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Electrically charged fluids with pressure in Newtonian gravitation and general relativity in *d* spacetime dimensions: Theorems and results for Weyl type systems

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Previous theorems concerning Weyl type systems, including Majumdar-Papapetrou systems, are generalized in two ways, namely, we take these theorems into d spacetime dimensions ($d \geq 4$), and we also consider the very interesting Weyl-Guilfoyle systems, i.e., general relativistic charged fluids with nonzero pressure. In particular within the Newton-Coulomb theory of charged gravitating fluids, a theorem by Bonnor (1980) in three-dimensional space is generalized to arbitrary $(d-1) > 3$ space dimensions. Then, we prove a new theorem for charged gravitating fluid systems in which we find the condition that the charge density and the matter density should obey. Within general relativity coupled to charged dust fluids, a theorem by De and Raychaudhuri (1968) in four-dimensional spacetime is rendered into arbitrary $d > 4$ dimensions. Then a theorem, new in $d = 4$ and $d > 4$ dimensions, for Weyl-Guilfoyle systems, is stated and proved, in which we find the condition that the charge density, the matter density, the pressure, and the electromagnetic energy density should obey. This theorem comprises, in particular cases, a theorem by Gautreau and Hoffman (1973) and results in four dimensions by Guilfoyle (1999). Upon connection of an interior charged solution to an exterior Tangherlini solution (i.e., a Reissner-Nordström solution in d dimensions), one is able to give a general definition for gravitational mass for this kind of relativistic systems and find a mass relation with several quantities of the interior solution. It is also shown that for sources of finite extent the mass is identical to the Tolman mass.

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I. INTRODUCTION

A. Weyl type systems: Definition and overview

Oddly, there is no noticeable mention of results for charged fluids in Newtonian gravitation in the 19th century, an epoch when such studies could certainly be performed with ease. Not even the useful monograph of Ramsey of 1940 "Newtonian attraction" [1] attempts such an incursion. The first work treating charged fluids in Newtonian gravitation we are aware of is Bonnor's work [2], which is inspired in relativistic gravitational works. This work [2] is divided into two distinct parts. The first one deals with charged matter theorems in Newtonian gravitation, and some interesting singular solutions are displayed. The second one studies axisymmetric solutions for charged systems in general relativity. Although the displaying of exact solutions with matter for axisymmetric systems in general relativity is of great interest, here we are only interested in the first part of Bonnor's work. In the study of Newtonian systems, Bonnor was inspired by previous works in general relativity; indeed no previous paper in Newtonian theory is cited. Bonnor [2] observed that in Newtonian mechanics and classical electrostatics an ensemble of N particles of masses and charges m_j and q_j will be in equilibrium in any configuration if $q_j = \epsilon\sqrt{G}m_j$ ($j = 1, \ldots, N$), where $\epsilon = \pm 1$ and G is the Newton's gravitational constant. In the case of continuous distributions of charged matter, with mass density ρ_m and charge density ρ_e, there will be equilibrium everywhere if $\rho_e = \epsilon\sqrt{G}\rho_m$. Such a neutral equilibrium is possible due to the exact balancing of the gravitational and electric forces on every fluid particle. Thus, a static distribution of charged dust, i.e., a perfect fluid with zero pressure, of any shape can in principle be built. Using the properties of the Newton-Coulomb system of equations, Bonnor showed that all Newton-Coulomb nontrivial solutions with closed equipotentials not satisfying the relation $\rho_e = \epsilon\sqrt{G}\rho_m$ are singular. As a byproduct he showed that a relation between the Newtonian gravitational potential V and the electric potential ϕ,

$$V = V(\phi), \tag{1}$$

should have a simple form, namely,

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$$V = -\epsilon\sqrt{G}\phi + \text{const.} \tag{2}$$

These types of general theorems and results for the coupling between charged matter and gravitation were originally attempted within the theory of general relativity. It was Weyl [3], while studying electric fields in vacuum Einstein-Maxwell theory in four-dimensional spacetime, who first perceived that it is interesting to consider a functional relation between the metric component $g_{tt} \equiv W^2(x^i)$ and the electric potential $\phi(x^i)$ (where x^i represent the spatial coordinates, $i = 1, 2, 3$) given by the ansatz

$$W = W(\phi). \tag{3}$$

Systems, either vacuum or matter, in which the ansatz (3) holds generically will be called Weyl type systems. If it is assumed the system is vacuum and axisymmetric then Weyl [3] found that such a relation must be of the form

$$W^2 = (-\epsilon\sqrt{G}\phi + b)^2 + c, \tag{4}$$

where b and c are arbitrary constants, and we use units such that the speed of light equals unity. The metric may be written as

$$ds^2 = -W^2dt^2 + h_{ij}dx^idx^j, \qquad i, j = 1, 2, 3, \tag{5}$$

where h_{ij} is also a function of the spatial coordinates x^i only and it was assumed by Weyl [3] to represent an axisymmetric space. Majumdar [4] extended this result by showing that it holds for a large class of static spacetimes with no particular spatial symmetry, axial or otherwise. Moreover, by choosing $c = 0$ in which case the potential W^2 assumes the form of a perfect square and so

$$W = -\epsilon\sqrt{G}\phi + b, \tag{6}$$

Majumdar [4] was able to show that the Einstein-Maxwell equations in the presence of charged dust, i.e., a perfect fluid with zero pressure, imply exactly the same condition of the Newtonian theory, namely, $\rho_e = \epsilon\sqrt{G}\rho_m$, with both the gravitational potential W and the electric potential ϕ satisfying a Poisson-like equation. As in the Newtonian case, the relativistic solutions are static configurations of charged dust and need not have any spatial symmetry. Majumdar [4] also showed that in the case W is as in Eq. (6) the metric of the three-space is conformally flat with the conformal factor given by $1/W^2$, and in such a case all the stresses in the charged matter vanish. Similar results were found independently by Papapetrou [5], who assumed as a starting point a perfect square relation among W and ϕ, in a dust filled spacetime, and showed further that the charge density ρ_e and the relativistic energy density ρ_m are related by $\rho_e = \epsilon\sqrt{G}\rho_m$. Relation (6) is called the Majumdar-Papapetrou relation. Solutions in which the condition $\rho_e = \epsilon\sqrt{G}\rho_m$ and the relation (6) hold are called Majumdar-Papapetrou solutions. These kinds of charged dust fluids were studied by other authors. For instance, Das [6] showed that if the ratio $\rho_e/\rho_m = \epsilon\sqrt{G}$ holds, then the relation between potentials must be as in Eq. (6). De and Raychaudhuri [7] generalized this by showing that if there is a closed equipotential within the charged dust fluid with no singularities, holes, or alien matter inside it, where alien matter was an expression used to indicate anything other than charged dust, then the charged dust fluid corresponds to a Majumdar-Papapetrou solution.

A further advance was performed by Gautreau and Hoffman [8]. They investigated the structure of the sources that produce Weyl type fields, which satisfy the Weyl quadratic relation (4), in the case the matter stresses, i.e., the pressures, do not vanish. They found that when there is pressure, for W being given by the Weyl relation then the fluid obeys the condition $b\rho_e = \epsilon\sqrt{G}(\rho_m + \frac{d-1}{d-3}p)W + \epsilon\sqrt{G}\phi\rho_e$, or equivalently $\rho_e(\epsilon\sqrt{G}\phi - b) = -\epsilon\sqrt{G}(\rho_m + \frac{d-1}{d-3}p)W$, in the same spirit of the Majumdar-Papapetrou condition. If, instead of a Weyl relation, one has a Majumdar-Papapetrou relation, but still keeping the pressure, then the condition is simpler, $\rho_e = \epsilon\sqrt{G}(\rho_m + \frac{d-1}{d-3}p)$.

Another interesting study was performed by Guilfoyle [9] who considered charged fluid distributions and made the hypothesis that the functional relation between the gravitational and the electric potential, $W = W(\phi)$, is slightly more general than the one given in (4). This Weyl-Guilfoyle relation has the form

$$W^2 = a(-\epsilon\sqrt{G}\phi + b)^2 + c, \tag{7}$$

where a, b, and c are arbitrary constants. Guilfoyle [9] investigated several general properties of such systems. In particular, he showed that if the fields satisfy the Majumdar-Papapetrou perfect square relation, then the fluid pressure is proportional to the gravitational potential $p = kW^2$. In addition, a set of exact spherically symmetric solutions were also analyzed.

Now, global analyses are important. To connect these general local results just mentioned to general global results one needs an exterior solution. For Weyl and Weyl-Guilfoyle relations, fluids which respect spherical symmetry can be joined to an exterior Reissner-Nordström solution [10,11] characterized by a mass m and charge q. Guilfoyle [9] by applying junction conditions found how the global mass m and the global charge q are linked to the interior fluid parameters. In these spacetimes, whose fluids obey a Weyl-Guilfoyle relation, the total mass m is not, in general, proportional to the total electric charge of the system q. In the perfect square case of the Majumdar-Papapetrou relation, it can be shown that the mass m of the solution is equal to its charge q, $\sqrt{G}m = \epsilon q$, and the exterior is extremal Reissner-Nordström. One can also follow the approach of Gautreau and Hoffman [8], an approach which does not use junction conditions. These authors have showed that, for a source of finite extent, not necessarily spherically symmetric, whose fluid admits pressure and a Weyl relation, and whose spacetime is asymptotically Reissner-Nordström, the total gravitational mass is proportional to the total electric charge of the source, and the ratio between them is equal to the constant b. Here, in this approach, the mass definition follows from Tolman [12] and can be applied to charged spacetimes (see Whittaker [13] and Florides [14,15], see also [16]). We further note that the complete global understanding of a single Majumdar-Papapetrou particle, i.e., the extremal Reissner-Nordström vacuum solution, was achieved by Carter [17], through a Carter-Penrose diagram, and the generalization to the vacuum solution with many extremal black holes scattered around was performed by Hartle and Hawking [18].

All the theorems and results mentioned above were performed in four dimensions. It is important nowadays to have results in d dimensions. Indeed, string theory, the AdS/CFT conjecture, brane world scenarios, and many other instances, point to the possible existence of a d-dimensional world, with $d \geq 4$. For charged fluids a first attempt in the study of d-dimensional spacetimes was performed by Lemos and Zanchin [19] where the beautiful four-dimensional results of Majumdar [4] were put into higher dimensions. It is thus of interest to render the general theorems and results mentioned above into d dimensions. We will see that some of the theorems are trivially extended, but in the process, we will get really new interesting theorems and results. For instance, when connecting the Gautreau-Hoffman work [8] to Guilfoyle's [9] we step upon a nontrivial generalization and find an important new relation for the matter and field quantities for charged fluids with pressure, which obey the Weyl-Guilfoyle relation, both in Newtonian gravitation and general relativity. In this setting the notion of alien matter has to be generalized to indicate anything other than a charged fluid of Weyl type, guaranteeing that when the pressure is zero one recovers the definition in [7]. We will also connect the general local results to general global results, using as exterior solution the d-dimensional Reissner-Nordström solution, also called the Tangherlini solution [20]. We find the d-dimensional version of the mass relation found by Guilfoyle [9] as well as the generalization of the Gautreau-Hoffman mass [8]. Summing up, we will enounce 12 theorems and two corollaries and analyze some global issues.

The literature in charged fluids in general relativity is vast and we can only mention some works more related to our paper. Inspired in the work of De and Raychaudhuri [7], some papers dealt with scalar charge rather than Maxwell charge, see, e.g., [21–24]. For fluids with pressure of Majumdar-Papapetrou type it is worth mentioning here a study by Ida [25]. By assuming a linear equation of state for the fluid, i.e. $\rho_{\rm m} = {\rm const} \times p (= {\rm const} \times \phi^{-2})$, he found that the resulting equation for the electric potential is simply a Helmholtz equation in a space of constant curvature, and he investigated some particular solutions in that case. In four dimensions, many other works have been performed for charged fluids with nonzero pressure without considering Weyl or other types of relations, see, e.g., [26] for a list of references. For a study of d-dimensional Majumdar-Papapetrou fluids and related Bonnor stars, with a thorough review of the subject, see [27].

B. Nomenclature

We consider a d-dimensional spacetime, both in Newtonian gravitation and general relativity. The number n of space dimensions is then $n = (d - 1)$. Throughout the paper we set the speed of light equal to unity. For the d-dimensional spacetime gravitational constant we write G_d, see, e.g., [27] for the definition of G_d. In $d = 4$ dimensions we write $G_4 \equiv G$.

1. Newton-Coulomb charged fluids

Here we set out the nomenclature related to the potentials for the Newton-Coulomb theory with charged matter. This nomenclature is inspired in the nomenclature for the general relativity theory with charged matter.

(i) The gravitational Newtonian potential is represented by V, and the electric Coulombian potential is denoted by ϕ. $\rho_{\rm m}$ is the mass density, p is the fluid pressure, and $\rho_{\rm e}$ is the electric charge density.

(ii) $V = V(\phi)$ is, inspired in the relativistic theory, the Weyl ansatz for the Newton-Coulomb theory. Systems bearing this hypothesis are Newton-Coulomb Weyl type systems, or simply Newtonian Weyl type systems.

(iii) $V = -\epsilon\beta\sqrt{G_d}\phi + \gamma$ is, inspired in the relativistic theory, what we call the original Weyl relation in the Newton-Coulomb theory, or simply Weyl relation. The Weyl relation is, in the Newtonian theory, the same as the Weyl-Guilfoyle relation.

(iv) $V = -\epsilon\sqrt{G_d}\phi + \gamma$ is, inspired in the relativistic theory, what we call the Majumdar-Papapetrou relation for the Newton-Coulomb theory (see [27]). The Majumdar-Papapetrou relation is a particular case of the Weyl-Guilfoyle relation, one sets $\beta = 1$.

Here we set out the nomenclature related to the fluid quantities. For the densities the nomenclature is more complicated than for the potentials, since it depends on whether there is pressure or not in the fluid.

(i) When there is no pressure, the only static solutions for $V(\phi)$ are of the form of the Majumdar-Papapetrou relation, which for the Newton-Coulomb theory is a particular case of the Weyl-Guilfoyle relation with $\beta = 1$. In such a case, the fluid densities obey the Majumdar-Papapetrou condition, $\rho_{\rm e} = \epsilon\sqrt{G_d}\rho_{\rm m}$.

(ii) When there is pressure, for V obeying the Weyl-Guilfoyle relation, the fluid densities obey the condition $\beta\rho_{\rm e} = \epsilon\sqrt{G_d}\rho_{\rm m}$, a condition given here for the first time. Inspired in the relativistic case, we call it the Gautreau-Hoffman condition for the Newton-Coulomb theory. Now, in the particular case where $\beta = 1$ the potential V obeys the Majumdar-Papapetrou relation for the Newton-Coulomb theory, the pressure vanishes, and the densities are related by the Majumdar-Papapetrou condition for the Newton-Coulomb theory.

2. Relativistic charged fluids

Here we set out the nomenclature related to the potentials for relativistic fluids.

(i) In d spacetime dimensions, we use the same notation for the metric as in four dimensions, cf. Eq. (5), and the symbols for electric and fluid quantities are the same as in the Newton-Coulomb case. So, W is the metric potential related to the time coordinate, which can be interpreted as the relativistic gravitational potential, ϕ is the relativistic electric potential, i.e., the only nonzero component of the electromagnetic gauge potential, ρ_{m} is the relativistic mass-energy density, p is the fluid relativistic pressure, and ρ_{e} is the relativistic electric charge density.

(ii) $W = W(\phi)$ is the Weyl ansatz. Systems bearing this hypothesis are Weyl type systems.

(iii) $W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c$ is the Weyl-Guilfoyle relation, where a, b, and c are constant parameters.

(iv) $W^2 = (-\epsilon\sqrt{G_d}\phi + b)^2 + c$ is the original Weyl relation, or simply Weyl relation. The Weyl relation is a particular case of the Weyl-Guilfoyle relation.

(v) $W^2 = (-\epsilon\sqrt{G_d}\phi + b)^2$, or $W = -\epsilon\sqrt{G_d}\phi + b$, is the Majumdar-Papapetrou relation, a particular case of the Weyl relation.

Here we set out the nomenclature related to the fluid quantities. For the densities the nomenclature is more complicated than for potentials, since it depends on whether there is pressure or not in the relativistic fluid.

(1) When there is no pressure, the only possible static solutions satisfy the Majumdar-Papapetrou relation, and the fluid variables are related by the equation $\rho_{\mathrm{e}}(-\epsilon\sqrt{G_d}\phi + b) = \epsilon\sqrt{G_d}\rho_{\mathrm{m}}W$, which can be cast into the form $\rho_{\mathrm{e}} = \epsilon\sqrt{G_d}\rho_{\mathrm{m}}$. This equation between the densities is called the Majumdar-Papapetrou condition.

(a) When there is pressure, for W obeying the Weyl-Guilfoyle relation, the fluid quantities satisfy the condition $ab\rho_{\mathrm{e}} = \epsilon\sqrt{G_d}[(\rho_{\mathrm{m}} + \frac{d-1}{d-3}p)W + \phi\rho_{\mathrm{e}}] + \epsilon\sqrt{G_d}(a-1)(\phi\rho_{\mathrm{e}} - W\rho_{\mathrm{em}})$, or equivalently, $a\rho_{\mathrm{e}}(-\epsilon\sqrt{G_d}\phi + b) = \epsilon\sqrt{G_d}[\rho_{\mathrm{m}} + \frac{d-1}{d-3}p + (1-a)\rho_{\mathrm{em}}]W$, which is given here for the first time. This condition can be considered as an equation of state.

(b) When there is pressure, in the particular case where $a = 1$, W obeys the original Weyl relation and the fluid quantities are related by $b\rho_{\mathrm{e}} = \epsilon\sqrt{G_d}[(\rho_{\mathrm{m}} + \frac{d-1}{d-3}p)W + \phi\rho_{\mathrm{e}}]$, or equivalently, $\rho_{\mathrm{e}}(-\epsilon\sqrt{G_d}\phi + b) = \epsilon\sqrt{G_d}(\rho_{\mathrm{m}} + \frac{d-1}{d-3}p)W$. This is called the Gautreau-Hoffman condition.

(c) When there is pressure, in the particular case where $a = 1$ and $c = 0$, W obeys the Majumdar-Papapetrou relation, $W = -\epsilon\sqrt{G_d}\phi + b$, and the fluid variables are related by $\rho_{\mathrm{e}}(-\epsilon\sqrt{G_d}\phi + b) = \epsilon\sqrt{G_d}(\rho_{\mathrm{m}} + \frac{d-1}{d-3}p)W$, or equivalently, $\rho_{\mathrm{e}} = \epsilon\sqrt{G_d}(\rho_{\mathrm{m}} + \frac{d-1}{d-3}p)$. This equation is a particular case of the Gautreau-Hoffman condition.

In [7] the expression alien matter was used to indicate anything other than charged dust. Here, since we treat charged fluids with pressure, the expression alien matter is generalized to indicate anything other than a charged fluid of Weyl type. When the pressure is zero one recovers the definition of [7].

The nomenclature will become clearer during the paper.

C. Structure of the paper

In Sec. II a self-gravitating static charged fluid in Newtonian physics is analyzed. We set up the Eulerian formulation of fluid dynamics in $n = (d-1)$-dimensional Euclidean space. We then analyze the general properties of a charged static Newton-Coulomb zero pressure fluid, or charged dust, and state two theorems rendering into higher dimensions previous results in three-dimensional spaces by other authors. We perform an analysis of charged Newton-Coulomb fluids with nonzero pressure reporting new results and stating and proving four theorems. We define the total mass and total electric charge in terms of the respective densities and find a mass-charge relation for Newton-Coulomb Weyl type fluids in $(d-1)$-dimensional Euclidean spaces.

In Sec. III a self-gravitating static charged fluid in general relativity is analyzed. The general formalism and setup is presented, followed by a study of a zero pressure charged fluid, or charged dust, where two theorems are stated and proved, rendering into higher dimensions results by Majumdar [4], Papapetrou [5], Das [6], and De and Raychaudhuri [7]. The study of a relativistic charged pressure fluid of Weyl type is divided into six parts. In the first part the basic equations are written and general results are reported. A theorem rendering into higher dimensions some results obtained in $d = 4$ by Guilfoyle [9] is stated there. In the second part the Weyl ansatz is imposed and its consequences to the relativistic fluid are analyzed. Another theorem rendering into higher dimensions some other results obtained in $d = 4$ by Guilfoyle [9] is stated in this part. For a Weyl-Guilfoyle relation, a generalized condition obeyed by the mass density, pressure, electric charge density, and electromagnetic energy density of a fluid in four and higher dimensions is found. This is done in the third part where a new theorem is proved. The fourth part contains the same new theorem on Weyl type fluids in d-dimensional spacetimes, the proof being now inspired in the work by De and Raychaudhuri [7]. In the fifth part we study spherically symmetric Weyl type systems and compare with the work in four dimensions done by Guilfoyle [9]. In the sixth part an analysis is performed where we show that the relativistic Weyl type charged pressure fluids have a correct Newtonian limit consistent with what was found in Sec. II. Then closing Sec. III, the mass and charge definitions for relativistic Weyl type fluids in asymptotically Tangherlini spacetimes are given, and the particular case of spherical symmetry is studied in some detail.

In Sec. IV we finally state our conclusions.

II. NEWTON-COULOMB CHARGED FLUID WITH PRESSURE IN $d = n + 1$ SPACETIME DIMENSIONS

A. Basic equations

Consider a d-dimensional Newtonian spacetime, with the number of space dimensions n being $n = d - 1$. Let us study the dynamics of a gravitating Newtonian charged pressure fluid in a $(d-1)$-dimensional Euclidean space according to the Euler description. The basic equations are the continuity and the Euler equations, which may be written as

$$\frac{\partial \rho_{\rm m}}{\partial t} + \nabla_i(\rho_{\rm m} v^i) = 0, \tag{8}$$

$$\rho_{\rm m}\frac{\partial v_i}{\partial t} + \rho_{\rm m} v^j \nabla_j v^i + \nabla_i p = -\rho_{\rm m}\nabla_i V - \rho_{\rm e}\nabla_i \phi, \tag{9}$$

where t is the time coordinate, ∇_i is the $(d-1)$-dimensional gradient operator. The fluid quantities appearing in the above equations are the $(d-1)$-dimensional fluid velocity v_i, the matter density $\rho_{\rm m}$, the electric charge density $\rho_{\rm e}$, and the fluid pressure p. Finally, V and ϕ are, respectively, the Newton gravitational and the Coulomb electric potentials, given by the respective Poisson equation

$$\nabla^2 V = S_{d-2} G_d \rho_{\rm m}, \tag{10}$$

$$\nabla^2 \phi = -S_{d-2}\rho_{\rm e}, \tag{11}$$

where the operator ∇^2 is the Laplace operator in $(d-1)$-dimensional Euclidean space, $S_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ is the area of the unit sphere $\mathbf{S}^{d-2}$, Γ is the usual gamma function, and G_d is the Newton's gravitational constant in d dimensions. S_{d-2} reduces to 4π in three space (four spacetime) dimensions, and Eqs. (10) and (11) are the natural generalizations of the corresponding three-dimensional Poisson equations for the potentials V and ϕ to $(d-1)$-dimensional space.

We will consider only static systems, so all of the quantities are functions of the space coordinates only, and the fluid's velocity can be made equal to zero, $v_i = 0$. Then, the continuity equation is identically satisfied and the Euler equation for the charged pressure fluid in static equilibrium reads

$$\rho_{\rm m}\nabla_i V + \rho_{\rm e}\nabla_i\phi + \nabla_i p = 0. \tag{12}$$

The important equations for the problem are Eqs. (10)–(12). Equations (10) and (11) are the field equations that determine the gravitational and the electric potentials once the mass and charge densities are given, while Eq. (12) is the equilibrium equation for the system. Some particular cases of these equations are considered below.

B. Zero pressure: Weyl type systems in the Newton-Coulomb theory and the Bonnor theorem in higher dimensions

A special but very interesting case of charged matter is the dust fluid, for which $p = 0$, where Eq. (53) reduces to

$$\nabla_i V + \frac{\rho_{\rm e}}{\rho_{\rm m}}\nabla_i\phi = 0. \tag{13}$$

From Eq. (13) we can render into $(d-1)$-dimensional spaces a theorem by Bonnor in $d = 4$. Bonnor himself was inspired in the relativistic analysis by De and Raychaudhuri [7].

Theorem II.1. (Bonnor 1980).

For any charged static dust distribution in the Newton-Coulomb theory, the $(d-2)$-dimensional hypersurfaces of constant V coincide with the $(d-2)$-dimensional hypersurfaces of constant ϕ, and V is a function of ϕ alone.

Proof. Note that V and ϕ are scalar functions in a $(d-1)$-dimensional Euclidean space, so any one of the conditions of constant V or ϕ defines a $(d-2)$-dimensional space. Now contracting Eq. (13) with dx^i it gives $\rho_{\rm m}(\nabla_i V)dx^i + \rho_{\rm e}(\nabla_i\phi)dx^i = 0$, which means $\rho_{\rm m} dV + \rho_{\rm e} d\phi = 0$. This equation implies that if any one of dV or $d\phi$ is null, then both of them are. In other words, it results in $dV/d\phi = -\rho_{\rm e}/\rho_{\rm m}$, which means that $dV/d\phi = \partial V/\partial\phi$. □

This results means that for a Newton-Coulomb charged dust fluid in which $p = 0$ there is a functional relation between the gravitational potential V and the electric potential ϕ:

$$V = V(\phi). \tag{14}$$

Equation (14) is the Weyl ansatz which tells also that ϕ and V share the same equipotential surfaces. Theorem (II.1) is the $(d-1)$-dimensional version of a theorem by Bonnor [2] in three-dimensional space. The theorem by Bonnor [2] is the Newtonian version of a theorem stated by De and Raychaudhuri for four-dimensional relativistic charged dust fluids [7] (see Sec. III B). The inclusion of pressure for relativistic fluids was done by Guilfoyle [9] (see Sec. III C 1).

We then work out the basic equations for the dust fluid case using the general form (14). To begin with, let us rewrite Eq. (10) by taking the Weyl ansatz (14) into account. It assumes the form

$$V'\nabla^2\phi + V''(\nabla_i\phi)^2 = S_{d-2}G_d\rho_m, \qquad (15)$$

where the primes stand for derivatives with respect to ϕ. Of course, Eq. (11) does not change form. On the other hand, the equilibrium equation (12) now reads

$$\rho_m V' + \rho_e = 0. \qquad (16)$$

Substituting ρ_m from the last equation, Eq. (16), into Eq. (15), we find $(V')^2\nabla^2\phi + V'V''(\nabla_i\phi)^2 = -S_{d-2}G_d\rho_e$. Then with the help of Eq. (11) we find

$$[V'^2 - G_d]\nabla^2\phi + V'V''(\nabla_i\phi)^2 = 0. \qquad (17)$$

This equation can be cast into the form

$$Z\nabla_i(Z\nabla^i\phi) = 0, \qquad (18)$$

where we have assumed $V'^2 > G_d$ and defined

$$Z = \sqrt{V'^2 - G_d}, \qquad (19)$$

and, otherwise, if $V'^2 - G_d < 0$, one just needs to redefine Z as $Z = G_d - V'^2$. Now we can state a theorem rendering into higher dimensions a theorem by Bonnor [2] in three-dimensional space. The analysis by Bonnor is inspired in the relativistic analysis of Papapetrou [5], Das [6], and De and Raychaudhuri [7].

Theorem II.2. (Bonnor 1980).

(i) In the Newton-Coulomb theory, if the surfaces of any static charged dust distribution are closed equipotential hypersurfaces and inside these hypersurfaces there are no singularities, holes, or alien matter, then the function $V(\phi)$ must satisfy the relation

$$V = -\epsilon\sqrt{G_d}\phi + \gamma, \qquad (20)$$

with γ being an integration constant, and in such a case it follows that

$$\rho_e = \epsilon\sqrt{G_d}\rho_m. \qquad (21)$$

(ii) In the Newton-Coulomb theory, if the ratio ρ_e/ρ_m equals a constant K, and there are no singularities, holes, or alien matter in that region, then it follows the relation (20) for the potentials, and also $K = \epsilon\sqrt{G_d}$.

Proof. The proof of (i) is as follows: Assuming $Z \neq 0$, one obtains $\nabla_i(Z\nabla^i\phi) = 0$. Then define a new field ψ by $\nabla^i\psi = Z\nabla^i\phi$, which due to Eq. (18) is divergenceless, i.e., $\nabla^2\psi = 0$. Now integrate the quantity $\nabla_i(\psi\nabla^i\psi)$ over a finite volume $\mathcal{V}_S$ in $(d-1)$-dimensional space to get

$$\int_{\mathcal{V}_S}\nabla_i(\psi\nabla^i\psi)d\mathcal{V} = \int_{\mathcal{V}_S}(\nabla_i\psi)^2 d\mathcal{V} = \int_{\mathcal{S}}\psi(\nabla_i\psi)n^i d\mathcal{S}, \qquad (22)$$

$\mathcal{S}$ being the boundary of $\mathcal{V}_S$, n^i being the unit vector normal to $\mathcal{S}$, and we have used the Gauss theorem. If there exists a closed surface which is an equipotential surface for ϕ, then by identifying such a surface with $\mathcal{S}$ one finds

$$\int_{\mathcal{V}_S}(\nabla_i\psi)^2 d\mathcal{V} = \int_{\mathcal{V}_S}Z^2(\nabla_i\phi)^2 d\mathcal{V} = 0. \qquad (23)$$

Since the integrand is a positive definite function in which $\nabla_i\phi \neq 0$, the integral in Eq. (23) gives zero only if $Z = 0$ everywhere within the region bounded by $\mathcal{S}$. The condition $Z = 0$ implies in $V' = -\epsilon\sqrt{G_d}$, with $\epsilon = \pm 1$. This result substituted into Eq. (16) furnishes Eq. (21) and after integration it gives Eq. (20), completing the proof of (i). The given proof follows standard proofs in potential theory where the uniqueness of the solutions of the Poisson equation under Dirichlet or Neumann boundary conditions is discussed. The proof of (ii) is as follows: Assuming the ratio ρ_e/ρ_m is constant, then Eq. (16) implies $V' =$ constant and from (18) one gets $Z\nabla^2\phi = 0$. Since, in view of Eq. (11), $\nabla^2\phi \neq 0$, this leads to $Z = 0$, or $V'^2 = G_d$, resulting in the same relations among V and ϕ, and among ρ_m and ρ_m as in Eqs. (20) and (21) above. □

The results just presented generalize the theorem by Bonnor [2] to $(d-1)$-dimensional spaces, which is the Newtonian version of the results found in part by Papapetrou [5], in part by Das [6], and fully by De and Raychaudhuri [7] (see Sec. III B). Equations (20) and (21) are called the Majumdar-Papapetrou relation and the Majumdar-Papapetrou condition, respectively.

C. Nonzero pressure: Weyl type systems in the Newton-Coulomb theory and new theorems in higher dimensions

1. Static charged pressure fluid: General properties

Equation (12) can be written in terms of total derivatives. For contracting it with dx^i yields $\rho_m(\nabla_i V)dx^i + \rho_e(\nabla_i\phi)dx^i + (\nabla_i p)dx^i = 0$, or $\rho_m dV + \rho_e d\phi + dp = 0$. This implies that V, ϕ, and p are functionally related, viz., $p = p(V, \phi)$, with $\rho_m = -(\partial p/\partial V)_\phi$ and $\rho_e = -(\partial p/\partial\phi)_V$. With this, we can state a theorem, whose relativistic version in four dimensions can be found in Guilfoyle [9].

Theorem II.3. (Newtonian version of Guilfoyle 1999).

For any static charged pressure fluid in the Newton-Coulomb theory, if any two of the $(d-2)$-dimensional hypersurfaces of constant V, ϕ, or p coincide, then the third also coincides.

Proof. First observe that V, ϕ, and p are scalar functions in a $(d-1)$-dimensional Euclidean space, so any one of the conditions of constant V, ϕ, or p defines a $(d-2)$-dimensional space. Moreover, from the continuity equation one gets $\rho_m dV + \rho_e d\phi + dp = 0$. This equation implies that if any two of dV, $d\phi$, or dp are simultaneously null, then all of them are null. □

Theorem II.3 generalizes theorem II.1, since one is now including pressure. For $d = 4$ it corresponds to the Newtonian version of a theorem by Guilfoyle on relativistic charged pressure fluids of Weyl type (see Sec. III C). Further consequences of this theorem are explored next.

2. Weyl and Majumdar-Papapetrou relations for generic pressure in the Newton-Coulomb theory in higher dimensions

Doing for d-dimensional Newtonian gravitation what Weyl did for general relativity [3] even in the presence of matter with pressure, assume a functional relation between the gravitational and the electric potential, as in Eq. (14), $V = V(\phi)$. With this ansatz that we call here the Weyl ansatz, Weyl originally worked out the Einstein-Maxwell vacuum equations that would follow and found that the relativistic potential is a quadratic function of the electric potential. Doing the same here in Newtonian gravitation, we find that the ansatz (14), when substituted into Eqs. (11) and (10), gives in vacuum, $\rho_{\rm m} = 0$, $p = 0$, and $\rho_{\rm e} = 0$, that, $V'' = 0$, i.e.,

$$V(\phi) = -\epsilon\beta\sqrt{G_d}\phi + \gamma, \tag{24}$$

where β and γ are arbitrary constants. This is the Weyl relation for the Newton-Coulomb theory in vacuum. When $\beta = 1$ one has $V(\phi) = -\epsilon\sqrt{G_d}\phi + \gamma$, which is the Majumdar-Papapetrou relation, see Eq. (20). In [27] it was stated that the Weyl and the Majumdar-Papapetrou relations are the same in the Newton-Coulomb theory, but this is only true for $\beta = 1$. In fact, $\beta \neq 1$ is important when one considers systems with pressure.

Let us now work out the basic equations for matter with nonzero pressure using the Weyl ansatz (14). It is convenient to analyze first Eq. (12), which now reads $(\rho_{\rm m}V' + \rho_{\rm e})\nabla_i\phi + \nabla_i p = 0$. It follows that p is also a function of ϕ, $p = p(\phi)$. So, the two fields V and ϕ have the same equipotential surfaces, which are also surfaces of constant pressure. Since we consider $\nabla_i\phi \neq 0$, Eq. (12) is then equivalent to

$$\rho_{\rm m}V' + \rho_{\rm e} + p' = 0, \tag{25}$$

where again the prime stands for the derivative with respect to ϕ. Let us now state these results in a compact form.

Theorem II.4. (Newtonian version of Guilfoyle 1999).

(i) If the Newton-Coulomb charged pressure fluid is of Weyl type and is in equilibrium, then the equipotentials are also hypersurfaces of constant pressure, and vice versa.

(ii) If the Newton-Coulomb charged pressure fluid is of Weyl type and is in equilibrium, then either the pressure gradient vanishes at the surface of the fluid or the surface is an equipotential.

Proof. Using the Weyl ansatz $V = V(\phi)$ and the equilibrium equation (12) we have $(\rho_{\rm m}V' + \rho_{\rm e})d\phi + dp = 0$. Then the hypersurfaces of constant ϕ and V coincide, and theorem II.3 implies that the third, of constant p, also coincides. On the other hand, a surface of constant p implies $d\phi = 0$, and thus it is also a surface of constant ϕ and of constant V. This proves (i). To prove (ii) we note that continuity conditions establish that the pressure is zero at the surface of the fluid, which means it is a surface of constant pressure, $dp = (\nabla_i p)dx^i = 0$. Then, unless the pressure gradient vanishes at the surface, $\nabla_i p = 0$ for all i, by (i) the surface of the fluid is an equipotential surface. □

Guilfoyle [9] has shown an analogous theorem for relativistic charged pressure fluid of Weyl type in four-dimensional spacetimes. The above theorem II.4 is the Newtonian version of Guilfoyle's theorem extended to higher dimensional spaces (see Sec. III C 2).

Now we rewrite the basic equation on the basis of theorems II.3 and II.4. From Eqs. (10) and (11), together with (25), we can find an equation for ϕ in terms of V' and p':

$$[(V')^2 - G_d]\nabla^2\phi + V'V''(\nabla_i\phi)^2 = -S_{d-2}G_d p'. \tag{26}$$

Once $V(\phi)$ and $p(\phi)$ are supplied, this is the final equation to be solved for ϕ. Such an analysis, however, is not done in the present work.

3. New theorem in four and higher dimensions and the Weyl-Guilfoyle relation: Following Gautreau-Hoffman

We turn once again to Eqs. (10) and (11) and observe that multiplying the second of those equations by $-\epsilon\beta\sqrt{G_d}$, with $\epsilon = \pm 1$ and β being an arbitrary parameter, and subtracting the result form the first equation we get

$$\nabla^2(V + \epsilon\beta\sqrt{G_d}\phi - \gamma) = S_{d-2}\sqrt{G_d}(\sqrt{G_d}\rho_{\rm m} - \epsilon\beta\rho_{\rm e}), \tag{27}$$

where γ is a constant. Now, if $V + \epsilon\beta\sqrt{G_d}\phi - \gamma = 0$ everywhere inside matter, it follows that $\rho_{\rm e}$ is proportional to $\rho_{\rm m}$, and, conversely, assuming the right-hand side of Eq. (27) is zero, i.e., if $\rho_{\rm e}$ is proportional to $\rho_{\rm m}$, then $\nabla^2(V + \epsilon\beta\sqrt{G_d}\phi) = 0$. Thus one can state the following theorem:

Theorem II.5. (New).

(i) In a Newton-Coulomb charged pressure fluid, if the potentials are such that $V + \epsilon\beta\sqrt{G_d}\phi - \gamma = 0$, with constant β and γ, then it follows the condition

$$\beta\rho_{\rm e} = \epsilon\sqrt{G_d}\rho_{\rm m}. \tag{28}$$

(ii) In a Newton-Coulomb charged pressure fluid, if the ratio $\rho_{\rm e}/\rho_{\rm m}$ equals a constant and there is a closed surface, with no singularities, holes, or alien matter inside it, where $V + \epsilon\beta\sqrt{G_d}\phi - \gamma$ vanishes, then it follows

$$V = -\epsilon\beta\sqrt{G_d}\phi + \gamma \tag{29}$$

everywhere.

Proof. The proof of (i) follows straightforwardly from Eq. (27), because the hypothesis of the theorem implies that the right-hand side of such an equation must be zero. For (ii), we take $\rho_{\rm e}/\rho_{\rm m} = \epsilon\sqrt{G_d}/\beta$, with constant β, and then Eq. (27) implies in $\nabla^2(V + \epsilon\beta\sqrt{G_d}\phi - \gamma) = 0$. Let $F = V + \epsilon\beta\sqrt{G_d}\phi - \gamma$, so that $\nabla^2 F = 0$. Hence, in view of this condition one has $\nabla_i(F\nabla^i F) = (\nabla_i F)^2$. Integrate this equation over a volume $\mathcal{V}_{\mathcal{S}}$ in $(d-1)$-dimensional space to get

$$\int_{\mathcal{V}_S} \nabla_i (F \nabla^i F) d\mathcal{V} = \int_{\mathcal{V}_S} (\nabla_i F)^2 d\mathcal{V} = \int_{\mathcal{S}} F(\nabla_i F) n^i d\mathcal{S}, \tag{30}$$

$\mathcal{S}$ being the boundary of $\mathcal{V}_S$, n^i being the unit vector normal to $\mathcal{S}$, and the Gauss theorem has been used. If there exists a closed surface on which F vanishes, then by identifying such a surface with $\mathcal{S}$ one finds $\int_{\mathcal{V}_S} (\nabla_i F)^2 d\mathcal{V} = 0$, which is satisfied only if $\nabla_i F = 0$ everywhere inside $\mathcal{S}$. This means $F =$ constant everywhere in the region bounded by $\mathcal{S}$ and the stated result follows. □

Our theorem II.5 is a Newtonian version of the results we find in the relativistic section (see Sec. III C 3). Our relativistic theorem for fluids obeying a Weyl-Guilfoyle relation, in turn, was inspired on the analysis by Gautreau and Hoffman [8], who studied relativistic charged pressure fluids obeying a Weyl relation in four-dimensional spacetimes. Equation (28) may be thought as the most general condition relating the densities and the pressure in the Newton-Coulomb theory with matter for fluids obeying a Weyl type relation. We call it the Gautreau-Hoffman condition for the Newtonian theory. Note also that Eq. (29) is identical to the Weyl relation for the Newton-Coulomb theory in vacuum, given by Eq. (24). However, upon comparison with the relativistic case, it is found that it is also a Weyl-Guilfoyle relation, so, in the case of pressure, it can be said that the Weyl and the Weyl-Guilfoyle relations coincide in the Newtonian case. Unlike the relativistic case, the most general relation between potentials in the Newton-Coulomb theory coincides with the relation obtained in vacuum. When one puts $\beta = 1$ in Eqs. (28) and (29) one gets the Majumdar-Papapetrou condition and relation, respectively. As we will see $\beta = 1$ means no pressure, while arbitrary β, $\beta \neq 1$, implies fluids with nonzero pressure.

4. The same new theorem as the last subsection: Following Bonnor-De-Raychaudhuri

We now proof in a different way the theorem stated in the last subsection. Here we follow Bonnor [2], who in turn followed the relativistic theorem of De and Raychaudhuri [7]. Indeed, the result found above can also be obtained following the Bonnor approach [2], in which the equilibrium equation is used as a subsidiary condition. As we shall see below, this strategy brings the pressure into play and, after some hypotheses concerning the relation among the pressure gradient and mass and charge densities, a result that is similar to theorem II.5 is found.

Notice that Eq. (25) displays explicitly an analogy between the pressure gradient p' and the charge density ρ_e, in fact ρ_e can be considered as some sort of pressure gradient, both act to balance the gravitational attraction. Such a similarity becomes even more striking by substituting ρ_m from the last equation, Eq. (25), into Eq. (10), to find $\nabla^2 V = (V')^2 \nabla^2 \phi + V' V'' (\nabla_i \phi)^2 = -S_{d-2}(\rho_e + p')$. Thus, the derivative of the pressure with respect to the electric potential ϕ acts as a source for the electric field, in much the same way as the charge density does. It is then natural to assume a relation in the form $p' = \chi \rho_e$. However, since the Weyl ansatz (14) tell us that the potentials V and ϕ are related to each other, so are the charge and mass densities. It is then reasonable to try a more general relation among p', ρ_e, and ρ_m. Namely,

$$p' = \chi \rho_e + \lambda V' \rho_m, \tag{31}$$

χ and λ being arbitrary functions of the coordinates. After this definition, using Eqs. (11), (15), and (26) one finds $[(1+\lambda)V'^2 - (1+\chi)G_d]\nabla^2\phi + (1+\lambda)V'V''(\nabla_i\phi)^2 = 0$. This equation can be cast into the form

$$Z_p \nabla_i (Z_p \nabla^i \phi) = -\tfrac{1}{2}(V'^2 \lambda' - G_d \chi')(\nabla_i \phi)^2, \tag{32}$$

where we have defined

$$Z_p = \sqrt{(1+\lambda)V'^2 - (1+\chi)G_d}. \tag{33}$$

Since a complete analysis of the general solutions to Eq. (32) for arbitrary λ and χ is not an easy task, here we will study some particular cases and leave the general case to be considered in future work.

Consider thus the particular case of a charged pressure fluid satisfying (31) with constant χ and λ. This condition implies $\chi' = 0$ and $\lambda' = 0$, and from Eqs. (25) and (32) we get

$$V' = -\frac{(1+\chi)\rho_e}{(1+\lambda)\rho_m}, \tag{34}$$

$$Z_p \nabla_i (Z_p \nabla^i \phi) = 0. \tag{35}$$

From this we can prove statements that generalize theorem II.2 to include pressure, and whose results are equivalent to what was found in connection to theorem II.5.

Theorem II.6. (Same as theorem II.5, following other path).

(i) If the surfaces of any Newton-Coulomb charged pressure fluid distribution are closed equipotential hypersurfaces and inside these hypersurfaces there are no singularities, holes, or alien matter, and the fluid is of Weyl type, whose pressure satisfies the condition given in Eq. (31) with constant λ and χ, then the relations

$$V = -\epsilon \sqrt{\frac{1+\chi}{1+\lambda} G_d}\, \phi + \gamma, \tag{36}$$

$$\rho_e = \epsilon \sqrt{\frac{1+\lambda}{1+\chi} G_d}\, \rho_m, \tag{37}$$

with constant λ, γ, and χ, hold everywhere inside the fluid distributions.

(ii) In a Newton-Coulomb charged pressure fluid, if the relation between mass and charge densities is as in Eq. (37), with constant λ and χ, and there are no singularities, holes, or alien matter in the considered region, then the relation (36) holds.

Proof. First we show that, with the above conditions, Z_p must be zero. This is done by following the same reasoning as in the case of theorem II.2, where now the new field ψ is defined by $\nabla_i\psi = Z_p\nabla_i\phi$, with Z_p given by Eq. (33). Once it is proved that $Z_p = 0$, which means $(1+\lambda)V'^2 = G_d(1+\chi)$, Eq. (36) is obtained by direct integration of this result, while Eq. (37) follows from this and from Eq. (34). This proves (i). The proof of (ii) is equivalent to theorem II.2 (ii) and we do not give it here. □

This theorem II.6 is inspired in the analysis of a relativistic charged dust fluid in four-dimensional spacetimes by De and Raychaudhuri [7] (see Sec. III B). It generalizes theorem II.2 to include pressure, which in turn generalizes results by Bonnor [2] for a charged pressureless Newton-Coulomb fluid in $d = 4$.

As seen above, in order to have nonsingular solutions of Weyl type in $(d-1)$-dimensional Euclidean spaces filled by charged matter with pressure satisfying the condition $dp/d\phi = \chi\rho_e + \lambda V'\rho_m$, with constant χ and λ, the function $V(\phi)$ must be a linear function of ϕ. Moreover, the ratio between mass and charge densities is a constant. As expected from theorem II.2, the case $\chi = \lambda$ gives $\rho_e = \sqrt{G_d}\rho_m$ and the result is the dust fluid $p = 0$. These results are essentially the same as what was found in connection with theorem II.5, but now we have a two parameter solution, one of them connecting the pressure to the charge density and the other one relating the pressure to the mass density. To see that explicitly define $\beta = \sqrt{\frac{1+\chi}{1+\lambda}}$ to find that the relations (36) and (37) reproduce, respectively, Eqs. (28) and (29), the corresponding results of theorem II.5. The two parameters χ and λ are convenient for the comparison of the present results with the relativistic case.

A consequence of the above analysis is that the pressure gradient p' is proportional to the charge density (or to the mass density). Moreover, if the fluid satisfies the Majumdar-Papapetrou relation for the potentials, the pressure is zero. This can be stated as a corollary to theorems II.5 and III.6.

Corollary II.7. (New).

For any static charged pressure fluid distribution in the Newton-Coulomb theory, if the potentials V and ϕ satisfy the relation (29), or equivalently, the relation (36), then the pressure is given by

$$p = \epsilon\frac{\sqrt{G_d}}{\beta}(1-\beta^2)\int\rho_m(\phi)d\phi + p_0$$
$$= \epsilon\frac{\chi-\lambda}{1+\lambda}\sqrt{\frac{1+\lambda}{1+\chi}G_d}\int\rho_m(\phi)d\phi + p_0, \tag{38}$$

p_0 being an integration constant, and in the case $\beta^2 = 1$, or equivalently, in the case $\lambda = \chi$, the pressure is zero.

Proof. Consider the equilibrium equation in the form of Eq. (25) and use the fact that from theorem II.5 one has $V' = -\epsilon\beta\sqrt{G_d}$ and $\rho_e = \epsilon\beta\sqrt{G_d}\rho_m$, to find $p' = (\beta^2 - 1)\rho_e = -\epsilon\frac{\sqrt{G_d}}{\beta}(\beta^2-1)\rho_m$, which integrates to (38). □

In the case $\beta^2 = 1$ the pressure equals a constant, which in the Newtonian theory is equivalent to zero pressure, in accordance to theorem II.2. This result corresponds to the Newtonian limit of the relativistic relation between pressure and the metric potential for a charged fluid satisfying the Majumdar-Papapetrou relation for the potentials, cf. Corollary III.7.

All of the fluid quantities can now be given in terms of only one variable, the mass density, for instance, and with only one free parameter. Once the mass density ρ_m is specified, the gravitational potential is determined by the Poisson equation, $\nabla^2 V = S_{d-2}G\rho_m$ and all of the other quantities follow from V and ρ_m.

D. The mass and charge, and the mass-charge relation of the global solution

For completeness and for comparison with the relativistic theory let us define mass and charge in the Newton-Coulomb theory. These quantities are obtained by integration of the respective densities over the whole volume of the source $\mathcal{V}_S$

$$m = S_{d-2}\int_{\mathcal{V}_S}\rho_m d\mathcal{V}, \tag{39}$$

$$q = S_{d-2}\int_{\mathcal{V}_S}\rho_e d\mathcal{V}. \tag{40}$$

For zero pressure, Eq. (21) implies that the total mass and total charge of the source are proportional to each other. For Weyl type fluids with nonzero pressure, m and q are proportional only in the case where $dp/d\phi$ is constant. In such a case, one has the mass-charge relation

$$\beta q = \epsilon\sqrt{G_d}m, \tag{41}$$

which follows from Eq. (28).

III. GENERAL RELATIVISTIC CHARGED FLUID WITH PRESSURE IN *d* SPACETIME DIMENSIONS

A. Basic equations

Einstein-Maxwell equations in d spacetime dimensions are written as

$$G_{\mu\nu} = \frac{d-2}{d-3}S_{d-2}G_d(T_{\mu\nu}+E_{\mu\nu}), \tag{42}$$

$$\nabla_\nu F^{\mu\nu} = S_{d-2}J^\mu, \tag{43}$$

where Greek indices μ, ν, etc., run from 0 to $d-1$. $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, with $R_{\mu\nu}$ being the

Ricci tensor, $g_{\mu\nu}$ the metric, and R the Ricci scalar. S_{d-2} and G_d have the same definitions as in the Newton-Coulomb theory (See Sec. II A, see also [27]). We have put the speed of light equal to unity throughout. Note the singular behavior of the lower dimensional cases, $d = 2$ and $d = 3$, which shall not be treated here. $E_{\mu\nu}$ is the electromagnetic energy-momentum tensor, which can be written as

$$S_{d-2}E_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{44}$$

where the Maxwell tensor is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{45}$$

∇_μ being the covariant derivative, and A_μ the electromagnetic gauge field. In addition,

$$J_\mu = \rho_{\rm e} U_\mu \tag{46}$$

is the current density, $\rho_{\rm e}$ is the d-dimensional electric charge density, and U_μ is the fluid velocity. $T_{\mu\nu}$ is the material energy-momentum tensor given by

$$T_{\mu\nu} = (\rho_{\rm m} + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{47}$$

where $\rho_{\rm m}$ is the fluid matter energy density in the d-dimensional spacetime, and p is the fluid pressure.

We assume the spacetime is static and that the metric can be written in the form

$$ds^2 = -W^2dt^2 + h_{ij}dx^idx^j, \qquad i, j = 1, \ldots, d-1, \tag{48}$$

a direct extension of the Majumdar-Papapetrou metric to extra dimensions. The gauge field A_μ and four-velocity U_μ are then given by

$$A_\mu = -\phi\delta^0_\mu, \tag{49}$$

$$U_\mu = -W\delta^0_\mu. \tag{50}$$

The spatial metric tensor h_{ij}, the metric potential W, and the electrostatic potential ϕ are functions of the spatial coordinates x^i alone. Initially, we are interested in the equations determining the metric potential W and the electric potential ϕ. These are obtained, respectively, from the tt component of Einstein equations (42) and from the t component of Maxwell equations (43). Such equations give

$$\nabla^2 W = \frac{G_d}{W}(\nabla_i\phi)^2 + S_{d-2}G_dW\left(\rho_{\rm m} + \frac{d-1}{d-3}p\right), \tag{51}$$

$$\nabla^2\phi - \frac{1}{W}\nabla_iW\nabla^i\phi = -S_{d-2}W\rho_{\rm e}, \tag{52}$$

where ∇_i denotes the covariant derivative with respect to the coordinate x^i, with connection coefficients given in terms of the metric h_{ij}.

Equations (51) and (52) determine the potentials W and ϕ in terms of a set of unknown quantities. Namely, the $(d-1)(d-2)/2$ spatial metric coefficients h_{ij} and the fluid variables, energy density $\rho_{\rm m}$, electric charge density $\rho_{\rm e}$, and pressure p. There are exactly $(d-1)(d-2)/2$ additional equations that come from the Einstein equations, which in principle determine the h_{ij} metric components in terms of $\rho_{\rm m}$, p, and $\rho_{\rm e}$. Hence, to complete the system of equations it is necessary to provide the energy and charge density functions, $\rho_{\rm m}$ and $\rho_{\rm e}$, and also to specify the pressure p or an equation of state for the fluid. In the present analysis, we will not need the explicit form of the space metric h_{ij} and so the corresponding Einstein equations will not be written here. Additional equations that can be used are the conservation equations, $\nabla_\nu T^{\mu\nu} = 0$, which are sometimes useful in replacing a subset of Einstein's equations. In the present case the conservation equations yield

$$(\rho_{\rm m} + p)\frac{\nabla_iW}{W} + \rho_{\rm e}\frac{\nabla_i\phi}{W} + \nabla_ip = 0. \tag{53}$$

This is the relativistic analogue to the Euler equation and carries the information of how the pressure gradients balance the equilibrium of the system. It also shows that p, W, and ϕ are functionally related, e.g., $p = p(W, \phi)$ or $W = W(p, \phi)$.

B. Zero pressure: Weyl type systems in the Einstein-Maxwell theory and the De-Raychaudhuri theorem in higher dimensions

A special interesting case of charged matter is the dust fluid, for which $p = 0$ and Eq. (53) reduces to

$$\nabla_iW + \frac{\rho_{\rm e}}{\rho_{\rm m}}\nabla_i\phi = 0. \tag{54}$$

Given Eq. (54), we can render into d dimensions a theorem stated in four dimensions, in part by De and Raychaudhuri [7], and fully by Guilfoyle [9] where pressure is included. The theorem is the relativistic version of our Newtonian theorem II.1, which in turn generalizes a theorem by Bonnor [2].

Theorem III.1. (De-Raychaudhuri 1968, Guilfoyle 1999).

For any charged static dust distributions in the Einstein-Maxwell theory, the $(d-2)$-dimensional hypersurfaces of constant W coincide with the $(d-2)$-dimensional hypersurfaces of constant ϕ, and W is a function of ϕ alone, $W = W(\phi)$.

Proof. First observe that even though W and ϕ are scalar functions in a d-dimensional spacetime, since they do not depend upon time, each one of the conditions of constant W and ϕ indeed defines a $(d-2)$-dimensional hypersurface. Moreover, from the conservation equation (54) one gets $\rho_{\rm m}dW + \rho_{\rm e}d\phi = 0$. This equation implies that if any one of dW or $d\phi$ is null, then both of them are. It also results in $dW/d\phi = -\rho_{\rm e}/\rho_{\rm m}$ and the theorem follows. □

The result just stated ensures that in the presence of static dust fluid distributions the Weyl ansatz (14) is not necessary; it is a consequence of the equilibrium equations.

It is now convenient to rewrite the preceding equations taking the condition $W = W(\phi)$ into account. With this, Eqs. (51) and (52) read, respectively,

$$W'\nabla^2\phi + \left(W'' - \frac{G_d}{W}\right)(\nabla_i\phi)^2 = S_{d-2}G_d W\rho_{\rm m}, \quad (55)$$

$$\nabla^2\phi - \frac{W'}{W}(\nabla_i\phi)^2 = -S_{d-2}W\rho_{\rm e}, \quad (56)$$

where we have defined $W' = \frac{dW}{d\phi}$ and $W'' = \frac{d^2W}{d\phi^2}$. Substituting the functional relation $W = W(\phi)$ into the conservation equations (54) it follows

$$W' = -\frac{\rho_{\rm e}}{\rho_{\rm m}}. \quad (57)$$

The basic system of equations to be solved is now composed of Eqs. (55)–(57). Such a system can be thought of as determining the fluid variables $\rho_{\rm m}$, and $\rho_{\rm e}$ once the potentials and metric functions are known. On the other hand, $\rho_{\rm m}$ and $\rho_{\rm e}$ may be eliminated from Eqs. (55)–(57), to obtain

$$\bar{Z}\nabla_i(\bar{Z}\nabla^i\phi) = 0, \quad (58)$$

where

$$\bar{Z} = \sqrt{W'^2 - G_d}. \quad (59)$$

Equation (58) has the same form of Eq. (18) (see [7] and also [2]). Hence the conditions of theorem II.2 hold here, and we can render into d dimensions a theorem initiated in part by Papapetrou [5] and stated in part by Das [6] and in part by De and Raychaudhuri [7].

Theorem III.2. (Das 1962, De-Raychaudhuri 1968).

(i) In the Einstein-Maxwell theory, if the surfaces of any charged dust distribution are closed equipotential hypersurfaces and inside these hypersurfaces there are no singularities, holes, or alien matter, then the function $W(\phi)$ must satisfy the relation

$$W = -\epsilon\sqrt{G_d}\phi + b, \quad (60)$$

with b being an integration constant, $\epsilon = \pm 1$ as before, and in such a case it follows

$$\rho_{\rm e} = \epsilon\sqrt{G_d}\rho_{\rm m}. \quad (61)$$

(ii) In the Einstein-Maxwell theory, if in a spacetime region the ratio $\rho_{\rm e}/\rho_{\rm m}$ equals a constant K, and there are no singularities, holes, or alien matter in that region, then the relation (60) for the potentials follows, and also $K = \epsilon\sqrt{G_d}$.

Proof. The proof of (i) is obtained by defining a new variable ψ, such that $\nabla^i\psi = \bar{Z}\nabla^i\phi$, with $\bar{Z}$ given by Eq. (59). Following the same steps as in the case of the Newtonian theorem II.2 (i) it can be shown that $\bar{Z}$ must be zero. Then Eqs. (60) and (61) are obtained immediately. The proof of (ii) follows once we assume that the ratio $\rho_{\rm e}/\rho_{\rm m}$ is constant and use Eqs. (57) and (59). Then $\bar{Z} = 0$, resulting in the same relations among W and ϕ, and among $\rho_{\rm m}$ and $\rho_{\rm e}$ as in Eqs. (60) and (61) above. □

Papapetrou found the relation (61) by assuming *a priori* the relation between the potentials to be a perfect square, $W^2 = (-\epsilon\sqrt{G}\phi + 1)^2$. The theorem stated by Das [6] assumes *a priori* that the level surfaces are the same $W = W(\phi)$, which is not necessary according to [7]; see also our theorems II.1 and II.2 for the Newton-Coulomb theory.

C. Nonzero pressure: Weyl type systems in the Einstein-Maxwell theory and new theorems in higher dimensions

1. Relativistic static charged pressure fluid: General results

As in the Newton-Coulomb case, Eq. (53) can be written in terms of total derivatives. For contracting it with dx^i yields $(\rho_{\rm m} + p)(\nabla_i W)dx^i + \rho_{\rm e}(\nabla_i\phi)dx^i + W(\nabla_i p)dx^i = 0$, or $(\rho_{\rm m} + p)dW + \rho_{\rm e}d\phi + Wdp = 0$. This implies that W, ϕ, and p are functionally related, and we can state a theorem rendering into d spacetime dimensions a result by Guilfoyle in $d = 4$ [9]:

Theorem III.3. (Guilfoyle 1999).

For any static charged pressure fluid in the Einstein-Maxwell theory, if any two of the $(d-2)$-dimensional hypersurfaces of constant W, ϕ, or p coincide, then the third also coincides.

Proof. As shown above (see theorem III.1) the conditions of constant W, ϕ, or p define a $(d-2)$-dimensional hypersurface. Moreover, from the conservation equation (53) one gets $(\rho_{\rm m} + p)dW + \rho_{\rm e}d\phi + Wdp = 0$. This equation implies that if any two of dW, $d\phi$, or dp are simultaneously null, then all the three are. This proves the theorem. □

A further interesting consequence of the conservation equation is that, thinking of p as a function of W and ϕ, and observing that Eq. (53) implies $dp = -(\rho_{\rm m} + p)dW/W - \rho_{\rm e}d\phi/W$, we find $(\partial p/\partial W)_\phi = -(\rho_{\rm m} + p)/W$, and $(\partial p/\partial\phi)_W = -\rho_{\rm e}/W$. Comparing to the Newton-Coulomb case, these relations confirm the fact that in relativistic theory the pressure itself acts against the pressure gradient, as the energy density, and that there are extra couplings between energy and charge densities to the gravitational (metric) field.

2. Weyl and Majumdar-Papapetrou relations for charged pressure fluids in higher dimensions

Doing in d-dimensional spacetimes what Weyl did in four-dimensional general relativity [3] even in the presence of matter with pressure, we make the assumption on Einstein-Maxwell charged fluids in d-dimensional spacetimes of being Weyl type fluids, where the metric potential $g_{tt} \equiv -W^2$ is a functional of the gauge potential ϕ, $W = W(\phi)$, that we call here the Weyl ansatz. With such an ansatz, Eqs. (51) and (52) read, respectively,

$$W'\nabla^2\phi + \left(W'' - \frac{G_d}{W}\right)(\nabla_i\phi)^2 = S_{d-2}G_d W\left(\rho_{\rm m} + \frac{d-1}{d-3}p\right), \tag{62}$$

$$\nabla^2\phi - \frac{W'}{W}(\nabla_i\phi)^2 = -S_{d-2}W\rho_{\rm e}. \tag{63}$$

For completion, and for comparison to the Newton-Coulomb case, let us write here the result of the d-dimensional Weyl ansatz in vacuum (see [19,27]). Taking into account the conditions $\rho_{\rm m} = 0$, $p = 0$, and $\rho_{\rm e} = 0$, Eqs. (51) and (52) give $WW'' + W'^2 - G_d = 0$, i.e.,

$$W^2(\phi) = (-\epsilon\sqrt{G_d}\phi + b)^2 + c, \tag{64}$$

where b and c are constant parameters. This is the d-dimensional version of the original Weyl relation, which is the analog of Eq. (24) in the Newton-Coulomb theory. For $c = 0$ one gets $W = -\epsilon\sqrt{G_d}\phi + b$ which is the Majumdar-Papapetrou relation, see Eq. (60). We shall now work out the basic equations for relativistic matter with nonzero pressure using the Weyl ansatz $W = W(\phi)$. From Eq. (53) one has $p = (W, \phi)$ and, by assuming further that W is a function of ϕ alone, it follows from theorem III.3 that the pressure p is also a function of the electric potential only, $p = p(\phi)$. Hence, Eq. (53) may be written as

$$(\rho_{\rm m} + p)\frac{W'}{W} + \frac{\rho_{\rm e}}{W} + p' = 0. \tag{65}$$

We may now state more formally these results. In $d = 4$ this was shown by Guilfoyle [9].

Theorem III.4. (Guilfoyle 1999).

(i) If the Einstein-Maxwell charged pressure fluid is of Weyl type and is in equilibrium, then the equipotentials are also hypersurfaces of constant pressure, and vice versa.

(ii) If the Einstein-Maxwell charged pressure fluid is of Weyl type and is in equilibrium, then either the pressure gradient vanishes at the surface of the fluid or the surface is an equipotential.

Proof. A relativistic Weyl type fluid satisfies the relativistic Weyl ansatz $W = W(\phi)$ and Eq. (65) holds. From that equation we have $(\rho_{\rm m} + p)dW + \rho_{\rm e}d\phi + Wdp = 0$. Then since the hypersurfaces of constant W and ϕ coincide, theorem III.3 implies the third, the one of constant p also coincides. Conversely, a surface of constant p implies $d\phi = 0$, and thus it is also a surface of constant ϕ and of constant W. This proves (i). To prove (ii) we first note that junction conditions establish that the pressure is zero at the surface of the fluid, which means it is a surface of constant pressure, $dp = (\nabla_i p)dx^i = 0$. Then, unless the pressure gradient vanishes at the surface, $\nabla_i p = 0$ for all i, by (i) the surface of the fluid is an equipotential surface. □

We now go back to Eqs. (62)–(65) and use them to eliminate $\rho_{\rm m}$ and $\rho_{\rm e}$ in terms of the other quantities. The resulting equation, which can be thought of as an equation for p, may be cast into the form

$$\bar{Z}\nabla_i(\bar{Z}\nabla^i\phi) = -S_{d-2}G_d W^{2(d-2)/(d-3)}(W^{-2/(d-3)}p)', \tag{66}$$

where $\bar{Z}$ is given by (59). Usually, a relativistic charged fluid problem is completely set out once we furnish the energy and charge densities and an equation of state for the fluid. However, as it can be seen from Eq. (66), for a Weyl type system, the problem is in a position to be solved once the pressure gradient p' and the metric potential W are given in terms of ϕ. In such a case, Eq. (66) can in principle be integrated for ϕ, from what all of the other fluid variables would follow. We shall analyze some particular cases of this system next.

3. New theorem in four and higher dimensions and the Weyl-Guilfoyle relation: Following Gautreau-Hoffman

Here we follow the approach by Gautreau and Hoffman [8] in order to find the general properties of a charged pressure fluid satisfying a Weyl-Guilfoyle relation, rather than the Weyl relation alone as assumed in [8]. The results found in this section generalize previous results in two ways. First, we render the theorem given in [8] to higher dimensions, and, second, we find, for $d = 4$ and $d > 4$, the conditions that source matter distributions must obey in order to satisfy the most general quadratic form for the potentials $W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c$, with arbitrary constants a, b, and c.

By defining the electromagnetic energy density as

$$\rho_{\rm em} = \frac{1}{S_{d-2}}\frac{(\nabla_i\phi)^2}{W^2}, \tag{67}$$

one may cast Eqs. (51) and (52) into the form

$$\nabla^2 W = S_{d-2}G_d W\left(\rho_{\rm m} + \frac{d-1}{d-3}p + \rho_{\rm em}\right), \tag{68}$$

$$\nabla_i\left(\frac{1}{W}\nabla^i\phi\right) = -S_{d-2}\rho_{\rm e}. \tag{69}$$

Also, using the identities $\nabla_i(\frac{1}{W}\nabla^i W^2) = 2\nabla^2 W$ and $\nabla_i(\frac{1}{W}\nabla^i\phi^2) = 2\frac{\phi}{W}\nabla^2\phi - 2\frac{\phi}{W^2}\nabla_i W\nabla^i\phi + 2S_{d-2}W\rho_{\rm em}$, we see that Eqs. (68) and (69) may be rearranged as

$$\nabla_i\left(\frac{1}{W}\nabla^i W^2\right) = 2S_{d-2}G_d W\left(\rho_{\rm m} + \frac{d-1}{d-3}p + \rho_{\rm em}\right), \tag{70}$$

$$\nabla_i\left(\frac{1}{W}\nabla^i\phi^2\right) = -2S_{d-2}(\rho_e\phi - \rho_{em}W). \tag{71}$$

Then, multiply (69) by $-2ab\epsilon\sqrt{G_d}$ and add to (71) multiplied by aG_d, with constant a and b, and subtract the result from Eq. (70) to find

$$\nabla_i\left(\frac{1}{W}\nabla^i[W^2 - a(-\epsilon\sqrt{G_d}\phi + b)^2 - c]\right) = 2S_{d-2}G_d\left[\left(\rho_m + \frac{d-1}{d-3}p + \rho_{em}\right)W + a(\phi\rho_e - W\rho_{em}) - \epsilon\frac{ab}{\sqrt{G_d}}\rho_e\right]. \tag{72}$$

On the basis of this equation some interesting conclusions can be drawn.

Theorem III.5. (New).

(i) In any Einstein-Maxwell charged pressure fluid, if the potentials are such that $W^2 - a(-\epsilon\sqrt{G_d}\phi + b)^2 - c$ vanishes everywhere, i.e., if $W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c$, then the charged pressure fluid quantities satisfy the constraint

$$ab\rho_e = \epsilon\sqrt{G_d}\left[\left(\rho_m + \frac{d-1}{d-3}p\right)W + \phi\rho_e + (a-1)(\phi\rho_e - W\rho_{em})\right], \tag{73}$$

or

$$a\rho_e(-\epsilon\sqrt{G_d}\phi + b) = \epsilon\sqrt{G_d}\left(\rho_m + \frac{d-1}{d-3}p + \epsilon(1-a)\rho_{em}\right)W, \tag{74}$$

which can be considered the equation of state satisfied by the charged fluid.

(ii) Conversely, in any Einstein-Maxwell charged pressure fluid, if the fluid quantities are such that Eq. (73) holds and there is a closed surface, with no singularities, holes, or alien matter inside it, where $W^2 - a(-\epsilon\sqrt{G_d}\phi + b)^2 - c$ vanishes, then

$$W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c \tag{75}$$

holds everywhere inside the surface.

Proof. The proof of this theorem is obtained in the exact same way as for the Newton-Coulomb case of theorem II.5. □

Theorem III.5 establishes that the generalized quadratic Weyl-Guilfoyle relation, i.e., Eq. (75) with $a \neq 1$, leads to a constraint among the fluid quantities which includes the electromagnetic energy density ρ_{em} and also the binding energy density $\phi\rho_e$. It generalizes the analysis by Gautreau and Hoffman [8] in two ways. First it holds for arbitrary values of constant a, while in [8] $a = 1$, and second it holds in an arbitrary number of spacetime dimensions $d \geq 4$, rendering into higher dimensions the results of Gautreau and Hoffman [8]. When a equals unity, the d-dimensional Weyl relation is recovered and, furthermore, if $c = 0$, Eq. (73) is reduced to the form found in [19]. Moreover, for $a = 1$ and in $d = 4$ it results in what was found by Gautreau and Hoffman [8]. Comparing our theorem to the theorem by Gautreau and Hoffman we see that in their analysis only the binding energy was taken into account, and hence the constant a was forced to be equal to unity. Furthermore, all the solutions found by Guilfoyle in [9] obey the equation of state, a constraint, provided by Eq. (73).

The functional form (75) is more general than the original quadratic Weyl type form. As pointed out by Guilfoyle [9], a reparametrization of the metric potential as $W \to \sqrt{a}W$, with $a > 0$ enables us to write Eq. (75) as $W^2 = (-\epsilon\sqrt{G_d}\phi + b)^2 + c/a$, which is indeed the original Weyl quadratic form. However, one needs to observe that this reparametrization, which corresponds to rescaling the time coordinate as $t \to t/\sqrt{a}$, leads to a new system where the mass density, the pressure, and the electric charge density of the fluid are rescaled as $\rho_e \to \rho_e/a$, $p \to p/a$, and $\rho_e \to \rho_e/\sqrt{a}$, respectively, which changes the balancing relation among the mass and charge densities and the pressure gradient. We explore this fact in the next section.

4. The same new theorem as the last subsection: Following De-Raychaudhuri

We shall analyze here the Weyl type charged pressure fluid now following the strategy of De and Raychaudhuri [7], in which the conservation equation is used to try to find a differential equation for the potentials, by eliminating all the fluid quantities from the system. This strategy has already been adopted in the Newton-Coulomb case of Sec. II C 4. The strategy is different from the Gautreau-Hoffman one, but the results are similar. Nevertheless it is interesting to see where this strategy leads.

A comparison between Eqs. (26) and (66) reveals that the quantity $W^{-2/(d-3)}p$ plays, in the Einstein-Maxwell with matter theory, a similar role to the one played by the fluid pressure p alone in the Newton-Coulomb with matter theory. Hence, in order to write the relativistic equations in a form that resembles the Newton-Coulomb case, we shall define an effective matter density $\bar{\rho}_m$ and an effective pressure $\bar{p}$, respectively, by $\bar{\rho}_m = \rho_m + \frac{d-1}{d-3}p$ and $\bar{p} = W^{-2/(d-3)}p$. With these definitions, Eq. (65) assumes the form $\bar{\rho}_m W' + \rho_e + W^{(d-1/d-3)}\bar{p}' = 0$. This result suggests that the product of the effective pressure gradient $\bar{p}'$ and some power of the metric potential W, such as $W^\delta p$, for some number δ, plays the role of the charge density analogously to the Newtonian case. Notice, however, that in the relativistic case the electromagnetic energy density $\sim(\nabla_i\phi)^2$ is also a source to the gravitational field W (cf. Eq. (51)). Then, it is seen that the pressure gradient $\bar{p}'$, besides being connected to the charge and mass densities, is also connected to the electromagnetic energy density. In fact, after a careful analysis, one finds that by defining two new quantities $\bar{\lambda}$ and $\bar{\chi}$ through the relation $W^{(d-1/d-3)}\bar{p}' = \frac{\bar{\chi}}{W^2}\rho_e + \bar{\lambda}W'(\bar{\rho}_m + \rho_{em})$, with ρ_{em} defined in Eq. (67), it is possible to put Eq. (66) into a similar form to Eq. (32). Going back to the original quantities, ρ_m, p, ρ_e, and ρ_{em}, we find the equation

$$\left(p' - \frac{2}{d-3}\frac{W'}{W}p\right)W = \frac{\bar{\chi}}{W^2}\rho_e + \bar{\lambda}W'\left(\rho_m + \frac{d-1}{d-3}p + \rho_{em}\right), \tag{76}$$

which is similar in form to the corresponding equation, Eq. (31), of the Newtonian theory. Here $\bar{\lambda}$ and $\bar{\chi}$ are arbitrary functions of ϕ alone. Then, Eq. (66) assumes the form

$$\bar{Z}_p\nabla_i(\bar{Z}_p\nabla^i\phi) = \frac{G_d}{2}\left(W'^2\bar{\lambda}' - \frac{\bar{\chi}'}{W^2}\right)(\nabla_i\phi)^2, \tag{77}$$

where we have defined

$$\bar{Z}_p = \sqrt{(1+\bar{\lambda})W'^2 - \left(1 + \frac{\bar{\chi}}{W^2}\right)G_d}. \tag{78}$$

As mentioned above, the study done in the present section was inspired in the work by De and Raychaudhuri [7] on the relativistic charged dust fluid in four-dimensional spacetimes. Besides performing the analysis in arbitrary dimensions, we have also generalized the analysis by including nonzero pressure and found that the pressure gradient plays the role of electric charge density. Then Eq. (77) is the fundamental equation to be solved for ϕ, noting that it is still needed to furnish the functions $W(\phi)$, $\bar{\lambda}(\phi)$, and $\bar{\chi}(\phi)$. There is, however, a particularly interesting case that deserves further analysis. In fact, as in the Newton-Coulomb case, for constant $\bar{\lambda}$ and $\bar{\chi}$, results that are equivalent to those stated in the theorem III.5 can be found.

Theorem III.6. (Same as theorem III.5, following another path).

(i) If the surfaces of any Einstein-Maxwell charged pressure fluid distribution are closed equipotential hypersurfaces and inside these hypersurfaces there are no singularities, holes, or alien matter, and the fluid is of Weyl type, whose pressure satisfies the condition given in Eq. (76) with constant $\bar{\lambda}$ and $\bar{\chi}$, then the function $W(\phi)$ must satisfy the relation

$$W^2 = \frac{1}{1+\bar{\lambda}}(-\epsilon\sqrt{G_d}\phi + b)^2 - \bar{\chi}, \tag{79}$$

b being an arbitrary integration constant, and it follows

$$b\rho_e = \epsilon(1+\bar{\lambda})\sqrt{G_d}\left[\left(\rho_m + \frac{d-1}{d-3}p\right)W + \phi\rho_e - \frac{\bar{\lambda}}{1+\bar{\lambda}}(\phi\rho_e - W\rho_{em})\right], \tag{80}$$

or equivalently

$$\rho_e(-\epsilon\sqrt{G_d}\phi + b) = \epsilon(1+\bar{\lambda})\sqrt{G_d}\left(\rho_m + \frac{d-1}{d-3}p + \frac{\bar{\lambda}}{1+\bar{\lambda}}\rho_{em}\right)W. \tag{81}$$

(ii) If condition (80) holds everywhere inside the Einstein-Maxwell charged pressure fluid, with constant $\bar{\lambda}$ and $\bar{\chi}$, and there are no singularities, holes, or alien matter in that region, then the function W is of the form given in Eq. (79).

Proof. The proof of (i) may be given by defining a new variable ψ by $\nabla_i\psi = \bar{Z}_p\nabla_i\phi$, with $\bar{Z}_p\nabla_i\phi$ given by (78), and then following the same steps as done in the case of theorem II.2. In the end, it is found that $\bar{Z}_p\nabla_i\phi$ vanishes everywhere in the fluid, which means $\sqrt{1+\bar{\lambda}}W' = -\epsilon\sqrt{(1+\bar{\chi}/W^2)G_d}$. By integrating this equation one gets the relation (79), and then using this result and Eqs. (65) and (76) one gets the constraint (80). On the other hand, part (ii) follows straightforwardly from Eqs. (76), (77), and (80). □

Even though we have followed different strategies in each case, the results stated in theorem III.6 are equivalent to what is stated in theorem III.5. Namely, the obtained relations among fluid quantities and potentials given in Eqs. (79) and (80) (or (81)) are equivalent to the relation given by Eqs. (73) and (75), respectively. This can be shown explicitly through the appropriate identifications of the arbitrary constants, namely, $a = 1/(1+\bar{\lambda})$ and $c = -\bar{\chi}$. Let us now comment on three interesting particular cases: (i) $\bar{\chi} = 0$, $\bar{\lambda} \neq 0$, (ii) $\bar{\chi} \neq 0$, $\bar{\lambda} = 0$, and (iii) $\bar{\chi} \neq 0$ and $\bar{\lambda} = 0$. In the case (i) $\bar{\chi} = 0$, for arbitrary $\bar{\lambda}$, the metric potential is a perfect square function of the electric potential, $W = (-\epsilon\sqrt{G_d}\phi + b)/\sqrt{(1+\bar{\lambda})}$. This is closely related to the Newton-Coulomb case studied in Sec. II C 4, as it can be seen from its Newtonian limit. Case (ii) corresponds to the Weyl original quadratic form between the potentials, for which several studies in $d = 4$ have been done. The particular case (iii) in which both of the free parameters are zero, $\bar{\lambda} = 0$ and $\bar{\chi} = 0$ reproduce the result found in [19], and in [9] in four-dimensional spacetimes. This is a special case where the conditions for theorem III.2 hold even for nonzero pressure.

As seen above, Eq. (66) is the fundamental equation and its validity is guaranteed also in the case $\bar{\chi} = \bar{\lambda} = 0$. In fact, from Eq. (76) one gets

$$W^{2/(d-3)}(W^{-2/(d-3)}p)' = p' - \frac{2}{d-3}p\frac{W'}{W} = 0, \tag{82}$$

and then the right-hand side of Eq. (66) vanishes identically so that the conditions of theorem III.2 hold. In other words, if $\bar{\lambda} = 0$ and $\bar{\chi} = 0$ the Majumdar-Papapetrou relation holds. So one may state the following corollary, which was shown as a theorem for four-dimensional spacetime by Guilfoyle [9], see also Lemos and Zanchin [19] for the d-dimensional generalization. We repeat it here for completeness and because it follows as corollary of the previous theorems, rather than being a theorem itself.

Corollary III.7. (Guilfoyle 1999, Lemos-Zanchin 2005). In a region of a static spacetime, filled by a charged pressure fluid of Weyl type, the relation

$$p = kW^{2/(d-3)} \tag{83}$$

holds if and only if

$$W = -\epsilon\sqrt{G_d}\phi + b, \tag{84}$$

and there is a closed equipotential surface in that region of the spacetime with no singularities, holes, or alien matter inside it. In such a case it follows the relation

$$\rho_e = \epsilon\sqrt{G_d}\left(\rho_m + \frac{d-1}{d-3}p\right). \tag{85}$$

Proof. The proof is given by observing that if the relation (83) holds, the right-hand side of Eq. (66) vanishes, and then from theorem III.2 the results (84) and (85) follow. On the contrary, if (84) holds, Eq. (83) follows directly from Eq. (66), clearly there is no need for a closed surface here, and the result given in Eq. (85) follows from Eq. (81) when one puts $\bar{\lambda} = 0$. □

The Newtonian limit of this relativistic solution is a pressureless fluid, as will be shown below.

5. Spherically symmetric spacetimes

The presence of the constant $\bar{\lambda} \neq 0$, or, in the notation of Sec. III C 3, the presence of the constant $a \neq 1$, in the formula for the potentials is related to the work by Guilfoyle [9] in four-dimensional spacetimes. In fact, it can be shown that the Weyl type solutions found in [9] satisfy the conditions given by Eqs. (73) and (75), or equivalently, by Eqs. (79) and (81). To show that, and also to verify that for $d = 4$ our results are consistent with previous work, let us analyze one of the spherical solutions of [9]. The spacetime metric, in Schwarzschild coordinates, is taken in the form $ds^2 = -W^2(r)dt^2 + dr^2/\sqrt{1 - r^2/R^2} + r^2 d\Omega^2$, with r being the radial coordinate and R being a constant. Take, for instance, the solution found in [9] for $A = 3/2$, which corresponds to $\bar{\lambda} = 1/2$, or $a = 2/3$, according to our notation,

$$W^2(r) = \frac{3\bar{\chi}^3F^2(r)}{3\bar{\chi}^2F^2(r) - 2}, \tag{86}$$

$$8\pi\rho_m(r) = \frac{3}{R^2} - \frac{9\bar{\chi}^2k^2r^2}{(3\bar{\chi}^2F^2(r) - 2)^2}, \tag{87}$$

$$Q(r) = \frac{-3\epsilon\bar{\chi}kr^3}{3\bar{\chi}^2F^2(r) - 2}, \tag{88}$$

$$8\pi p(r) = -\frac{1}{R^2} + \frac{9\bar{\chi}^2k^2r^2}{(3\bar{\chi}^2F^2(r) - 2)^2} - \frac{4k\sqrt{1 - \frac{r^2}{R^2}}}{F(r)(3\bar{\chi}^2F^2(r) - 2)}, \tag{89}$$

where k is an integration constant, and $F(r)$ and $Q(r)$ are defined by

$$F(r) = c_0 - kR^2\sqrt{1 - \frac{r^2}{R^2}}, \tag{90}$$

$$Q(r) = 4\pi\int_0^r \rho_e(r)\frac{r^2dr}{\sqrt{1 - \frac{r^2}{R^2}}} = \frac{r^2}{W}\sqrt{1 - \frac{r^2}{R^2}}\left(\frac{d\phi(r)}{dr}\right)^2, \tag{91}$$

with c_0 being another integration constant, and we have put $G_4 \equiv G = 1$.

From Eqs. (88) and (91) we get both the electric charge density ρ_e and the electromagnetic energy density ρ_{em},

$$8\pi\rho_e(r) = \frac{-18\epsilon\bar{\chi}k}{3\bar{\chi}^2F^2(r) - 2}\left(\sqrt{1 - \frac{r^2}{R^2}} - \frac{2\bar{\chi}^2kr^2F(r)}{3\bar{\chi}^2F^2(r) - 2}\right), \tag{92}$$

$$8\pi\rho_{em}(r) = \frac{18\bar{\chi}^2k^2r^2}{(3\bar{\chi}^2F^2(r) - 2)^2}. \tag{93}$$

In the case $d = 4$ and $\bar{\lambda} = 1/2$, Eq. (81) reads

$$\rho_e = \sqrt{\frac{3}{2}}\left(\rho_m + 3p + \frac{\rho_{em}}{3}\right)\frac{\epsilon W}{\sqrt{W^2 + \bar{\chi}}}. \tag{94}$$

Hence, in order to check if the solution satisfies this constraint, we use Eqs. (87), (89), and (93) to obtain

$$8\pi\sqrt{\frac{3}{2}}\left(\rho_m + 3p + \frac{\rho_{em}}{3}\right) = -\frac{12k\sqrt{3/2}}{3\bar{\chi}^2F^2(r) - 2}\left(\frac{1}{F(r)} \times\sqrt{1 - \frac{r^2}{R^2}} - \frac{2\bar{\chi}^2kr^2}{3\bar{\chi}^2F^2(r) - 2}\right), \tag{95}$$

where we have multiplied both sides of the last equation by 8π. Now, from Eq. (86) we find $\frac{W}{\sqrt{W^2+\bar{\chi}}} = \sqrt{\frac{3}{2}}\bar{\chi}F(r)$, and then the right-hand side of Eq. (94) corresponds to

$$8\pi\sqrt{\frac{3}{2}}\left(\rho_m + 3p + \frac{\rho_{em}}{3}\right)\frac{\epsilon W}{\sqrt{W^2 + \bar{\chi}}} = -\frac{18\epsilon\bar{\chi}k}{3\bar{\chi}^2F^2(r) - 2}\left(\sqrt{1 - \frac{r^2}{R^2}} - \frac{2\bar{\chi}^2kr^2F(r)}{3\bar{\chi}^2F^2(r) - 2}\right), \tag{96}$$

agreeing with the solutions for $\rho_e(r)$ as in Eq. (92), and showing that this particular solution by Guilfoyle [9] really matches the conditions established in the present work. With confidence we can state that all Guilfoyle's solutions [9] obey the condition (73) or the other equivalent ones.

6. The Newtonian limit

The Newtonian limit can be obtained from the relativistic quantities by considering Eq. (66) with the following approximations: (i) Write $W = 1 + F$ and consider $F = F(\phi)$ small compared to unity, F is to be compared to the Newtonian potential V; (ii) Neglect p with respect to $\rho_{\rm m}$; (iii) The product $p'W = p'(1+F)$ is approximated by p' since $p'F$ is also a second order term; (iv) Neglect pW' when compared to $p'W$, because it gives $pW' = pF'$ which is a second order term too; (v) The product $\rho_{\rm e}W = \rho_{\rm e}(1+F)$ is approximated to $\rho_{\rm e}$. It then follows that $Wp' - \frac{1}{d-3}pW' \simeq p' - \frac{1}{d-3}pF'$, and from Eq. (59), that $\bar{Z} = F'^2 - G_d$. Therefore, Eq. (66) reduces to

$$\sqrt{F'^2 - G_d}\,\nabla_i\big(\sqrt{F'^2 - G_d}\,\nabla^i\phi\big) = S_{d-2}G_d p', \qquad (97)$$

which, with the identifications $F = V$ and $\bar{Z} = Z = \sqrt{F'^2 - G_d}$, gives exactly Eq. (18), that is the basic equation for the Newton-Coulomb theory (see also [2]). From this result one then finds the corresponding results for zero pressure, cf. Eqs (20) and (21). The Newtonian results for nonzero pressure in which $p' = \chi\rho_{\rm e} + \lambda V'\rho_{\rm m}$, with constant χ and λ, also follow immediately (cf. Eqs (36)–(38)).

Alternatively, one may obtain the Newtonian limit directly from the relativistic solutions found above. For instance, considering the case of Sec. III C 4 the Newtonian limit can be found by writing Eq. (75) in the form $W^2 = aG_d\phi^2 - 2\epsilon ab\sqrt{G_d}\phi + ab^2 + c$, so that to first order approximation in ϕ one gets $W^2 = ab^2 + c - 2\epsilon ab\sqrt{G_d}\phi$. Now writing $W \simeq 1 + 2V$ and redefining appropriately the constants a, b, and c in terms of λ, γ, and χ, Eq. (36) follows. Also, the two remaining Newtonian relations, Eqs. (37) and (38), are immediately obtained as the Newtonian limit of the corresponding relativistic equations. Namely, Eq. (37) follows from Eq. (80), with $W^2 \sim 1$ and neglecting the pressure p when compared to the matter density $\rho_{\rm m}$, and Eq. (38) is obtained after integration of the Newtonian approximation of Eq. (76), which is $p' = \bar{\chi}\rho_{\rm e} + \bar{\lambda}V'\rho_{\rm m}$. Note that the relation among the constants $\beta = \sqrt{(1+\chi)/(1+\lambda)}$ and γ used in the Newton-Coulomb theory and the constants $a = 1/(1+\bar{\lambda})$, b and $c = -\bar{\chi}$ used in the Einstein-Maxwell theory is not unique, since it depends on how the Newtonian limit is calculated. By taking $W^2 = 1 + 2V$, we find $\beta = ab$ and $\gamma = (ab^2 + c - 1)/2$.

Additionally, the relativistic relation $p = kW^{2/(d-3)}$ found in Sec. III C 4, which to first order approximation is $p = k = $ constant, and so $p' = 0$, corresponds to the zero pressure Newton-Coulomb fluid studied in Sec. II B (cf. Corollary II.7). In fact, in the Eulerian description of a Newton-Coulomb charge fluid the conditions $p = 0$ and $p = $ constant are equivalent.

D. Asymptotically Tangherlini spacetimes: The mass and charge, and the mass-charge type relation of the global solution

Here we particularize the analysis to asymptotically Tangherlini spacetimes, i.e., Reissner-Nordström d-dimensional spacetimes, filled by charged fluids of Weyl-Guilfoyle type. Such spacetimes have an asymptotically electrovacuum region, which implies strong constraints on the fluid distributions. Namely, the source fluid has to be of finite extent with a well-defined boundary, or the fluid quantities must approach zero in a sufficiently fast form, in such a way to guarantee the existence of an asymptotically vacuum region. We show below that in the case the fluid distribution has no definite boundary, or whenever a Weyl-Guilfoyle type relation of the form $W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c$ is imposed in the whole spacetime, the existence of an asymptotically electrovacuum region requires that $a = 1$, $c + b^2 = 1$ throughout, and the mass and electric charge are related by $bq = \epsilon\sqrt{G_d}m$.

The case of a fluid distribution with a boundary is more interesting, since different Weyl-Guilfoyle type relations may be assumed in each region of the spacetime. In fact, solutions of this kind were studied by Guilfoyle himself [9], who had found a more general mass to charge relation. Below we extend the Guilfoyle analysis to d-dimensional spacetimes. Following Gautreau and Hoffman [8] we also give a more formal definition of mass and charge for d-dimensional asymptotically Reissner-Nordström spacetimes of Weyl-Guilfoyle type. Finally, we show that in asymptotically Reissner-Nordström spacetimes the mass definition given here is identical to the Tolman mass [12].

1. Spherically symmetric interiors joined to an exterior Tangherlini spacetime: Relation between the mass, charge, and other parameters

Let us consider the case of spherically symmetric fluid distributions (see Guilfoyle [9] for solutions in $d = 4$). The metric is written as

$$ds^2 = -W(r)^2dt^2 + U(r)^2dr^2 + r^2d\Omega_{d-2}, \qquad (98)$$

where r is the radial coordinate in $(d-1)$ spatial dimensions, W and U are a function of r only, and $d\Omega_{d-2}$ is the metric of the unit sphere $\mathbf{S}^{d-2}$. The charged pressure fluid is bounded by a spherical surface of radius $r = \mathfrak{a}$, and for $r > \mathfrak{a}$ the metric is given by the Tangherlini solution [20]

$$ds^2 = -\left(1 - \frac{2G_d}{d-3}\frac{m}{r^{d-3}} + \frac{G_d}{(d-3)^2}\frac{q^2}{r^{2(d-3)}}\right)dt^2 + \frac{dr^2}{1 - \frac{2G_d}{d-3}\frac{m}{r^{d-3}} + \frac{G_d}{(d-3)^2}\frac{q^2}{r^{2(d-3)}}} + r^2d\Omega_{d-2}. \qquad (99)$$

Furthermore, in the asymptotic region the electric potential is given by

$$\phi = \frac{1}{d-3}\frac{q}{r^{d-3}} + \phi_0, \tag{100}$$

ϕ_0 being an arbitrary constant which defines the zero of the electric potential. Notice that no Weyl type relation of any kind is imposed to the potentials in the exterior region.

Our aim here is to find the explicit relation between q and m. This can be done, for instance, by direct integration of Eq. (73). However, in such a case we need to know the explicit form of the solution. Alternatively, such a relation can be obtained considering appropriate junction conditions on the boundary surface $r = \mathfrak{a}$. In fact, in the spherically symmetric case, we can integrate the Maxwell equation (71), which furnishes

$$Q(r) = r^{d-2}\frac{\phi'}{WU}, \tag{101}$$

where the prime denotes the derivative with respect to the radial coordinate r. We then use the Weyl-Guilfoyle relation to write ϕ in terms of W,

$$\sqrt{G_d}\phi = \epsilon b - \epsilon\sqrt{\frac{W^2}{a} - \frac{c}{a}}, \tag{102}$$

where $\epsilon = \pm 1$. With this, Eq. (101) reads

$$Q(r) = \epsilon\frac{1}{\sqrt{G_d}}\frac{r^{d-2}W'}{U\sqrt{aW^2 - ac}}. \tag{103}$$

Now we use the fact that $Q(r = \mathfrak{a}) = q$, and that the continuity of the metric coefficients on the surface $r = \mathfrak{a}$ implies in $W(\mathfrak{a})^2 = 1/U(\mathfrak{a})^2 = 1 - \frac{2G_d}{d-3}\frac{m}{\mathfrak{a}^{d-3}} + \frac{G_d}{(d-3)^2}\times\frac{q^2}{\mathfrak{a}^{2(d-3)}}$. With this, Eq. (103) yields

$$ac = a\left[1 - \frac{G_d m^2}{q^2}\right] + (a-1)\left(\frac{m}{q} - \frac{q}{(d-3)\mathfrak{a}^{d-3}}\right)^2 G_d, \tag{104}$$

which for $d = 4$ coincides with Eq. (25) of [9] and holds for all spherically symmetric charged pressure fluid distribution whose boundary is the spherical surface of radius $r = \mathfrak{a}$. The proportionality between q and m is recovered whenever $a = 1$, and, moreover, if one imposes further that $c = 1 - b^2$ it gives $\sqrt{1-c}q = bq = \epsilon\sqrt{G_d}m$. From Eq. (104) one sees there is also the possibility $m\mathfrak{a}^{d-3} = q^2/(d-3)$, for $a \neq 1$, and still holding $q \propto m$, more precisely $\sqrt{1-c}q = \epsilon\sqrt{G_d}m$. The extremal relation $q = \sqrt{G_d}m$ holds when the relation among W^2 and ϕ is a perfect square, for which the parameter c must vanish, $c = 0$, besides $a = 1$, implying also in $b = 1$.

2. Relation between the mass and charge for a fluid distribution with no symmetry a priori, linked to a vacuum that asymptotes the Tangherlini solution

Let us assume that the charge fluid distribution is such that the spacetime is asymptotically Tangherlini. This is the case, e.g., of sources of finite extent with a boundary, or of fluid distributions without a boundary in which the fluid quantities vanish smoothly in the asymptotic region in a sufficiently fast way. For these kinds of spacetimes, the metric in the asymptotic region can be taken as the Tangherlini metric (99), and the electric potential is given by Eq. (100).

Assuming that in the asymptotic region the relation between the gravitational potential W and the electric potential ϕ is of Weyl-Guilfoyle type (cf. Eq. (75)), $W^2 = \bar{a}(-\epsilon\sqrt{G_d}\phi + \bar{b})^2 + \bar{c}$, where a bar here means we are working in the asymptotic region, we find the conditions

$$\bar{a} = 1, \qquad \bar{a}(-\epsilon\sqrt{G_d}\phi_0 + \bar{b})q = \epsilon\sqrt{G_d}m,$$
$$\bar{a}(-\epsilon\sqrt{G_d}\phi_0 + \bar{b})^2 + \bar{c} = \frac{1}{\bar{a}}\frac{G_d m^2}{q^2} + \bar{c} = 1. \tag{105}$$

From these results an interesting conclusion can be drawn: If it is assumed that a Weyl-Guilfoyle relation between potentials holds throughout an asymptotically Tangherlini spacetime then the resulting relation is in fact a Weyl relation, $W^2 = (-\epsilon\sqrt{G_d}\phi + b)^2 + c$. A further simplification can be done considering that the arbitrary constant ϕ_0 in the electric potential can be put to zero without loss of generality. With $\phi_0 = 0$ we find $\bar{b}^2 + \bar{c} = 1$, and therefore

$$\bar{b}q = \epsilon\sqrt{G_d}m. \tag{106}$$

That the charge to mass relation is given by Eq. (106) in a four-dimensional asymptotically Reissner-Nordström spacetime of Weyl type was first found by Gautreau and Hoffman [8], using a formal method. Inspired in this work [8], let us see how it can be generalized to d-dimensional asymptotically Tangherlini spacetimes of Weyl-Guilfoyle type. Following closely Gautreau and Hoffman [8] we assume there is a fluid distribution of finite extent, not necessarily spherically symmetric, linked to a vacuum that faraway asymptotes to the Tangherlini solution, and moreover, the parameters a, b, and c defining the Weyl-Guilfoyle relation inside the fluid are the same as the parameters of the vacuum region $\bar{a}$, $\bar{b}$, and $\bar{c}$, i.e., $a = \bar{a}$, $b = \bar{b}$, and $c = \bar{c}$. Since in the vacuum region $\bar{a} = 1$, one also has $a = 1$. Take now the quantity $\nabla_i(\frac{1}{W}\nabla^i W^2) - a\nabla_i(\frac{1}{W}\nabla^i\phi^2)$ with $a = 1$ (see Eqs. (70) and (71)), integrate it over a volume $\mathcal{V}$, and use the Gauss theorem to find

$$\int_{\mathcal{V}}\left[\nabla_i\left(\frac{1}{W}\nabla^i W^2\right) - \nabla_i\left(\frac{1}{W}\nabla^i\phi^2\right)\right]d\mathcal{V}$$
$$= \int_{\mathcal{S}}\left(\frac{1}{W}\nabla_i[W^2 - G_d\phi^2]\right)n^i d\mathcal{S}, \tag{107}$$

where, as before, $d\mathcal{V}$ is the invariant volume element of the $(d-1)$-dimensional spatial section ($t =$ constant) of the spacetime, $\mathcal{S}$ is the boundary of $\mathcal{V}$, and n^i is the unit vector orthogonal to the hypersurface $\mathcal{S}$ pointing outwards. Now we take $\mathcal{S}$ at infinity and denote $\mathcal{S} = \mathcal{S}_\infty$, so that $\mathcal{V}$ stands for the whole space volume $\mathcal{V}_\infty$, and it results that the surface integration on the right-hand side of Eq. (107) is done at spatial infinity. Since the spacetime is asymptotically flat, the metric on $\mathcal{S}_\infty$ can be taken as the Tangherlini metric (99), with the electric potential given by Eq. (100). Under these conditions the right-hand side of Eq. (107) gives $2S_{d-2}G_d m$, i.e.,

$$\int_{\mathcal{V}_\infty}\left[\nabla_i\left(\frac{1}{W}\nabla^i W^2\right) - \nabla_i\left(\frac{1}{W}\nabla^i\phi^2\right)\right]d\mathcal{V} = \int_{\mathcal{S}_\infty}\left(\frac{1}{W}\nabla_i[W^2 - G_d\phi^2]\right)n^i d\mathcal{S} = 2G_d m. \tag{108}$$

Now the field equations (70) and (71) are used to write the first integral on the left-hand side of Eq. (108) in terms of the fluid quantities and of the electromagnetic energy density $\rho_{\rm em}$. In the case where the fluid distribution has no boundary the analysis is straightforwardly performed and it results in $\int_{\mathcal{V}_\infty}[\nabla_i(\frac{1}{W}\nabla^i W^2) - \nabla_i(\frac{1}{W}\nabla^i\phi^2)]d\mathcal{V} = 2S_{d-2}G_d\int_{\mathcal{V}_\infty}(\rho_{\rm m} + \frac{d-1}{d-3}p + \frac{\phi}{W}\rho_{\rm e})Wd\mathcal{V}$. On the other hand, if the source is of finite extent with a boundary, the spacetime has two distinct regions: The interior region, bounded by the surface $\mathcal{S}_{\rm b}$ and whose volume we denote by $\mathcal{V}_{\rm b}$; and the exterior region, whose volume we denote by $\Delta\mathcal{V}$, so that we obtain $\int_{\mathcal{V}_\infty}[\nabla_i(\frac{1}{W}\nabla^i W^2) - \nabla_i(\frac{1}{W}\nabla^i\phi^2)]d\mathcal{V} = 2S_{d-2}G_d\int_{\mathcal{V}_{\rm b}}(\rho_{\rm m} + \frac{d-1}{d-3}p + \frac{\phi}{W}\rho_{\rm e})Wd\mathcal{V}$. Hence, using this result and Eq. (108) we finally obtain

$$m = \int_{\mathcal{V}}\left[\left(\rho_{\rm m} + \frac{d-1}{d-3}p\right)W + \phi\rho_{\rm e}\right]d\mathcal{V}, \tag{109}$$

where $\mathcal{V}$ stands for the whole volume of the fluid distribution. The parameter m is the total gravitational mass of the spacetime.

We now turn attention to the charge definition, for which one can always make use of the Maxwell equations. Hence, integrating Eq. (69) over the volume of the source and using the Gauss theorem it is found

$$\int_{\mathcal{S}}\frac{1}{W}\nabla_i\phi n^i d\mathcal{S} = -S_{d-2}\int_{\mathcal{V}_{\mathcal{S}}}\rho_{\rm e}d\mathcal{V}, \tag{110}$$

where $\mathcal{S}$ is the hypersurface bounding the volume $\mathcal{V}_{\mathcal{S}}$. Again the left-hand side integration can be done over an infinitely large spherical surface, with the metric on $\mathcal{S}$ being the Tangherlini metric (99), and with the electric potential being given by Eq. (100). Then, the left-hand side of Eq. (110) gives $-S_{d-2}q$. We may then identify the quantity on the right-hand side of Eq. (110) to the total charge of the source q,

$$q = \int_{\mathcal{V}_{\mathcal{S}}}\rho_{\rm e}d\mathcal{V} = \int_{\mathcal{V}_\infty}\rho_{\rm e}d\mathcal{V}. \tag{111}$$

Finally, the constraint (73) implies that the total charge of a spacetime containing a charged pressure fluid distribution of finite size satisfying the conditions of theorem III.5 is proportional to the Arnowitt-Deser-Misner (ADM) mass of the spacetime. This can be shown as follows: Integrating Eq. (73) over the whole volume of the source, $\mathcal{V}_{\rm b}$, and using Eqs. (109) and (111), we find

$$bq = \epsilon\sqrt{G_d}m, \tag{112}$$

which is the same as Eq. (106), since we are assuming $b = \bar{b}$. This result shows that for a compact fluid distribution with no symmetry *a priori*, in which the spacetime tends asymptotically to the Tangherlini solution, with the potentials W and ϕ being related by $W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c$, and with the mass density $\rho_{\rm m}$, the pressure p, the charge density $\rho_{\rm e}$ being related by $b\rho_{\rm e} = \epsilon\sqrt{G_d}[(\rho_{\rm m} + \frac{d-3}{d-1}p)W + \phi\rho_{\rm e}]$, the total mass and the total electric charge of the spacetime are proportional to each other with the proportionality constant being exactly b. For a fluid mass distribution with spherical symmetry, the relation (106) can be derived directly from Eq. (104). Indeed, under the Gautreau-Hoffman assumptions, one has $b = \bar{b}$, and thus $\bar{b}^2q^2 = G_d m^2$, so that one finds immediately from Eq. (104) that $a = 1$ and $b^2 + c = 1$ is a solution. So the Gautreau-Hoffman mass relation is, in the spherically symmetric case, a particular instance of the Guilfoyle relation.

We can also show that the integral given by Eq. (109) is the total gravitational energy, i.e., the total mass, of a source and is equal to the Tolman mass [12]. Using the Tolman formula we find

$$M = \int_{\mathcal{V}_\infty}\left(\rho_{\rm m} + \frac{d-1}{d-3}p + \rho_{\rm em}\right)Wd\mathcal{V}, \tag{113}$$

where we assumed that the charged fluid distribution is of finite extent. Using Eqs. (109) and (113) we get

$$M - m = \int_{\mathcal{V}_{\rm b}}(W\rho_{\rm em} - \phi\rho_{\rm e})d\mathcal{V} + \int_{\Delta\mathcal{V}}W\rho_{\rm em}d\mathcal{V}, \tag{114}$$

where $\Delta\mathcal{V}$ stands for the volume outside the matter distribution. We can get rid of the term containing the product $\phi\rho_{\rm e}$ on the right-hand side of Eq. (112). Multiplying Eq. (69) by ϕ, using the identity $\phi\nabla_i(\frac{1}{W}\nabla^i\phi) = \nabla_i(\frac{\phi}{W}\nabla^i\phi) - (\nabla^i\phi)^2/W$, integrating over a space volume $\mathcal{V}_{\mathcal{S}}$, and using the Gauss theorem, it is obtained that $\int_{\mathcal{S}}(\frac{\phi}{W}\nabla_i\phi)n^i d\mathcal{S} - \int_{\mathcal{V}_{\mathcal{S}}}\frac{1}{W}(\nabla^i\phi)^2 d\mathcal{V} = -S_{d-2}\int_{\mathcal{V}_{\mathcal{S}}}\phi\rho_{\rm e}d\mathcal{V}$. Then, using Eq. (67) to bring in the electromagnetic energy density, $(\nabla^i\phi)^2/W = S_{d-2}W\rho_{\rm em}$, it is finally found that $S_{d-2}\int_{\mathcal{V}_{\rm b}}(\phi\rho_{\rm e} - W\rho_{\rm em})d\mathcal{V} = -\int_{\mathcal{S}_{\rm b}}(\frac{\phi}{W}\nabla_i\phi)n^i d\mathcal{S}$ where $\mathcal{S}_{\rm b}$ is the boundary surface of the fluid distribution.

Substituting this result into Eq. (114), it gives

$$M - m = \frac{1}{S_{d-2}} \int_{\mathcal{S}_b} \left(\frac{\phi}{W} \nabla_i \phi\right) n^i d\mathcal{S} + \int_{\Delta \mathcal{V}} \rho_{em} W d\mathcal{V}, \tag{115}$$

where $\mathcal{S}_b$ is the surface which bounds the source. It seems that the mass m is in general different from the Tolman mass M. However, as verified above, for d-dimensional spherically symmetric systems of Weyl-Guilfoyle type it results in $m = M$. In fact, one can show that under certain conditions the two integral terms on the right-hand side of Eq. (115) cancel each other out. We use the vacuum field equations and the Gauss theorem to transform the volume integration over $\Delta\mathcal{V}$ into a surface integration, viz., $S_{d-2}\int_{\Delta\mathcal{V}} \rho_{em} W d\mathcal{V} = \int_{\mathcal{S}_\Delta} (\frac{\phi}{W}\nabla_i\phi)\bar{n}^i d\mathcal{S}$, where $\mathcal{S}_\Delta$ is the (closed) surface boundary to the volume $\Delta\mathcal{V}$ and $\bar{n}^i$ stands for the unit vector orthogonal to $\mathcal{S}_\Delta$ pointing outwards. Note that the boundary S_Δ is composed of two closed surfaces. The external boundary $\mathcal{S}_\infty$ and inner boundary, which coincides with the boundary of the source, $\mathcal{S}_b$. The unit vector $\bar{n}^i$ on $\mathcal{S}_b$ pointing outwards with respect to $\Delta\mathcal{V}$ is parallel to n^i but points inwards with respect to $\mathcal{V}_b$, namely, $\bar{n}^i = -n^i$ where n^i is the same as in Eq. (115). Therefore, we have the identity $\int_{\mathcal{S}_\Delta}(\frac{\phi}{W}\nabla_i\phi)\bar{n}^i d\mathcal{S} = -\int_{\mathcal{S}_b}(\frac{\phi}{W}\nabla_i\phi)n^i d\mathcal{S} + \int_{\mathcal{S}_\infty}(\frac{\phi}{W}\nabla_i\phi)\bar{n}^i d\mathcal{S}$. Since the source is of finite extent the spacetime is asymptotically Tangherlini, cf. Eqs (99) and (100), the integral over $\mathcal{S}_\infty$ vanishes and, after substituting the result into Eq. (115), we find

$$M = m, \tag{116}$$

i.e., the total mass is the Tolman mass [12] (for the Tolman mass and mass in charged matter see [13–15], and also [16]).

IV. FURTHER COMMENTS AND CONCLUSIONS

We have studied the structure of the sources that produce Weyl type systems, including systems obeying a Weyl-Guilfoyle relation, both in the Newton-Coulomb theory with matter in $d-1$ space dimensions and in the Einstein-Maxwell theory with matter in d spacetime dimensions.

In the Newton-Coulomb case, we have rendered theorems by Bonnor for charged dust fluids into higher dimensions and obtained new results for charged pressure fluids. For zero pressure fluids, it follows that the gravitational potential V is a function of the electric potential ϕ alone, $V = V(\phi)$. In the case of a nonzero pressure fluid, the equations of fluid dynamics in Eulerian description together with the Poisson equations for the potentials with the assumption of a Weyl type ansatz for the potentials, $V = V(\phi)$, implies that the pressure is also a function of ϕ alone, and then all fluid quantities are given in terms of the gravitational potential. If one assumes further that the relation between the pressure gradient $dp/d\phi$ is proportional to the charge density, then the gravitational potential is given by $V(\phi) = -\epsilon\beta\sqrt{G_d}\phi + \gamma$, with β and γ being arbitrary constants and $\epsilon = \pm 1$, with the matter and charge densities being proportional to each other too.

In the case of the relativistic theory things are more intricate and more interesting. First, with the Weyl ansatz $W = W(\phi)$, the d-dimensional Einstein-Maxwell theory in vacuum yields the Weyl quadratic relation between the metric potential W^2 and the electric gauge potential ϕ. Then, a series of results for the Einstein-Maxwell with matter theory for fields of Weyl type in four spacetime dimensions can be rendered into higher dimensional spacetimes. The most important new result of our analysis is the generalization of the Gautreau-Hoffman relation among the fluid quantities when the potentials are related through the Weyl-Guilfoyle relation (75). This analysis, done in Sec. III C 4, shows that the most general charged pressure fluid in which the metric gravitational potential W satisfying the Weyl-Guilfoyle quadratic relation $W^2 = a(-\epsilon\sqrt{G_d}\phi + b)^2 + c$, with constants a, b, and c, corresponds to different systems when compared to the original Weyl quadratic relation as Eq. (4), in which $a = 1$. To see that, consider the following reparametrization of the fields $W \to \alpha_0 W$, $\phi \to \alpha_1\phi$, with constant α_0 and α_1. Taking these transformations into the system of equations given by Eqs. (51)–(53), we can conclude that the Einstein-Maxwell system of equations is invariant only if $\alpha_0 = \alpha_1$, as expected (this corresponds to a rescaling of the time coordinate, $t \to t/\alpha_0$). Therefore, any rescaling of the potentials W and ϕ for which $\alpha_0 \neq \alpha_1$ leads to a different system. Hence, a new relation among fluid quantities and the electromagnetic energy density is found in four and higher dimensions, cf. Eq. (73). Upon connection of an interior charged solution to an exterior Tangherlini solution, we found a relation between the mass, the charge, and the several quantities of the interior solution. It was also shown that for sources of finite extent the mass is identical to the Tolman mass.

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