Topological black holes in Hořava-Lifshitz gravity

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We find topological (charged) black holes whose horizon has an arbitrary constant scalar curvature $2k$ in Hořava-Lifshitz theory. Without loss of generality, one may take $k = 1$, 0, and -1. The black hole solution is asymptotically anti–de Sitter with a nonstandard asymptotic behavior. Using the Hamiltonian approach, we define a finite mass associated with the solution. We discuss the thermodynamics of the topological black holes and find that the black hole entropy has a logarithmic term in addition to an area term. We find a duality in Hawking temperature between topological black holes in Hořava-Lifshitz theory and Einstein's general relativity: the temperature behaviors of black holes with $k = 1$, 0, and -1 in Hořava-Lifshitz theory are, respectively, dual to those of topological black holes with $k = -1$, 0, and 1 in Einstein's general relativity. The topological black holes in Hořava-Lifshitz theory are thermodynamically stable.

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I. INTRODUCTION

Recently a field theory model for a UV complete theory of gravity was proposed by Hořava [1], which is a nonrelativistic renormalizable theory of gravity and reduces to Einstein's general relativity at large scales. This theory is named Hořava-Lifshitz theory in the literature since at the UV fixed point of the theory space and time have different scalings. Since then much attention has been attracted to this gravity theory [2–9], including its implications in cosmology [3–5,7–9]. In [7] the authors find some static spherically symmetric black hole solutions in Hořava-Lifshitz theory.

In the (3 + 1)-dimensional Arnowitt-Deser-Misner formalism, where the metric can be written as

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1.1)$$

and for a spacelike hypersurface with a fixed time, its extrinsic curvature K_{ij} is

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (1.2)$$

where a dot denotes a derivative with respect to t and covariant derivatives defined with respect to the spatial metric g_{ij}. The action of Hořava-Lifshitz theory is [1]

$$I = \int dt d^3x \sqrt{g} N\bigg(\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda R - 3\Lambda^2)}{8(1-3\lambda)} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} + \frac{\kappa^2\mu}{2\omega^2}\epsilon^{ijk}R_{il}\nabla_j R^l{}_k - \frac{\kappa^2}{2\omega^4}C_{ij}C^{ij}\bigg), \quad (1.3)$$

where κ^2, λ, μ, ω, and Λ are constant parameters and the Cotten tensor, C_{ij}, is defined by

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R^j_l - \tfrac{1}{4}R\delta^j_l\right) = \epsilon^{ikl}\nabla_k R^j_l - \tfrac{1}{4}\epsilon^{ikj}\partial_k R. \quad (1.4)$$

In (1.3), the first two terms are the kinetic terms, while the others give the potential of the theory in the so-called "detailed-balance" form.

Comparing the action to that of general relativity, one can see that the speed of light, Newton's constant, and the cosmological constant are

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda}{1-3\lambda}}, \qquad G = \frac{\kappa^2 c}{32\pi}, \qquad \tilde{\Lambda} = \frac{3}{2}\Lambda, \quad (1.5)$$

respectively. Let us notice that when $\lambda = 1$, the first three terms in (1.3) could be reduced to the usual ones of Einstein's general relativity. However, in Hořava-Lifshitz theory, λ is a dynamical coupling constant, susceptible to quantum correction [1]. In addition, we see from (1.5) that when $\lambda > 1/3$, the cosmological constant Λ must be negative. However, the cosmological constant can be positive if we make an analytic continuation $\mu \to i\mu$, $w^2 \to -iw^2$ [7]. In this paper, we consider the former case with negative cosmological constant.

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For later convenience, we rewrite the action (1.3) as follows [7]:

$$I = \int dt d^3x(\mathcal{L}_0 + \mathcal{L}_1),$$
$$\mathcal{L}_0 = \sqrt{g}N\Big\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda R - 3\Lambda^2)}{8(1-3\lambda)}\Big\},$$
$$\mathcal{L}_1 = \sqrt{g}N\Big\{\frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2}{2\omega^4}\Big(C_{ij} - \frac{\mu\omega^2}{2}R_{ij}\Big) \times \Big(C^{ij} - \frac{\mu\omega^2}{2}R^{ij}\Big)\Big\}. \quad (1.6)$$

The equations of motion for the action are given in [5,7], but they are very lengthy and we will not reproduce them here.

In this note we are interested in black hole solutions in the action (1.6). Considering the static, spherically symmetric solutions with the metric ansatz

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.7)$$

Without the term $\mathcal{L}_1$, the solution is the just the (anti–)de Sitter Schwarzschild black hole solution with metric functions [7]

$$N^2(r) = f(r) = 1 - \tfrac{\Lambda}{2}r^2 - \tfrac{m}{r}. \quad (1.8)$$

With the term $\mathcal{L}_1$, a general static, spherically symmetric black hole solution with an arbitrary λ is also found in [7], but the solution is elusive. We discuss the general solution in the appendix. Of particular interest is the case with $\lambda = 1$, on which we focus in the following. The solution is then given by

$$N^2 = f = 1 + x^2 - \alpha\sqrt{x}, \quad (1.9)$$

where $x = \sqrt{-\Lambda}r$ and α is an integration constant. This solution is asymptotically AdS_4 and has a singularity at $x = 0$ if $\alpha \neq 0$. The singularity could be covered by black hole horizon at x_+; the largest root of the equation $f = 0$ if $\alpha > 0$. The Hawking temperature of the black hole horizon is easily given by [7]

$$T = \frac{3x_+^2 - 1}{8\pi x_+}\sqrt{-\Lambda}. \quad (1.10)$$

Note that here we have corrected a typo in [7]. One can see from (1.10) that there exists an extremal limit, $x_+ = 1/\sqrt{3}$, where the temperature vanishes. Another remarkable point one can see by comparing the solution (1.8) and (1.9) is that general relativity is not always recovered at large distance [7]. In addition, one may naively expect that the mass of the black hole solution (1.9) is divergent due to the square root term.

The black hole solution (1.9) is obtained from the action (1.6) in the detailed balance [1]. The authors in [7] also considered black hole solution in Hořava-Lifshitz theory without the condition of the detailed balance, namely, in the theory given by

$$\mathcal{L} = \mathcal{L}_0 + (1 - \epsilon^2)\mathcal{L}_1, \quad (1.11)$$

where ϵ is a constant. In this theory, the black hole solution they found turns to be

$$N^2 = f = 1 + \frac{x^2}{1-\epsilon^2} - \frac{\sqrt{\alpha^2(1-\epsilon^2)x + \epsilon^2 x^4}}{1-\epsilon^2}. \quad (1.12)$$

In the large distance limit, the solution reduces to

$$f = 1 + \frac{x^2}{1+\epsilon} - \frac{\alpha^2}{2\epsilon x} + \mathcal{O}(x^{-4}). \quad (1.13)$$

The authors in [7] suggest that the solution has a finite mass for nonvanishing ϵ, while it becomes divergent as $\epsilon = 0$. In the latter case, the solution goes back to the one (1.9). Furthermore, when $\epsilon = 1$, the solution becomes the (anti–)de Sitter Schwarzschild black hole solution (1.8).

In this note we are going to discuss thermodynamics of the black hole solutions (1.9) and (1.12), which have not been studied in [7]. Since the solutions (1.9) and (1.12) are asymptotically AdS, we will generalize those solutions to the case of topological black holes with any constant scalar curvature horizon [10–13]. We will also discuss the topological charged black holes in Hořava-Lifshitz theory by including Maxwell field.

II. TOPOLOGICAL BLACK HOLES AND THERMODYNAMICS

In this section we first generalize the spherically symmetric black hole solution (1.9) to the topological black hole case with arbitrary constant scalar curvature horizon. The black hole solution is of the metric ansatz

$$ds^2 = -\tilde{N}^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \quad (2.1)$$

where $d\Omega_k^2$ denotes the line element for a two-dimensional Einstein space with constant scalar curvature $2k$. Without loss of generality, one may take $k = 0, \pm 1$, respectively. Following [7], substituting the metric (2.1) into (1.6), we find

$$I = \frac{\kappa^2\mu^2\Lambda\Omega_k}{8(1-3\lambda)}\int dt dr\tilde{N}\Big\{-3\Lambda r^2 - 2(f-k) - 2r(f-k)' + \frac{(\lambda-1)f'^2}{2\Lambda} + \frac{(2\lambda-1)(f-k)^2}{\Lambda r^2} - \frac{2\lambda(f-k)}{\Lambda r}f'\Big\}, \quad (2.2)$$

where a prime denotes the derivative with respect to r and Ω_k is the volume of the two-dimensional Einstein space. Again, we consider the solution in the case of $\lambda = 1$. In that case, we have

$$I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\int dtdx\tilde{N}\left(x^3 - 2x(f-k) + \frac{(f-k)^2}{x}\right)'. \tag{2.3}$$

Note that here $x = \sqrt{-\Lambda}r$ and a prime becomes the derivative with respect to x. From the action, we obtain the equations of motion

$$0 = \tilde{N}', \qquad 0 = \left(x^3 - 2x(f-k) + \frac{(f-k)^2}{x}\right)'. \tag{2.4}$$

From the first equation, we have $\tilde{N} = N_0$, a constant. One can set $N_0 = 1$ by rescaling the time coordinate t. From the second one, one can obtain $x^3 - 2x(f-k) + \frac{(f-k)^2}{x} = c_0$; here c_0 is an integration constant. Solving this yields

$$f(r) = k + x^2 - \sqrt{c_0 x}. \tag{2.5}$$

Note that c_0 should be positive here. When $k = 1$, the solution reduces to the one given by [7]. Thus we generalize the solution in [7] to the case of topological black holes with arbitrary k. In addition, let us stress here that although we have obtained the black hole solution through the minisuperspace approach, it has been checked that the solution (2.5) with $N_0 = 1$ indeed satisfies the equations of motion given in [7].

A remarkable property of black holes is that they are associated with thermodynamics. Now we are going to discuss thermodynamics of the black hole solution (2.5), which has not yet been discussed. Comparing to the AdS Schwarzschild black hole solution, one may naively expect that the mass of the solution (2.5) is divergent and one could not define a finite mass for this solution. However, this conclusion is not true. In fact, such nonstandard asymptotic behavior also appears for the black hole solutions in the so-called dimensionally continued gravity [13,14]. For the dimensionally continued black hole solutions, a finite mass can be obtained by using the Hamiltonian approach. We find that this approach also works for the Hořava-Lifshitz theory. Note that the action (2.3) can be written as

$$I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}(t_2 - t_1)\int dx\tilde{N}\left(x^3 - 2x(f-k) + \frac{(f-k)^2}{x}\right)' + B, \tag{2.6}$$

where B is a surface term, which must be chosen so that the action has an extremum under variations of the fields with appropriate boundary conditions. One demands that the fields approach the classical solutions at infinity. Varying the action (2.6), one finds the boundary term

$$\delta B = -(t_2 - t_1)N_0\delta M. \tag{2.7}$$

The boundary term B is the conserved charge associated to the "improper gauge transformations" produced by time evolution [15]. Here M and N_0 are a conjugate pair. Therefore when one varies M, N_0 must be fixed. Thus the boundary term should be in the form

$$B = -(t_2 - t_1)N_0 M + B_0, \tag{2.8}$$

where B_0 is an arbitrary constant, which should be fixed by some physical consideration; for example, mass vanishes when black hole horizon goes to zero. For details, see [14]. According to this Hamiltonian approach, we get the mass of the solution (2.5) as

$$M = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}c_0. \tag{2.9}$$

Note that here Λ is negative, therefore the black hole mass is always positive because we have already set $c_0 > 0$. One can easily obtain the Hawking temperature of the black hole, either by directly calculating the surface gravity at the horizon, or by requiring the absence of the conical singularity at the horizon of the Euclidean black hole. Both methods give the same result:

$$T = \frac{3x_+^2 - k}{8\pi x_+}\sqrt{-\Lambda}. \tag{2.10}$$

The next step is to get the entropy associated with the topological black hole. In Einstein's general relativity, entropy of black hole is always given by one quarter of black hole horizon area. But in higher derivative gravities, in general, the area formula breaks down. Here we will obtain the black hole entropy by using the first law of black hole thermodynamics with assumption that as a thermodynamical system [11–14], the first law always keeps valid: $dM = TdS$. Integrating this relation yields

$$S \equiv \int T^{-1}dM + S_0 = \int T^{-1}\frac{dM}{dx_+}dx_+ + S_0, \tag{2.11}$$

where S_0 is an integration constant, which should be fixed by physical consideration. Through (2.11), we obtain

$$S = \frac{\pi\kappa^2\mu^2\Omega_k}{4}(x_+^2 + 2k\ln x_+) + S_0 = \frac{c^3}{4G}\left(A - \frac{k\Omega_k}{\Lambda}\ln\frac{A}{A_0}\right), \tag{2.12}$$

where the Newton's constant and speed of light are given in (1.5), $A = \Omega_k r_+^2$ is the black hole horizon area, and A_0 is a constant of dimension of length squared. The leading term is just one quarter of horizon area in units of $c = G = 1$, which should be the contribution from the $\mathcal{L}_0$ term. The second term is a logarithmic function, therefore we cannot fix the integration constant S_0 or A_0, unfortunately, by some physical consideration, for example, black hole entropy should vanish when black hole horizon goes to zero. The integration constant S_0 could be fixed by counting micro degrees of freedom in some quantum theory of gravity like string theory. An interesting fact is that such a term often appears in the quantum correction of black hole entropy. In addition, when $k = 0$, namely, for black hole with Ricci flat horizon, the logarithmic term disappears. Thus, the area formula of black hole entropy is recovered in this case. It might be a universal result that the area formula still holds for Ricci flat black holes in higher derivative gravity theories [11–13].

Two additional points are worth stressing here. One is on the temperature (2.10). For $k = 1$, as pointed out in [7], there is an extremum at $x_+ = 1/\sqrt{3}$, where the temperature vanishes, and it corresponds to an extremal black hole. For $k = 0$, the temperature $T = 3x_+\sqrt{-\Lambda}/8\pi$. In these two cases, the temperature always monotonically increases as the horizon x_+ grows. For $k = -1$, the inverse temperature starts from zero at $x_+ = 0$, monotonically increases and reaches a maximal value, $\beta = 1/T = 4\pi/\sqrt{-3\Lambda}$ at $x_+ = 1/\sqrt{3}$, then monotonically decreases as x_+ grows. It is interesting to compare these temperature behaviors of the topological black holes in Hořava-Lifshitz theory with those for topological black holes in Einstein's general relativity [the latter could be obtained by replacing 1 by k in (1.8))]. The temperature for the topological black holes in Einstein's general relativity is

$$T_{\rm TSch} = \frac{\sqrt{-\Lambda}}{8\pi x_+}(3x_+^2 + 2k). \tag{2.13}$$

We see that except for the coefficient difference in front of the horizon curvature constant k, there is a duality relation in these two temperatures: the temperature behaviors of black holes in Hořova-Lifshitz theory in the cases of $k = 1$, 0, and -1, are dual to the cases of $k = -1$, 0, and 1 in Einstein's general relativity, respectively. Note that for topological black holes in Einstein's general relativity [10], in the cases of $k = 0$ and $k = -1$, the black holes are always thermodynamically stable, while in the case of $k = 1$, the small black hole with $x_+ < \sqrt{2/3}$ is thermodynamically unstable and it becomes thermodynamically stable for large horizon radius $x_+ > \sqrt{2/3}$.

However, a close check tells us that in the case of $k = -1$, there exists a minimal horizon at $x_+ = 1$ for the topological black hole in Hořava-Lifshitz theory, which can be seen from the metric function $f(r)$ in (2.5), namely, for the case of $c_0 = 0$. This is just the massless black hole in AdS space. Thus in the range $x_+ \in [1, \infty)$, the temperature of the topological black hole is also a monotonically increasing function of x_+. Thus the unstable phase for the topological black hole with $k = -1$ in Hořava-Lifshitz theory does not appear, and the black hole is always thermodynamically stable.

To see this more clearly, let us calculate heat capacity of black hole, defined as $C = dM/dT$. The heat capacity of the black hole in Hořava-Lifshitz gravity is

$$C = \frac{\pi\kappa^2\mu^2\Omega_k}{2}\frac{(3x_+^2 - k)(x_+^2 + k)}{3x_+^2 + k}. \tag{2.14}$$

We see that for the cases $k = 1$ and $k = 0$, the heat capacity is always positive, which implies that the black hole is locally thermodynamically stable, while in the case of $k = -1$, if $x_+ > 1$, it is also positive. For comparison, we give the heat capacity for the topological AdS black hole in Einstein's general relativity

$$C_{\rm TSch} = \frac{\pi\kappa^2\mu^2\Omega_k}{2}\frac{3x_+^2 + 2k}{3x_+^2 - 2k}x_+^2. \tag{2.15}$$

When $k = 0$ and -1, it is always positive while when $k = 1$, it is negative for $x_+^2 < 2/3$, positive for $x_+^2 > 2/3$, and diverges at $x_+^2 = 2/3$.

Another interesting question is whether there exists the Hawking-Page phase transition associated with the black holes in Hořava-Lifshitz gravity. It is well known that there is a Hawking-Page transition for static, spherically symmetric AdS-Schwarzschild black hole (the case of $k = 1$) between a large AdS black hole and thermal gas in AdS space [16]. On the other hand, for the cases of $k = 0$ and $k = -1$ topological black hole in Einstein's general relativity, the Hawking-Page phase transition does not exist. To discuss the Hawking-Page transition, one has to calculate the Euclidean action or free energy of the black hole. The Euclidean action has a relation to the free energy by $I = \beta F$, here β is the inverse temperature of the black hole. By definition, the free energy F is given by $F = M - TS$. By using (2.9), (2.10), and (2.12), we find

$$F = \frac{\kappa^2\mu^2\Omega_k\sqrt{-\Lambda}}{32x_+}(-x_+^4 + 5kx_+^2 + 2k^2 - 6kx_+^2\ln x_+ + 2k^2\ln x_+) - TS_0. \tag{2.16}$$

Because of the uncertainty of S_0, we cannot determine the signature of the free energy. However, if one can neglect the term S_0, we see the free energy is negative for large enough horizon radius, which means that large black holes in Hořava-Lifshitz gravity is thermodynamically stable globally.

Now we turn to the case without the detailed balance condition, namely, $\epsilon^2 \neq 0$. Replacing (2.3) we have

$$I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\int dtdx\tilde{N}\bigg(x^3 - 2x(f - k) + (1 - \epsilon^2)\frac{(f - k)^2}{x}\bigg)'. \tag{2.17}$$

In this case, one has the solution

$$\tilde{N} = N_0, \qquad f(r) = k + \frac{x^2}{1 - \epsilon^2} - \frac{\sqrt{\epsilon^2x^4 + (1 - \epsilon^2)c_0x}}{1 - \epsilon^2}. \tag{2.18}$$

Again, c_0 is an integration constant and N_0 could be set to one. Similar to the case of $\epsilon^2 = 0$, we find the mass of the solution is

$$M = \frac{\kappa^2 \mu^2 \Omega_k \sqrt{-\Lambda}}{16} c_0, \tag{2.19}$$

and c_0 can be expressed in terms of black hole horizon radius x_+,

$$c_0 = \frac{x_+^4 + 2kx_+ + (1-\epsilon^2)k^2}{x_+}. \tag{2.20}$$

The Hawking temperature of the black hole is found to be

$$T = \frac{\sqrt{-\Lambda}}{8\pi} \frac{3x_+^4 + 2kx_+^2 - (1-\epsilon^2)k^2}{x_+(x_+^2 + (1-\epsilon^2)k)}. \tag{2.21}$$

With the mass and temperature, we obtain the entropy of the black hole

$$\begin{aligned} S &= \frac{\pi\kappa^2\mu^2\Omega_k}{4}(x_+^2 + 2k(1-\epsilon^2)\ln x_+) + S_0 \\ &= \frac{c^3}{4G}\left(A - (1-\epsilon^2)\frac{k\Omega_k}{\Lambda}\ln\frac{A}{A_0}\right). \end{aligned} \tag{2.22}$$

When $\epsilon^2 = 0$, it goes back to (2.12), while it reduces to the well-known area formula for $\epsilon^2 = 1$, as expected, since in that case, the effect of higher derivative terms disappears.

Now let us discuss the behavior of the temperature (2.21).

(i) When $k = 0$, the temperature is independent of ϵ^2, given by

$$T = \frac{3\sqrt{-\Lambda}}{8\pi} x_+. \tag{2.23}$$

Clearly it is a monotonically increasing function of x_+

(ii) When $k = -1$ and $\epsilon^2 < 1$, an extremal black hole with $T = 0$ is obtained at $x_+^2 = (1 + \sqrt{1 + 3(1-\epsilon^2)})/3$. While to keep the denominator in (2.21) positive, one has to have $x_+^2 > (1-\epsilon^2)$, which is always smaller than $(1 + \sqrt{1 + 3(1-\epsilon^2)})/3$. This indicates that there does exist an extremal black hole in this case with the minimal horizon radius $x_{+\,\mathrm{min}}^2 = (1 + \sqrt{1 + 3(1-\epsilon^2)})/3$. When $\epsilon^2 > 1$, according to (2.18), the minimal horizon radius is $x_+^2 = 1 + \epsilon$. In both cases of $\epsilon^2 > 1$ and <1, the temperature of the black hole is a monotonically increasing function of x_+ in the physical regime.

(iii) When $k = 1$, let us first consider the case of $\epsilon^2 < 1$. A vanishing temperature happens at $x_{+\,\mathrm{min}}^2 = (-1 + \sqrt{1 + 3(1-\epsilon^2)})/3$. When $\epsilon^2 > 1$, there does not exist an extremal black hole, but to keep the temperature positive, a physical horizon radius must obey $x_+^2 > \epsilon^2 - 1$. As the case of $k = -1$ with any ϵ^2, the temperature of the black hole is a monotonically increasing function of x_+ in the physical regime, again.

In summary, the case with $\epsilon^2 \neq 0$ is similar to the case with $\epsilon^2 = 0$; the Hawking temperature of the black holes with any k is always a monotonically increasing function of horizon radius x_+ in the physical regime. This implies that the topological black holes in Hořava-Lifshitz theory are thermodynamically stable. Note that when $\epsilon^2 = 1$, the situation is reduced to the case of the well-known topological AdS Schwarzschild black holes [10].

III. TOPOLOGICAL CHARGED BLACK HOLES

In this section we consider the charged generalization of the topological black hole found in Sec. II. To give a universal result, we assume $\epsilon^2 \neq 0$. Following [13,14], the Hamiltonian action for the Maxwell field can be written as

$$\begin{aligned} I_{\mathrm{em}} = \int dt d^3x \bigg[& p^i \dot{A}_i - \frac{1}{2} N\left(\alpha g^{-1/2} p^i p_i + \frac{g^{1/2}}{2\alpha} F_{ij} F^{ij}\right) \\ & + \varphi p^i{}_{,i} \bigg] + B_{\mathrm{em}}, \end{aligned} \tag{3.1}$$

where p^i is the momentum conjugate of the spatial components of the Maxwell field A_i, $\varphi = A_0$, B_{em} is a boundary term, N is the lapse function, and α is a parameter to be fixed shortly. Considering the static topological black hole solution with the metric ansatz (2.1), the action (3.1) is reduced to

$$I_{\mathrm{em}} = \frac{\Omega_k}{\alpha} \int dt dr \left(-\frac{1}{2} \tilde{N} r^2 p^2 + \varphi (r^2 p)' \right) + B_{\mathrm{em}}, \tag{3.2}$$

where $p = \alpha p^r / r^2 \gamma^{1/2}$ and γ is the determinant of the two-dimensional Einstein space $d\Omega_k^2$. Note that here the solution without magnetic charge $F_{ij} = 0$ has been assumed. To be consistent with (2.17), we set $x = \sqrt{-\Lambda} r$. The action (3.2) then becomes

$$I_{\mathrm{em}} = \frac{\Omega_k}{\alpha\sqrt{-\Lambda}} \int dt dx \left(-\frac{1}{2} \tilde{N} x^2 \tilde{p}^2 + \varphi (x^2 \tilde{p})' \right) + B_{\mathrm{em}}, \tag{3.3}$$

where a prime denotes derivative with respective to x and $\tilde{p} = p/\sqrt{-\Lambda}$. Now we set

$$\alpha^{-1} = -\frac{\kappa^2 \mu^2 \Lambda}{16}. \tag{3.4}$$

Combining (2.17) and (3.4), we have

$$\begin{aligned} I = \frac{\kappa^2 \mu^2 \sqrt{-\Lambda} \Omega_k}{16} \int dt dx \left(\tilde{N}\left(U' - \frac{1}{2} x^2 \tilde{p}^2 \right) + \varphi (x^2 \tilde{p})' \right) \\ + B, \end{aligned} \tag{3.5}$$

where

$$U = x^3 - 2x(f-k) + (1-\epsilon^2)\frac{(f-k)^2}{x}.$$

From the action (3.5) we obtain the equations of motion

$$U' = \tfrac{1}{2}x^2\tilde{p}^2, \qquad (x^2\tilde{p})' = 0, \qquad \varphi' = -\tilde{N}\,\tilde{p}, \qquad \tilde{N}' = 0, \tag{3.6}$$

which have the solution

$$\tilde{N} = N_0, \qquad \varphi = \frac{N_0 q}{x} + \varphi_0, \qquad \tilde{p} = \frac{q}{x^2}, \qquad U = -\frac{q^2}{2x} + c_0. \tag{3.7}$$

Here N_0, φ_0, c_0, and q are integration constants, their physical meanings are clear. Physical electric charge and mass of the solution are

$$Q = \frac{\kappa^2\mu^2\Omega_k\sqrt{-\Lambda}}{16}q, \qquad M = \frac{\kappa^2\mu^2\Omega_k\sqrt{-\Lambda}}{16}c_0, \tag{3.8}$$

respectively, and the metric function f is given by

$$f(r) = k + \frac{x^2}{1-\epsilon^2} - \frac{\sqrt{\epsilon^2 x^4 + (1-\epsilon^2)(c_0 x - q^2/2)}}{1-\epsilon^2}, \tag{3.9}$$

while $\tilde{N} = N_0$ could be set to one. Taking the limit $\epsilon \to 1$, the solution is reduced to

$$f(r) = k + \frac{x^2}{2} - \frac{c_0}{2x} + \frac{q^2}{4x^2}, \tag{3.10}$$

as expected, it is just the AdS Reissner-Nordström black hole solution. The Hawking temperature of the black hole is

$$T = \frac{\sqrt{-\Lambda}(3x_+^4 + 2kx_+^2 - (1-\epsilon^2)k^2 - q^2/2)}{8\pi x_+(x_+^2 + (1-\epsilon^2)k)}. \tag{3.11}$$

Putting the temperature (3.11) and mass (3.8) into the first law of black hole thermodynamics, it is easy to check that one reproduces the entropy (2.22), the charge q does not appear explicitly in the expression of black hole entropy in terms of horizon radius. This is consistent with the fact that black hole entropy is a function of horizon geometry. The behavior of the temperature can be analyzed as the case without the electric charge, but we do not repeat here. Instead we only point out that due to the appearance of the electric charge, extremal black holes with vanishing temperature always exist within reasonable parameter regime.

IV. CONCLUSION

In this paper we found topological (charged) black hole solutions with arbitrary constant scalar curvature horizon in Hořava-Lifshitz theory, generalizing the static, spherically symmetric black hole solutions in [7]. Although there is a square root term in the metric function $f(r)$, we can define a finite mass associated with the black hole solution by use of the Hamiltonian approach. We have calculated the Hawking temperature of the black hole and the black hole entropy by using the first law of black hole thermodynamics, and found that, except for the well-known horizon area term, the black hole entropy has a logarithmic term. Such a logarithmic term often occurs on the occasion of considering quantum corrections to black hole entropy. In our entropy expression, there is an undetermined constant S_0. To fix the constant entropy S_0, one has to invoke quantum theory of gravity.

We find that the temperature behavior of the topological black holes in Hořava-Lifshitz theory is very interesting. Indeed there is a duality for temperature between topological black holes in Hořava-Lifshitz theory and topological black holes in Einstein's general relativity. The temperatures of topological black holes with $k = 1$, 0, and -1 in Hořava-Lifshitz theory are dual to those of black holes with $k = -1$, 0, and 1 in Einstein's general relativity, respectively.

In this paper we have only considered thermodynamics of topological black holes in Hořava-Lifshitz theory with $\lambda = 1$. It is of great interest to see whether one can find a way to study thermodynamics for the general topological black holes in the theory with $\lambda \neq 1$.

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APPENDIX: TOPOLOGICAL BLACK HOLES FOR GENERAL λ

Here we briefly discuss topological black hole solution with a general λ. In terms of the new function F defined by

$$F(r) = k - \Lambda r^2 - f(r), \tag{A1}$$

the action (2.2) takes the form

$$I = \frac{\kappa^2\mu^2\Omega_k}{8(1-3\lambda)}\int dtdr\tilde{N}\Big\{\frac{(\lambda-1)}{2}F'^2 - \frac{2\lambda}{r}FF' + \frac{(2\lambda-1)}{r^2}F^2\Big\}. \tag{A2}$$

The equations of motion are then

$$0 = \Big(\frac{2\lambda}{r}F - (\lambda-1)F'\Big)\tilde{N}' + (\lambda-1)\Big(\frac{2}{r^2}F - F''\Big)\tilde{N}, \tag{A3}$$

$$0 = (\lambda-1)r^2F'^2 - 4\lambda rFF' + 2(2\lambda-1)F^2. \tag{A4}$$

The latter is easily solved to give [7]

$$F(r) = \alpha r^{(2\lambda \pm \sqrt{2(3\lambda-1)})/(\lambda-1)}, \quad \text{(A5)}$$

and then the first gives

$$\tilde{N} = \beta r^{-(1+3\lambda \pm \sqrt{2(3\lambda-1)})/(\lambda-1)}, \quad \text{(A6)}$$

where α and β are both integration constants. When $\alpha = 0$ or $F = 0$, Eq. (A3) does not restrict $\tilde{N}$. Note that the exponent of Eq. (A5) for the negative branch is always less than 2 for positive λ, and thus the r^2 term in the metric function (A1) dominates at large distances. The other branch gives a power larger than 2. We are interested in the solutions with asymptotic AdS behavior. In that case, we should look at the negative branch with constant $\tilde{N}$. It follows from Eq. (A3) that either $\lambda = 1$ or $F'' = \frac{2}{r^2} F$. The latter leads to $F \sim r^2$ or $1/r$; the first one does not satisfy (A4), and the second solution requires $\lambda = 1/3$, which may be of some interest [1], but the action (1.3) appears singular. So we discuss the $\lambda = 1$ case mainly in this paper.

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