lambda d_{j_{03}}) = \frac{(\frac{\lambda}{2}(d_{j_{01}} + d_{j_{02}} - d_{j_{03}}) - \frac{1}{2})!(\frac{\lambda}{2}(d_{j_{01}} - d_{j_{02}} + d_{j_{03}}) - \frac{1}{2})!(\frac{\lambda}{2}(-d_{j_{01}} + d_{j_{02}} + d_{j_{03}}) - \frac{1}{2})!}{(\frac{\lambda}{2}(d_{j_{01}} + d_{j_{02}} + d_{j_{03}}) - \frac{1}{2})!}.
$$

The quantity $\tilde{v}_K$ gives the perimeter of the triangle K while the $\tilde{p}_k$'s are the perimeters of (nonplanar) quadrilaterals.

III. PERTURBATIVE EXPANSION OF THE 6J-SYMBOL

In this section, we will give a procedure to obtain the full perturbative expansion of the $\{6j\}$-symbol in term of the length scale λ and we compute explicitly the leading order (Ponzano-Regge formulas) then the next-to-leading order analytically.

A. General procedure

We give all the necessary formulas to obtain the Ponzano-Regge corrections at any order. But calculations are only performed explicitly at the next-to-leading order for a generic $\{6j\}$-symbol. We start from Eq. (2).

a. First approximation: factorials. The factorial can be expanded in a series:

$$
n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \cdots\right) \tag{3}
$$

In Eq. (2), there are factorials of the form: $n!$, $(n + 1/2)!$ and $(n - 1/2)!$, which are rigorously defined through Euler's Γ function. From (3) we deduce asymptotic expansions for $(n + 1/2)!$ and $(n - 1/2)!$ (see the details in Appendix B). In order to get the next-to-leading order (NLO) in the $1/\lambda$ expansion of the $\{6j\}$-symbol, we replace the factorials in Eq. (2) by their respective asymptotic expansion:

$$
\begin{aligned}
n! &\sim \sqrt{2\pi} e^{(n+(1/2))\ln(n)-n}\left(1 + \frac{1}{12n}\right) \\
\left(n + \frac{1}{2}\right)! &\sim \sqrt{2\pi} e^{(n+1)\ln(n)-n}\left(1 + \frac{11}{24n}\right) \\
\left(n - \frac{1}{2}\right)! &\sim \sqrt{2\pi} e^{n\ln(n)-n}\left(1 - \frac{1}{24n}\right).
\end{aligned} \tag{4}
$$

Then, Eq. (2) reads at first order as:

$$
\begin{aligned}
&\begin{Bmatrix} \lambda d_{j_{01}}/2 - 1/2 & \lambda d_{j_{02}}/2 - 1/2 & \lambda d_{j_{03}}/2 - 1/2 \\ \lambda d_{j_{23}}/2 - 1/2 & \lambda d_{j_{13}}/2 - 1/2 & \lambda d_{j_{12}}/2 - 1/2 \end{Bmatrix} \\
&\quad = \frac{1}{2\pi} e^{(\lambda/2)h(d_{j_{IJ}})}\left(1 - \frac{1}{24\lambda}H(d_{j_{IJ}}) + O\left(\frac{1}{\lambda^2}\right)\right)\Sigma.
\end{aligned} \tag{5}
$$

The first factor is given by:

$$h(d_{j_{IJ}}) = \sum_{I<J} d_{j_{IJ}} h_{d_{j_{IJ}}} \quad \text{with}$$
$$h_{d_{j_{IJ}}} = \frac{1}{2}\ln\left(\frac{(d_{j_{IJ}} - d_{j_{IK}} + d_{j_{IL}})(d_{j_{IJ}} + d_{j_{IK}} - d_{j_{IL}})(d_{j_{IJ}} - d_{j_{JK}} + d_{j_{JL}})(d_{j_{IJ}} + d_{j_{JK}} - d_{j_{JL}})}{(d_{j_{IJ}} + d_{j_{IK}} + d_{j_{IL}})(-d_{j_{IJ}} + d_{j_{IK}} + d_{j_{IL}})(d_{j_{IJ}} + d_{j_{JK}} + d_{j_{JL}})(-d_{j_{IJ}} + d_{j_{JK}} + d_{j_{JL}})}\right), \tag{6}$$

where (KL) is the opposite side to (IJ), that is $K \neq L$ and $K, L \neq I, J$. The second factor is due to the NLO of the factorials:

$$H(d_{j_{IJ}}) = 2\sum_{j,K}\frac{1}{\tilde{p}_j - \tilde{v}_K} - 2\sum_K \frac{1}{\tilde{v}_K} = \sum_I\left[\frac{-r^I + \sum_{K\neq I} r^I_K}{2A_I}\right], \tag{7}$$

where A_I is the area of triangle I, r^I is the radius of the incircle of triangle I and r^I_K is the radius of the excircle to the triangle I tangent to the side $d_{j_{IK}}$ of the triangle I. Finally, Σ is a Riemann sum:

$$\Sigma = \frac{1}{\lambda^2}\sum_{x=\max\tilde{v}_I/2}^{\min\tilde{p}_j/2} e^{F(x)}\left(1 - \frac{1}{12\lambda}G(x) + O\left(\frac{1}{\lambda^2}\right)\right)e^{\lambda f(x)} \tag{8}$$

with the prefactor and the action given by:

$$f(x) = i\pi x + x\ln(x) - \sum_K (x - \tilde{v}_K)\ln(x - \tilde{v}_K) - \sum_j (\tilde{p}_j - x)\ln(\tilde{p}_j - x),$$
$$F(x) = \frac{1}{2}\ln\left(\frac{x^3\prod_j(\tilde{p}_j - x)^3}{\prod_K (x - \tilde{v}_K)^4}\right),$$
$$G(x) = -\frac{13}{x} + \frac{47}{2}\sum_K\frac{1}{x - \tilde{v}_K} + 13\sum_j\frac{1}{\tilde{p}_j - x}. \tag{9}$$

The details of the computation are given in Appendix C.

b. Second approximation: Riemann sum. The second approximation consists in replacing the Riemann sum Σ of (5) by an integral. One k^{-1} factor of Σ plays the role of dx. We can then rewrite Eq. (5) as:

$$\{6j\} \sim \frac{1}{2\pi}\left(1 - \frac{1}{24\lambda}H(d_{j_{IJ}}) + O\left(\frac{1}{\lambda}\right)\right)e^{(\lambda/2)h(d_{j_{IJ}})}\frac{1}{\lambda} \times \int_{\max(\tilde{v}_I/2)}^{\min(\tilde{p}_j/2)} dx e^{F(x)}\left(1 - \frac{1}{12\lambda}G(x) + O\left(\frac{1}{\lambda^2}\right)\right)e^{\lambda f(x)}. \tag{10}$$

This approximation does not generate any corrections at leading order and at first order. It will nevertheless enter at second order in terms in $1/\lambda^2$.

c. Third approximation: saddle point approximation. We have to study an integral of the form $I = \int_a^b dx g(x)e^{\lambda f(x)}$ where λ is a large parameter. The asymptotic expansion of such an integral is given by contributions around the stationary points of the action f which are points, denoted x_0, of the complex plane such that $f'(x_0) = 0$. We expand the action $f(x)$ and the function $g(x)$ around the stationary points x_0 in term of $\delta x = x - x_0$:

$$f(x) = \sum_{j=0}^{\infty}\frac{f(x_0)^{(j)}}{j!}(\delta x)^j = f(x_0) + \frac{f''(x_0)}{2}(\delta x)^2 + f^{\geq 2}_{x_0}(\delta x)$$
$$\text{and} \quad g(x) = \sum_{j=0}^{\infty}\frac{g(x_0)^{(j)}}{j!}(\delta x)^j = g(\delta x).$$

We then expand $K(\delta x) = g(\delta x)e^{kf^{\geq 2}_{x_0}(\delta x)}$ in power of δx. Following the standard stationary phase approximation, we extend the integration domain to the whole $\mathbb{R}$. The integrals are then "generalized Gaussians" which can easily be computed. We group the resulting terms according to their dependence on $1/\lambda$, being careful because of the function $g(x)$ which depends on $1/\lambda$. We recall that $g(x)$ was obtained by replacing the factorials in (2) by their series expansion and we write $g(x)$ under the general form:

$$g(x) = \sum_{i=1}^{\infty}\frac{g_i(x)}{i!\lambda^i}.$$

Then the complete perturbative expansion of I can be written as:

$$I = \sum_{x_0} e^{\lambda f(x_0)}\sqrt{\frac{2\pi}{-f''(x_0)\lambda}}\left(1 + \sum_{n=1}^{\infty}\frac{1}{\lambda^n}\left[\sum_{p=0}^{n-1}\tilde{N}_p\frac{(2p-1)!!}{(-f''(x_0))^p} + \sum_{p=0}^{2n} N_p\frac{(2n+2p-1)!!}{(-f''(x_0))^{n+p}}\right]\right) \tag{11}$$

where

$$\tilde{N}_p = \sum_{i=1}^{E[2p/3]} \frac{1}{i!(n-p+i)!} \sum_{l_1\cdots l_i=3}^{E[2p/i]} \frac{g_{n-p+i}^{(2p-\sum_{j=1}^{i} l_j)}(x_0)}{(2p-\sum_{j=1}^{i} l_j)!} \prod_{j=1}^{i} \frac{f^{l_j}(x_0)}{(l_j)!}$$

$$N_0 = \frac{g_0^{(2n)}(x_0)}{(2n)!} + \sum_{i=1}^{E[2n/3]} \frac{1}{(i!)^2} \sum_{l_1\cdots l_i=3}^{E[2n/i]} \frac{g_i^{(2n-\sum_{j=1}^{i} l_j)}(x_0)}{(2p-\sum_{j=1}^{i} l_j)!} \prod_{j=1}^{i} \frac{f^{l_j}(x_0)}{(l_j)!} \tag{12}$$

$$N_p = \sum_{i=p}^{E[2(p+n)/3]} \frac{1}{i!(i-p)!} \sum_{l_1\cdots l_i=3}^{E[2(n+p)/i]} \frac{g_{i-p}^{(2(n+p)-\sum_{j=1}^{i} l_j)}(x_0)}{(2(n+p)-\sum_{j=1}^{i} l_j)!} \prod_{j=1}^{i} \frac{f^{l_j}(x_0)}{(l_j)!} \quad \text{for } p \geq 1$$

The details of the computation are given in Appendix D. From this expansion and adjusting the first approximation to get the proper dependence on λ for g and the prefactors, it is possible to compute analytically the whole asymptotic expansion of the $\{6j\}$-symbol.

Here to get explicitly the next-to-leading order of the $\{6j\}$-symbol asymptotic expansion, we only need the next-to-leading order of the $1/\lambda$ expansion of I, so we cut the previous formulas at $n = 1$, then

$$I \sim \sum_{x_0} e^{\lambda f(x_0)} \sqrt{\frac{2\pi}{-f''(x_0)\lambda}} \left(1 + \frac{1}{\lambda}\left(\tilde{N}_0 + \frac{N_0}{-f''(x_0)} + \frac{3N_1}{(-f''(x_0))^2} + \frac{15N_2}{(-f''(x_0))^3}\right)\right)$$

with the expansion coefficients given by

$$\tilde{N}_0 = g_1(x_0), \qquad N_0 = \frac{g_0''(x_0)}{2},$$
$$N_1 = \frac{f^{(3)}(x_0)g_0'(x_0)}{3!} + \frac{f^{(4)}(x_0)g_0(x_0)}{4!},$$
$$N_2 = \frac{g_0(x_0)}{2}\left(\frac{f^{(3)}(x_0)}{3!}\right)^2.$$

We recall that $g(x) = e^{F(x)}(1 - \frac{G(x)}{12\lambda})$; that is: $g_0(x) = e^{F(x)}$ and $g_1(x) = -\frac{G(x)}{12} e^{F(x)}$. Finally, we obtain the approximation:

$$I \sim \sum_{x_0} \sqrt{\frac{2\pi}{-f''(x_0)\lambda}} e^{F(x_0)+\lambda f(x_0)} \left[1 + \frac{1}{\lambda}\left(-\frac{G(x_0)}{12} - \frac{F''(x_0) + (F'(x_0))^2}{2f''(x_0)} + \frac{f^{(4)}(x_0) + 4f^{(3)}(x_0)F'(x_0)}{8(f''(x_0))^2} - \frac{5(f^{(3)}(x_0))^2}{24(f''(x_0))^3}\right) + O\left(\frac{1}{\lambda^2}\right)\right] \tag{13}$$

This gives us the following expression for the asymptotic expansion of the $\{6j\}$-symbol at second order:

$$\begin{Bmatrix} \lambda d_{j_{01}}/2 - 1/2 & \lambda d_{j_{02}}/2 - 1/2 & \lambda d_{j_{03}}/2 - 1/2 \\ \lambda d_{j_{23}}/2 - 1/2 & \lambda d_{j_{13}}/2 - 1/2 & \lambda d_{j_{12}}/2 - 1/2 \end{Bmatrix}$$
$$\sim \sum_{x_0} \sqrt{\frac{1}{-f''(x_0)2\pi\lambda^3}} \exp(F(x_0) + \lambda f(x_0)) \times \exp\left(\sum_{I<J} \frac{\lambda d_{j_{IJ}}}{2} h_{d_{j_{IJ}}}\right)\left[1 + \frac{1}{\lambda}\left(-\frac{H(j_{IJ})}{24} - \frac{G(x_0)}{12} - \frac{F''(x_0) + (F'(x_0))^2}{2f''(x_0)} + \frac{f^{(4)}(x_0) + 4f^{(3)}(x_0)F'(x_0)}{8(f''(x_0))^2} - \frac{5(f^{(3)}(x_0))^2}{24(f''(x_0))^3}\right) + O\left(\frac{1}{\lambda^2}\right)\right] \tag{14}$$

where x_0 are the stationary points of the phase, i.e. $f'(x_0) = 0$. The next step is to identify these stationary points.

B. Contributions of the stationary points

The phase $f(x)$ is an analytical function given by:

$$f(x) = i\pi x + x\ln(x) - \sum_K \left(x - \frac{\tilde{v}_K}{2}\right)\ln\left(x - \frac{\tilde{v}_K}{2}\right) - \sum_j \left(\frac{\tilde{p}_j}{2} - x\right)\ln\left(\frac{\tilde{p}_j}{2} - x\right), \tag{15}$$

therefore the stationary points x_0 satisfy the following equation as shown in [13]:

$$f'(x) = i\pi + \ln(x) - \sum \ln(x - \tilde{v}_K/2) + \sum \ln(\tilde{p}_j/2 - x) = 0, \tag{16}$$

which is equivalent to

$$x\prod_j (p_j - x) = -\prod_K (x - v_K). \tag{17}$$

The previous equation reduces to a quadratic equation $Ax^2 - Bx + C = 0$ with

$$A = -\sum_{j<l}\tilde{p}_k\tilde{p}_l + \sum_{K<L}\tilde{v}_K\tilde{v}_L = \frac{1}{2}\Big(\sum_{I<J,K<L,(I,J)\neq(K,L)} d_{j_{IJ}}d_{j_{KL}}\Big)$$
$$B = -\tilde{p}_1\tilde{p}_2\tilde{p}_3 + \sum_{I<J<K}\tilde{v}_I\tilde{v}_J\tilde{v}_K = \frac{1}{4}\Big[\Big(\sum_{I<J,K<L,(I,J)\neq(K,L)} d_{j_{IJ}}d_{j_{KL}}\Big)\Big(\sum_{I<J} d_{j_{IJ}}\Big) + \sum_J\Big(\prod_{K\neq J} d_{j_{JK}}\Big)\Big] \qquad C = \prod_K v_K. \tag{18}$$

As shown in [13], the discriminant $\Delta = -(B^2 - 4AC)$ is given in terms of the $d_{j_{IJ}}$ by:

$$\Delta = \frac{1}{16}\Big[\sum_{I<J,K<L,(I,J)\neq(K,L)} d_{j_{IJ}}d_{j_{KL}}\Big(\sum_{M<N,(M,N)\neq(I,J),(M,N)\neq(K,L)} d^2_{j_{MN}} - d^2_{j_{IJ}} - d^2_{j_{KL}}\Big) - \sum_K\prod_{L\neq K} d^2_{j_{KL}}\Big]$$
$$= 2\begin{vmatrix} 0 & (\frac{d_{j_{23}}}{2})^2 & (\frac{d_{j_{13}}}{2})^2 & (\frac{d_{j_{12}}}{2})^2 & 1 \\ (\frac{d_{j_{23}}}{2})^2 & 0 & (\frac{d_{j_{03}}}{2})^2 & (\frac{d_{j_{02}}}{2})^2 & 1 \\ (\frac{d_{j_{13}}}{2})^2 & (\frac{d_{j_{03}}}{2})^2 & 0 & (\frac{d_{j_{01}}}{2})^2 & 1 \\ (\frac{d_{j_{12}}}{2})^2 & (\frac{d_{j_{02}}}{2})^2 & (\frac{d_{j_{01}}}{2})^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 2^4(3!)^2V^2, \tag{19}$$

where V is the volume of the tetrahedron of edge length $d_{j_{IJ}}/2$. In the following we will focus on the case where $\Delta > 0$, i.e. $V^2 > 0$, which corresponds to tetrahedra in flat Euclidean space. The other case $\Delta < 0$ corresponds to tetrahedra admitting an embedding in the $2 + 1d$ Minkowski space-time. And so, we get two stationary points:

$$x_\pm = \frac{B \pm i\sqrt{\Delta}}{2A}. \tag{20}$$

The geometrical interpretation of the stationary points is not clear yet. We have shown that Δ is related to the volume of the tetrahedron. B and A are also related to invariant of the tetrahedron:

$$B = \sum_I \frac{v_I}{2}A + 24V\cot\theta,$$

where we recall that v_I is the perimeter of the triangle I of the tetrahedron. The angle θ is the Brocard angle of the tetrahedron. Indeed, $\frac{d_{j_{01}}}{2}\frac{d_{j_{02}}}{2}\frac{d_{j_{03}}}{2} :: \frac{d_{j_{03}}}{2}\frac{d_{j_{23}}}{2}\frac{d_{j_{13}}}{2} :: \frac{d_{j_{12}}}{2} \times \frac{d_{j_{02}}}{2}\frac{d_{j_{23}}}{2} :: \frac{d_{j_{01}}}{2}\frac{d_{j_{12}}}{2}\frac{d_{j_{13}}}{2}$ are the barycentric coordinates of the second Lemoine point of the tetrahedron denoted L. This point is such that the distance from L to the face I of the tetrahedron is equal to $R_I\tan\theta$ where R_I is the radius of the circumscribed circle of the triangle I and θ is then defined by $\sum_J(\prod_{K\neq J}\frac{d_{j_{JK}}}{2}) = 12V\cot\theta$.

The geometrical significance of the stationary points still has to be understood. However, we can now give the explicit form of the leading order and of the next-to-leading order of the $\{6j\}$-symbol.

Leading order. We first focus on the leading order and on the x_+ contribution. This analysis has already been done in [13] and we just recall the main steps and give the notations:

$$f(x_+) = \sum_{I<J}\frac{d_{j_{IJ}}}{2}f_{d_{j_{IJ}}} \text{ where}$$
$$f_{d_{j_{0i}}} = \ln\Big[\frac{(x_+ - \tilde{v}_0)(x_+ - \tilde{v}_i)}{\prod_{j\neq i}(\tilde{p}_j - x_+)}\Big] \quad \text{for } i, j \in \{1, \cdots, 3\}$$
$$f_{d_{j_{ik}}} = \ln\Big[\frac{(x_+ - \tilde{v}_k)(x_+ - \tilde{v}_i)}{(\tilde{p}_k - x_+)(\tilde{p}_i - x_+)}\Big] \quad \text{for } i, k \in \{1, \cdots, 3\}. \tag{21}$$

The second derivative of f is given by:

$$-f''(x_+) = \sum_K\frac{1}{x_+ - \tilde{v}_K} + \sum_j\frac{1}{\tilde{p}_j - x_+} - \frac{1}{x_+}$$
$$= -i\sqrt{\Delta}\exp\Big(-\ln\Big(x_+\prod_j(\tilde{p}_j - x_+)\Big)\Big),$$

where we have used the Eq. (17) which gives $x_+\prod_j(\tilde{p}_j - x_+) = -\prod_K(x_+ - \tilde{v}_K)$. In the same way, we can simplify $F(x_+) = -\frac{1}{2}\ln(x_+\prod_j(\tilde{p}_j - x_+))$. The exponential piece of $f''(x_+)$ and $e^{F(x_+)}$ compensate and we get:

$$\frac{1}{\sqrt{-f''(x_+)}}e^{F(x_+)} = \frac{1}{\sqrt{-i\sqrt{\Delta}}}.$$

Collecting these different results yields the following contribution of the x_+ stationary point:

$$\sqrt{\frac{1}{-f''(x_+)2\pi\lambda^3}}\exp(F(x_+) + \lambda f(x_+))$$
$$\times \exp\Big(\sum_{I<J}\frac{\lambda d_{j_{IJ}}}{2}h_{d_{j_{IJ}}}\Big)$$
$$= \frac{1}{\sqrt{2\pi\lambda^3\sqrt{\Delta}}}\exp\Big[i\frac{\pi}{4} + \sum_{IJ}(\lambda d_{j_{IJ}}/2)(h_{d_{j_{IJ}}} + f_{d_{j_{IJ}}})\Big]. \tag{22}$$

The same analysis for the x_- contribution yields the same contribution as the previous one with an opposite phase:

$$\sqrt{\frac{1}{-f''(x_-)2\pi\lambda^3}}\exp(F(x_-)+\lambda f(x_-)) \times \exp\Big(\sum_{I<J}\frac{\lambda d_{j_{IJ}}}{2}h_{d_{j_{IJ}}}\Big) = \frac{1}{\sqrt{2\pi\lambda^3\sqrt{\Delta}}}\exp\Big[-i\frac{\pi}{4}+\sum_{IJ}(\lambda d_{j_{IJ}}/2)(h_{d_{j_{IJ}}}+\bar{f}_{d_{j_{IJ}}})\Big]. \tag{23}$$

We must now compute $f_{d_{j_{IJ}}}$ which is a complex logarithm. We recall that the principal value of the logarithm is defined by $\mathrm{Log}z := \ln|z| + i\,\mathrm{Arg}z$. Therefore, we have to compute $\Im(f_{d_{j_{IJ}}}) = \theta_{IJ}$. From (21), we can write that:

$$\theta_{0i} = \mathrm{Arg}(x_+ - \tilde{v}_0) + \mathrm{Arg}(x_+ - \tilde{v}_i) - \sum_{j\neq i}\mathrm{Arg}(\tilde{p}_j - x_+)$$
$$\theta_{ik} = \mathrm{Arg}(x_+ - \tilde{v}_k) + \mathrm{Arg}(x_+ - \tilde{v}_i) - \mathrm{Arg}(\tilde{p}_k - x_+) - \mathrm{Arg}(\tilde{p}_i - x_+). \tag{24}$$

The analysis done in [13] shows that θ_{IJ} can be identified as the (exterior) dihedral angles of the tetrahedron. Moreover,

$$\Re(f_{d_{j_{0i}}}) = \ln\left|\frac{(x_+ - \tilde{v}_0)(x_+ - \tilde{x}_i)}{\prod_{j\neq i}(\tilde{p}_j - x_+)}\right|$$
$$\Re(f_{d_{j_{ik}}}) = \ln\left|\frac{(x_+ - \tilde{v}_k)(x_+ - \tilde{v}_i)}{(\tilde{p}_i - x_+)(\tilde{p}_k - x_+)}\right|. \tag{25}$$

A tedious (but interesting) computation shows that:

$$\Re(f_{d_{j_{IJ}}}) + h_{d_{j_{IJ}}} = 0. \tag{26}$$

Then, summing the contributions of x_+ and x_- we get the leading order of the 6j-symbol:

$$\begin{Bmatrix} \lambda d_{j_{01}}/2-1/2 & \lambda d_{j_{02}}/2-1/2 & \lambda d_{j_{03}}/2-1/2 \\ \lambda d_{j_{23}}/2-1/2 & \lambda d_{j_{13}}/2-1/2 & \lambda d_{j_{12}}/2-1/2 \end{Bmatrix} \overset{\text{L.O.}}{\sim} \sqrt{\frac{1}{12\pi\lambda^3 V}}\cos\Big[\frac{\pi}{4}+S_R\Big], \tag{27}$$

where $S_R = \sum_{I<J}\frac{\lambda d_{j_{IJ}}}{2}\theta_{IJ}$ is the Regge action. This is the well-known limit given by Ponzano and Regge [1] and which has justified their state sum model for 3d Euclidean gravity where the $\{6j\}$-symbol is the spin-foam amplitude for a single tetrahedron.

Next-to-leading order. The next-to-leading order is then given by the term in $\frac{1}{\lambda^{5/2}}$ in Eq. (14). Using Eqs. (6)–(8), we rewrite the leading order in terms of $x_\pm$, $\tilde{v}_I$, $\tilde{p}_j$ and Δ:

$$\frac{1}{\sqrt{48\pi\lambda^5 V}}\{A(x_+,\tilde{v}_I,\tilde{p}_j,\Delta)e^{i(S_R+(\pi/4))} + A(x_-,\tilde{v}_I,\tilde{p}_j,\Delta)e^{-i(S_R+(\pi/4))}\}, \tag{28}$$

where

$$\begin{aligned} A(x_+,\tilde{v}_I,\tilde{p}_j,\Delta) = &-\frac{H(d_{j_{IJ}})}{24} + \frac{1}{24i\sqrt{\Delta}\Delta\prod_I(x_+-\tilde{v}_I)}\Big[-\Delta^2 - 3i\Big(\sum_K\prod_{L\neq K}(x_+-\tilde{v}_L)\Big)\Delta\sqrt{\Delta} + \Big(9\sum_K\prod_{L\neq K}(x_+-\tilde{v}_L)^2 \\ &+ 6\prod_I(x_+-\tilde{v}_I)\sum_{K<L}(x_+-\tilde{v}_K)(x_+-\tilde{v}_L)\Big)\Delta - 6i\Big(-\prod_j(\tilde{p}_j-x_+)^3 - \sum_K\prod_{L\neq K}(x_+-\tilde{v}_L)^3 \\ &+ \sum_j x_+^3\prod_{l\neq j}(\tilde{p}_l-x_+)^3 - \Big(\sum_K\prod_{L\neq K}(x_+-\tilde{v}_L)\Big)\Big(-\prod_j(\tilde{p}_j-x_+)^2 + \sum_K\prod_{L\neq K}(x_+-\tilde{v}_L)^2 \\ &- \sum_j x_+^2\prod_{l\neq j}(\tilde{p}_l-x_+)^2\Big)\Big)\sqrt{\Delta} - 5\Big(-\prod_j(\tilde{p}_j-x_+)^2 + \sum_K\prod_{L\neq K}(x_+-\tilde{v}_L)^2 - \sum_j x_+^2\prod_{l\neq j}(\tilde{p}_l-x_+)^2\Big)^2\Big]. \end{aligned} \tag{29}$$

Since $x_\pm$ are conjugated to each other, we obviously have $A(x_+) = \overline{A(x_-)}$. Moreover, numerical computations shows that $\Re(A(x_\pm,\tilde{v}_I,\tilde{p}_j,\Delta)) = 0$, and in particular $A(x_+) = -A(x_-)$. This is *a priori* a nontrivial result to obtain from the previous formulas. Nevertheless, we tested it numerically for various choices of spin and it always turned out true. Thus we believe that there should be a way to show it analytically. We can then give an explicit formula of the NLO of the $\{6j\}$-symbol:

$$\{6j\}\underset{\lambda\to\infty}{\sim}\{6j\}_{\mathrm{NLO}} = \frac{1}{\sqrt{12\pi\lambda^3 V}}\cos\Big[\frac{\pi}{4}+S_R\Big] - \frac{1}{\sqrt{12\pi\lambda^5 V}}\Im(A(x_+,\tilde{v}_I,\tilde{p}_j,\Delta)) \times \sin(S_R+\pi/4). \tag{30}$$

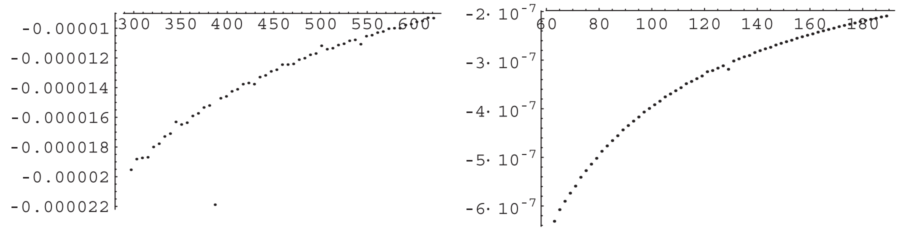

FIG. 2. Plots of the difference $\delta_{\rm NLO}$ between the $\{6j\}$-symbol and its analytical approximation up to NLO. On the left, we look at the $\{6j\}$-symbol for $d_1 = 5\lambda$, $d_2 = 7\lambda$, $d_3 = 9\lambda$, $d_4 = 7\lambda$, $d_5 = 9\lambda$, $d_6 = 9\lambda$ with the x-coordinate standing for 3λ. On the right, we have plotted the case $d_1 = 15\lambda$, $d_2 = 17\lambda$, $d_3 = 19\lambda$, $d_4 = 19\lambda$, $d_5 = 21\lambda$, $d_6 = 17\lambda$ with λ running from 60 to 200.

This result is confirmed by numerical simulations. The plots Fig. 2 represent numerical simulations of the $\{6j\}$-symbol minus its approximation given above (30). Moreover, to enhance the comparison, we have multiplied by $\lambda^{5/2}$ to see how the coefficient of the next-to-leading order is approached and we have divided by $\cos(S_R + \pi/4)$ (oscillations of the next-to-next-to-leading order) to suppress the oscillations; that is we have plotted:

$$\delta_{\rm NLO} \equiv \lambda^{5/2} \frac{\{6j\} - \{6j\}_{\rm NLO}}{\cos(S_R + \pi/4)}. \tag{31}$$

As expected, the numerical simulations show that this rescaled difference $\delta_{\rm NLO}$ goes to 0 as $1/\lambda$ when λ goes to ∞. Moreover, the data for $\delta_{\rm NLO}$ without any oscillation suggest that we correctly divided by $\cos(S_R + \pi/4)$ and thus the NNLO of the $\{6j\}$-symbol should oscillate in $\cos(S_R + \pi/4)$. Therefore, this strongly suggest that the asymptotic expansion of the $\{6j\}$-symbol in term of the inverse length scale λ^{-1} is given by a series of alternating sines and cosines. We strongly underline that this is true because we have rescaled the edge lengths $d_{j_{IJ}}$. If we had instead rescaled the spins j_{IJ} as usually done, we would have found an oscillatory behavior controlled by a mixing of cos and sin at each order (as shown explicitly for the case of the isosceles tetrahedron in [8]). This suggests that the $d_{j_{IJ}}$ are indeed the right parameter to consider when studying the semiclassical behavior of the $\{6j\}$-symbol.

The only thing left to do in the present analysis is to provide the NLO coefficient $\Im(A(x_+))$ with a geometrical interpretation and to show rigorously that $\Re(A(x_+))$ vanishes.

Finally, we rewrite the approximation up to NLO of the $\{6j\}$-symbol in a slightly different manner:

$$\{6j\} \sim \frac{1}{\sqrt{12\pi\lambda^3 V}} \cos\left[\frac{\pi}{4} + S_R + \frac{1}{\lambda}\Im(A(x_+)) + O\left(\frac{1}{\lambda^2}\right)\right]. \tag{32}$$

This shows that the next-to-leading corrections to the $\{6j\}$-symbol can be directly considered as corrections to the Regge action for (3d) gravity:

$$S_R^{\rm effective} \equiv S_R + \frac{1}{\lambda}\Im(A(x_+)).$$

This quantum gravity correction $\frac{1}{\lambda}\Im(A(x_+))$ should have an interpretation in the continuum limit (or large scale limit). It would be interesting to understand to which kind of terms it corresponds in an effective action for 3d (quantum) gravity. We point out that an expansion in $1/\lambda$ with alternating cos and sin could be similarly reabsorbed as corrections to the Regge action. This would define in the spin-foam framework the quantum gravity corrections to classical 3d gravity due to the fundamental discreteness of the theory. Such correction would enter the gravitational correlations (of the "graviton propagator" type) at second order as suggested in [7].

IV. SOME PARTICULAR CASES

A. The equilateral tetrahedron

For the equilateral tetrahedron, all the edges have the same length: that is $\forall\ I, J$, $d_{j_{IJ}} = d$. The tetrahedron with edge length $d/2$ has a volume $V = (d/2)^3\sqrt{2}/12$ and has all equal dihedral angles $\theta = \arccos(-1/3)$. In this case, the expressions greatly simplify. For instance, the stationary points are $x_\pm = \frac{11 \pm i\sqrt{\frac{1}{2}}}{6} d$. Equations (27) and (28) reduce to:

$$\{6j\}^{\rm NLO}_{\rm equi} = \frac{2^{5/4}}{\sqrt{\pi d^3}} \cos\left(S_R + \frac{\pi}{4}\right) - \frac{31}{72\sqrt{\sqrt{2}\pi d^5}} \sin\left(S_R + \frac{\pi}{4}\right), \tag{33}$$

where the Regge action is $S_R = 3d\theta$. The result was already obtained in [8]. We confirm it by numerical simulations. The plot in Fig. 3 gives the equilateral $\{6j\}$-symbol minus its NLO approximation (33). Like for the previous plots, we have multiplied by $\lambda^{5/2}$ to see how the coefficient of the next-to-leading order is approached and we have divided by $\cos(S_R + \pi/4)$ (oscillations of the next to next-to-leading order) to suppress the oscillations. This gives a curve converging to 0 as $1/\lambda$ asymptotically.

B. The isosceles tetrahedron

We now consider an isosceles tetrahedron that is a tetrahedron which has two opposite edges of length equal

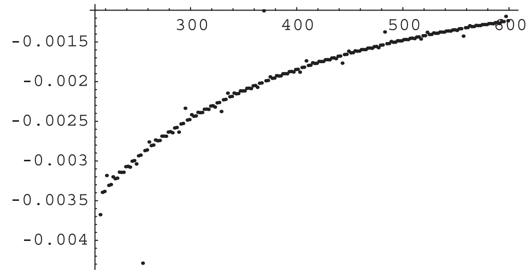

FIG. 3. Difference between the equilateral $\{6j\}$-symbol and the analytical result (33). The x-axis stands for d and d goes from 200 to 600.

to $\frac{d_1}{2}$ and $\frac{d_2}{2}$ and the remaining four edges of the same length equal to $\frac{d}{2}$ (see Fig. 4). The volume of the tetrahedron is:

$$V^2 = \frac{1}{2^8(3!)^2} d_1^2 d_2^2 (4d^2 - d_1^2 - d_2^2),$$

and the dihedral angles are:

$$\theta = \arccos\left(\frac{-d_1 d_2}{\sqrt{4d^2 - d_1^2}\sqrt{4d^2 - d_2^2}}\right),$$

$$\theta_{1,2} = 2\arccos\left(\frac{d_{2,1}}{\sqrt{4d^2 - d_{1,2}^2}}\right).$$

Once again, Eqs. (27) and (28) simplify and we get :

$$\{6j\}^{(\mathrm{iso})}_{\mathrm{NLO}} = \frac{1}{\sqrt{12\pi V \lambda^3}} \cos\left(S_R + \frac{\pi}{4}\right) - \frac{F(d, d_1, d_2)}{24 V \lambda \sqrt{12\pi V \lambda^3}} \sin\left(S_R + \frac{\pi}{4}\right), \quad (34)$$

where $F(d, d_1, d_2) = 768 d^6 (d^2 - d_1^2 - d_2^2) + 736 d^4 d_1^2 d_2^2 + 240 d^4 (d_1^4 + d_2^4) - 176 d^2 d_1^2 d_2^2 (d_1^2 + d_2^2) \times \frac{-24d^2(d_1^6 + d_2^6) + 10 d_1^2 d_2^2 (d_1^4 + d_2^4) + 25 d_1^4 d_2^4}{96(4d^2 - d_1^2)(4d^2 - d_2^2)(4d^2 - d_1^2 - d_2^2)},$ and the Regge action $S_R = 2d\theta + \frac{d_1}{2}\theta_1 + \frac{d_2}{2}\theta_2$. Let us point out that the volume increases as λ^3 while F goes as λ^2, so that the NLO scales properly as $\lambda^{-5/2}$.

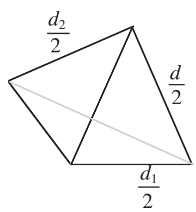

FIG. 4. The isosceles tetrahedron.

This reproduces the result previously obtained in [8]. We can easily check that this reduces to the previous equilateral case when $d_1 = d_2 = d$ and we further confirm it by numerical simulations. The plots in Fig. 5 represents numerical simulations of an isosceles $\{6j\}$-symbol minus the analytical formula (34). Like for the previous plot, we have multiplied the data by $\lambda^{5/2}$ to see how the coefficient of the NNLO order is approached and we have divided by $\cos(S_R + \pi/4)$ (NNLO oscillations) to suppress the oscillations. This rescaled difference goes asymptically as λ^{-1} as in the previous plots (the apparent change of convergence rate is simply due to the *a priori* different factors in front of λ^{-1} and the different scales used on the x and y axis). Finally, the geometrical interpretation of the term $F(d, d_1, d_2)$ remains to be understood. If we cannot provide it with a geometrical meaning, there is little hope to interpret the NLO coefficient $\Im(A(x_+))$ in the generic case. Nevertheless, we give a more compact expression for the denominator of F:

$$96(4d^2 - d_1^2)(4d^2 - d_2^2)(4d^2 - d_1^2 - d_2^2) = 96^3 \frac{V^2}{\cos^2\theta}. \quad (35)$$

We still need to express the numerator of F in term of geometrical objects. For instance, we could express it in term of d^2, $(4d^2 - d_1^2)(4d^2 - d_2^2)$ and $(4d^2 - d_1^2 - d_2^2)$, which would provide a formula in term of the volume and the dihedral angles. Nevertheless, we have not been able to find such a useful rewriting of this NLO coefficient.

V. CONCLUSION

We investigated the asymptotical behavior of the $\{6j\}$-symbol. Starting from its expression as a (finite) sum over (half-)integers of algebraic combinations of factorials, we followed the footsteps of [13] and showed that

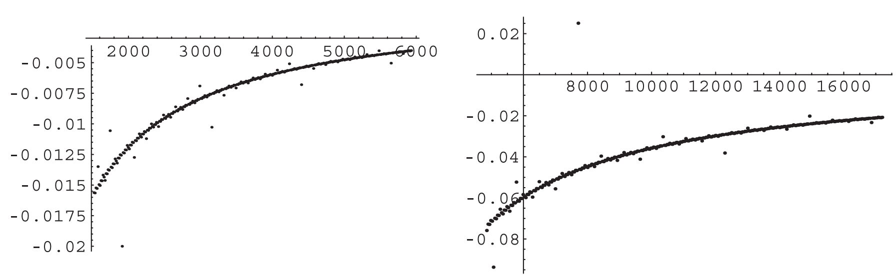

FIG. 5. Differences between isosceles $\{6j\}$-symbols and their analytical approximation (34). The x-axis stands for d with λ goes from 200 to 600. On the left hand side, we consider isosceles tetrahedra with $d_1 = 3\lambda$, $d_2 = 3\lambda$, $d = 7\lambda$. On the right hand side, we have plotted the case $d_1 = 9\lambda$, $d_2 = 3\lambda$, $d = 21\lambda$.

one can derive systematically the corrections to the leading order formula at any order. The method relies on three steps. First, we use the Stirling formula (with the appropriate corrections) to approximate the factorials. Second, we consider the sum as a Riemann sum and approximate it by an integral (over the real line). Finally, we perform a saddle point approximation to compute the behavior of the $\{6j\}$-symbol for (homogeneously) large spins.

Using this framework, we showed that we recover an oscillating leading order (LO) with frequency given by the Regge action as is well-known and was already proved in [13]. Then we computed analytically the next-to-leading (NLO) corrections. The formula that we obtain is explicit, although not compact, and we could not interpret it geometrically in a clear way. Nevertheless, we performed two simple checks. First, we checked that our complicated formula reduces to the known expression for the NLO for isosceles tetrahedra [8]. Second, we checked it numerically in various cases and found a perfect fit. These numerical simulations also confirmed that the NLO is a $\frac{\pi}{2}$-phase shift with respect to the LO (the NLO is given by a sin instead of a cos) and that the NNLO is back in phase with the LO (again a cos), which confirms our expectation of an alternating asymptotical series in $\cos + \frac{1}{j}\sin + \frac{1}{j^2}\cos + \frac{1}{j^3}\times \sin + \ldots$.

We point out that we computed in details the corrections due to the Stirling formula and to the saddle point approximation. However we did not study the Riemann sum approximation. It does not contribute to the LO and NLO. It will only enter at the level of the NNLO.

This work is mainly technical and can be applied to the computation of gravitational correlations for 3d quantum gravity following [6–9]. It will enter the quantum corrections to the propagator/correlations at second order, as was shown in [7]. Indeed, the first order corrections are derived from the path integral of the Regge action, while the deviations from the Regge action as computed here enter at second order (as two-loop corrections). From this perspective, this NLO of the $\{6j\}$-symbol describes the leading order deviation of quantum gravity with respect to the classical gravity, or in other words the first quantum gravity corrections to the classical 3d Regge action.

Beyond the technicality of the paper, our purpose was to show that computing such corrections is indeed possible (although it does lead to complicated expressions) and that similar methods could be used for 4d spin-foam gravity. Although these methods allow straightforward (but lengthy) analytical calculations, which might be handled by a computer program, their drawback is the loss of the geometrical meaning of the expressions obtained. An alternative way to proceed is to use the exact recursion relations satisfied by the $\{6j\}$-symbol (see [10]) and other spin-foam amplitudes (see [16]) to probe the asymptotic behavior and the induced corrections of the correlations. This is left to future investigation [15].

ACKNOWLEDGMENTS

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APPENDIX A: THE $\{6j\}$-SYMBOL—RECOUPLING THEORY

The $\{6j\}$-symbol is a real number and it is obtained by combining four normalized Clebsh-Gordan coefficients along the six edges of a tetrahedron, with edge lengths given by $j_{IJ} + \frac{1}{2} = \frac{d_{JIJ}}{2}$ $(0 \le I < J \le 3)$. We usually express the $6j$-symbol in term of the Wigner $3j$-symbols :

$$\begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{Bmatrix} = \sum_{\alpha} (-1)^{j_{01}+j_{03}+j_{01}-\alpha_{01}-\alpha_{03}-\alpha_{01}} \times \begin{pmatrix} j_{01} & j_{12} & j_{13} \\ \alpha_{01} & \alpha_{12} & -\alpha_{13} \end{pmatrix} \times \begin{pmatrix} j_{13} & j_{23} & j_{03} \\ \alpha_{13} & \alpha_{23} & \alpha_{03} \end{pmatrix} \times \begin{pmatrix} j_{03} & j_{02} & j_{01} \\ \alpha_{03} & \alpha_{02} & -\alpha_{01} \end{pmatrix} \times \begin{pmatrix} j_{02} & j_{23} & j_{12} \\ \alpha_{02} & \alpha_{23} & \alpha_{12} \end{pmatrix}. \tag{A1}$$

The Wigner $3j$ symbols are very simply related to the Clebsh-Gordan coefficients $\langle j_{01} j_{12} \alpha_{01} \alpha_{12} | j_{13} \alpha_{13} \rangle$ by:

$$\langle j_{01} j_{12} \alpha_{01} \alpha_{12} | j_{13} \alpha_{13} \rangle = (-1)^{j_{01}-j_{12}+\alpha_{13}} (2j_{13} + 1/2)^{1/2} \times \begin{pmatrix} j_{01} & j_{12} & j_{13} \\ \alpha_{01} & \alpha_{12} & -\alpha_{13} \end{pmatrix}.$$

And Racah gave a general formulas for the Clebsh-Gordan coefficient:

$$\langle j_{01} j_{12} \alpha_{01} \alpha_{12} | j_{13} \alpha_{13} \rangle = \delta(\alpha_{01} + \alpha_{12}, \alpha_{13}) \Delta(j_{01} j_{12} j_{13}) \times \sqrt{(2j_{13}+1)(j_{01}+\alpha_{01})!(j_{01}-\alpha_{01})!(j_{12}+\alpha_{12})!(j_{12}-\alpha_{12})!(j_{13}+\alpha_{13})!(j_{13}-\alpha_{13})!} \times \sum_{\mu} \frac{(-1)^{\mu}}{(j_{01}-\alpha_{01}-\mu)!(j_{13}-j_{12}+\alpha_{01}+\alpha)!(j_{12}+\alpha_{12}-\mu)!(j_{13}-j_{01}-\alpha_{12}+\alpha)!\mu!(j_{01}+j_{12}-j_{13}-\mu)!},$$

where $\Delta(j_{01}, j_{12}, j_{13}) = \frac{(j_{01}+j_{12}-j_{13})!(j_{01}-j_{12}+j_{13})!(-j_{01}+j_{12}+j_{13})!}{(j_{01}+j_{12}+j_{13}+1)!}$ From these, Racah gave a tensorial formulas for the 6j-symbol, the Racah's single sum formulas:

$$\begin{Bmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{Bmatrix} = \sqrt{\Delta(j_{01}, j_{02}, j_{03})\Delta(j_{23}, j_{02}, j_{12})\Delta(j_{23}, j_{13}, j_{03})\Delta(j_{01}, j_{13}, j_{12})} \sum_{\max v_I}^{\min p_j} (-1)^t \frac{(t+1)!}{\prod_i (t - v_I)! \prod_j (p_j - t)!} \quad \text{(A2)}$$

with $v_K = \sum_{I \neq K} j_{IK}\ \forall\ K \in \{0, \cdots, 3\}$ and $p_k = \sum_{i \neq 0,k} (j_{0i} + j_{ki})\ \forall\ k \in \{1 \cdots 3\}$.

APPENDIX B: FACTORIALS

The factorial $n!$ is defined for a positive integer n as:

$$n! \equiv n(n-1)\cdots 2\cdot 1 = \Gamma(n+1),$$

where $\Gamma(n)$ is the gamma function for integers n. This definition is generalized to noninteger values. Using the identities for the Γ function, we write explicitly the values for half-integers:

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}, \quad \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}, \quad \left(n - \frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2^n}(2n-1)!! = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}, \quad \left(n + \frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2^{n+1}}(2n+1)!! = \frac{\sqrt{\pi}(2n+1)!}{2^{2n+1} n!},$$

where $n!!$ is the double factorial :

$$n!! \equiv \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n > 0 \text{ odd}, \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n > 0 \text{ even}, \\ 1 & \text{if } n = -1 \text{ or } 0. \end{cases}$$

Using the asymptotic expansion of a large factorial $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n(1 + \frac{1}{12n} = \frac{1}{288n^3} - \frac{139}{51840n^3} - \frac{571}{2488320n^4}\cdots)$, we can get an asymptotic expansion for:

$$\begin{aligned} (n+1/2)! &\sim \sqrt{2\pi} e^{(n+1)\ln(n)-n}\left(1 + \frac{1}{2n}\right)\left(1 + \frac{11}{12(2n)} + \frac{1}{288(2n)^2} - \frac{139}{51840(2n)^3} - \frac{571}{2488320(2n)^4} + \cdots\right) \\ &\quad \times \left(1 - \frac{1}{12n} - \frac{1}{288n^2} + \frac{139}{51840n^3} + \frac{571}{2488320n^4} - \cdots\right), \\ \left(n - \frac{1}{2}\right)! &\sim \sqrt{2\pi} e^{n\ln(n)-n}\left(1 + \frac{11}{12(2n)} + \frac{1}{288(2n)^2} - \frac{139}{51840(2n)^3} - \frac{571}{2488320(2n)^4} + \cdots\right) \\ &\quad \times \left(1 - \frac{1}{12n} - \frac{1}{288n^2} + \frac{139}{51840n^3} + \frac{571}{2488320n^4} - \cdots\right), \end{aligned} \quad \text{(B1)}$$

or more simply, at the next-to-leading order:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n}\right), \quad \left(n + \frac{1}{2}\right)! \sim \sqrt{2\pi} e^{(n+1)\ln(n)-n}\left(1 + \frac{11}{24n}\right), \quad \left(n - \frac{1}{2}\right)! \sim \sqrt{2\pi} e^{n\ln(n)-n}\left(1 - \frac{1}{24n}\right). \quad \text{(B2)}$$

APPENDIX C: FIRST APPROXIMATION: FACTORIALS → NEXT-TO-LEADING ORDER OF THE STIRLING FORMULA

In this section, all computations are done at the next-to-leading order. We replace the factorials in Eq. (2) by their respective asymptotic expansion.

(i) Then, a typical triangle coefficient:

$$\Delta(\lambda d_{j_{01}}, \lambda d_{j_{02}}, \lambda d_{j_{03}}) = \frac{(\frac{\lambda}{2}(d_{j_{01}} + d_{j_{02}} - d_{j_{03}}) - \frac{1}{2})!(\frac{\lambda}{2}(d_{j_{01}} - d_{j_{02}} + d_{j_{03}}) - \frac{1}{2})!(\frac{\lambda}{2}(-d_{j_{01}} + d_{j_{02}} + d_{j_{03}}) - \frac{1}{2})!}{(\frac{\lambda}{2}(d_{j_{01}} + d_{j_{02}} + d_{j_{03}}) - \frac{1}{2})!}$$

will be

$$\Delta(\lambda d_{j_{01}}, \lambda d_{j_{02}}, \lambda d_{j_{03}}) = 2\pi[e^{-(\lambda/2)(d_{j01}+d_{j02}+d_{j03})\ln[(\lambda/2)(d_{j01}+d_{j02}+d_{j03})]+(\lambda/2)(d_{j01}+d_{j02}+d_{j03})}$$
$$\times\left(1+\frac{1}{12\lambda(d_{j_{01}}+d_{j_{02}}+d_{j_{03}})}\right)e^{(\lambda/2)(d_{j01}+d_{j02}-d_{j03})\ln[(\lambda/2)(d_{j01}+d_{j02}-d_{j03})]-(\lambda/2)(d_{j01}+d_{j02}-d_{j03})}$$
$$\times\left(1-\frac{1}{12\lambda(d_{j_{01}}+d_{j_{02}}-d_{j_{03}})}\right)e^{(\lambda/2)(d_{j01}-d_{j02}+d_{j03})\ln[(\lambda/2)(d_{j01}-d_{j02}+d_{j03})]-(\lambda/2)(d_{j01}-d_{j02}+d_{j03})}$$
$$\times\left(1-\frac{1}{12\lambda(d_{j_{01}}-d_{j_{02}}+d_{j_{03}})}\right)e^{(\lambda/2)(-d_{j01}+d_{j02}+d_{j03})\ln[(\lambda/2)(-d_{j01}+d_{j02}+d_{j03})]-(\lambda/2)(-d_{j01}+d_{j02}+d_{j03})}$$
$$\times\left(1-\frac{1}{12\lambda(-d_{j_{01}}+d_{j_{02}}+d_{j_{03}})}\right),$$

which simplifies

$$\Delta(\lambda d_{j_{01}}, \lambda d_{j_{02}}, \lambda d_{j_{03}}) = 2\pi e^{(\lambda/2)[(-d_{j01}+d_{j02}+d_{j03})\ln(-d_{j01}+d_{j02}+d_{j03})+(d_{j01}-d_{j02}+d_{j03})\ln(d_{j01}-d_{j02}+d_{j03})]}$$
$$\times e^{-(\lambda/2)[(d_{j01}+d_{j02}-d_{j03})\ln(d_{j01}+d_{j02}-d_{j03})+(d_{j01}+d_{j02}+d_{j03})\ln(d_{j01}+d_{j02}+d_{j03})]}$$
$$\times\left[1-\frac{1}{12\lambda}\left(\frac{1}{-d_{j_{01}}+d_{j_{02}}+d_{j_{03}}}+\frac{1}{d_{j_{01}}-d-j02+d_{j_{03}}}+\frac{1}{d_{j_{01}}+d_{j_{02}}-d-j03}\right.\right.$$
$$\left.\left.-\frac{1}{d_{j_{01}}+d_{j_{02}}+d_{j_{03}}}\right)\right]. \tag{C1}$$

The factor $\sqrt{\Delta(\lambda d_{j_{01}}, \lambda d_{j_{02}}, \lambda d_{j_{03}})\Delta(\lambda d_{j_{23}}, \lambda d_{j_{02}}, \lambda d_{j_{12}})\Delta(\lambda d_{j_{23}}, \lambda d_{j_{13}}, \lambda d_{j_{03}})\Delta(\lambda d_{j_{01}}, \lambda d_{j_{13}}, \lambda d_{j_{12}})}$ in Eq. (2) can then easily be put into the form:

$$(2\pi)^2 e^{(\lambda/2)h(d_{j_{IJ}})}\left(1-\frac{1}{24\lambda}H(d_{j_{IJ}})\right) \tag{C2}$$

where

$$h(d_{j_{IJ}}) = \sum_{I<J} d_{j_{IJ}} h_{d_{j_{IJ}}} \quad \text{with}$$
$$h_{d_{j_{IJ}}} = \frac{1}{2}\ln\left(\frac{(d_{j_{IJ}}-d_{j_{IK}}+d_{j_{IL}})(d_{j_{IJ}}+d_{j_{IK}}-d_{j_{IL}})(d_{j_{IJ}}-d_{j_{JK}}+d_{j_{JL}})(d_{j_{IJ}}+d_{j_{JK}}-d-j_{JL})}{(d-j_{IJ}+d_{j_{IK}}+d_{j_{IL}})(-d_{j_{IJ}}+d_{j_{IK}}+d_{j_{IL}})(d_{j_{IJ}}+d_{j_{JK}}+d_{j_{JL}})(-d_{j_{IJ}}+d_{j_{JK}}+d_{j_{JL}})}\right)$$
$$K\neq L \quad \text{and} \quad K,L\neq I,J \qquad H(d_{j_{IJ}}) = 2\sum_{j,K}\frac{1}{\tilde{p}_j-\tilde{v}_K} - 2\sum_K\frac{1}{\tilde{v}_K} \quad \text{where } K\in\{0,\cdots,3\} \quad \text{and} \quad j\in\{1,\cdots,3\} \tag{C3}$$

and we recall that $\tilde{v}_K = \sum_{I\neq K}\frac{d_{j_{IK}}}{2}\ \forall\ K\in\{0,\cdots,3\}$, $\tilde{p}_k = \sum_{i\neq 0,k}\frac{(d_{j_{0i}}+d_{j_{ki}})}{2}\ \forall\ k\in\{1\cdots 3\}$.

(ii) We now replace the factorials in the sum of (2) by their approximations and we change of variables: $t=\lambda x$:

$$\Sigma(\lambda d_{j_{IJ}}) = \sum_{x=\max\tilde{v}_I}^{\min\tilde{p}_j}(-1)^{\lambda x}\frac{(\lambda x+1)(\lambda x)!\prod_j(\lambda(\tilde{p}_j-x))\prod_j(\lambda(\tilde{p}_j-x)-1)}{\prod_I(\lambda(x-\tilde{v}_I)+3/2)(\lambda(x-\tilde{v}_I)+1/2)!\prod_j(\lambda(\tilde{p}_j-x))!} = \frac{1}{(2\pi)^3}\sum_{x=\max v_I}^{\min p_j} e^{G_1(x)}G_2(x), \tag{C4}$$

where

$$G_1(x) = i\pi\lambda x + 3\ln\lambda + \ln x + 2\sum_j\ln(\tilde{p}_j-x) - \sum_I\ln(x-\tilde{v}_I) + (\lambda x+1/2)(\ln x+\ln\lambda) - \lambda x + \sum_I\lambda(x-\tilde{v}_I)$$
$$-\sum_I(\lambda(x-\tilde{v}_I)+1)(\ln\lambda+\ln(x-\tilde{v}_I)) - \sum_j(\lambda(\tilde{p}_j-x)+1/2)(\ln\lambda+\ln(\tilde{p}_j-x)) + \sum_j\lambda(\tilde{p}_j-x), \tag{C5}$$

which can be simplified using the fact that $\sum_I\tilde{v}_I = \sum_j\tilde{p}_j$:

$$G_1(x) = -2\ln\lambda + \frac{1}{2}\ln\frac{x^3\prod_j(\tilde{p}_j - x)^3}{\prod_I(x-\tilde{v}_I)^4} + \lambda\Big[i\pi x + x\ln x - \sum_I (x - v_I)\ln(x - v_I) - \sum_j (p_j - x)\ln(p_j - x)\Big], \tag{C6}$$

and

$$G_2(x) = \frac{1+\frac{1}{12\lambda x}}{(1+\frac{3}{2\lambda(x-\tilde{v}_I)})\prod_I(1+\frac{11}{24\lambda(x-\tilde{v}_I)})\prod_j(1+\frac{1}{12\lambda(p_j-x)})} = 1 - \frac{1}{\lambda}\Big(-\frac{13}{12x} + \sum_I\frac{47}{24(x-v_I)} + \sum_j\frac{13}{12(p_j-x)} + O\Big(\frac{1}{\lambda}\Big)\Big). \tag{C7}$$

Moreover,

$$e^{G_1(x)} = \frac{1}{\lambda^2}e^{F(x)+\lambda f(x)}, \tag{C8}$$

where

$$f(x) = i\pi x + x\ln(x) - \sum_I (x - v_K)\ln(x - v_I) - \sum_j (p_j - x)\ln(p_j - x)$$

$$F(x) = \frac{1}{2}\ln\Big(\frac{x^3\prod_j(p_j - x)^3}{\prod_I(x-v_I)^4}\Big). \tag{C9}$$

Then the sum can be approximated by:

$$\Sigma(\lambda d_{j_{IJ}}) = \frac{1}{(2\pi)^3\lambda^2}\sum_{x=\max v_I}^{\min p_j} e^{\lambda f(x)+F(x)} \times\Big(1 - \frac{1}{12\lambda}G(x) + O\Big(\frac{1}{\lambda}\Big)\Big)e^{\lambda f(x)}, \tag{C10}$$

where

$$G(x) = -\frac{13}{x} + \sum_K\frac{47}{24(x-v_K)} + \sum_j\frac{13}{p_j - x}. \tag{C11}$$

APPENDIX D: THIRD APPROXIMATION: THE STATIONARY PHASE METHOD

We are interested in the $1/\lambda$ expansion of the integral:

$$I = \int_{\max\tilde{v}_I/2}^{\min\tilde{p}_j/2} dx e^{F(x)}\Big(1 - \frac{1}{12\lambda}G(x) + O\Big(\frac{1}{\lambda}\Big)\Big)e^{\lambda f(x)}.$$

We do not give here the proof of the whole expansion (Eq. (11)) because of the heavy formalism but we directly prove the next-to-leading order formula (Eq. (13)); the procedure is the same but the computations are easier. The asymptotic expansion of such an integral is given by contributions around the stationary points of the phase denoted x_0. We expand the phase $f(x)$ around the stationary points x_0 at fourth order and the function $g(x) = e^{F(x)}(1 - \frac{1}{12\lambda}G(x))$ at second order and we extend the integration to infinity.

$$I \sim \sum_{x_0}\int_{-\infty}^{+\infty} d(\delta x)(g(x_0) + g'(x_0)\delta x + \frac{1}{2}g''(x_0)(\delta x)^2)e^{\lambda(f(x_0)+(1/2)f''(x_0)(\delta x)^2)} \times\Big(1 + \lambda\Big(\frac{1}{3!}f^{(3)}(x_0)(\delta x)^3 + \frac{1}{4!}f^{(4)}(x_0)(\delta x)^4\Big) + \frac{\lambda^2}{2}\Big(\frac{1}{3!}f^{(3)}(x_0)(\delta x)^3\Big)^2 + O(\lambda^2)\Big), \tag{D1}$$

where in our case, $g(x) = e^{F(x)}(1 - \frac{1}{12\lambda}G(x))$ and then the integration are "generalized" Gaussians:

$$I \sim \sum_{x_0} e^{F(x_0)+\lambda f(x_0)}\Big[\Big(1 - \frac{1}{12\lambda}G(x_0)\Big)\int_{-\infty}^{+\infty} d(\delta x)e^{-\lambda((-f''(x_0))/2)(\delta x)^2} + \frac{1}{2}((F'(x_0))^2 + F''(x_0)) \times\int_{-\infty}^{+\infty} d(\delta x)(\delta x)^2 e^{-\lambda((-f''(x_0))/2)(\delta x)^2} + \lambda\Big(\frac{f^{(4)}(x_0)}{4!} + \frac{f^{(3)}(x_0)}{3!}F'(x_0)\Big)\int_{-\infty}^{+\infty} d(\delta x)(\delta x)^4 e^{-\lambda((-f''(x_0))/2)(\delta x)^2} + \frac{\lambda^2}{2}\Big(\frac{f^{(3)}(x_0)}{3!}\Big)^2\int_{-\infty}^{+\infty} d(\delta x)(\delta x)^6 e^{-\lambda((-f''(x_0))/2)(\delta x)^2} + O\Big(\frac{1}{\lambda^{3/2}}\Big)\Big] \tag{D2}$$

which can easily be computed:

$$I \sim \sum_{x_0}\sqrt{\frac{2\pi}{-f''(x_0)\lambda}}e^{F(x_0)+\lambda f(x_0)}\Big[1 + \frac{1}{\lambda}\Big(-\frac{G(x_0)}{12} - \frac{F''(x_0)+(F'(x_0))^2}{2f''(x_0)} + \frac{f^{(4)}(x_0)+4f^{(3)}(x_0)F'(x_0)}{8(f''(x_0))^2} - \frac{5(f^{(3)}(x_0))^2}{24(f''(x_0))^3}\Big) + O\Big(\frac{1}{\lambda}\Big)\Big]. \tag{D3}$$

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