Note on stability of de Sitter solutions of $f(R)$ theories

Israel Quiros,[1,2,*] Yoelsy Leyva,[1,†] and Yunelsy Napoles[3,‡]

[1]*Departamento de Física, Universidad Central de Las Villas, 54830 Santa Clara, Cuba*
[2]*Instituto de Física de la Universidad de Guanajuato, A.P. 150, 37150, León, Guanajuato, México*
[3]*Departamento de Matemática, Universidad Central de Las Villas, 54830 Santa Clara, Cuba*

(Received 9 June 2009; revised manuscript received 17 June 2009; published 20 July 2009)

The consequences of the constraints which stability of de Sitter solutions of $f(R)$ theories imposes on the Lagrangian's parameters are investigated within the metric formalism. It is shown, in particular, that several common $f(R)$ Lagrangians do not actually admit matching of local solutions with background de Sitter spaces. Otherwise, asymptotic matching of local solutions of the corresponding models with maximally symmetric spaces of constant curvature is either unstable or anti–de Sitter space is the only stable asymptotic solution. Additional arguments are given in favor of a previous claim that a class of $f(R)$ models comprising both positive and negative powers of R (two different mass scales) could be a nice scenario in which to address, in a united picture, both early-time inflation and late-time accelerated expansion of the Universe. The approach undertaken here is used, also, to check ghost freedom of a Dirac-Born-Infeld modification of general relativity previously studied in the literature.

DOI: 10.1103/PhysRevD.80.024022 PACS numbers: 04.50.-h, 04.20.Cv, 04.20.Fy, 04.90.+e

I. INTRODUCTION

Attempts to modify the Einstein-Hilbert (EH) action of general relativity (GR)

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}(R - 2\Lambda),$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci curvature scalar, and Λ-the cosmological constant ($\kappa^2 = m_{Pl}^{-2} = 8\pi G$), have been motivated by a number of reasons. In particular, renormalization at one loop demands that the Einstein-Hilbert action be supplemented by higher-order curvature terms [1].[1] Besides, when quantum corrections or string theory are taken into account, the effective low-energy action for pure gravity admits higher-order curvature invariants [3].

More recently it has been suggested that the present cosmic speed-up could have its origin in—among other possibilities—corrections to the GR equations of motion, generated by nonlinear contributions of the scalar curvature R in the pure gravity Lagrangian of $f(R)$ theories [4–7]. It has been demonstrated, however, that Solar System experiments seem to rule out $f(R)$ theories that are able to accommodate present accelerated expansion of the Universe [8,9] (for recent reviews see [10–13]). The demonstration relies on the weak-field limit expansion of the $f(R)$ Lagrangian, and the consequent calculation of post-Newtonian contributions to the metric coefficients [8,9,11].

Nonetheless, even if $f(R)$ theories are not a viable alternative to explain current acceleration of the expansion, their relevance to study early-time inflation [14] might fuel further interest in these alternatives to general relativity.

In the present article we aim at investigating the stability of matching of local solutions of $f(R)$ theories with the ambient background space-time, an issue that is central for the study of the weak-field limit and of the post-Newtonian metric of these theories. Our study will rely on the metric variational formalism. We shall focus, in particular, on the constraint's existence and stability of de Sitter solutions of $f(R)$ theories [15], imposed on the overall parameters of the $f(R)$ Lagrangian.[2] Although the conditions for existence and stability of these solutions are usually assumed to be obeyed, thorough consideration of these conditions in standard calculations of the relevant parameters is mostly lacking. Thorough consideration of the conditions that make the matching of local solutions with maximally symmetric de Sitter spaces possible may save additional (mostly unnecessary) assumptions on the cosmic dynamics to reach to robust physical conclusions.

In order to judge the theoretical viability of a given $f(R)$ gravity model, the constraints imposed by matching of the local solutions of the equations of the theory with asymptotic solutions of constant curvature $R = R_c$—in a maximally symmetric background—are used to fix the effective parameters that characterize the nonlinearities of $f(R)$ theories, such as the effective gravitational coupling and the effective mass of the additional scalar degree of freedom $\phi = f'(R)$.

It will be shown, in particular, that several common $f(R)$ Lagrangians either do not actually admit stable de Sitter (dS) solutions, or admit a stable asymptotic matching of

*israel@uclv.edu.cu
†yoelsy@uclv.edu.cu
‡yna@uclv.edu.cu

[1]Higher-order actions are indeed renormalizable (but not unitary) [2]).

[2]Stability of de Sitter space as the gravitational theory varies was studied for the first time in the pioneering work of Ref. [16].

local solutions with the ambient metric of an anti–de Sitter (AdS) space instead.

II. $f(R)$ EQUATIONS OF MOTION

In this article we shall focus on gravity theories that can be derived from the general action

$$S_{\rm tot} = S_g + S_{\rm mat}(g_{\mu\nu}, \chi), \tag{1}$$

where $S_{\rm mat}$ is the action of the matter degrees of freedom—collectively denoted by χ—while the pure gravity action S_g has the form of an $f(R)$ theory [11]:

$$S_g = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}f(R). \tag{2}$$

The field equations that can be derived from the action (1) through the standard metric variational procedure are the following:

$$f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f' = \kappa^2 T_{\mu\nu}, \tag{3}$$

where $f' \equiv df/dR$, and, as customary, the matter stress-energy tensor $T_{\mu\nu}$ is defined through

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta S_{\rm mat}}{\delta g^{\mu\nu}}.$$

The trace of Eq. (3) yields to a dynamical equation for determining the curvature scalar:

$$f'R - 2f + 3\Box f' = \kappa^2 T. \tag{4}$$

For purposes of comparison with canonical (EH) general relativity, Eq. (3) can be rearranged in the following way:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_{\rm eff}(T_{\mu\nu} + T^{\rm eff}_{\mu\nu}),$$
$$\kappa^2 T^{\rm eff}_{\mu\nu} = (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f' + \frac{1}{2}(f - f'R)g_{\mu\nu}, \tag{5}$$

where $8\pi G_{\rm eff} = \kappa^2/f'$ is the effective gravitational coupling. The right-hand side (RHS) of (5) can now be seen as the source terms for the metric, meaning that the metric is generated by the matter and by nonlinear terms related to the scalar curvature [9]. Additionally, the scalar curvature R satisfies a second-order differential equation (Eq. (4)) with the trace $T = g^{\mu\nu}T_{\mu\nu}$ of the matter stress-energy tensor and other (nonlinear) curvature terms acting as sources. In other words, the Ricci scalar curvature is now a dynamical quantity whose dynamics is determined by the trace Eq. (4).

III. BOUNDARY CONDITIONS

Here we shall focus on the embedding of a local (noncompact) object in the ambient background space-time, within the metric approach to $f(R)$ theories. This issue is motivated by the fact that, in order to have a complete description of the local physical system, one has to take into account, also, its interaction with the environment—in the present case the rest of the Universe [9].

The boundary conditions for the metric are imposed by a suitable choice of coordinates. For instance, one might consider that asymptotically (far away from the local source) the metric is Minkowski and fix its first derivatives to zero [17]. Since, according to the field Eqs. (3) and (4) $f'(R)$ is a dynamical quantity, one has to impose additional boundary conditions on f' also. In other words, the local system will interact with the background cosmology via the boundary value $f'(R_c)$ (R_c is the cosmic value of the Ricci curvature scalar) and its time derivative. To fix the latter boundary condition one may invoke the adiabatic approximation, according to which the evolution of the Universe is very slow when compared with the local dynamics, so that we can ignore terms such as $\dot{f}'(R_c)$ (the dot means derivative with respect to the cosmic time).

In order to derive solutions of Eq. (3) (or, alternatively, of Eq. (5)), one has to expand the metric about the asymptotic Minkowski metric $\eta_{\mu\nu}$:[3]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(t, x),$$

where $h_{\mu\nu}$ represent small perturbations about the background ($|h_{\mu\nu}| \ll 1$), and, at the same time, we expand the scalar field degree of freedom $\phi = f'$: $\phi = \phi_c + \varphi(t, x)$. The trace equation tends to

$$3_c\Box f'_c + f'_c R_c - 2f_c = \kappa^2 T_c,$$

where, for a vacuum background $T_c = 0$. Here $f_c = f(R_c)$, $f'_c = f'(R_c)$, etc. One may calculate the effective mass squared of the scalar field perturbation propagating in the background [8] (see also [18,19]):

$$m^2_\varphi = \frac{f'_c - R_c f''_c}{3f''_c}, \tag{6}$$

which will be used here as a criterion to judge the physical viability of the theory. Only scalar field perturbations with positive $m^2_\varphi > 0$ are physical (the perturbations exponentially damp). Otherwise, for negative $m^2_\varphi < 0$, the perturbation of the scalar field degree of freedom oscillates. It has been demonstrated that the corresponding solutions are unphysical [8].

[3]Expanding about the Minkowski metric does not actually entail existence of global Minkowskian solutions. In fact, the general solutions to the present problem turn out to be asymptotically de Sitter space-times [9]. Expansion around the Minkowski metric is local, with de Sitter space providing the background. Otherwise, only perturbation modes with wavelengths much smaller than the Hubble radius H_0^{-1} are to be taken into account.

IV. STABILITY ISSUE

An independent criterion to evaluate the viability of a given asymptotic matching of local solutions is based on Ricci stability of the $f(R)$ theories. The stability condition $f''(R_c) \geq 0$ expresses the fact that the scalar degree of freedom $\phi(R) = f'(R)$ is not a ghost [20]. Additionally, studies of linear stability of the scalar perturbation lead to the following condition [11]:[4]

$$f'_c \geq R_c f''_c. \tag{7}$$

Simultaneous requirement of linear and of Ricci stability

$$f'_c \geq R_c f''_c, \qquad f''_c \geq 0,$$

leads to positivity of the effective mass squared of the scalar field degree of freedom ($m_\varphi^2 \geq 0$). Notice, however, that the contrary statement is not true in general. Actually, the mass squared is positive also if, simultaneously, $f'_c < R_c f''_c$, and $f''_c < 0$.

In the present article, to avoid misleading results, we will use as independent criteria to judge the existence and stability of de Sitter solutions the following requirements: i) positivity of the mass squared $m_\varphi^2 \geq 0$, and ii) Ricci stability $f''_c \geq 0$. Simultaneous fulfillment of the above requirements means that linear stability is also granted.

V. EXISTENCE AND STABILITY OF DE SITTER SOLUTIONS

Our goal will be to fix the effective parameters that characterize the nonlinearities of $f(R)$ theories (e.g., the effective gravitational coupling and the effective mass m_φ^2) around solutions with constant curvature $R = R_c$ in a maximally symmetric background [22]:

$$R_{\mu\nu\sigma\upsilon} = \frac{R_c}{12}(g_{\mu\sigma}g_{\nu\upsilon} - g_{\mu\upsilon}g_{\nu\sigma}) \Rightarrow R_{\mu\nu} = \frac{R_c}{4}g_{\mu\nu}. \tag{8}$$

According to the trace Eq. (4) the condition for the existence of such a embedding can be written in the following way [15]:

$$f'_c R_c - 2f_c = 0 \Rightarrow R_c = 2\frac{f_c}{f'_c}. \tag{9}$$

The above condition,[5] together with the requirement of positivity of the mass squared $m_\varphi^2 \geq 0$, and of Ricci stability $f''_c \geq 0$, are central in the subsequent discussion since, in general, Eq. (9) reduces to an algebraic constraint on the overall parameters of the $f(R)$ Lagrangian.

[4]The condition of stability of de Sitter solutions in $f(R)$ gravity was first obtained in [21].

[5]The importance of condition (9) to exit from matter dominance with positive conclusion about subsequent acceleration in many $f(R)$ theories was studied in [23]. This result contradicts the claim in [24] stating that it is impossible in general.

Actually, take, for instance, $f(R)$ theories that can be written in the form $f(R) = R + \alpha g(R)$, where α is a small parameter.[6] The condition (9) translates into the following constraint on the boundary value of g':

$$g'_c - 2\frac{g_c}{R_c} = \frac{1}{\alpha}. \tag{10}$$

To see how the above condition constrains the overall parameters of the given theory, as an illustration, let us consider the theory $f(R) = R + \alpha R^2$ [14]. In this case $g(R) = R^2 \Rightarrow g'_c = 2R_c$, while $g_c/R_c = R_c$, so that the condition (10) implies that

$$g'_c - 2\frac{g_c}{R_c} = 0 = \frac{1}{\alpha} \Rightarrow \alpha = \infty,$$

contrary to the requirement that α be a small quantity. This means that matching of local solutions with asymptotic vacuum solutions of constant curvature in the model of Ref. [14] is not allowed. Otherwise stated: de Sitter solutions do not exist in the above model.

A somewhat similar conclusion is attained for generic models with $g(R) = R^n$ ($n > 2$). In this case, however, the condition for matching of local solutions with background spaces of constant curvature leads to

$$R_c = \left[\frac{1}{(n-2)\alpha}\right]^{1/n-1},$$

so that, since in general α is a very small mass scale, the matching can be consistent only at high curvature.

VI. GENERIC $f(R)$ THEORIES

As additional illustrations of the importance of the condition (9) to judge the existence and stability of de Sitter solutions, in this section we shall explore several classes of generic $f(R)$ theories that have been extensively studied, for instance, in Refs. [6,8,11,24,25].

A. Positive powers of R

Let us consider theories given by [6,8]:

$$f(R) = R + \frac{\epsilon R^n}{M^{2n-2}}, \tag{11}$$

where M represents a very large mass scale, $\epsilon \equiv \pm 1 \Rightarrow \epsilon^2 = 1$, and $n \geq 0$. The function $f(R)$ in Eq. (11) comprises several cases formerly studied, for instance, in Refs. [14,24]. The model of [14], for instance, is recovered if in (11) one sets $n = 2$.

In this case the constraint (9) can be written as

$$2\frac{f_c}{f'_c} = \frac{2M^{2n-2}R_c + 2\epsilon R_c^n}{M^{2n-2} + \epsilon n R^{n-1}} = R_c,$$

which leads to the following relationship of the boundary

[6]GR is obtained as the limit $\alpha \to 0$ of these theories.

value R_c with the parameter M:

$$\left(\frac{M^2}{R_c}\right)^{n-1} = \epsilon(n-2). \tag{12}$$

On the other hand, for the effective mass squared of the scalar field perturbation (6), one obtains [8]

$$m_\varphi^2 = \frac{R_c}{3\epsilon(n-1)}\left[\frac{1}{n}\left(\frac{M^2}{R_c}\right)^{n-1} - \epsilon(n-2)\right]. \tag{13}$$

By substituting (12) in (13), one is led to the following expression for the mass squared:

$$m_\varphi^2 = -\left(\frac{n-2}{3n}\right)R_c, \tag{14}$$

while, on the other hand

$$f_c'' = \frac{\epsilon n(n-1)}{R_c}\left(\frac{R_c}{M^2}\right)^{n-1} = \frac{n}{R_c}\left(\frac{n-1}{n-2}\right). \tag{15}$$

Notice that in the above expressions for m_φ^2 and f_c'' there is no explicit dependence on ϵ. Otherwise, the results of the stability study are independent of the sign of the second term in the right-hand side of Eq. (11).

As clearly seen, for $n > 2$, asymptotic matching of local solutions with de Sitter space ($R_c > 0$) is unstable since the requirements of positivity of mass squared and of Ricci stability cannot be simultaneously met. The same is true for anti–de Sitter solutions ($R_c < 0$). To state it more clearly, for the region of parameter space $n > 2$, (vacuum) maximally symmetric spaces of constant curvature cannot be a solution of $f(R)$ models given by (11). It is worth noticing that the same conclusion holds true for $1 < n < 2$ since, in this case, m_φ^2 and f_c'' are of opposite sign.

The above matching is possible only for the region of parameter space $0 \leq n < 1$. This is an important result since, as discussed in [11], theories with $f(R)$ given by (11) are compatible with the observations, precisely, in the region of the parameter space $0 < n \leq 0.25$. In this case, however, M has to be a sufficiently small mass scale.

B. Negative powers of R

An interesting alternative to (11) can be given by [8]:

$$f(R) = R + \frac{\epsilon\mu^{2n+2}}{R^n}, \tag{16}$$

where, as before $\epsilon = \pm 1$, and μ^2 is a tiny mass scale ($n \geq 0$). The above expression for $f(R)$ contains as a particular case, for instance, the one studied in [25]. In this case the constraint (9) translates into

$$2\frac{f_c}{f_c'} = \frac{2R_c + 2\epsilon\mu^{2n+2}/R_c^n}{1 - \epsilon\mu^{2n+2}/R_c^{n+1}} = R_c. \tag{17}$$

This, in turn, yields to the following relationship:

$$\left(\frac{R_c}{\mu^2}\right)^{n+1} = -\epsilon(n+2), \tag{18}$$

which, in general, holds true for the negative sign $\epsilon = -1$. When the above constraint is substituted in the expression for the effective mass squared of the scalar perturbation [8],

$$m_\varphi^2 = \frac{R_c}{3\epsilon(n+1)}\left[\frac{1}{n}\left(\frac{R_c}{\mu^2}\right)^{n+1} - \epsilon(n+2)\right], \tag{19}$$

one gets the following expression:

$$m_\varphi^2 = -\left(\frac{n+2}{3n}\right)R_c, \tag{20}$$

while, on the other hand,

$$f_c'' = -\left(\frac{n+1}{n+2}\right)\frac{n}{R_c}. \tag{21}$$

Notice, as in the former case, that the expressions for m_φ^2 and f_c'' do not contain ϵ, meaning that existence and stability of the de Sitter solutions do not depend on the sign of the second term in the RHS of Eq. (16).

From Eqs. (20) and (21) considering the requirements $m_\varphi^2 \geq 0$, $f_c'' \geq 0$—it is evident that a dS solution is unstable, while an asymptotic AdS matching of local solutions is stable instead. In consequence, local solutions of the equations of $f(R)$ theory given by (16) can be consistently matched only with asymptotic AdS background space.

The latter result represents an additional argument in favor of previous claims that $f(R)$ theories do not seem a good alternative to explain the late-time cosmic speed-up [8](see the related discussion in Sec. VII).

C. Combined powers of R

A natural generalization of models given by (11) and (16), can be based on the following form of the $f(R)$ function [19]:

$$f(R) = R \pm \frac{\mu^{2n+2}}{R^n} + \frac{R^m}{M^{2m-2}}, \tag{22}$$

where, as before, μ and M are a small and a large mass scales, respectively, and we shall consider that $n \geq 0$, $m \geq 2$. The above expression contains as a particular case the function [11]

$$f(R) = R - \frac{\mu^4}{R} + \alpha R^2.$$

The condition (9) for the above $f(R)$ model can be written in the following:

$$\left(\frac{R_c}{M^2}\right)^{m-1} = \frac{1 \pm (n+2)(\mu^2/R_c)^{n+1}}{m-2}. \tag{23}$$

On the other hand, the effective mass squared of the scalar perturbation is given by

$$m_\varphi^2 = \frac{1 \pm n(n+2)(\frac{\mu^2}{R_c})^{n+1} + m(m-2)(\frac{R_c}{M^2})^{m-1}}{3f''(R_c)},$$

where

$$f''(R_c) = \frac{\pm n(n+1)}{R_c}\left(\frac{\mu^2}{R_c}\right)^{n+1} + \frac{m(m-1)}{R_c}\left(\frac{R_c}{M^2}\right)^{m-1}.$$

When the condition (23) is substituted in the above expressions one gets:

$$m_\varphi^2 = \frac{m + 1 \pm (m-n)(n+2)(\mu^2/R_c)^{n+1}}{3f''(R_c)}, \tag{24}$$

$$f''(R_c) = \frac{m(m-1)}{R_c(m-2)}\left[1 \pm k\left(\frac{\mu^2}{R_c}\right)^{n+1}\right], \tag{25}$$

where, for short, we have introduced a constant parameter:

$$k \equiv \frac{n(n+1)(m-2) + m(m-1)(n+2)}{m(m-1)},$$

and we shall consider only situations where $m > n$.

It is evident from the above equations that, when the plus sign in the second term of the RHS of Eq. (22) is considered, the de Sitter solution of the $f(R)$ theory is stable.

If one chooses, instead, the negative sign in (22):

$$f(R) = R - \frac{\mu^{2n+2}}{R^n} + \frac{R^m}{M^{2m-2}},$$

then, since as it can be demonstrated,

$$k > \frac{(m-n)(n+2)}{m+1},$$

it follows that stability of de Sitter solutions of the corresponding $f(R)$ theory can be achieved only if $(R_c/\mu^2)^{n+1} \geq k$.

Notice that, in the above discussion, the case with $m = 2$ cannot be considered due to lack of definition of several expressions. Notwithstanding, this case (with $n = 1$) has been studied in Ref. [11].

VII. DIRAC-BORN-INFELD MODIFICATION OF GENERAL RELATIVITY

To be phenomenologically viable, nonlinear modifications of general relativity have to satisfy several physically motivated requirements [26]:

(1) Reduction to EH action at small curvature,
(2) ghost freedom,
(3) regularization of some singularities (as, for instance, the Coulomb-like Schwarzschild singularity), and
(4) supersymmetrizability.

The later requirement is quite stringent and, for most purposes, might be excluded. A theory that fulfills the above requirements can be based on the following Dirac-Born-Infeld-type action [27]:

$$S = \frac{1}{\kappa^4}\int d^4x\sqrt{|g|}(1 - \sqrt{1 - \alpha\kappa^2 R + \beta\kappa^4 \mathcal{G}}), \tag{26}$$

where $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\upsilon}R^{\mu\nu\sigma\upsilon}$ is the Gauss-Bonnet term. It has been demonstrated in Ref. [27] that this action has the EH leading term at small curvature, is ghost-free, and, for an appropriate region in the parameter space, it shows indications for the cancellation of the Coulomb-like Schwarzschild singularity.

In order to straightforwardly apply the approach undertaken here, let us first focus on the following simplified Dirac-Born-Infeld (DBI) modification of the Einstein-Hilbert action:

$$S_g = \frac{1}{\kappa^4}\int d^4x\sqrt{|g|}(1 - \sqrt{1 - \alpha\kappa^2 R}). \tag{27}$$

Notice that this modification of the Einstein-Hilbert action is a particular case of (26) for $\beta = 0$. In consequence (27) fulfills the requirements 1–4 above. In particular, as it was demonstrated in Ref. [27], it has the correct EH limit at low curvature and is free of ghosts.

Through using the approach of [8,9] (see also [11]), we will show here that the above statement about ghost freedom is wrong for positive values of the parameter $\alpha > 0$. Actually, the theory given by (27) can be recast, alternatively, into the form of an $f(R)$ theory of the kind in Eq. (2), with

$$f(R) = \frac{2}{\kappa^2}(1 - \sqrt{1 - \alpha\kappa^2 R}). \tag{28}$$

The condition (9) for existence and stability of asymptotic matching of local solutions of the above $f(R)$ theory with a maximally symmetric (vacuum) background space of constant curvature leads to the following relationship between the boundary value of the curvature R_c and the parameters α, κ^2:

$$2\frac{f_c}{f_c'} = R_c \Rightarrow R_c = \frac{8}{9}\frac{1}{\alpha\kappa^2}. \tag{29}$$

When this boundary R-value is substituted into the definition of effective mass squared of the scalar field degree of freedom:

$$m_\varphi^2 = \frac{f_c' - R_c f_c''}{3f_c''} = -\frac{R_c}{4}\left(= -\frac{2}{9}\frac{1}{\alpha\kappa^2}\right), \tag{30}$$

where we have taken into account that $f_c'' = 27\alpha^2\kappa^2/2$; the theory is Ricci stable for any value of the overall parameters, and it can be seen from (30) that the theory (27) can be consistently (asymptotically) matched only with AdS background ($\alpha < 0$). Otherwise, for positive $\alpha > 0$, the scalar degree of freedom carrying the nonlinear R-contribution is a (unphysical) ghost degree of freedom. This fact rules out this type of modification of general relativity as a candidate to explain the late-time cosmic speed-up.

We have to point out, however, that inclusion of a Gauss-Bonnet term as, for instance, in the more general action (26), could change this pessimistic picture. Indications of this for simple general models of the kind

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}f(R,\mathcal{G}) = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}f(R,P,Q), \tag{31}$$

where $P \equiv R_{\mu\nu}R^{\mu\nu}$, $Q \equiv R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}$, have been given, for instance, in [28].[7] The condition for existence of vacuum maximally symmetric solutions of constant curvature R_c in this more general case can be written in the following way ($f_R \equiv \partial_R f$, $f_P \equiv \partial_P f$, $f_Q \equiv \partial_Q f$):

$$\left[2f - Rf_R - \frac{R^2}{2}f_P - \frac{R^2}{3}f_Q\right]_{R_c} = 0. \tag{32}$$

For maximally symmetric spaces of constant curvature, $P = R_c^2/4$, while $Q = R_c^2/6$, so that $\mathcal{G} = Q = R_c^2/6$, and, for the case (26) where

$$f(R,P,Q) = \frac{2}{\kappa^2}(1-s), \tag{33}$$

$$s \equiv \sqrt{1 - \alpha\kappa^2 R + \beta\kappa^4\mathcal{G}}, \tag{34}$$

we obtain that

$$s_c = \sqrt{1 - \alpha\kappa^2 R_c + \beta\kappa^4 R_c^2/6}. \tag{35}$$

Hence, the condition for existence of a maximally symmetric solution of constant curvature (32) leads to the following constraint equation on R_c:

$$\beta\kappa^4 R_c^2 - 3\alpha\kappa^2 R_c = 12 s_c(s_c - 1),$$

or alternatively,

$$\beta\kappa^4 R_c^2 - 9\alpha\kappa^2 R_c + 12 = 12 s_c. \tag{36}$$

It has been demonstrated in [27] that there are no ghosts associated with the additional spin-two mode of theory (26); however nothing concrete has been said about the scalar mode. The effective mass squared of the latter mode can be written in the following way:

$$m_0^2 = \frac{[f_R - Rf_{RR}]_{R_c}}{\Sigma_0^c}, \tag{37}$$

where

$$\Sigma_0^c \equiv \left[3f_{RR} + 2(f_P + f_Q) + R(3f_{RP} + 2f_{RQ}) + R^2\left(\frac{3}{4}f_{PP} + f_{PQ} + \frac{1}{3}f_{QQ}\right)\right]_{R_c}. \tag{38}$$

If one substitutes the constraint (36) into the above expression for the effective mass of the spin-zero mode, one obtains

$$m_0^2 = \frac{3R_c(2\beta\kappa^4 R_c - \alpha\kappa^2)^2}{23\beta^2\kappa^8 R_c^2 - 18\alpha\beta\kappa^6 R_c - \alpha^2\kappa^4}. \tag{39}$$

Positivity of the effective mass of the spin-zero mode requires that, either:

$$R_c > \frac{(9 + 2\sqrt{6})\alpha}{23\beta\kappa^2},$$

or

$$R_c < \frac{(9 - 2\sqrt{6})\alpha}{23\beta\kappa^2}.$$

Here we are assuming positive α and β. Therefore, for the theory depicted by the action (26), given that the above conditions hold true, there are no ghost scalar degrees of freedom. There is only a gap in the values of the curvature scalar:

$$\Delta R_c = \frac{4\sqrt{6}\alpha}{23\beta\kappa^2},$$

where the theory given by (26) has a spin-zero ghost state.

VIII. DISCUSSION

An accurate handling of boundary conditions in $f(R)$ theories requires us to solve Eqs. (3) (or (5)), and (4), for the metric and the scalar field $\phi = f'(R)$ respectively, in the cosmic regime, where homogeneity and isotropy lead to a Friedman-Robertson-Walker metric $g^c_{\mu\nu} = \mathrm{diag}(-1, a(t)^2\delta_{ij})$, and to a space-independent value of the scalar field $\phi_c = \phi_c(t)$ [8]. At smaller scales local deviations from the cosmic values of the fields become appreciable. Then, usually, one invokes the weak-field limit and treats these deviations as small perturbations around the background boundary values $g^c_{\mu\nu}$, ϕ_c [8,9,17].

Hence, it is clear the role existence and stability of de Sitter solutions play in the phenomenology of the $f(R)$ models: they act as a selection rule to choose a theory whose local solutions can be asymptotically matched with phenomenologically viable cosmological models. This is the reason why we chose to look for stable matching of local solutions with asymptotic—maximally symmetric—background spaces of constant (positive) curvature $R_c > 0$: In a cosmological setting such spaces fit well with the existing observational evidence on late-time accelerated pace of the cosmic expansion.

In the present article we investigate the constraint's conditions for existence and stability of de Sitter (vacuum) background, imposed on the parameters of the $f(R)$ Lagrangian. Although these conditions are usually assumed to be obeyed, their thorough consideration in standard calculations of the relevant parameters is mostly lacking. In Ref. [8], for instance, the expressions for the effective mass squared of the scalar field degree of freedom are given, but the conditions for existence and stability of

[7] In terms of the invariants P and Q, the Gauss-Bonnet invariant can be written as $\mathcal{G} = R^2 - 4P + Q$.

background de Sitter spaces are not substituted into these expressions so that, in consequence, the author was able to make physical conclusions only after assuming a given cosmological solution (for a matter-dominated era, in particular). On the contrary, thorough consideration of the condition (9) may save additional (mostly unnecessary) assumptions on the cosmic dynamics, to reach to robust physical conclusions.

$f(R)$ models were primarily intended to explain late-time cosmic acceleration. The reasoning behind this effect is that, typically, during the course of the cosmic expansion, the curvature dilutes and the term nonlinear in R starts dominating the late-time cosmic dynamics, acting as a dark energy source encoded in the effective stress-energy tensor $T^{\rm eff}_{\mu\nu}$ in the RHS of Eq. (5). However, there are several $f(R)$ models that fail to be compatible with late-time dynamics. The approach undertaken by us in this article allows one to have an additional check of consistency. Take, for instance, $f(R)$ models given by (16):

$$f(R) = R + \epsilon\frac{\mu^{2n+2}}{R^n},$$

where $\epsilon = \pm 1$. The above models have been shown in Sec. VI B to have a stable anti–de Sitter solution. Otherwise, local solutions of the corresponding theories cannot be asymptotically matched with de Sitter space, which means, in turn, that any solution of the equations of the theory that is compatible with cosmic acceleration either does not exist or has to be necessarily unstable. This result may be considered as an additional criterion to judge the phenomenological viability of $f(R)$ theories given by (16).

For models where the function $f(R)$ is given by (11):

$$f(R) = R + \epsilon\frac{R^n}{M^{2n-2}},$$

one may find a region in parameter space $0 \leq n < 1$ where the background de Sitter solution is stable. As far as we know, no such conclusive argument has been given before on $f(R)$ theories of the kind (11). A lucky circumstance is related to the fact that the region of parameter space where the model of (11) is compatible with the cosmological observations $0 < n \leq 0.25$ is contained within the region that allows stable de Sitter solutions.

In both cases—$f(R)$ theories given by either (11) or (16)—the existence (and stability) of de Sitter solutions is independent on the sign ϵ of the nonlinear R-term. This latter result is a robust conclusion on the parameter space of the above theories.

There are other $f(R)$ models as, for instance, the one of Ref. [14]:

$$f(R) = R + \alpha R^2,$$

which do not admit matching of local solutions with a maximally symmetric background space of constant curvature.[8]

Additional comments deserve the $f(R)$ models given by Eq. (22) [19], which contain combined powers of R. As properly noted, as long as one considers only the positive sign in front of the second term in the RHS of (22):

$$f(R) = R + \frac{\mu^{2n+2}}{R^n} + \frac{R^m}{M^{2m-2}},$$

the corresponding $f(R)$ models admit stable de Sitter solutions, so that these could be appropriated to address late-time accelerated expansion. Because of the existence of two mass scales: a large one fixed by M, which sets the scale at which early-time inflation happens, and a tiny one fixed by μ, which sets the (inverse of the) length scale at which late-time accelerated expansion occurs, the above models could represent interesting alternatives to address unified description of early-time inflation (without the inflaton), and late-time accelerated expansion of the Universe (without dark energy). At high enough energy the third term $\propto R^m$ in the RHS of the above equation causes the Universe to inflate, while the second term $\propto R^{-n}$ dominates at late times.

The approach undertaken in this article permitted us to demonstrate also that the DBI modification of general relativity proposed in [27] can be free of ghosts given that certain constraints on the de Sitter curvature hold true, while the particular case given by the action (27)) is free of ghosts only for negative values of the parameter $\alpha < 0$. The possible conflict of this result with the result of Ref. [27] can be based on the fact that, in order to warrant ghost freedom, the author of [27] only demanded that the mass of the additional spin-two degree of freedom (not present in our simpler case) $m_2 \to \infty$. No additional criterion was given on the mass of the spin-zero degree of freedom m_0^2, which is central in the present discussion.

We want to point out that if realistic (viable) $f(R)$ gravity models are considered [18,31], the constraint (9) becomes a higher-order algebraic equation whose roots cannot be found algebraically. In this case, numerical methods like, for instance Sturm's theorem and Descartes's rule of signs, need to be invoked. An example of the kind of problems that could arise because of the highly nonlinear structure of these kinds of $f(R)$ theories is studied in [32], where the dynamical system tools were used to investigate the issue of stability of de Sitter solutions for several realistic $f(R)$ models. In the latter reference the de Sitter universes had to be constructed with numerical tools mainly and, in order to simplify the problem, the authors did not consider the matter contributions.

[8]Besides leading to early-time inflation and satisfying the Solar System observational constraints, the R^2-inflation model of Ref. [14] also seems compatible with CMBR observations [29]. The correct slope of the spectrum of scalar perturbations in this model was found in [30].

IX. CONCLUSIONS

In this article we have explored the phenomenological consequences of the thorough consideration of the conditions for existence and stability of de Sitter solutions for a wide class of $f(R)$ theories. Otherwise, the interaction of the local system with the background cosmology imposes boundary conditions which, when taken into account, impose constraints on the overall parameters of the $f(R)$ Lagrangian. Although it is a well-known fact, it is not thoroughly used to constrain the parameter space of the models.

It has been shown here that a wide variety of models do not actually pass the de Sitter stability test. There are models which admit matching of local solutions with anti–de Sitter background cosmology. Others, in general, do not admit stable (asymptotic) matching of local solutions with maximally symmetric backgrounds of constant curvature $R_{\mu\nu} = R_c g_{\mu\nu}/4$. Additionally, it has been shown here that ghost freedom of a DBI modification of general relativity formerly studied in Ref. [27] demands certain constraints on the curvature of the de Sitter space. For a simpler model where the Gauss-Bonnet invariant is set to zero, ghost freedom arises only for negative values of one of the parameters of the Lagrangian ($\alpha < 0$). The latter result conflicts with the one in [27], claiming that theories based on Lagrangians of this kind (26)—which include as a particular case the one given by (27)—are ghost-free in general.

There is a class of $f(R)$ models where nonlinear R-terms have positive as well as negative powers of R (see Eq. (22)). Because of coexistence of two different mass scales in (22), for an appropriated region in the parameter space, these models could provide a nice scenario in which to address in a unified picture both early-time inflation and late-time speed-up.

ACKNOWLEDGMENTS

Our thanks to S. D. Odintsov and A. A. Starobinsky for their useful comments and for calling our attention to various relevant references, including several seminal papers. We acknowledge the MES of Cuba for partial support of the research. The work of I. Q. was partly supported by CONACyT México Grant No. I0101/131/07 C-234/07, Instituto Avanzado de Cosmologia (IAC) Collaboration.

[1] R. Utiyama and B. S. DeWitt, J. Math. Phys. (N.Y.) **3**, 608 (1962).
[2] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).
[3] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Spacetime* (Cambridge University Press, Cambridge, 1982); I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Actions in Quantum Gravity* (IOP Publishing, Bristol, 1992); G. A. Vilkovisky, Classical Quantum Gravity **9**, 895 (1992).
[4] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007).
[5] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002); D. N. Vollick, Phys. Rev. D **68**, 063510 (2003).
[6] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
[7] S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden, and M. S. Turner, Phys. Rev. D **71**, 063513 (2005).
[8] G. J. Olmo, Phys. Rev. Lett. **95**, 261102 (2005); Phys. Rev. D **72**, 083505 (2005).
[9] G. J. Olmo, Phys. Rev. D **75**, 023511 (2007).
[10] S. Nojiri and S. D. Odintsov, arXiv:0807.0685.
[11] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726.
[12] V. Faraoni, arXiv:0810.2602.
[13] T. P. Sotiriou, arXiv:0810.5594.
[14] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
[15] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, J. Cosmol. Astropart. Phys. 02 (2005) 010.
[16] J. D. Barrow and A. C. Ottewill, J. Phys. A **16**, 2757 (1983).
[17] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, UK, 1993).
[18] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, and S. Zerbini, Phys. Rev. D **77**, 046009 (2008).
[19] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003); Gen. Relativ. Gravit. **36**, 1765 (2004).
[20] V. Faraoni, Phys. Rev. D **75**, 067302 (2007).
[21] V. Muller, H. J. Schmidt, and A. A. Starobinsky, Phys. Lett. B **202**, 198 (1988).
[22] D. Wands, Classical Quantum Gravity **11**, 269 (1994); A. Hindawi, B. A. Ovrut, and D. Waldram, Phys. Rev. D **53**, 5597 (1996); A. Nunez and S. Solganik, Phys. Lett. B **608**, 189 (2005); T. Chiba, J. Cosmol. Astropart. Phys. 03 (2005) 008.
[23] S. Capozziello, S. Nojiri, S. D. Odintsov, and A. Troisi, Phys. Lett. B **639**, 135 (2006); S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006).
[24] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007); L. Amendola, D. Polarski, and S. Tsujikawa, Phys. Rev. Lett. **98**, 131302 (2007); Int. J. Mod. Phys. D **16**, 1555 (2007).
[25] A. L. Erickcek, T. L. Smith, and M. Kamionkowski, Phys. Rev. D **74**, 121501(R) (2006).
[26] S. Deser and G. W. Gibbons, Classical Quantum Gravity **15**, L35 (1998).
[27] D. Comelli, Phys. Rev. D **72**, 064018 (2005).
[28] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S.

Zerbini, Phys. Rev. D **73**, 084007 (2006).
[29] A. A. Starobinsky, Sov. Astron. Lett. **9**, 302 (1983); J. Hwang and H. Noh, Phys. Lett. B **506**, 13 (2001).
[30] V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532 (1981).
[31] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007); S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007).
[32] G. Cognola, E. Elizalde, S. D. Odintsov, P. Tretyakov, and S. Zerbini, Phys. Rev. D **79**, 044001 (2009).