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Interior of a charged distorted black hole

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We study the interior of a charged, nonrotating distorted black hole. We consider static and axisymmetric black holes, and focus on a special case when an electrically charged distorted solution is obtained by the Harrison-Ernst transformation from an uncharged one. We demonstrate that the Cauchy horizon of such a black hole remains regular, provided the distortion is regular at the event horizon. The shape and the inner geometry of both the outer and inner (Cauchy) horizons are studied. We demonstrate that there exists a duality between the properties of the horizons. Proper time of a free fall of a test particle moving in the interior of the distorted black hole along the symmetry axis is calculated. We also study the property of the curvature in the inner domain between the horizons. Simple relations between the 4D curvature invariants and the Gaussian curvature of the outer and inner horizon surfaces are found.

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I. INTRODUCTION

In this paper, we study how the distortion of a charged, static black hole generated by axisymmetric, static matter distribution in its exterior region affects its interior. This paper is a direct generalization of a similar study for the distorted neutral black hole interior performed in [1].

Structure and properties of the charged and/or rotating black hole interior is a subject that has attracted a lot of interest during the past 30 years (see, e.g., [2] and references therein). Analytic continuation of the Reissner-Nordström (RN) and Kerr solutions results in the existence of infinitely many new ''universes'' in the black holes interior. However, the region containing these new universes lies in the future of the Cauchy horizon, a null hypersurface beyond which predictability breaks down. A natural question is whether these universes are accessible to an observer traveling in the interior of the black hole. That is why the issue of the Cauchy horizon stability is so important. Observers traveling along a timelike world line receive an infinitely blueshifted radiation when they approach the horizon. Penrose [3] used these facts to argue that small perturbations produced in the black hole exterior grow infinitely near the Cauchy horizon. The evolution of small perturbations inside charged black holes was analyzed in [4–6]. These results confirm Penrose's intuitive arguments.

If one considers ingoing radiation only and neglects backscattered radiation, then the resulting Cauchy horizon singularity is weak. Namely, the Kretschmann invariant calculated on the Cauchy horizon is finite. A freely falling observer detects an infinite increase of energy density, but tidal forces remain finite as the observer crosses the Cauchy horizon [7,8]. Such singularity is called the whimper singularity. However, in a realistic situation, when both incoming and outgoing radiation are present, the curvature grows infinitely near the Cauchy horizon. This was demonstrated by Poisson and Israel [9] who considered the outgoing and ingoing radiation simulated as two noninteracting radial streams of ingoing and outgoing lightlike particles following null geodesics. Poisson and Israel showed that such radiation results in an infinite growth of the black hole internal mass parameter and divergence of the Weyl scalar. They called this effect the *mass inflation*. Mass inflation for a slowly rotating, charged black hole was discussed in [10]. Later, Ori constructed an exact, simplified solution describing this effect [11]. Using his solution Ori showed that the *mass inflation* singularity is weak enough. Namely, the tidal forces calculated at the Cauchy horizon diverge in the reference frame of a freely falling observer, but their integral along the world line of the observer remains finite. It means that freely falling observers might in fact cross the Cauchy horizon. For a more detailed discussion, see, e.g., [12–16]. Early numerical analysis of the Cauchy horizon stability predicted its destruction as a result of classical instability [17]. Later, analytical [18–20], and numerical [20] discussions did not confirm this result. The mass inflation phenomenon may shed light on the Cauchy horizon stability problem. However, further investigation is necessary.

Although rotating black holes are of real astrophysical interest, charged black holes are often considered in the publications. The reason for this is simple: a charged black hole also has a Cauchy horizon, but its spherical geometry makes an analysis easier. However, even in this case such a model is very simplified, for in the realistic world there always exists some matter outside the black hole. This matter distorts the gravitational field of the black hole. What is important is that this distortion generated by the matter distribution in the exterior of the black hole occurs not only outside the black hole, but also affects its interior. Since the region near the Cauchy horizon is ''fragile'' and

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"vulnerable," it is interesting to analyze how such external matter affects the properties of the black hole Cauchy horizon. This is one of the questions we address in our paper. We shall make several assumptions simplifying the analysis. Namely, we assume that the distortions of the black hole are static and axisymmetric. Moreover, we consider a special class of charged distorted black hole solutions which can be generated by the Harrison-Ernst transformation [21,22] from a neutral distorted black hole metric. This class includes a large variety of solutions which can be presented in an explicit form.

We always assume that in the vicinity of the black hole and in its interior the Einstein-Maxwell equations are satisfied, and the matter disturbing the black hole is located in the black hole exterior. The matter sources are described by the corresponding energy-momentum tensor which has to be included in the Einstein-Maxwell equations. To avoid this one can "move" these sources to infinity. The "price" for this is that the corresponding space-time is not asymptotically flat anymore. In our description of a distorted black hole we follow [23] and adopt that approach.

Our main problem is to study how the black hole interior is distorted by the external fields. In particular, we shall study distortion of the inner (Cauchy) horizon and its relation to the distortion of the outer (event) horizon. Let us emphasize that our consideration is completely classical, and we do not consider quantum effects which may play an important role in the charged black hole interior. Discussion of these effects can be found, e.g., in [24–27].

It should be emphasized that the study of the black hole interior is a dynamical problem. The geometry of the black hole interior is similar to the geometry of a contracting, anisotropic, homogeneous universe. To study how the evolution of this universe is modified by an external influence, one must study first the modification of the external geometry of the black hole and use these results to find the corresponding modification of the geometry of the event horizon. This gives the initial data which determines the evolution of the black hole interior. In this paper, we study a simple case when the distortion of the black hole in the exterior region is both stationary and axisymmetric. A similar problem for the neutral black hole was studied earlier in [1].

This paper is organized as follows. Section II collects the results concerning the charged distorted black hole solution generated by the Harrison-Ernst transformation technique. We remind the reader of these results mainly in order to fix the notations we use in the main part of the paper. In Sec. III, we establish special duality relations between properties of the inner and outer horizons for the charged distorted black hole. In Secs. IV and V, we study the Gaussian curvature of the horizon surfaces and present their isometric embedding diagrams. In Sec. VI, we discuss how the black hole distortion affects the maximal proper time of a free fall of a test particle moving along the axis of symmetry in the black hole interior. In Sec. VII, we establish a relation between the space-time curvature invariants near the horizons and their Gaussian 2D curvatures. We summarize and discuss our results in Sec. VIII. Necessary details are included in the Appendix. In this paper, we use the units where $G = c = 1$, and the sign conventions adopted in [28].

II. METRIC OF A DISTORTED RN BLACK HOLE

A. Static, axisymmetric Einstein-Maxwell space-time

In this section following [29–31], we present a solution for a static, axisymmetric distorted charged black hole. This solution is obtained by applying the Harrison-Ernst transformation [21,22] to the Weyl metric of a distorted vacuum black hole. Here we reproduce the basic relations, mainly in order to explain notations we shall use later.

The metric of a charged distorted black hole is a special solution of the Einstein-Maxwell equations

$$R_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1}$$

$$\nabla_\beta F^{\alpha\beta} = 0, \qquad \nabla_{[\alpha} F_{\beta\gamma]} = 0, \tag{2}$$

$$8\pi T_{\alpha\beta} = 2F_\alpha{}^\gamma F_{\beta\gamma} - \tfrac{1}{2} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}. \tag{3}$$

Here, $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$, and A_α is the electromagnetic 4-potential. The nabla stands for the covariant derivative defined with respect to the metric $g_{\alpha\beta}$.

Before we proceed with the description of a charged distorted black hole, let us make a few remarks about the charged black hole solution in the absence of distortions. This is the well-known Reissner-Nordström solution (see, e.g., [32])

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

$$F = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad A_\alpha = -\Phi_0 \delta_\alpha{}^t, \qquad \Phi_0 = \frac{Q}{r}. \tag{5}$$

Here, M is the black hole mass, and Q is its electric charge. We shall consider nonextremal black holes with $|Q| < M$. The space-time is static and asymptotically flat. It has a timelike singularity at $r = 0$. The black hole horizons are defined by $r_\pm = M \pm \sqrt{M^2 - Q^2}$, where the upper sign stands for the event horizon, and the lower sign stands for the Cauchy horizon. Correspondingly, we denote these horizons as $\mathcal{H}^{(\pm)}$.

It is convenient to make the following coordinate transformation

$$r = M(1 + p\eta), \qquad p = \frac{\sqrt{M^2 - Q^2}}{M}, \qquad \eta \in (-1/p, \infty), \tag{6}$$

and to rewrite the Reissner-Nordström solution in the following form

$$ds^2 = -\frac{p^2(\eta^2-1)}{(1+p\eta)^2}dt^2 + M^2(1+p\eta)^2 \times\left[\frac{d\eta^2}{\eta^2-1} + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{7}$$

$$\Phi_0 = \frac{\sqrt{1-p^2}}{(1+p\eta)}. \tag{8}$$

In these new coordinates, $\eta = \eta_\pm = \pm 1$ corresponds to the horizons of metric (7), and $\eta = -1/p$ corresponds to the black hole singularity.

The general form of the static, axisymmetric metric in prolate spheroidal coordinates $(\eta, \cos\theta)$ reads

$$ds^2 = -e^{2U}dt^2 + M^2p^2e^{-2U}\left[e^{2V}(\eta^2-\cos^2\theta)\times\left(\frac{d\eta^2}{\eta^2-1}+d\theta^2\right) + (\eta^2-1)\sin^2\theta d\phi^2\right], \tag{9}$$

where the metric functions U and V depend on (η, θ) coordinates. The corresponding electrostatic 4-potential is

$$A_\alpha = -\Phi(\eta,\theta)\delta_\alpha{}^t. \tag{10}$$

B. The Harrison-Ernst transformation

The Einstein-Maxwell equations for U and Φ are the Ernst equations [22], which in our case of static space-time (9) take the following form:

$$\nabla(e^{-2U}\nabla\mathcal{E}) = 0, \qquad \nabla(e^{-2U}\nabla\Phi) = 0. \tag{11}$$

Here, $\mathcal{E} = e^{2U} - \Phi^2$ is the Ernst potential, and ∇ is the nabla operator defined with respect to the 3D flat metric

$$dl^2 = (\eta^2-\cos^2\theta)\left[\frac{d\eta^2}{\eta^2-1}+d\theta^2\right] + (\eta^2-1)\sin^2\theta d\phi^2. \tag{12}$$

There exists a special class of solutions where the Ernst potential $\mathcal{E}$ is an analytic function of Φ. Under this assumption Eqs. (11) imply

$$\frac{d^2\mathcal{E}}{d\Phi^2} = 0. \tag{13}$$

If space-time is asymptotically flat, we choose $U = \Phi = 0$ at infinity. In this case a general solution of (13) can be written as

$$\mathcal{E} = 1 - \frac{2}{\sqrt{1-p^2}}\Phi. \tag{14}$$

We shall keep this relation in our consideration. Following [22] it is convenient to parametrize $\mathcal{E}$ and Φ as follows:

$$\mathcal{E} = \frac{\xi-1}{\xi+1}, \qquad \Phi = \frac{\sqrt{1-p^2}}{\xi+1}, \tag{15}$$

where ξ is the auxiliary Ernst potential. Using (11) one obtains the following equation for ξ

$$(\xi^2-p^2)\nabla^2\xi - 2\xi\nabla\xi\cdot\nabla\xi = 0. \tag{16}$$

In the absence of an electric field, $\Phi = 0$, the Ernst equation (11) is

$$\bar{\mathcal{E}}\nabla^2\bar{\mathcal{E}} = \nabla\bar{\mathcal{E}}\cdot\nabla\bar{\mathcal{E}}, \tag{17}$$

where $\bar{\mathcal{E}} = e^{2\bar{U}}$, and $\bar{U}$ corresponds to the vacuum uncharged solution. In this case one can also use parametrization (15) which gives

$$\bar{\mathcal{E}} = \frac{\bar{\xi}-1}{\bar{\xi}+1}, \tag{18}$$

and the Ernst equation (17) takes the form

$$(\bar{\xi}^2-1)\nabla^2\bar{\xi} - 2\bar{\xi}\nabla\bar{\xi}\cdot\nabla\bar{\xi} = 0. \tag{19}$$

Comparing (16) and (19) we can derive the relation between the vacuum and the electrostatic Ernst potentials. This is the Harrison-Ernst transformation:

$$\xi = p\bar{\xi}. \tag{20}$$

Thus, if we know a solution to vacuum Einstein equations $\bar{U}$, we can apply (20) and (15) to obtain the corresponding solution U, and the electrostatic potential Φ obeying the Einstein-Maxwell equations. Namely, using expressions (20), (18), and (15) we derive

$$e^{2U} = \frac{4p^2e^{2\bar{U}}}{[1+p-(1-p)e^{2\bar{U}}]^2}, \qquad \Phi = \frac{\sqrt{1-p^2}(1-e^{2\bar{U}})}{1+p-(1-p)e^{2\bar{U}}}. \tag{21}$$

These expressions determine the charged version of an electrically neutral, vacuum static solution. For example, starting with the Schwarzschild black hole solution, we can derive the Reissner-Nordström black hole. If the Schwarzschild black hole is distorted by neutral exterior matter, these expressions electrically charge both, the black hole and the matter.

In the next subsection, we apply this "charging" procedure to the Weyl static metric describing a vacuum, axisymmetric distorted black hole, and obtain an electrically charged distorted black hole. We discuss the corresponding metric in the next subsection.

C. Charged distorted black hole

Now we are ready to present a solution for a charged, axisymmetric distorted black hole. Following the procedure presented in the previous subsection, we start with the vacuum solution representing an axisymmetric distorted

Schwarzschild black hole, which we write in the form [1,33]

$$ds^2 = -e^{2\bar{U}}dt^2 + M^2 e^{-2\bar{U}}\Big(e^{2\bar{V}}(\eta^2 - \cos^2\theta) \times\left[\frac{d\eta^2}{\eta^2-1} + d\theta^2\right] + (\eta^2-1)\sin^2\theta d\phi^2\Big), \tag{22}$$

$$e^{2\bar{U}} = \frac{\eta-1}{\eta+1}e^{2\hat{U}}, \qquad e^{2\bar{V}} = \frac{\eta^2-1}{\eta^2-\cos^2\theta}e^{2\hat{V}}. \tag{23}$$

For an undistorted Schwarzschild solution $\hat{U} = \hat{V} = 0$. For the distorted metric, the vacuum Einstein equations for $\hat{U}$ and $\hat{V}$ distortion fields imply

$$(\eta^2-1)\hat{U}_{,\eta\eta} + 2\eta\hat{U}_{,\eta} + \hat{U}_{,\theta\theta} + \cot\theta\hat{U}_{,\theta} = 0, \tag{24}$$

$$\hat{V}_{,\eta} = N(\eta[(\eta^2-1)\hat{U}^2_{,\eta} - \hat{U}^2_{,\theta}] + 2(\eta^2-1)\cot\theta\hat{U}_{,\eta}\hat{U}_{,\theta} + 2\eta\hat{U}_{,\eta} + 2\cot\theta\hat{U}_{,\theta}), \tag{25}$$

$$\hat{V}_{,\theta} = -N((\eta^2-1)\cot\theta[(\eta^2-1)\hat{U}^2_{,\eta} - \hat{U}^2_{,\theta}] - 2\eta(\eta^2-1)\hat{U}_{,\eta}\hat{U}_{,\theta} + 2(\eta^2-1)\hat{U}_{,\eta} - 2\eta\hat{U}_{,\theta}). \tag{26}$$

Here, $N = \sin^2\theta(\eta^2 - \cos^2\theta)^{-1}$, and the comma stands for a partial derivative. Once the solution to Eq. (24) is found, $\hat{V}$ can be determined by integration of (25) and (26). Details of derivation of $\hat{U}$ can be found, for example, in [1,32]. Regularity of the distorted black hole horizon implies that $\hat{U}$ can be decomposed over the Legendre polynomials of the first kind

$$\hat{U} = \sum_{n\geq 0} a_n P_n(\eta)P_n(\cos\theta). \tag{27}$$

Thus, $\hat{U}$ and its derivatives are everywhere regular. Using this decomposition one can write the distortion field in equivalent form [29,30]

$$\hat{U} = \sum_{n\geq 0} c_n R^n P_n, \tag{28}$$

$$P_n = P_n(\eta\cos\theta/R), \qquad R = (\eta^2 - \sin^2\theta)^{1/2}. \tag{29}$$

Here, the constant coefficients c_n's define the distortion field. We call these coefficients the *multipole moments* [35]. The multipole moments uniquely characterize the distortion. Later we discuss some examples illustrating the nature of distortion defined by the lowest multipole moments.

The distortion field $\hat{V}$ can be written in a closed form as a sum of two terms $\hat{V} = \hat{V}_1 + \hat{V}_2$ (see, e.g., [29,30]). The first term, $\hat{V}_1$, is linear, and the second one, $\hat{V}_2$, is quadratic in c_n's

$$\hat{V}_1 = \sum_{n\geq 1} c_n \sum_{l=0}^{n-1}[\cos\theta - \eta - (-1)^{n-l}(\eta + \cos\theta)]R^l P_l, \tag{30}$$

$$\hat{V}_2 = \sum_{n,k\geq 1}\frac{nkc_nc_k}{n+k}R^{n+k}(P_nP_k - P_{n-1}P_{k-1}). \tag{31}$$

An equilibrium of the black hole with respect to the distortion fields means that the distortion field $\hat{U}$ takes the same values at the points of the symmetry axis on the black hole outer horizon (see, e.g., [23]),

$$\hat{U}(\eta = 1, \theta = 0) = \hat{U}(\eta = 1, \theta = \pi) \equiv u_0. \tag{32}$$

We can rewrite this condition in terms of the multipole moments. Using (28) and (29) and the property of the Legendre polynomials,

$$P_n(\pm 1) = (\pm 1)^n, \tag{33}$$

the equilibrium condition reads

$$\sum_{n\geq 0} c_{2n+1} = 0, \tag{34}$$

and one has

$$u_0 = \sum_{n\geq 0} c_n = \sum_{n\geq 0} c_{2n}. \tag{35}$$

Thus, a static, axisymmetric, distorted black hole is at equilibrium if the sum of odd multipole moments of the distortion vanishes. The equilibrium condition implies the local flatness (absence of conical singularities) along the symmetry axis of the black hole. Namely,

$$\hat{V}(\eta, \theta = 0) = \hat{V}(\eta, \theta = \pi) = 0. \tag{36}$$

To obtain a charged version of the distorted black hole it is sufficient to derive U and Φ from $\bar{U}$ [see, (23) and (28)] using the Harrison-Ernst transformation (21). We have

$$e^{2U} = \frac{4p^2(\eta^2-1)e^{2\hat{U}}}{[(1+p)(\eta+1) - (1-p)(\eta-1)e^{2\hat{U}}]^2}, \tag{37}$$

$$\Phi = \frac{\sqrt{1-p^2}[\eta + 1 - (\eta-1)e^{2\hat{U}}]}{(1+p)(\eta+1) - (1-p)(\eta-1)e^{2\hat{U}}}. \tag{38}$$

Remarkably, the Harrison-Ernst transformation does not alter Eqs. (25) and (26). Thus, U and Φ, given by (37) and (38), and $\bar{V}$, which is determined by (23), (30), and (31), solve the corresponding Einstein-Maxwell equations. The axisymmetric distorted RN solution is given by (9) and (10) with (37) and (38) and $V = \bar{V}$. A more general case of a distorted, electrically charged, rotating black hole is considered in [30].

D. Dimensionless form of the metric

The black hole metric contains only one essential dimensional parameter, say its mass, while all other parameters can be presented in dimensionless form. It is convenient to write metric (9) in the following dimensionless form adopted to the black hole horizons $\mathcal{H}^{(\pm)}$:

$$ds^2 = \Omega_\pm^2 dS_\pm^2, \tag{39}$$

$$dS_\pm^2 = -\frac{\eta^2 - 1}{\Delta_\pm} e^{2\mathcal{U}} dT_\pm^2 + \frac{\Delta_\pm}{\eta^2 - 1} e^{-2\mathcal{U}+2\hat{V}} d\eta^2 + \Delta_\pm e^{-2\mathcal{U}}(e^{2\hat{V}} d\theta^2 + \sin^2\theta d\phi^2), \tag{40}$$

$$\Omega_\pm = M(1 \pm p)e^{\mp u_0} = M'(1 \pm p'). \tag{41}$$

For the dimensionless metric $dS_\pm^2$ we defined $T_\pm = \kappa_\pm t$, where $\kappa_\pm$ is the surface gravity, which is given by

$$\kappa_\pm = \frac{(1 + p')e^{u_0} - (1 - p')e^{-u_0}}{2M'(1 \pm p')^2}. \tag{42}$$

We also use the following expressions for the metric functions $\Delta_\pm$ and $\mathcal{U}$

$$\Delta_\pm = \frac{\delta^{\pm 1}}{4\delta}[\eta + 1 - \delta e^{2\mathcal{U}}(\eta - 1)]^2, \tag{43}$$

$$\mathcal{U} = \hat{U} - u_0, \qquad \delta = \delta_0 e^{2u_0} = \frac{1 - p}{1 + p} e^{2u_0} = \frac{1 - p'}{1 + p'}. \tag{44}$$

Together with the original parameters M and p it is convenient to use the related parameters

$$M' = \frac{M}{2}[(1 + p)e^{-u_0} + (1 - p)e^{u_0}], \tag{45}$$

$$p' = \frac{\sqrt{M'^2 - Q^2}}{M'}. \tag{46}$$

In the absence of distortion $M' = M$ is the Komar mass of the RN black hole measured at asymptotic infinity. In the case $Q = 0$, M' is the local mass of a distorted Schwarzschild black hole defined in [23].

The coordinate η changes from $\eta = \infty$ (a spatial infinity) to the region of $\eta < -1$ where the space-time singularity is located (see, subsec. II E). As in the case of the RN black hole (7), the horizons of metric (40) are defined by $\eta = \eta_\pm = \pm 1$. As we mentioned earlier, we shall use the notation $\mathcal{H}^{(\pm)}$ for the outer (+), and for the inner (−) horizons. To indicate that a dimensional quantity $(\dots)$ is calculated at the black hole horizons $\mathcal{H}^{(\pm)}$, we shall use a superscript (±), and denote this quantity as $(\dots)^{(\pm)}$ [40].

As we shall see in the next section, the form of metric (39) is convenient for the analysis and comparison of the properties of the inner and outer black hole horizons. 2D metrics on the horizon surfaces can be obtained by taking $T =$ const, and $\eta = \eta_\pm = \pm 1$ in the metric. In the next section, we show that the surface area of the outer (event) horizon calculated for the dimensionless metric dS_+^2 is equal to 4π. Similarly, the surface area of the inner (Cauchy) horizon calculated for the metric dS_-^2 is also equal 4π. These normalization conditions specify the form of the conformal factor $\Omega_\pm$ in (39). The "real" (dimensional) areas of the horizon surfaces are

$$\mathcal{A}^{(\pm)} = 4\pi\Omega_\pm^2, \tag{47}$$

and the ratio of these areas is

$$\mathcal{A}^{(+)}/\mathcal{A}^{(-)} = (\Omega_+/\Omega_-)^2 = \left(\frac{1 + p'}{1 - p'}\right)^2 \equiv \delta^{-2}. \tag{48}$$

In what follows, we shall discuss different geometrical objects, such as the Kretschmann invariant $\mathcal{K}$, the Weyl scalar $\mathcal{C}^2$,

$$\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \qquad \mathcal{C}^2 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \tag{49}$$

and the Gaussian curvature of the 2D horizon surface K. We shall use the same notations with an index ± for an object calculated for the metric $dS_\pm^2$. One has

$$\mathcal{K} = \Omega_\pm^{-4}\mathcal{K}_\pm, \qquad \mathcal{C}^2 = \Omega_\pm^{-4}\mathcal{C}_\pm^2, \qquad K = \Omega_\pm^{-2}K_\pm. \tag{50}$$

To study the interior region we can use any of these two forms of the dimensionless metric $dS_\pm^2$. Certainly, the "physical" result, calculated for the metric ds^2 will be the same.

The dimensionless electrostatic potential for metric (40) is given by

$$\Phi_\pm = \frac{\sqrt{\delta}\Delta_\pm^{-1/2}}{(e^{2u_0} - \delta)}[\eta + 1 - (\eta - 1)e^{2\mathcal{U}+2u_0}]. \tag{51}$$

It is related to the electrostatic potential (38) as follows

$$\Phi = \Omega_\pm \kappa_\pm \Phi_\pm. \tag{52}$$

The nonvanishing dimensionless components of the electromagnetic field $F_{\mu\nu}$ are defined by

$$F_{\pm T_\pm \eta} = \Phi_{\pm,\eta} = \frac{\delta^{\pm 1/2}}{\Delta_\pm} e^{2\mathcal{U}}[(1 - \eta^2)\mathcal{U}_{,\eta} - 1], \tag{53}$$

$$F_{\pm T_\pm \theta} = \Phi_{\pm,\theta} = \frac{\delta^{\pm 1/2}}{\Delta_\pm} e^{2\mathcal{U}}(1 - \eta^2)\mathcal{U}_{,\theta}. \tag{54}$$

E. Singularities

In this paper, we mainly focus on the study of the horizons $\mathcal{H}^{(\pm)}$, and the inner domain located between the horizons. Since one cannot trust the metric obtained by the analytical continuation of the exterior metric beyond the inner (Cauchy) horizon, it is reasonable to postpone study of the regions close to the space-time singularity until the classical and quantum (in)stability will be proved. For this reason, we give only a couple of remarks about properties of the singularities in the analytic continuation of the charged distorted black hole solution.

The curvature and the electromagnetic field invariants diverge for $\Delta_\pm = 0$, i.e., for

$$\eta = -\frac{1 + \delta_0 e^{2\hat{U}}}{1 - \delta_0 e^{2\hat{U}}}, \tag{55}$$

indicating the space-time singularity. For the RN black hole, the singularity is located at $\eta = -1/p$, $p \in (0, 1]$, corresponding to $r = 0$. Analyzing expression (55) we see that for $\hat{U} \leq 0$ the singularity is located in the region $\eta < -1$, whereas for $\hat{U} > 0$ the space-time singularity is naked and located outside the outer horizon, $\eta > 1$. Thus, if the distortion field $\hat{U}$ satisfies the strong energy conditions, i.e., $\hat{U} \leq 0$, the space-time outside the black hole outer horizon is regular, and the singularity is located behind the inner (Cauchy) horizon.

III. DUALITY RELATIONS BETWEEN THE INNER AND OUTER HORIZONS

In this section, we describe special symmetry relations between the inner and outer horizons. Consider a 2D subspace $T_\pm = \text{const}$, $\phi = \text{const}$ orthogonal to the corresponding Killing vectors. In the coordinates

$$\eta = \cos\psi, \qquad \psi \in [0, \pi] \tag{56}$$

the subspace metric is

$$d\Sigma_\pm^2 = \Delta_\pm e^{-2\mathcal{U}+2\hat{V}}[-d\psi^2 + d\theta^2]. \tag{57}$$

Figure 1 illustrates the Carter-Penrose diagram for these metrics. Lines $\psi \pm \theta = \text{const}$ are null rays propagating from the outer to the inner horizon within the 2D subspace. One of such null rays is shown in the figure. It starts at point A on the outer horizon $\mathcal{H}^{(+)}$, goes through the "north pole" at $\theta = \pi$, and reaches point B at the inner horizon $\mathcal{H}^{(-)}$.

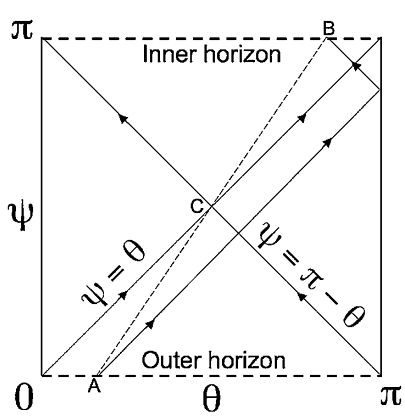

FIG. 1. The Carter-Penrose diagram for (ψ, θ) subspace of the charged distorted black hole interior. The arrows illustrate propagation of future directed null rays. Points A and B are symmetric with respect to the central point $C(\pi/2, \pi/2)$.

Consider a transformation R_C representing the reflection of coordinates (ψ, θ) with respect to the "central point" C in the interior region

$$R_C\colon (\psi, \theta) \to (\pi - \psi, \pi - \theta). \tag{58}$$

This transformation determines a map R_C^* between functions defined in the inner domain and on its boundaries

$$f^* = R_C^*(f), \qquad f^*(\psi, \theta) = f(\pi - \psi, \pi - \theta). \tag{59}$$

Using the relations (28)–(31) we obtain

$$\mathcal{U}^*(\psi, \theta) \equiv \mathcal{U}(\pi - \psi, \pi - \theta) = \mathcal{U}(\psi, \theta), \tag{60}$$

$$\hat{V}_1^*(\psi, \theta) \equiv \hat{V}_1(\pi - \psi, \pi - \theta) = -\hat{V}_1(\psi, \theta), \tag{61}$$

$$\hat{V}_2^*(\psi, \theta) \equiv \hat{V}_2(\pi - \psi, \pi - \theta) = \hat{V}_2(\psi, \theta). \tag{62}$$

It is easy to see that points A and B connected by a null ray (see Fig. 1) are related by the reflection R_C. Thus, the transformation R_C^* determines a map between functions on the inner and outer horizons. Now we demonstrate that for $\mathcal{U}$ and $\hat{V}$ this is a symmetry transformation. In other words, the values of $\mathcal{U}$ and $\hat{V}$ on the inner horizon, $\psi = \pi$, are determined by their values on the outer horizon, $\psi = 0$.

Using (60), (28), and (29) and the properties of the Legendre polynomials (33) we derive

$$\mathcal{U}(\pi, \pi - \theta) = \mathcal{U}(0, \theta) = \sum_{n\geq 0} c_n \cos^n\theta - u_0. \tag{63}$$

Expressions (29)–(31) and (33) give

$$\begin{aligned} \hat{V}_1(0, \theta) &= -(1 - \cos\theta)\sum_{n\geq 1} c_n \sum_{l=0}^{n-1} \cos^l\theta \\ &\quad - (1 + \cos\theta)\sum_{n\geq 1} (-1)^n c_n \sum_{l=0}^{n-1} (-\cos\theta)^l \\ &= 2\mathcal{U}(0, \theta), \\ \hat{V}_2(0, \theta) &= 0. \end{aligned} \tag{64}$$

Thus, using (61) and (62) we have

$$\hat{V}(\pi, \pi - \theta) = -\hat{V}(0, \theta) = -2\mathcal{U}(0, \theta). \tag{65}$$

The above expressions (63) and (65) allow one to establish special symmetry relations between the geometric properties of the inner and outer horizons. We call relations (63) and (65) the *duality relations*.

Let us denote

$$u_\pm(\theta) = \sum_{n\geq 0} (\pm 1)^n c_n \cos^n\theta - u_0. \tag{66}$$

As we shall see below, this function defines boundary values of the distortion fields, and as a result, the metric on the black hole horizons. It is easy to check that

$$u_{\pm}(\theta) = u_{\mp}(\pi - \theta), \qquad u_{\pm}(0) = u_{\pm}(\pi) = 0. \tag{67}$$

Expression (66) implies that the functions $u_+(\theta)$ and $u_-(\theta)$ transform into each other under reflection with respect to the point $\theta = \pi/2$. This transformation property is directly related to the properties of the distortion field $\mathcal{U}$. Namely, using (63), (67), (28), and (29) we derive the following *boundary values* of $\mathcal{U}$

$$\mathcal{U}(0, \theta) = u_+(\theta), \qquad \mathcal{U}(\pi, \theta) = u_-(\theta), \tag{68}$$

$$\mathcal{U}(\psi, 0) = u_+(\psi), \qquad \mathcal{U}(\psi, \pi) = u_-(\psi). \tag{69}$$

Analogously, using (65), (68), (67), and (36) we derive the *boundary values* of $\hat{V}$

$$\hat{V}(0, \theta) = 2u_+(\theta), \qquad \hat{V}(\pi, \theta) = -2u_-(\theta), \tag{70}$$

$$\hat{V}(\psi, 0) = 0, \qquad \hat{V}(\psi, \pi) = 0. \tag{71}$$

Thus, the distortion fields calculated on the inner horizon are expressed through those calculated on the outer horizon. This fact allows one to make important conclusions about the distortion of the Cauchy horizon.

The boundary values of the distortion fields $\mathcal{U}$ and $\hat{V}$ define symmetry properties of the metrics on the black hole horizon surfaces. The surface of the outer and the inner horizon is defined by $T_\pm = \text{const}$ and $\eta = \eta_\pm = \pm 1$, respectively. The corresponding dimensionless metrics derived from metric (40) by applying (56) and the boundary conditions (68) and (70) are

$$d\sigma_\pm^2 = e^{\pm 2u_\pm} d\theta^2 + e^{\mp 2u_\pm} \sin^2\theta d\phi^2. \tag{72}$$

The dimensional metrics on the horizon surfaces are [see (39)]

$$d\sigma^{(\pm)2} = \Omega_\pm^2 d\sigma_\pm^2. \tag{73}$$

Here, and in what follows $u_\pm \equiv u_\pm(\theta)$. The metric $d\sigma_+^2$ coincides with the metric on the distorted Schwarzschild black hole horizon surface [1]. The dimensionless areas of the horizon surfaces are

$$\mathcal{A}^{(+)} = \mathcal{A}^{(-)} = 4\pi. \tag{74}$$

The metrics $d\sigma_+^2$ and $d\sigma_-^2$ are related to each other by the transformation

$$u_+ \leftrightarrow -u_-, \tag{75}$$

which according to (66) implies the following *duality relations* between the outer and the inner horizons

$$c_{2n} \leftrightarrow -c_{2n}, \qquad c_{2n+1} \leftrightarrow c_{2n+1}. \tag{76}$$

Thus, the metrics $dS_\pm^2$ are identical for distortions which have only odd multipole moments. The derived duality relations imply, in particular, that the inner (Cauchy) horizon of a distorted charged black hole solution obtained by the Harrison-Ernst transformation is regular, if the outer horizon is regular. This conclusion and its generalization to the case of rotating and charged black holes was proven recently in [41,42].

IV. GAUSSIAN CURVATURE

In this section we discuss geometry of the distorted horizon surfaces. Gaussian curvature is a natural measure of intrinsic curvature of a 2D surface. It is equal to 1/2 of its scalar curvature. Gaussian curvature of a horizon surface was studied by several authors (e.g., [43–46]). For the metric (72) the Gaussian curvature is given by

$$K_\pm = e^{\mp 2u_\pm}[1 \pm u_{\pm,\theta\theta} \pm 3\cot\theta u_{\pm,\theta} - 2u_{\pm,\theta}^2]. \tag{77}$$

The dimensional Gaussian curvatures associated with metrics (73) are

$$K^{(\pm)} = \Omega_\pm^{-2} K_\pm. \tag{78}$$

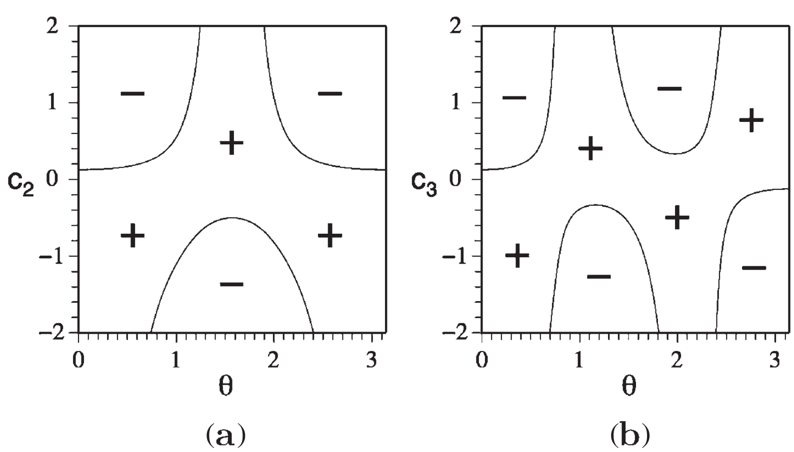

FIG. 2. Regions of positive and negative Gaussian curvature for the outer horizon surface. Plot (a) illustrates the regions for different values of the quadrupole moment. Plot (b) illustrates the regions for different values of the octupole moment. Curves separating these regions correspond to zero Gaussian curvature.

We shall illustrate our analysis of the charged distorted black hole considering simple examples of the lowest order multipole distortions. Namely, we shall consider quadrupole and octupole distortions for which the corresponding functions $u_\pm$ read

$$u_\pm = -c_2\sin^2\theta, \qquad u_\pm = \mp c_3\sin^2\theta\cos\theta. \tag{79}$$

Here, c_2 and c_3 are the quadrupole and the octupole moments, respectively.

Regions of positive and negative Gaussian curvature for different values of the quadrupole and octupole moments, for the outer horizon surface, are presented in Fig. 2. From the figure we see that for the quadrupole distortion regions of negative Gaussian curvature near the black hole poles ($\theta = 0, \pi$) correspond to high positive values of c_2, and near its equator ($\theta = \pi/2$) to high negative values of c_2. Using (77) and (35) and the auxiliary expressions

$$u_{\pm,\theta}(\theta) = -\sum_{n\geq 0}(\pm 1)^n c_n n \sin\theta\cos^{n-1}\theta, \tag{80}$$

$$u_{\pm,\theta\theta}(\theta) = \sum_{n\geq 0}(\pm 1)^n c_n n\cos^{n-2}\theta[n\sin^2\theta - 1], \tag{81}$$

we derive

$$K_\pm|_{\theta=0} = 1 \pm 4u_\pm^{(2)}, \qquad K_\pm|_{\theta=\pi} = 1 \pm 4u_\mp^{(2)}, \tag{82}$$

$$K_\pm|_{\theta=\pi/2} = e^{\pm 2(u_0 - c_0)}(1 \pm 2c_2 - 2c_3^2). \tag{83}$$

Here,

$$u_\pm^{(2)} = -\sum_{n\geq 0}(\pm 1)^n c_n n. \tag{84}$$

Thus, the sign of the Gaussian curvature strictly depends on the distortion field. Using these expressions, we derive that for the quadrupole distortion Gaussian curvature of the outer horizon surface is positive at the poles for $c_2 < 1/8$, and on the equator for $c_2 > -1/2$. According to the duality relations (76), regions of positive and negative Gaussian curvature of the inner horizon surface can be constructed by mirror reflection of Fig. 2 with respect to the line $c_2 = 0$.

Figure 2(b) illustrates that there is a symmetry between the regions of positive and negative Gaussian curvature and signs of the octupole moment. Namely, the transformation $c_3 \to -c_3$, $\theta \to \pi/2 - \theta$ leave the figure unchanged. Using (82) we derive that for $c_3 > 1/8$ Gaussian curvature is negative on the "north" pole and positive on the "south" pole, whereas for $c_3 < -1/8$ it is negative on the south pole and positive on the north. In addition, there are the regions of negative Gaussian curvature near the "tropics" ($\pm 23°26'22''$ from the equator), i.e., near $\theta_- \approx 1.165$ (corresponding to $\approx 23°16'39''$ from the equator) for $c_3 < -0.333$, and $\theta_+ \approx 1.977$ (corresponding to $\approx -23°16'39''$ from the equator) for $c_3 > 0.333$. According to the duality relations (76) Gaussian curvature of the inner horizon surface is identical to that of the outer horizon surface. Dimensionless Gaussian curvature of the outer horizon surface for certain values of the quadrupole and octupole moments is plotted in Fig. 3.

As we shall see in Sec. VII, the curvature and the electromagnetic field invariants calculated on and at the vicinity of the black hole horizons are expressed in terms of the corresponding Gaussian curvatures and their derivatives.

V. EMBEDDING

To visualize the distorted horizon surfaces, we present their isometric embedding into a flat 3D space. To construct the embedding we consider an axisymmetric 2D surface parametrized as follows:

$$\rho = \rho(\theta), \qquad z = z(\theta). \tag{85}$$

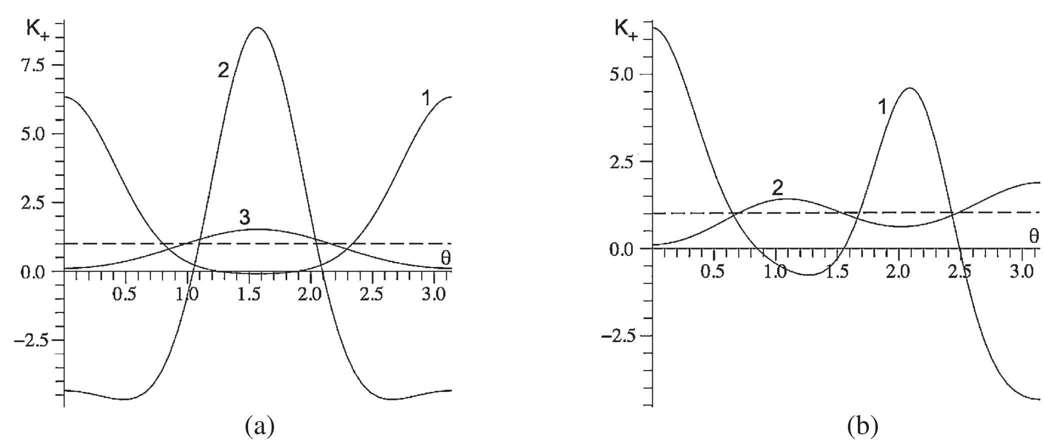

FIG. 3. Dimensionless Gaussian curvature K_+ of the outer horizon surface. (a) The quadrupole distortion: $c_2 = -2/3$ (line 1), $c_2 = 2/3$ (line 2), and $c_2 = 1/9$ (line 3). (b) The octupole distortion: $c_3 = -2/3$ (line 1), and $c_3 = 1/9$ (line 2). Dashed horizontal lines of $K_+ = 1$ correspond to the RN black hole.

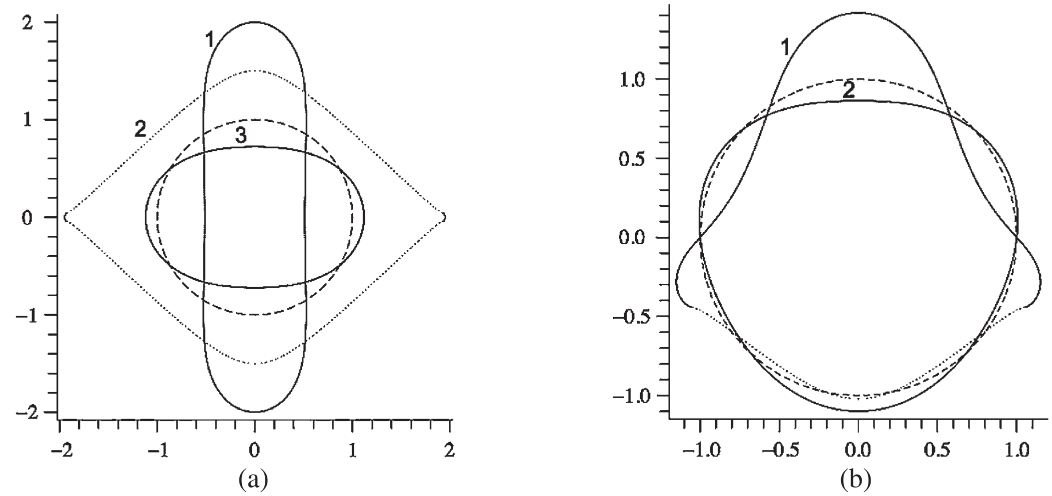

FIG. 4. The shape of the outer horizon surface. The shape curves are shown in the (ρ, z) plane. (a) The quadrupole distortion: $c_2 = -2/3$ (line 1), $c_2 = 2/3$ (line 2), and $c_2 = 1/9$ (line 3). (b) The octupole distortion: $c_3 = -2/3$ (line 1), and $c_3 = 1/9$ (line 2). Regions embedded into pseudo-Euclidian space are illustrated by doted lines. Dashed circles of radius 1 correspond to the RN black hole.

Let us embed this surface into a flat 3D space with the metric in cylindrical coordinates (z, ρ, ϕ):

$$dl^2 = \epsilon dz^2 + d\rho^2 + \rho^2 d\phi^2, \tag{86}$$

where for Euclidean space $\epsilon = 1$, and for pseudo-Euclidean space $\epsilon = -1$ [47]. The geometry induced on the surface is given by

$$dl^2 = (\epsilon z_{,\theta}^2 + \rho_{,\theta}^2)d\theta^2 + \rho^2 d\phi^2. \tag{87}$$

Matching metrics (72) and (87) we derive the following embedding map:

$$\rho = e^{\mp u_\pm}\sin\theta, \qquad z = \int_\theta^{\pi/2} \mathcal{Z} d\theta, \tag{88}$$

$$\mathcal{Z}^2 = \epsilon e^{\pm 2u_\pm}[1 - e^{\mp 4u_\pm}(\cos\theta \mp u_{\pm,\theta}\sin\theta)^2]. \tag{89}$$

From (89) we see that if the expression in the square brackets is negative, an isometric embedding into 3D Euclidean space is not possible, and we should take $\epsilon = -1$.

According to the duality relations (76) it is enough to consider embedding of the outer horizon surface only. The shape curves of the outer horizon surface are presented in Fig. 4. The embedding diagrams for the outer horizon surface can be obtained by rotation of the curves around the vertical axis of symmetry lying in the plane of the figure, parallel to the z axis. Note, that the change in sign from "+" to "−" of the quadrupole moment corresponds to deformation of the rotational curve from oblate to prolate and vice versa. This transformation corresponds to the duality relations (76) between the outer and inner horizon surfaces. The change in sign of the octupole moment corresponds to overturn of the rotational curve preserving its shape.

VI. FREE FALL FROM THE OUTER TO THE INNER HORIZON

It is interesting to check how the distortion changes the maximal proper time of a free fall of a test particle from the outer to the inner horizon. Let us consider motion of a test particle of zero angular momentum which moves from the outer to the inner horizon along the axis of symmetry. Free fall from the north pole corresponds to $\theta = 0$, and free fall from the south pole corresponds to $\theta = \pi$. We use metric (39) with dS_+^2. Using (36) we derive the proper time of the free fall:

$$\tau(E) = \Omega_+ \int_{-1}^{+1} \frac{\Delta_+^{1/2} e^{-\mathcal{U}} d\eta}{(\Omega_+^{-2}\Delta_+ e^{-2\mathcal{U}}E^2 + 1 - \eta^2)^{1/2}}\Bigg|_{\theta=0,\pi}, \tag{90}$$

where E is the energy of the particle,

$$E = \Omega_+^2 \frac{\eta^2 - 1}{\Delta_+} e^{2\mathcal{U}} \frac{dT_+}{d\tau}. \tag{91}$$

The maximal proper time corresponds to $E = 0$. Using the coordinate transformation (56) and applying (69) we derive the maximal proper time for the free fall

$$\tau_{\max} = \tau(0) = \tau_+\Omega_+, \tag{92}$$

where the dimensionless time τ_+ is

$$\tau_+ = \int_0^\pi \frac{d\psi}{2}[(\cos\psi + 1)e^{-u(\psi)} - \delta e^{u(\psi)}(\cos\psi - 1)]. \tag{93}$$

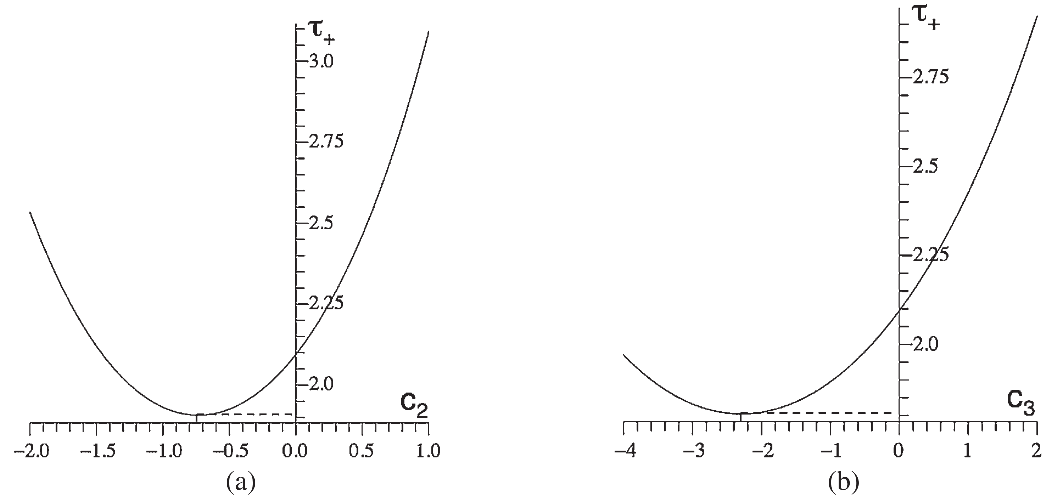

FIG. 5. The free fall along the axis of symmetry from the outer to the inner horizon surface for $p' = 1/2$. (a) The dimensionless proper time τ_+ for different values of the quadrupole moment c_2. Here, the minimal value of the dimensionless proper time $\tau_{+\,\mathrm{min}} \approx 1.907$ corresponds to $c_{2\,\mathrm{min}} \approx -0.734$. (b) The dimensionless proper time τ_+ for different values of the octupole moment c_3, for the fall from the north pole. Here, the minimal value of the dimensionless proper time $\tau_{+\,\mathrm{min}} \approx 1.804$ corresponds to $c_{3\,\mathrm{min}} \approx -2.292$, where $c_{3\,\mathrm{min}}$ does not depend on the value of p'. For the RN black hole $\tau_+ = 2\pi/3 \approx 2.094$.

Here, $u(\psi) = u_+(\psi)$ for the fall from the north pole, and $u(\psi) = u_-(\psi)$ for the fall from the south pole. For the RN black hole we have $\tau_+ = \pi/(1 + p)$, and $\tau_{\mathrm{max}} = \pi M$, that is exactly the same as the maximal proper time for a free fall from event horizon to the singularity of the Schwarzschild black hole of mass M ([28], p. 836).

In the case of the quadrupole distortion (79) the integral in (93) can be calculated analytically:

$$\tau_+ = \frac{\pi}{2} I_0(c_2/2)[e^{c_2/2} + \delta e^{-c_2/2}], \tag{94}$$

where $I_0(x)$ is the modified Bessel function. Note, that because of the reflection symmetry of the horizon surfaces with respect to the plane $\theta = \pi/2$ the proper time is the same for the fall from the north and south poles. For the octupole distortion we evaluate the integral numerically. From expressions (79) and (93) we see that the change in sign of the octupole moment corresponds to the change of the poles as the starting points of the fall. The dimensionless proper time calculated for $p' = 1/2$ is presented in Fig. 5

VII. THE SPACE-TIME INVARIANTS

For distorted vacuum black holes there exists a remarkable relation between the Kretschmann scalar calculated on the surface of the event horizon $\mathcal{K}^{(+)}$ and the Gaussian curvature of the horizon $K^{(+)}$ calculated at the same point

$$\mathcal{K}^{(+)} = 12 K^{(+)^2}. \tag{95}$$

The proof of this relation can be found in [1]. This relation shows that the 4D curvature invariant of the space-time calculated on the horizon is correlated with the shape of the horizon surface. In a region where the horizon is sharper the 4D curvature invariant is larger than in a region where the horizon is smoothed out. In order to prove the property (95) one uses the fact that the horizon $\mathcal{H}^{(+)}$ surface is a *totally geodesic* surface.

The general analysis by Boyer [49], and, in particular, his conclusion saying that a bifurcate Killing horizon contains a totally geodesic 2D surface, which is in fact independent of the field equations, can be applied to the case of the charged distorted black hole. For this reason one can expect the existence of a relation similar to (95) and generalizing the latter. In this section, we discuss this problem.

First of all, let us emphasize that in the presence of the electromagnetic field $F_{\alpha\beta}$ there exist an additional 4D invariant $F^2 = F_{\alpha\beta}F^{\alpha\beta}$ characterizing the strength of the field. For the distorted black hole the calculations give the following value of this invariant on the outer horizon [see (53), (54), (52), (68), and (70)]

$$F^{(+)^2} = -\frac{2}{M'^2}\frac{(1 - p')}{(1 + p')^3}. \tag{96}$$

The minus sign on the right-hand side reflects the fact that we are dealing with an electric (not magnetic) field. The Kretschmann scalar $\mathcal{K}$ and the Weyl invariant $\mathcal{C}^2$ are related as follows:

$$\mathcal{K} = \mathcal{C}^2 + 2(F^2)^2. \tag{97}$$

In the presence of matter, in order to characterize the "strength" of the gravitational field, it is more convenient to use the Weyl invariant. The calculations presented in the Appendix give for the Weyl invariant on the event horizon the following expression:

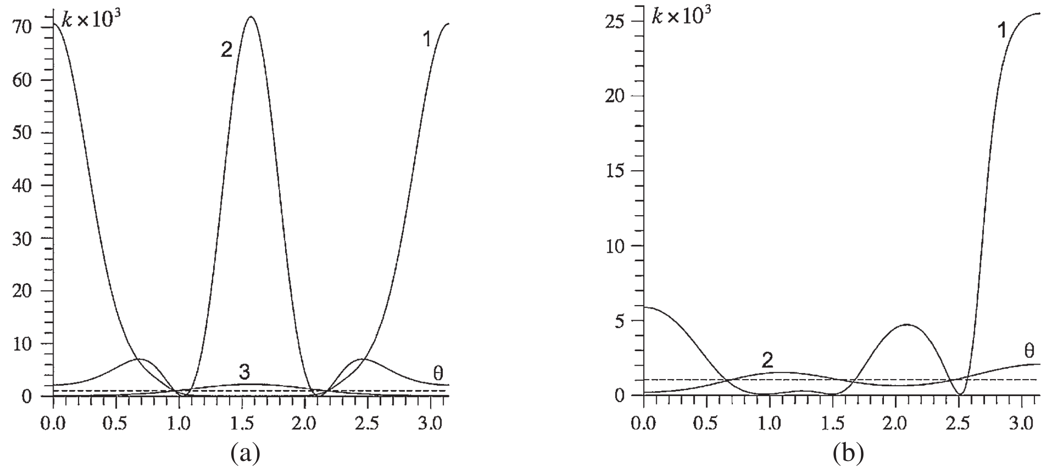

FIG. 6. The ratio k for $p' = 1/2$. Plot (a) illustrates the ratio for the quadrupole distortion of $c_2 = -2/3$ (line 1), $c_2 = 2/3$ (line 2), and $c_2 = 1/9$ (line 3). Plot (b) illustrates the ratio for the octupole distortion of $c_3 = -2/3$ (line 1), and $c_3 = 1/9$ (line 2). The dashed horizontal line corresponds to the RN black hole.

$$\mathcal{C}^{2(+)} = 12[K^{(+)} - \tfrac{1}{2}F^{(+)^2}]^2. \tag{98}$$

It is evident that in vacuum, when F^2 vanishes and the Kretschmann invariant coincides with the Weyl invariant, this relation reduces to (95). The second term in the square brackets is constant on the horizon [see the Appendix and Eq. (102) below]. Hence, in the presence of the electrostatic field, the Gaussian curvature of the horizon surface is, effectively, uniformly shifted by a positive value.

Similar relations are valid for the inner horizon

$$F^{(-)^2} = -\frac{2}{M'^2}\frac{(1+p')}{(1-p')^3}, \tag{99}$$

$$\mathcal{C}^{2(-)} = 12[K^{(-)} - \tfrac{1}{2}F^{(-)^2}]^2. \tag{100}$$

Using (97) we can calculate the ratio of the Kretschmann invariants on the black hole horizons:

$$k = \frac{\mathcal{K}^{(+)}}{\mathcal{K}^{(-)}} = \delta^4 \frac{3(K_+ + \delta)^2 + 2\delta^2}{3(K_- + \delta^{-1})^2 + 2\delta^{-2}}. \tag{101}$$

This ratio calculated for $p' = 1/2$ is presented in Fig. 6 below. The behavior of the curves is very similar to those for the Gaussian curvature illustrated in Fig. 3.

Finally, we present the expressions for the curvature and the electromagnetic field invariants at the vicinity of the black hole horizons. We use the results of the Appendix (A42), (A38), and (A43). The expansion of the electromagnetic field invariant near the black hole horizons reads

$$F_{\pm}^2 = -2\delta^{\pm 1} \pm 4\delta^{\pm 1}e^{\pm 2u_{\pm}}(K_{\pm} - \delta^{\pm 1})(\eta \mp 1) + \dots \tag{102}$$

The expansion of the Weyl invariant near the black hole horizons is

$$\mathcal{C}_{\pm}^2 = 12K_{e\pm}^2 \mp 4(3K_{e\pm}^2[3K_{\pm} - 2\delta^{\pm 1}]e^{\pm 2u_{\pm}} - 2[K_{\pm,\theta}]^2 + 3K_{e\pm}[K_{\pm,\theta\theta} + \cot\theta K_{\pm,\theta}])(\eta \mp 1) + \dots, \tag{103}$$

where $K_{e\pm} = (K_{\pm} - \delta^{\pm 1})$.

VIII. CONCLUSION

In this paper, we studied the interior of a distorted, static, axisymmetric, electrically charged black hole. The corresponding metric was derived by the Harrison-Ernst transformation applied to the metric of a distorted, static, axisymmetric vacuum black hole, whose interior was discussed in [1]. We established the special duality relations between the properties of the inner and outer horizons of the distorted charged black hole. These duality relations allow one to make a conclusion about the inner (Cauchy) horizon structure, which is based on the structure of the outer (event) horizon of the black hole. In particular, regions of positive and negative Gaussian curvature and its values on the outer horizon surface are correlated with those on the inner horizon surface. There is a correlation between the shapes of the horizon surfaces as well.

We derived expansion of the curvature and electromagnetic field invariants near the black hole horizons, which is expressed in terms of the Gaussian curvature, electrostatic field, and their derivatives calculated on the horizon surfaces. Thus, the established duality relations show that the space-time geometry near the inner (Cauchy) horizon is correlated with the space-time geometry near the outer (event) horizon. This implies that if the distortion leaves the outer horizon regular, the inner horizon remains regular as well.

The duality between the outer and inner horizons seems important. Apparently, according to the mass inflation phenomenon [9] such duality breaks in the case of dynamical perturbation of the RN black hole. Namely, due to the presence of the outgoing flux the inner apparent horizon and the Cauchy horizon become separated. The infinite grow of the mass parameter induced by the blueshift of the ingoing flux on the Cauchy horizon is not canceled by the redshift of the ingoing flux on the apparent horizon. As a result, the Cauchy horizon becomes singular. This does not happen in the case of static, axisymmetric distortion. One may think of the static distortion in the dynamical region between the black hole horizons as represented by standing waves. According to the duality relations between the horizons, initial and boundary values of the waves should be dual as well.

Quite possibly, the axisymmetric, static distortion due to remote charged masses and fields cannot affect much interior of the charged black hole. In such a situation nothing enters, or leaves (through the Cauchy horizon into other "universes") the black hole. Thus, the black hole inner horizon remains regular due to such a type of distortion. Nevertheless, as our analysis shows, such "serene" distortion can in fact deform the interior of the black hole to create regions of high local curvature. Moreover, the distortion noticeably affects the maximal proper time of a free fall of a test particle moving along the axis of symmetry in the black hole interior. An important question if the Cauchy horizon of an electrically charged black hole is regular for an arbitrary static, external distortion remains open.

ACKNOWLEDGMENTS

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APPENDIX: CALCULATION OF THE SPACE-TIME INVARIANTS NEAR THE BLACK HOLE HORIZONS

In this appendix, we obtain expressions for the curvature and electromagnetic field invariants near the black hole horizons. We start our construction in the regions where the Killing vector is timelike, namely, outside of the horizons. Final expressions of the invariants will be valid in the region between the horizons as well.

The simplest curvature invariant is the Kretschmann scalar, which for Einstein-Maxwell 4D space-time admits the following decomposition

$$\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \mathcal{C}^2 + 2R_{\alpha\beta}R^{\alpha\beta}, \tag{A1}$$

where $\mathcal{C}^2 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ is the Weyl scalar. The Weyl invariant characterizes properties of a "pure" gravitational field, while the square of the Ricci tensor $R_{\alpha\beta}R^{\alpha\beta}$ is determined in our case by the electrostatic field. In this appendix, we derive an expansion of these invariants near the black hole horizons for an arbitrary static, charged distorted black hole. In the main text of the paper, we shall use these results for a special case, when the static space-time is axisymmetric. A similar analysis for a vacuum distorted black hole can be found in [50].

It is convenient to start with the form of the metric proposed in [51]. Namely, we consider static space-time and denote timelike, hypersurface orthogonal Killing vector by $\boldsymbol{\xi}$. We assume that in the region under consideration $\nabla_\alpha(\boldsymbol{\xi}^2)$ does not vanish. Following [51] we write our metric, $g_{\alpha\beta}$ $(\alpha, \beta, \ldots = 0, \ldots, 3)$ in this region in the form

$$ds^2 = -k^2dt^2 + d\gamma^2,$$
$$d\gamma^2 = \gamma_{AB}dy^Ady^B = \kappa^{-2}(k, \theta^c)dk^2 + h_{ab}(k, \theta^c)d\theta^a d\theta^b. \tag{A2}$$

Here, $k = (-\xi_\alpha\xi^\alpha)^{1/2}$; $A, B, \ldots = 1, 2, 3$; $a, b, c, \ldots = 2, 3$,

$$\kappa^2 = -\tfrac{1}{2}(\nabla^\beta\xi^\alpha)(\nabla_\beta\xi_\alpha), \tag{A3}$$

and h_{ab} is the metric on "equipotential" 2D surfaces $k =$ const spanned by θ^a coordinates. At the horizon of a static black hole, that is for $k = 0$, κ coincides with the surface gravity. In a static space-time the Weyl invariant can be written as follows [52]:

$$\begin{aligned}\mathcal{C}^2 &\equiv C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\\ &= 8\Pi_{\alpha\beta}\Pi^{\alpha\beta} + 8\Pi_{\alpha\beta}\Lambda^{\alpha\beta} + 4\Lambda_{\alpha\beta}\Lambda^{\alpha\beta}\\ &\quad - (\Pi + \Lambda)^2 - 2R_{\alpha\beta}R^{\alpha\beta},\end{aligned} \tag{A4}$$

where

$$\Pi_{\alpha\beta} = R_{\alpha\gamma\delta\beta}\zeta^{\gamma\delta}, \qquad \Pi \equiv \Pi_\alpha{}^\alpha = -\zeta_{\alpha\beta}R^{\alpha\beta}, \tag{A5}$$

$$\Lambda_{\alpha\beta} = R_{\alpha\beta} + \zeta_{\alpha\beta}\Pi, \qquad \Lambda \equiv \Lambda_\alpha{}^\alpha = R + \Pi. \tag{A6}$$

Here $\zeta_{\gamma\delta} = -\xi_\gamma\xi_\delta/\boldsymbol{\xi}^2$. For a static space-time $\Pi_{00} = \Pi_{0A} = 0$. To calculate $\mathcal{C}^2$, it is convenient to use the Gauss-Codazzi equations

$$R_{ABCD} = \mathcal{R}_{ABCD} + \varepsilon[\mathcal{S}_{AD}\mathcal{S}_{BC} - \mathcal{S}_{AC}\mathcal{S}_{BD}], \tag{A7}$$

$$n^\alpha R_{\alpha BCD} = \mathcal{S}_{BC|D} - \mathcal{S}_{BD|C}, \tag{A8}$$

$$\begin{aligned}kR_{A\gamma\delta B}n^\gamma n^\delta &= -k\Pi_{AB}\\ &= \gamma_{AC}\mathcal{S}_B{}^C{}_{,t} + \varepsilon k_{|AB} + k\mathcal{S}_{AC}\mathcal{S}_B{}^C.\end{aligned} \tag{A9}$$

Here $n^\alpha = \xi^\alpha/k$ is the unit normal to hypersurface $t =$ const, $\varepsilon = \boldsymbol{n}^2 = -1$, $\mathcal{S}_{AB}$ is the extrinsic 3D curvature of a hypersurface $t =$ const, $\mathcal{R}_{ABCD}$ is its 3D intrinsic curvature defined with respect to the metric $d\gamma^2$, while $\mathcal{R}$ is the 3D scalar curvature. The stroke stands for a covariant derivative with respect to this metric.

Relations (A7)–(A9) imply

$$2G_{\alpha\beta}n^\alpha n^\beta = -\varepsilon\mathcal{R} - \mathcal{S}_{AB}\mathcal{S}^{AB} + \mathcal{S}^2, \tag{A10}$$

$$R_{\alpha\beta}n^\alpha n^\beta = -\mathcal{S}_{AB}\mathcal{S}^{AB} - \varepsilon k^{-1}k_{|A}{}^{|A} - k^{-1}\mathcal{S}_{,t}, \tag{A11}$$

$$G_{\alpha B}n^\alpha = R_{\alpha B}n^\alpha = -\mathcal{S}_{,B} + \mathcal{S}_B{}^C{}_{|C}, \tag{A12}$$

$$R_{AB} = \mathcal{R}_{AB} - \varepsilon\mathcal{S}\mathcal{S}_{AB} - k^{-1}k_{|AB} - \varepsilon k^{-1}\gamma_{AC}\mathcal{S}_B{}^C{}_{,t}. \tag{A13}$$

Here $\mathcal{S} = \gamma^{AB}\mathcal{S}_{AB}$ is twice the mean curvature. Since metric (A2) is static, the extrinsic curvature defined as

$$\mathcal{S}_{AB} = \tfrac{1}{2}k^{-1}\gamma_{AB,t} \tag{A14}$$

vanishes. Thus, (A7)–(A13) imply

$$\Pi_{AB} = k^{-1}k_{|AB}, \qquad \Pi = k^{-1}k_{|A}{}^{|A}, \tag{A15}$$

$$\Lambda_{AB} = \mathcal{R}_{AB} - k^{-1}k_{|AB}, \qquad \Lambda = \mathcal{R} - k^{-1}k_{|A}{}^{|A}, \tag{A16}$$

$$\Lambda_{00} = 0, \qquad \Lambda_{0A} = 0. \tag{A17}$$

The Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ give

$$\begin{aligned}\mathcal{R} &= 16k^{-2}\pi T_{00}, \qquad T_{0A} = 0,\\ G_{AB} &= 8\pi T_{AB} + k^{-1}k_{|AB} - k^{-1}\gamma_{AB}k_{|A}{}^{|A}.\end{aligned} \tag{A18}$$

Thus, the Weyl invariant (A4) written in terms of 3D objects related to hypersurface $t =$ const is

$$\mathcal{C}^2 = 2k^{-2}(k_{|AB}k^{|AB} - 3k_{|A}{}^{|A}k_{|B}{}^{|B}) + 2(\mathcal{R}_{AB} + 2k^{-1}k_{|AB})\mathcal{R}^{AB}. \tag{A19}$$

The next step is a (2 + 1) decomposition. We use the following expression for the 3D metric

$$d\gamma^2 = \kappa^{-2}(k, \theta^c)dk^2 + h_{ab}(k, \theta^c)d\theta^a d\theta^b. \tag{A20}$$

We denote a covariant derivative with respect to the 2D metric h_{ab} as $(\ldots)_{:a}$. A unit vector orthogonal to the equipotential 2D surface $k =$ const is $n^A = \kappa\delta^A{}_k$, $\varepsilon = \boldsymbol{n}^2 = 1$. The extrinsic curvature of the surface is

$$S_{ab} = \frac{\kappa}{2}h_{ab,k}. \tag{A21}$$

Using (A20) we derive

$$\begin{aligned}&k_{|kk} = \kappa^{-1}\kappa_{,k}, \qquad k_{|ka} = \kappa^{-1}\kappa_{:a}, \qquad k_{|ab} = \kappa S_{ab},\\ &k_{|A}{}^{|A} = \kappa S + \kappa\kappa_{,k}, \qquad S = h^{ab}S_{ab}.\end{aligned} \tag{A22}$$

To project the Einstein equations on the 2D surface we have to define the stress-energy tensor of the electrostatic field. The electrostatic potential is given by $\Phi = \Phi(k, \theta^a)$. The corresponding electric field vector defined with respect to Schwarzschild time t on hypersurface $t =$ const reads

$$E_A = -k^{-1}F_{0A} = -k^{-1}\Phi_{,A}. \tag{A23}$$

We are interested in deformation of equipotential 2D surfaces. Thus, it is convenient to define orthogonal to the surfaces component of the electric field vector separately. The electric field vector components in an orthonormal frame are

$$E_{\hat{k}} = -\kappa k^{-1}\Phi_{,k}, \qquad E_a = k^{-1}\Phi_{:a}. \tag{A24}$$

Thus, in our notations

$$\vec{E}^2 = E_{\hat{k}}^2 + k^{-2}\Phi_{:a}\Phi^{:a}. \tag{A25}$$

The energy-momentum tensor of the field is

$$8\pi T_{\alpha\beta} = 2\xi_\alpha\xi_\beta k^{-2}\vec{E}^2 - 2E_\alpha E_\beta + g_{\alpha\beta}\vec{E}^2. \tag{A26}$$

Using relations (A10)–(A13) for metric (A20) together with (A18) we derive the Einstein equations projected onto 2D equipotential surfaces:

$$\begin{aligned}\kappa^3 S_a{}^b{}_{,k} = {}&\kappa^2[K - E_{\hat{k}}^2 - k^{-2}\Phi_{:c}\Phi^{:c}]\delta_a{}^b - \kappa^3 k^{-1}S_a{}^b\\ &+ \kappa\kappa_{:a}{}^{:b} - 2\kappa_{:a}\kappa^{:b} - \kappa^2 SS_a{}^b + 2\kappa^2k^{-2}\Phi_{:a}\Phi^{:b},\end{aligned} \tag{A27}$$

$$\begin{aligned}\kappa^3 S_{,k} = {}&\kappa^2[\kappa k^{-1}S - S_a{}^bS_b{}^a - 2k^{-2}\Phi_{:a}\Phi^{:a}]\\ &+ \kappa\kappa_{:a}{}^{:a} - 2\kappa_{:a}\kappa^{:a},\end{aligned} \tag{A28}$$

$$k^{-1}\kappa\kappa_{,k} = -\kappa k^{-1}S + E_{\hat{k}}^2 + k^{-2}\Phi_{:a}\Phi^{:a}, \tag{A29}$$

$$\kappa^2 E_{\hat{k},k} = -\kappa SE_{\hat{k}} - \kappa_{:a}k^{-1}\Phi^{:a} + \kappa k^{-1}\Phi_{:a}{}^{:a}. \tag{A30}$$

The corresponding constraints are

$$0 = S^2 - S_a{}^bS_b{}^a - 2K + 2[\kappa k^{-1}S + E_{\hat{k}}^2 - k^{-2}\Phi_{:a}\Phi^{:a}], \tag{A31}$$

$$0 = [S_{:a} - S_a{}^b{}_{:b}]k + 2E_{\hat{k}}\Phi_{:a} + \kappa_{:a}. \tag{A32}$$

Here, K is the Gaussian curvature of a 2D equipotential surface $k =$ const.

The square of the Ricci tensor $R_{\alpha\beta}R^{\alpha\beta}$ is equal to the squared electromagnetic field invariant

$$R_{\alpha\beta}R^{\alpha\beta} = (F^2)^2 = (F_{\alpha\beta}F^{\alpha\beta})^2. \tag{A33}$$

According to (A23) and (A25) F^2 has the following form:

$$F^2 = -2\vec{E}^2 = -2[E_{\hat{k}}^2 + k^{-2}\Phi_{:a}\Phi^{:a}]. \tag{A34}$$

Using expressions (A18), (A24)–(A30) we derive

$$\mathcal{R}_{kk} = k^{-1}\kappa^{-1}\kappa_{:k} + \kappa^{-2}[\vec{E}^2 - 2E^2_{\hat{k}}],$$
$$\mathcal{R}_{ak} = k^{-1}\kappa^{-1}[\kappa_{:a} + 2E_{\hat{k}}\Phi_{:a}],$$
$$\mathcal{R}_{ab} = k^{-1}\kappa S_{ab} + h_{ab}\vec{E}^2 - 2k^{-2}\Phi_{:a}\Phi_{:b},$$
$$\mathcal{R} = 2\vec{E}^2 = 2k^{-1}\kappa[S + \kappa_{:k}]. \tag{A35}$$

Using (A19), (A1), and (A34) and expressions (A22) and (A35) we have

$$\tfrac{1}{8}\mathcal{C}^2 = [\kappa^2 S_a{}^b S_b{}^a + 2\kappa_{:a}\kappa^{:a} + \kappa^2 S^2 + 2\vec{E}^2\Phi_{:a}\Phi^{:a}]k^{-2} + 4E_{\hat{k}}\kappa_{:a}\Phi^{:a}k^{-2} - 2\kappa[S_a{}^b\Phi_{:b} + S\Phi_{:a}]\Phi^{:a}k^{-3}. \tag{A36}$$

The hypersurface orthogonal Killing vector field ξ^α by definition is null on the Killing horizon which is bifurcate ($\kappa \neq 0$). A bifurcate Killing horizon contains a 2D space-like, totally geodesic surface [49]. In our coordinates this equipotential surface is defined by $t =$ const and $k = 0$. On the other side, a necessary and sufficient condition that a hypersurface is totally geodesic is its vanishing extrinsic curvature defined in the corresponding enveloping space [53]. Thus, for the equipotential surfaces $t =$ const, $k = 0$ we have $S_{ab} = 0$. For a regular horizon its 2D surface has everywhere finite Gaussian curvature, and the electrostatic field on the surface is finite as well. Thus, we can deduce from the constraints (A31) and (A32) that on the horizon $\Phi_{:a} = \kappa_{:a} = 0$. Hence, the electrostatic field potential Φ and the surface gravity κ are constant on the horizon, as it has to be for a static black hole. This is nothing but the zeroth law of black hole thermodynamics [54].

Projecting the first (A27), and the second (A28) of the Einstein equations on the horizon, and using the first constraint (A31) we derive

$$2\kappa S_a{}^b k^{-1}|_H = \delta_a{}^b[K - E^2_{\hat{k}}]|_H. \tag{A37}$$

Here, $(\ldots)|_H$ means calculated on the horizon. Thus, from (A33), (A34), and (A36) we derive the following expressions for the space-time invariants calculated on the horizon:

$$F^4|_H = R_{\alpha\beta}R^{\alpha\beta}|_H = 4E^4_{\hat{k}}|_H, \tag{A38}$$

and

$$\mathcal{C}^2|_H = 12[K - E^2_{\hat{k}}]^2|_H. \tag{A39}$$

This expression generalizes the relation between Gaussian curvature and the Kretschmann scalar calculated on the event horizon surface of an arbitrary distorted Schwarzschild black hole [1,50].

We can expand the metric and the electrostatic field in a series near the horizon and substituting these expansions into (A34) and (A36) derive expressions of the space-time invariants near the horizon. There are two types of quantities, even and odd in k, which we denote by $A = \{\kappa, h_{ab}, K, \Phi, E_{\hat{k}}, F^2, \mathcal{C}^2\}$ and $B = \{S_a{}^b, S\}$, respectively. The series expansions of A and B read

$$A = \sum_{n\geq 0} A^{[2n]}k^{2n}, \qquad B = \sum_{n\geq 0} B^{[2n+1]}k^{2n+1}. \tag{A40}$$

The first term in A gives its value on the horizon. We can express higher order coefficients in the expressions in terms of these on the horizon substituting (A40) into the Einstein equations (A27)–(A32). The necessary coefficients to calculate the first order expansion of the space-time invariants are the following:

$$\kappa^{[2]} = \frac{1}{2\kappa^{[0]}}[2E^{[0]2}_{\hat{k}} - K^{[0]}], \qquad \Phi^{[2]} = -\frac{E^{[0]}_{\hat{k}}}{2\kappa^{[0]}},$$
$$S_a{}^{b[1]} = \frac{\delta_a{}^b}{2\kappa^{[0]}}[K^{[0]} - E^{[0]2}_{\hat{k}}], \qquad S^{[1]} = \frac{1}{\kappa^{[0]}}[K^{[0]} - E^{[0]2}_{\hat{k}}],$$
$$S_a{}^{b[3]} = \frac{1}{8\kappa^{[0]2}}[2\kappa^{[2]:b}_{:a} + \kappa^{[2]:a}_{:a}\delta_a{}^b - \kappa^{[0]}S^{[1]2}\delta_a{}^b] + \frac{1}{16\kappa^{[0]3}}[2E^{[0]}_{\hat{k}:a}E^{[0]:b}_{\hat{k}} - 3E^{[0]}_{\hat{k}:c}E^{[0]:c}_{\hat{k}}\delta_a{}^b],$$
$$S^{[3]} = \frac{1}{4\kappa^{[0]2}}[2\kappa^{[2]:a}_{:a} - \kappa^{[0]}S^{[1]2}] - \frac{1}{4\kappa^{[0]3}}E^{[0]}_{\hat{k}:a}E^{[0]:a}_{\hat{k}},$$
$$E^{[3]}_{\hat{k}} = -\frac{1}{4\kappa^{[0]2}}[2\kappa^{[0]}S^{[1]}E^{[0]}_{\hat{k}} + E^{[0]:a}_{\hat{k}:a}]. \tag{A41}$$

Finally, we derive the first order expansions of the space-time invariants near the horizon:

$$F^2 \approx -2E^2_{\hat{k}}|_H + \frac{1}{2\kappa^2}[4K_eE^2_{\hat{k}} + E^2_{\hat{k}:a}{}^{:a} - 3E_{\hat{k}:a}E_{\hat{k}}{}^{:a}]|_H k^2, \tag{A42}$$

$$\mathcal{C}^2 \approx 12K_e^2|_H - \frac{1}{\kappa^2}[6K_e^2[3K_e - 2E^2_{\hat{k}}] - [2K_e - E^2_{\hat{k}}]_{:a} \times [2K_e - E^2_{\hat{k}}]^{:a} + 6K_e[K_{e:a}{}^{:a} - 2E_{\hat{k}}E_{\hat{k}:a}{}^{:a}]]|_H k^2, \tag{A43}$$

where $K_e|_H = [K - E^2_{\hat{k}}]|_H$.

[1] V. P. Frolov and A. A. Shoom, Phys. Rev. D **76**, 064037 (2007).

[2] V. Frolov and I. Novikov, *Black Hole Physics: Basic Concepts and Recent Developments* (Kluwer Academic Publishers, Dodrecht-Boston-London, 1998), Chap. 14.

[3] R. Penrose, in *Battelle Rencontres*, edited by C. M. de Witt and J. A. Wheeler (Benjamin, New York, 1968).
[4] J. M. McNamara, Proc. R. Soc. A **358**, 499 (1978).
[5] R. A. Matzner, N. Zamorana, and V. D. Sandberg, Phys. Rev. D **19**, 2821 (1979).
[6] S. Chandrasekhar and J. B. Hartle, Proc. R. Soc. A **384**, 301 (1982).
[7] G. F. R. Ellis and A. R. King, Commun. Math. Phys. **38**, 119 (1974).
[8] A. R. King, Phys. Rev. D **11**, 763 (1975).
[9] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990).
[10] A. J. S. Hamilton, arXiv:0903.2021.
[11] A. Ori, Phys. Rev. Lett. **67**, 789 (1991).
[12] L. M. Burko, Int. J. Mod. Phys. D **11**, 1561 (2002).
[13] L. M. Burko, Phys. Rev. D **66**, 024046 (2002).
[14] L. M. Burko, Phys. Rev. Lett. **90**, 121101 (2003).
[15] L. M. Burko, Phys. Rev. Lett. **90**, 249902(E) (2003).
[16] A. J. S. Hamilton and P. P. Avelino, arXiv:0811.1926.
[17] M. L. Gnedin and N. Y. Gnedin, Classical Quantum Gravity **10**, 1083 (1993).
[18] A. Bonanno, S. Droz, W. Israel, and S. Morsink, Can. J. Phys. **72**, 755 (1994).
[19] E. Flanagan and A. Ori, Phys. Rev. D **54**, 6233 (1996).
[20] P. Brady and J. Smith, Phys. Rev. Lett. **75**, 1256 (1995).
[21] B. K. Harrison, J. Math. Phys. (N.Y.) **9**, 1744 (1968).
[22] F. J. Ernst, Phys. Rev. **168**, 1415 (1968).
[23] R. Geroch and J. B. Hartle, J. Math. Phys. (N.Y.) **23**, 680 (1982).
[24] Y. Gurtel, I. D. Novikov, V. D. Sandberg, and A. A. Starobinsky, Phys. Rev. D **20**, 1260 (1979).
[25] I. D. Novikov and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. **78**, 3 (1980) [Sov. Phys. JETP **51**, 1 (1980)].
[26] A. V. Frolov, K. R. Kristjansson, and L. Thorlacius, Phys. Rev. D **72**, 021501 (2005).
[27] A. V. Frolov, K. R. Kristjansson, and L. Thorlacius, Phys. Rev. D **73**, 124036 (2006).
[28] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., San Francisco, 1973).
[29] N. Bretón, T. E. Denisova, and V. S. Manko, Phys. Lett. A **230**, 7 (1997).
[30] N. Bretón, A. A. García, V. S. Manko, and T. E. Denisova, Phys. Rev. D **57**, 3382 (1998).
[31] S. Fairhurst and B. Krishnan, Int. J. Mod. Phys. D **10**, 691 (2001).
[32] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, 1983).
[33] Axisymmetric, distorted Schwarzschild black hole space-time is of type I, and type D on the horizon and its axis of symmetry [34]. It can be checked that the Harrison-Ernst transformation does not alter these properties, and in addition the corresponding space-time is of type D on the Cauchy horizon.
[34] D. Papadopoulos and B. C. Xanthopoulos, Nuovo Cimento Soc. Ital. Fis. B **83**, 113 (1984).
[35] Discussion of the relation of these multipole moments to the Newtonian and Geroch-Hansen relativistic multipole moments can be found in [36–39].
[36] R. Geroch, J. Math. Phys. (N.Y.) **11**, 1955 (1970).
[37] R. Geroch, J. Math. Phys. (N.Y.) **11**, 2580 (1970).
[38] R. O. Hansen, J. Math. Phys. (N.Y.) **15**, 46 (1974).
[39] H. Quevedo, Phys. Rev. D **39**, 2904 (1989).
[40] The metric (39) allows Wick's rotation $T_+ \longrightarrow iT_E$, which transforms the exterior region $\eta \geq 1$ into that of the corresponding distorted Euclidean charged black hole. The Euclidean horizon surface $\eta = 1$ is a regular 2D totally geodesic surface. The electric field transforms as $\Phi_+ \longrightarrow -i\Phi_E$.
[41] M. Ansorg and J. Hennig, Classical Quantum Gravity **25**, 222001 (2008).
[42] J. Hennig and M. Ansorg, arXiv:0904.2071.
[43] L. Smarr, Phys. Rev. D **7**, 289 (1973).
[44] W. J. Wild and R. M. Kerns, Phys. Rev. D **21**, 332 (1980).
[45] W. J. Wild, R. M. Kerns, and W. F. Drish, Jr., Phys. Rev. D **23**, 829 (1981).
[46] R. Kulkarni and N. Dadhich, Phys. Rev. D **33**, 2780 (1986).
[47] It is known that an axisymmetric 2D surface cannot be isometrically embedded into a 3D Euclidean space if its Gaussian curvature is negative at the vicinity of a fixed point of rotation [48]. However, a failure of the embedding does not generally imply that Gaussian curvature of the surface is negative.
[48] V. P. Frolov, Phys. Rev. D **73**, 064021 (2006).
[49] R. H. Boyer, Proc. R. Soc. A **311**, 245 (1969).
[50] V. P. Frolov and N. Sanchez, Phys. Rev. D **33**, 1604 (1986).
[51] W. Israel, Commun. Math. Phys. **8**, 245 (1968).
[52] V. P. Frolov and A. I. Zel'lnikov, Phys. Rev. D **35**, 3031 (1987).
[53] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1993).
[54] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).