Asymptotic behavior of dynamical variables and naked singularity formation in spherically symmetric gravitational collapse

Hayato Kawakami, Eiji Mitsuda,* Yasusada Nambu,† and Akira Tomimatsu‡

Department of Physics, Graduate School of Science, Nagoya University, Nagoya 464-8602, Japan
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In considering the gravitational collapse of matter, it is an important problem to clarify what kind of conditions leads to the formation of naked singularity. For this purpose, we apply the $1 + 3$ orthonormal frame formalism introduced by Uggla *et al.* to the spherically symmetric gravitational collapse of a perfect fluid. This formalism allows us to construct an autonomous system of evolution and constraint equations for scale-invariant dynamical variables normalized by the volume expansion rate of the timelike orthonormal frame vector. We investigate the asymptotic evolution of such dynamical variables towards the formation of a central singularity and present a conjecture that the steep spatial gradient for the normalized density function is a characteristic of the naked singularity formation.

I. INTRODUCTION

The investigation of the end states of gravitational collapse of sufficiently massive stars is a long-standing problem to be pursued in general relativity. One of the remarkable results is the naked singularity formation, and this is one of the possible end states in spherically symmetric gravitational collapse starting from regular initial data sets with nonzero measure. This was firstly shown by the detailed analysis of the Lemaître-Toleman-Bondi (LTB) solution describing the inhomogeneous dust gravitational collapse and has been subsequently confirmed in several other matter models including perfect fluid models [1–6]. In relation to the cosmic censorship conjecture, much attention therefore has been paid to the generality and the physical reasonability of the initial data sets leading to the naked singularity formation in the various matter models (see [7–11], for example).

Besides the initial data analysis, dynamical aspects of spherically symmetric gravitational collapse at later stages have been studied in terms of the shear function associated with the velocity vectors of the collapsing matter near the center [12–14], and it has been suggested that the shear can contribute to the delay of the apparent horizon formation. This role of the shear function may be important in considering the dynamical mechanism of the naked singularity formation. In this paper, however, we would like to focus on another dynamical approach to the end-state problem in spherically symmetric gravitational collapse. Our main concern is the relationship between asymptotic behavior of dynamical variables towards the formation of a central singularity and the causal structure of the arising singularity.

As a useful method to analyze such dynamical properties, we adhere to the $1 + 3$ orthonormal frame formalism

which was originally proposed by Elst and Uggla *et al.* [15,16] to study the dynamical behavior of the gravitational field variables such as the shear σ_{ab} and the expansion Θ associated with the timelike orthonormal frame vectors near the spatially inhomogeneous cosmological initial singularity. This formalism is based on the coordinate independent representation of Einstein's field equations in the form of an autonomous system of the first order evolution equations and constraints with the scale-invariant dimensionless variables normalized by the Hubble scalar $H \equiv \Theta/3$. By virtue of this Hubble normalization of dynamical variables, it is possible to remove the time-dependent factors due to the volume contraction given by the rate Θ measured in the local reference frame. As will be shown in this paper, the Hubble normalized density function Ω can approach zero or remain finite with the lapse of time towards singularity formation in the spherically symmetric inhomogeneous perfect fluid collapse, even though the central proper density becomes divergent at this final stage.

Our main purpose in this paper is to discuss the relation between the asymptotic behavior of Ω and the causal structure of the singularity. For the LTB solution, using the causal structure classified by the papers [1,2], we find that the growth of the steep spatial gradient of the profile of Ω near the center in the case $\Omega \to 0$ is a characteristic property leading to the naked singularity formation. If the density function Ω remains finite at the final stage, the end-state problem becomes more subtle. Nevertheless, we can discuss the critical value of the density contrast which gives a threshold of the transition from the black hole formation to the naked singularity formation.

This paper is organized as follows: In Sec. II, we begin with a brief review of the $1 + 3$ orthonormal frame formalism with the Hubble normalized variables and apply it to a spherically symmetric perfect fluid system. In this formalism, we adopt the separable volume gauge, which specifies the lapse function to be equal to the inverse of the Hubble

*emitsuda@gravity.phys.nagoya-u.ac.jp
†nambu@gravity.phys.nagoya-u.ac.jp
‡atomi@gravity.phys.nagoya-u.ac.jp

scalar H. In Sec. III, considering the inhomogeneous dust gravitational collapse described by the marginally bound LTB solution, we show that this gauge condition is useful to examine the asymptotic behavior of the Hubble normalized variables towards the central singularity formation. Then, we relate the asymptotic behavior of the Hubble normalized density function Ω to the arising causal structure of the end states of collapse. We propose a conjecture that the larger spatial gradient of the asymptotic profile of Ω is essential to the naked singularity formation. In Sec. IV, to support this conjecture, the asymptotic analysis is extended to gravitational collapse of perfect fluid with pressure. Taking account of the causal structure of spherically symmetric self-similar spacetimes which has been clarified by previous works [17–19], we numerically estimate the critical value of the density contrast using the asymptotic profile of Ω for the naked singularity formation. The results are summarized in Sec. V. Throughout this paper, the units in which $8\pi G = c = 1$ are used.

II. BASIC EQUATIONS FOR HUBBLE NORMALIZED VARIABLES

In this section, following the $1 + 3$ orthonormal frame formalism developed in [15,16], we present an autonomous system of evolution equations and the constrains for the scale-invariant variables in a spherically symmetric system with a perfect fluid source. We express an orthonormal frame as $\{e_0, e_\alpha\}$ (where $\alpha = 1, 2, 3$) with the unit vectors e_0 and e_α representing the timelike reference congruence and the rest of 3-spaces, respectively. The frame metric is given by $\eta_{\mu\nu} = \text{diag}[-1, 1, 1, 1]$. For simplicity, we specify the timelike frame vector e_0 to be hypersurface orthogonal, and the spacelike frame vectors e_α to be nonrotating Fermi-propagated along the integral curves of e_0.

Defining the four-velocity $u \equiv e_0$ for the unit timelike vector e_0 tangent to the reference congruence, we can introduce the basic geometrical quantities ($\dot{u}^\alpha$, Θ, $\sigma^{\alpha\beta}$, a^α, $n^{\alpha\beta}$) through the commutator relations as follows

$$[e_0, e_\alpha] = \dot{u}_\alpha e_0 - \left(\frac{\Theta}{3}\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta\right)e_\beta, \qquad (2.1)$$

$$[e_\alpha, e_\beta] = (2a_{[\alpha}\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta}n^{\delta\gamma})e_\gamma, \qquad (2.2)$$

where the square brackets denote the antisymmetric part of a tensor, and $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric three-dimensional permutation tensor. The scalar function Θ is the volume expansion rate, the vector $\dot{u}^\alpha$ and the tensor $\sigma^{\alpha\beta}$ (the trace-free symmetric tensor) are the acceleration rate and the shear rate of the frame vector e_0, respectively. They are calculated from the equations

$$\Theta = \nabla_\mu u^\mu, \qquad (2.3)$$

$$\dot{u}_\alpha = u^\mu \nabla_\mu u_\alpha, \qquad (2.4)$$

$$\sigma_{\alpha\beta} = \nabla_{(\beta}u_{\alpha)} - \frac{\Theta}{3}(\eta_{\alpha\beta} + u_\alpha u_\beta) + \dot{u}_{(\alpha}u_{\beta)}, \qquad (2.5)$$

where the round brackets denote the symmetric part of a tensor. (See [15] for the definition of the covariant derivative in the orthonormal frame formalism.) In addition, the quantities a^α and $n^{\alpha\beta}$ (the symmetric tensor) determine the connection on the one-parameter family spacelike hypersurfaces which can be defined by the assumption of the hypersurface-orthogonality of the timelike frame vector e_0.

Now let us turn our attention to the introduction of the basic variables characterizing the matter field. We consider a perfect fluid as the collapsing matter in this paper. We assume the equation of state to be

$$\tilde{p} = (\gamma - 1)\tilde{\mu} \qquad (2.6)$$

with the constant γ lying in the range $1 \le \gamma \le 2$. The pressure $\tilde{p}$ and the energy density $\tilde{\mu}$ are measured by a comoving observer who has the same velocity as the fluid four-velocity $\tilde{u}$. In general, the fluid four-velocity $\tilde{u}$ is not equal to the four-velocity u defined by the timelike frame vector e_0, thus we introduce the basic matter variables μ and v by decomposing the energy-momentum tensor of the perfect fluid with respect to u into the form

$$T_{\mu\nu} = \mu\{u_\mu u_\nu + \gamma G^{-1}(2v_{(\mu}v_{\nu)} + v_{<\mu}v_{\nu>})\}$$
$$+ p(g_{\mu\nu} + u_\mu u_\nu), \qquad (2.7)$$

where the angle brackets denote the symmetric trace-free part of a tensor. We have the following relations

$$\mu = \Gamma^2 G\tilde{\mu}, \qquad p = G^{-1}\left\{\gamma - 1 + \left(1 - \frac{2}{3}\gamma\right)v_\mu v^\mu\right\}\mu \qquad (2.8)$$

with the scalar functions G and Γ are defined by

$$G \equiv 1 + (\gamma - 1)v_\mu v^\mu, \qquad \Gamma \equiv \frac{1}{\sqrt{1 - v_\mu v^\mu}}. \qquad (2.9)$$

The vector v represents the peculiar fluid velocity relative to the rest 3-spaces of e_0, and defined through the relations

$$\tilde{u}^\mu \equiv \Gamma(u^\mu + v^\mu), \qquad u_\mu v^\mu = 0 \qquad (2.10)$$

with the fluid four-velocity $\tilde{u}$ normalized as $\tilde{u}_\mu \tilde{u}^\mu = -1$.

An important procedure of the formalism developed in [16] is to introduce the scale-invariant dimensionless variables by normalizing the geometrical and matter variables using the Hubble scalar

$$H \equiv \frac{\Theta}{3}. \qquad (2.11)$$

We denote the Hubble normalized quantities as

$$\partial_0 \equiv \frac{e_0}{H}, \qquad \partial_\alpha \equiv \frac{e_\alpha}{H}, \qquad (2.12)$$

$$\{\dot{U}^\alpha, \Sigma_{\alpha\beta}, A^\alpha, N^{\alpha\beta}\} \equiv \frac{1}{H}\{\dot{u}^\alpha, \sigma_{\alpha\beta}, a^\alpha, n^{\alpha\beta}\}, \quad (2.13)$$

$$\Omega \equiv \frac{\mu}{3H^2}, \quad (2.14)$$

where Ω is the Hubble normalized density function. The vector $\boldsymbol{v}$ is a dimensionless variable and the Hubble normalization is not necessary for this variable. To introduce a local coordinate system, we also define the Hubble normalized components of the frame vectors as

$$E_\mu{}^{\hat{a}} \equiv \frac{e_\mu{}^{\hat{a}}}{H}. \quad (2.15)$$

As expressed in Eq. (2.15), we hereafter attach the hat ^ to spacetime indices in order to distinguish them from the orthonormal frame indices. As physically interesting additional scale-invariant variables, we introduce the deceleration scalar q and the spatial Hubble gradient λ_α defined by

$$q \equiv -1 - \frac{1}{H}\boldsymbol{\partial}_0 H, \quad (2.16)$$

$$\lambda_\alpha \equiv -\frac{1}{H}\boldsymbol{\partial}_\alpha H. \quad (2.17)$$

They will be used to eliminate the Hubble scalar H appearing in the evolution equations and the constraints.[1]

Some gauge choice is still allowed within the framework of the $1 + 3$ orthonormal frame formalism, and in [15,16], the evolution equations and the constraints for the Hubble normalized variables have been given with the so-called separable volume gauge, which simplifies the temporal frame derivative $\boldsymbol{\partial}_0$ to

$$\boldsymbol{\partial}_0 = -\partial_t \quad (2.18)$$

using a nondimensional time coordinate t.[2] Further, the additional gauge constraint

$$\dot{U}_\alpha = \lambda_\alpha \quad (2.19)$$

is required for the separable volume gauge (see [16] for its details). Owing to this specification of the gauge, the matching of the four-velocity $\boldsymbol{u} = \boldsymbol{e}_0$ of the reference congruence with the fluid four-velocity $\tilde{\boldsymbol{u}}$ (i.e., $\boldsymbol{v} = \boldsymbol{0}$) as in [20] is not always permitted. The important geometrical

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[1]Although the spatial Hubble gradient was expressed as r_α in [16], we have changed its notation in order to prevent readers from confusing it with the radial coordinate r, which will be introduced later.

[2]While the lapse function is the positive definite function equal to the Hubble scalar H in the separable volume gauge of [16], in this paper we specify the lapse function to be $-H$ in order to keep the positivity of the lapse function. The minus sign in the right-hand side of Eq. (2.18), which does not appear in the corresponding equation in [16], comes from this gauge specification, which may be referred to as the separable volume gauge for gravitational collapse.

result of Eq. (2.18) with the commutator equation (2.1) is that the volume density $\mathcal{V}$ defined by $\mathcal{V}^{-1} \equiv \det(e_\alpha{}^i)$ has the form

$$\mathcal{V} = \mathcal{V}_0 \times e^{-3t}, \quad (2.20)$$

where $\mathcal{V}_0$ is an arbitrary function of spatial coordinates. From Eq. (2.20), in the limit $t \to \infty$, the volume density approaches zero to form a singularity. This is a useful property of the separable volume gauge to investigate the asymptotic dynamical behavior just before the singularity formation in gravitational collapse.

Now let us study spherically symmetric gravitational collapse with a perfect fluid using the Hubble normalized scale-invariant variables. By virtue of the Hubble normalization, the Hubble scalar H becomes the only variable carrying a physical reference scale and the analysis of the evolution equations and the constraints for these variables will allow us to observe dynamical behavior deviated from the time dependence of the volume contraction of the reference congruence. We consider the spherically symmetric line element of the form

$$ds^2 = -I^2(t, r)dt^2 + J^2(t, r)dr^2$$
$$+ R^2(t, r)(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.21)$$

The coordinate expressions for the orthonormal frame derivatives can be given by

$$\begin{aligned} \boldsymbol{e}_0 &= I^{-1}\partial_{\hat{t}}, & \boldsymbol{e}_r &= J^{-1}\partial_{\hat{r}}, \\ \boldsymbol{e}_\theta &= R^{-1}\partial_{\hat{\theta}}, & \boldsymbol{e}_\phi &= R^{-1}\sin^{-1}\theta\partial_{\hat{\phi}}. \end{aligned} \quad (2.22)$$

As is used in Eq. (2.22), hereafter we substitute the letters r, θ, ϕ (instead of the numbers 1,2,3) into the spacetime spatial indices $\hat{i}$ and the orthonormal frame spatial indices α (i.e., $i = \hat{r}, \hat{\theta}, \hat{\phi}$ and $\alpha = r, \theta, \phi$). The gauge conditions (2.18) and (2.20) are reduced to

$$I = -H^{-1} \quad (2.23)$$

and

$$J(t, r) = C(r)R^{-2}e^{-3t}, \quad (2.24)$$

where C is an arbitrary function of r.

An autonomous system of evolution equations and constraints for the Hubble normalized variables presented in [15,16] has been expressed with the orthonormal frame derivatives applicable to any spacetimes. It is a straightforward to apply the formalism to the spherically symmetric metric (2.21) with the separable volume gauge. The quantities $(E_r{}^{\hat{r}}, E_\theta{}^{\hat{\theta}}, A^r, \lambda^r, \Sigma^{rr}, \Omega, v^r, q)$ are independent Hubble normalized variables to be analyzed here and we arrive at the following evolution equations for these variables

$$\dot{E}_r{}^{\hat{r}} = -(q - \Sigma^{rr})E_r{}^{\hat{r}}, \quad (2.25)$$

$$\dot{E}_\theta{}^{\hat{\theta}} = -(q + \tfrac{1}{2}\Sigma^{rr})E_\theta{}^{\hat{\theta}}, \qquad (2.26)$$

$$\dot{A}^r = -(q - \Sigma^{rr})A^r - \tfrac{1}{2}E_r{}^{\hat{r}}(\Sigma^{rr})', \qquad (2.27)$$

$$\dot{\lambda}^r = -(q - \Sigma^{rr})\lambda^r - E_r{}^{\hat{r}}q', \qquad (2.28)$$

$$\dot{\Sigma}^{rr} = -(q - 2)\Sigma^{rr} + \tfrac{2}{3}E_r{}^{\hat{r}}(A^r)' - \tfrac{2}{3}E_r{}^{\hat{r}}(\lambda^r)'$$
$$- \tfrac{4}{3}\lambda^r A^r - \tfrac{2}{3}(E_\theta{}^{\hat{\theta}})^2 - 2\gamma G^{-1}\Omega(v^r)^2, \qquad (2.29)$$

$$\dot{\Omega} = -(2q - 1)\Omega + 3G^{-1}\{\gamma - 1 + (1 - \tfrac{2}{3}\gamma)(v^r)^2\}\Omega$$
$$+ \gamma E_r{}^{\hat{r}}(G^{-1}\Omega v^r)' + \gamma G^{-1}\Omega v^r(v^r\Sigma^{rr} - 2A^r), \qquad (2.30)$$

and the constraints

$$1 + \tfrac{1}{3}\{2E_r{}^{\hat{r}}(A^r)' - 2\lambda^r A^r - 3(A^r)^2$$
$$+ (E_\theta{}^{\hat{\theta}})^2\} - \tfrac{1}{4}(\Sigma^{rr})^2 - \Omega = 0, \qquad (2.31)$$

$$E_r{}^{\hat{r}}(\Sigma^{rr})' + 2\lambda^r - \Sigma^{rr}\lambda^r - 3A^r\Sigma^{rr} + 3\gamma G^{-1}\Omega v^r = 0, \qquad (2.32)$$

$$E_r{}^{\hat{r}}(E_\theta{}^{\hat{\theta}})' - (A^r + \lambda^r)E_\theta{}^{\hat{\theta}} = 0, \qquad (2.33)$$

where the dot and the prime mean the partial derivatives ∂_t and ∂_r, respectively. These equations correspond to the Einstein equations, the Jacobi identities and the contracted Bianchi identities. In addition to these equations, we can use the following Raychaudhuri equation for the deceleration scalar q:

$$q = \tfrac{1}{2}(\Sigma^{rr})^2 - \tfrac{1}{3}E_r{}^r\partial_r\lambda^r + \tfrac{2}{3}\lambda^r A^r$$
$$+ \tfrac{1}{2}[1 + 3G^{-1}\{\gamma - 1 + (1 - \tfrac{2}{3}\gamma)(v^r)^2\}]\Omega. \qquad (2.34)$$

It is remarkable that no time derivative of the radial velocity v^r appears in these set of the evolution equations and the constraint equation (2.32) can be used to determine v^r. We can also check that the remaining two constraints (2.31) and (2.33) are consistent with other six evolution equations for $(E_r{}^{\hat{r}}, E_\theta{}^{\hat{\theta}}, A^r, \lambda^r, \Sigma^{rr}, \Omega)$.

Finally, let us explicitly present the Hubble normalized variables using the metric functions and the matter fields:

$$E_r{}^{\hat{r}} = -\frac{I}{J}, \qquad E_\theta{}^{\hat{\theta}} = -\frac{I}{R}, \qquad (2.35)$$

$$\Sigma^{rr} = -\frac{2}{3}\left(\frac{\dot{J}}{J} - \frac{\dot{R}}{R}\right), \qquad (2.36)$$

$$A^r = \frac{I\partial_r R}{JR}, \qquad (2.37)$$

$$\lambda^r = -\frac{\partial_r I}{J}, \qquad q = -1 - \frac{\dot{I}}{I}, \qquad (2.38)$$

$$v^r = \frac{\tilde{u}^r}{\tilde{u}^t}, \qquad (2.39)$$

$$\Omega = \frac{\{1 + (\gamma - 1)(v^r)^2\}I^2\tilde{\mu}}{3\{1 - (v^r)^2\}}. \qquad (2.40)$$

These relations are useful to analyze the asymptotic behavior of dynamical variables. Although the description of time evolution with the Hubble normalized variables and the separable volume gauge is available until the singularity is formed at $t = \infty$, the spacetime region covered by the radial coordinate r may be too restricted to determine the nakedness of the arising singularity. Thus we will restrict our investigations to the models of gravitational collapse of which causal structure of the end states is already known. An example of such a model is the LTB solution, which is well known as a generic model of the inhomogeneous dust gravitational collapse. In the next section, we will examine the dynamical behavior of the Hubble normalized variables in the marginally bound LTB spacetime to discuss their key feature relevant to the naked singularity formation.

III. ASYMPTOTIC BEHAVIOR IN INHOMOGENEOUS DUST COLLAPSE

A. Brief review of the LTB solution

Let us begin with a brief review of the LTB solution describing inhomogeneous dust gravitational collapse (see [1–3] for its details). This solution includes the two arbitrary functions of radial coordinate usually denoted as F and f, which are related to the Misner-Sharp mass and the initial velocity, respectively. In this paper, we consider only the solution with $f = 0$, which is called the marginally bound solution. Using the comoving coordinate system $\{\tau, \rho, \theta, \phi\}$, the line element for this solution is

$$ds^2 = -\frac{4}{9B}d\tau^2 + (\partial_\rho R)^2 d\rho^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (3.1)$$

$$R(\tau, \rho) = \rho\left\{1 - \sqrt{\frac{F(\rho)}{B\rho^3}}(1 + \tau)\right\}^{2/3}, \qquad (3.2)$$

where B is an arbitrary positive constant.

The time $\tau = -1$ is usually interpreted as the initial time at which the area radius R becomes equal to the coordinate radius ρ. In addition, without loss of generality, we can choose $\tau = 0$ to be the time of the central singularity formation by specifying the leading order term of the arbitrary function $F(\rho)$ near the regular center $\rho = 0$ as $F \simeq B\rho^3$. In the comoving coordinate system, the components of the dust four-velocity are given by

$$\tilde{u}^{\hat{\tau}} = -\frac{3\sqrt{B}}{2}, \qquad \tilde{u}^{\hat{\rho}} = \tilde{u}^{\hat{\theta}} = \tilde{u}^{\hat{\phi}} = 0, \qquad (3.3)$$

and from the Einstein equations we obtain the proper

energy density $\tilde{\mu}$ measured by a comoving observer as

$$\tilde{\mu}(\tau, \rho) = \frac{\partial_\rho F}{R^2 \partial_\rho R}. \tag{3.4}$$

The approximate form of the function F near the regular center $\rho = 0$ can be written as

$$F(\rho) = B\rho^3(1 - 2F_n\rho^n + O(\rho^{n+1})) \tag{3.5}$$

with a positive integer n. The subleading term $F_n\rho^n$ in Eq. (3.5) represents the dominant inhomogeneity of an initial dust distribution near the center, and we assume $n \geq 2$ and $F_n > 0$ to require the regularity of the proper energy density at the center (i.e., $\partial_\rho \tilde{\mu} = 0$ at $\rho = 0$ and at $\tau = -1$). The causal structure of the end states of the collapse is closely related to the inhomogeneity given by the term $F_n\rho^n$. For $n = 2$, the shell-focusing naked singularity appears at the center $\rho = 0$ at the time $\tau = 0$. For $n \geq 4$, the arising singularity is hidden behind the event horizon. For $n = 3$, using the parameter

$$b \equiv \frac{F_3}{B^{3/2}}, \tag{3.6}$$

the condition for the naked singularity formation is given by [1,2]

$$b > b_c, \qquad b_c = \frac{26 + 15\sqrt{3}}{4}. \tag{3.7}$$

B. Asymptotic behavior of the Hubble normalized variables

In this subsection, we would like to clarify how the asymptotic behavior of the Hubble normalized variables leading to the naked singularity formation depends on the initial density inhomogeneity characterized by the integer n. With the help of the relation between the causal structure of the end states and the choice of n obtained in [1,2], we discuss what asymptotic behavior of the Hubble normalized variables characterize the causal structure of the singularity. As the formation of the central singularity occurs at the point $\tau = \rho = 0$ in the comoving coordinate system $\{\tau, \rho, \theta, \phi\}$, our strategy is to analyze the asymptotic t-dependence of the Hubble normalized variables in the limit $t \to \infty$ by using the coordinate system $\{t, r, \theta, \phi\}$ with the separable volume gauge condition. For this purpose, we consider the coordinate transformation between the two coordinate systems, which leads to the following partial differential equations for $\tau(t, r)$, $\rho(t, r)$, $I(t, r)$, and $J(t, r)$

$$\frac{4}{9B}\dot{\tau}^2 - (\partial_\rho R)^2\dot{\rho}^2 = I^2, \tag{3.8}$$

$$(\partial_\rho R)^2\rho'^2 - \frac{4}{9B}\tau'^2 = J^2, \tag{3.9}$$

$$(\partial_\rho R)^2\dot{\rho}\rho' = \frac{4}{9B}\dot{\tau}\tau'. \tag{3.10}$$

We also have Eq. (2.24) as the separable volume gauge condition. As will be shown, the four Eqs. (3.8), (3.9), and (3.10) with (2.24) demand that timelike curves with $r =$ const to converge to the singular point $\tau = \rho = 0$. The behavior of these coordinates is schematically shown in Fig. 1.

In particular, the exponential t-dependence of the function JR^2 in Eq. (2.24) significantly affects the asymptotic relation between the two coordinate systems $\{\tau, \rho\}$ and $\{t, r\}$, and the converging behavior of the $r =$ const timelike curves is useful to analyze the asymptotic dynamical features just before the central singularity formation. In this subsection, we analyze the asymptotic t-dependence of the Hubble normalized variables on such a congruence of the timelike curves. When the coordinate variables $\tau(t, r)$ and $\rho(t, r)$ approach zero in the limit $t \to \infty$ with a fixed value of r, the metric function $R(t, r)$ also goes to zero as

$$R(\tau, \rho) \simeq \rho(F_n\rho^n - \tau)^{2/3}. \tag{3.11}$$

The key issue to be analyzed here is which of the terms $F_n\rho^n$ and τ in Eq. (3.11) becomes dominant in the limit $t \to \infty$ along the timelike curves.

From Eq. (2.24), we assume that the metric functions J and R have exponential t-dependence in the limit $t \to \infty$. This assumption turns out to be compatible with Eqs. (3.8), (3.9), and (3.10) under the relations

$$R \sim J \sim \tau \sim I \sim \exp(-t), \tag{3.12}$$

if the exponential t-dependence of ρ is also derived from Eq. (3.11). To check this, we first assume that the ratio $F_n\rho^n/|\tau|$ approaches zero. In this case, Eq. (3.11) leads to
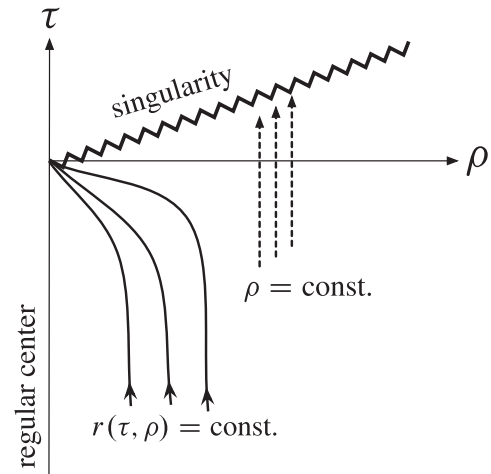

FIG. 1. A schematic diagram describing the relation between the comoving coordinate systems $\{\tau, \rho\}$ and the separable volume gauge coordinates $\{t, r\}$. In the limit $t \to \infty$, all the timelike curves labeled $r =$ const (solid curves) converge to the onset point $\tau = \rho = 0$ of the singularity formation.

$\rho \sim \exp(-t/3)$, which is consistent with the assumption $F_n\rho^n/|\tau| \to 0$ only for $n \geq 4$. If the ratio $F_n\rho^n/|\tau|$ is assumed to blow up, we have $\rho \sim \exp\{-3t/(3 + 2n)\}$. It is easy to check that this is allowed only for $n = 2$, which gives the exponential t-dependence as $\rho \sim \exp(-3t/7)$. For the remaining case $n = 3$, we obtain $\rho \sim \exp(-t/3)$ and the ratio $F_n\rho^n/|\tau|$ remains finite.

Now let us see the asymptotic behavior of the Hubble normalized variables using the asymptotic exponential forms of the metric functions I, J, R and the coordinate variables τ, ρ. Because we have the relation Eq. (3.12) irrespective of the choice of n, Eqs. (2.35), (2.36), (2.37), and (2.38) mean that the Hubble normalized variables ($E_r^{\hat{r}}$, $E_\theta^{\hat{\theta}}$, A^r, λ^r) become finite and their values can only depend on r. On the other hand, variables Σ^{rr} and q approach zero. Using the condition $\tilde{u}^{\hat{\rho}} = 0$ for the comoving coordinate system, the radial velocity v^r given by (2.39) can be rewritten into the form

$$(v^r)^2 = \frac{J^2\dot{\rho}^2}{I^2\rho'^2}. \tag{3.13}$$

Hence, the radial velocity v^r remains finite in the limit $t \to \infty$ irrespective of n. From Eqs. (3.8), (3.9), and (3.10), we have

$$1 - (v^r)^2 = \frac{9BI^2}{4\dot{\tau}^2} \tag{3.14}$$

and using this relation, the Hubble normalized density Ω given by Eq. (2.40) becomes

$$\Omega = \frac{4\dot{\tau}^2\rho^2}{9R^2\partial_\rho R}. \tag{3.15}$$

Applying the approximate form (3.11) of R and the asymptotic relation $\dot{\tau} \simeq -\tau$, we have the following asymptotic form of the density function

$$\Omega_{\text{asym}} = \frac{4}{3\{3 + (2n + 3)F_n\rho^n/|\tau|\}\{1 + F_n\rho^n/|\tau|\}}. \tag{3.16}$$

Owing to the term $F_n\rho^n/|\tau|$ contained in Eq. (3.16), an interesting difference of the asymptotic behavior of Ω appears according to the choice of n and this will be shown in the next subsection.

C. Relation between the asymptotic Hubble normalized density and the causal structure of the end state

We rewrite the term $F_n\rho^n/|\tau|$ in Eq. (3.16) as a function of t and r. In the limit $t \to \infty$, the metric functions can be written as $I = I_0(r)\exp(-t)$, $J = J_0(r)\exp(-t)$ and $R = R_0(r)\exp(-t)$. Because the choice of the spatial coordinate r remains arbitrary within the framework of the separable volume gauge, the arbitrary function $C(r)$ is included in Eq. (2.24) for J. To remove this ambiguity, we introduce the new spatial coordinate ζ by

$$-E_r^{\hat{r}}\frac{d\zeta}{dr} = 1, \tag{3.17}$$

where the Hubble normalized variable $E_r^{\hat{r}} = -I_0/J_0$ should be regarded as a function of r. This specification of the spatial coordinate ζ leads to the line element

$$ds^2 = e^{-2t}[-I_0^2(\zeta)(dt^2 - d\zeta^2) + R_0^2(\zeta)(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{3.18}$$

1. The $n \geq 4$ case: Approach to homogeneous dust dynamics and black hole formation

The $n \geq 4$ case corresponds to the black hole formation and the ratio $F_n\rho^n/|\tau|$ approaches zero towards the singularity formation. From Eq. (3.16), we have

$$\Omega_{\text{asym}} = \tfrac{4}{9}. \tag{3.19}$$

In addition, from Eqs. (3.8), (3.9), and (3.10), we obtain the solution for $n \geq 4$ as

$$I_0 = \frac{Bl^3}{12}\cosh^2\left(\frac{\zeta}{3}\right), \qquad R_0 = \frac{Bl^3}{4}\cosh^2\left(\frac{\zeta}{3}\right)\sinh\left(\frac{\zeta}{3}\right), \tag{3.20}$$

where l is an arbitrary constant and

$$\rho = l\sinh(\zeta/3) \times \exp(-t/3),$$
$$\tau^{2/3} = (Bl^2/4)\cosh^2(\zeta/3) \times \exp(-2t/3). \tag{3.21}$$

Although the metric tensor written by Eqs. (3.20) is derived as an asymptotic form in the limit $t \to \infty$, it is identical with an exact solution of the Einstein equations describing spherically symmetric homogeneous dust collapse. This unfamiliar form of the metric tensor is due to the separable volume gauge; the timelike congruence parametrized by the coordinates (t, r) (or ζ) covers only a limited region of the spacetime (see Fig. 1). In fact, from Eq. (3.14), the radial velocity is $v^r = \tanh(\zeta/3)$ and this family of the timelike curves has the null boundary $|v^r| = 1$ at $\zeta = \infty$ ($r \to \infty$).

2. The $n = 2$ case: Growth of the central density gradient and naked singularity formation

For $n = 2$, the ratio $F_2\rho^2/|\tau|$ grows with the lapse of time and the inhomogeneity due to the term $F_n\rho^n$ is significantly involved in the central naked singularity formation. Under the approximation $F_2\rho^2/|\tau| \gg 1$, we obtain the metric functions

$$I_0 = l, \qquad R_0 = l\sinh(\zeta) \tag{3.22}$$

and

$$F_2^{1/3}\rho^{7/3} = l\sinh\zeta \times \exp(-t),$$
$$\tau^2 = (9Bl^2/4)\cosh^2\zeta \times \exp(-2t). \tag{3.23}$$

This asymptotic form of I_0 and R_0 represents a flat metric written in the accelerating coordinate system $\{t, \zeta\}$. Using Eq. (3.14), we have $v^r = \tanh\zeta$. The spacetime region covered by the accelerating coordinate system also has the null boundary at $\zeta = \infty$.

The timelike congruence parametrized by (t, ζ) converges to the point $\tau = \rho = 0$ with the four-velocity $\boldsymbol{u} = \boldsymbol{e}_0$. The spatial distance between the two neighboring time-like curves ζ and $\zeta + d\zeta$ changes in proportion to $\exp(-t)d\zeta$. This change of the reference congruence may be larger than that of the matter contraction effect which increases the proper energy density $\tilde{\mu}$. Thus, the effect of matter on the metric becomes weak and we have the flat metric (3.22) as the dominant asymptotic behavior. We can find from Eq. (3.16) that the Hubble normalized density Ω asymptotically takes the form

$$\Omega_{\text{asym}} = \frac{3B}{7} F_2^{-6/7} l^{2/7} \cosh^2\zeta (\sinh\zeta)^{-12/7} e^{-2t/7} \to 0.$$
(3.24)

This exponential decay of Ω_{asym} is a remarkable feature of the asymptotic evolution for the $n = 2$ case. As the contribution of matter to the metric is negligible, one can call the behavior represented by Eq. (3.24) as the "vacuum-dominated" evolution [16].

It must be noted, however, that the approximation $F_2\rho^2/|\tau| \gg 1$ breaks down near the center $\rho = 0$ ($\zeta = 0$) which is regular during $\tau < 0$ ($t < \infty$). Let us denote the asymptotic value of Ω at the center as Ω_0. Then, we obtain $\Omega_0 = 4/9$ even for $n = 2$. Note that Eq. (3.24) shows an infinite increase of Ω in the limit $\zeta \to 0$. This increase of Ω with respect to the spatial coordinate ζ should be suppressed if the ratio $F_2\rho^2/|\tau|$ becomes smaller than unity in the vicinity of the regular center. Unfortunately, it is difficult to see analytically the smooth decrease of Ω from the central value Ω_0. Nevertheless, it is sure that the gradient of Ω near the center increases infinitely as t increases. Hence, the exponential decay (3.24) of Ω_{asym} should be rather regarded as the growth of the density contrast between the central region near $\zeta = 0$ and the outer region $\zeta \gg 1$. We expect such a profile of Ω_{asym} with a large gradient with respect to the coordinate ζ is essential to the naked singularity formation.

3. The $n = 3$ case: Existence of the critical density gradient between naked singularity formation and black hole formation

In this case, the ratio $F_3\rho^3/|\tau|$ remains finite in the limit $t \to \infty$. As $\tau \sim \rho^3 \sim \exp(-t)$ for this case, by introducing the ratio

$$k \equiv \frac{F_3\rho^3}{|\tau|}$$
(3.25)

as a function of the spatial coordinate ζ, we obtain from Eq. (3.16) the asymptotic profile of Ω as follows

$$\Omega_{\text{asym}}(\zeta) = \frac{4}{9(1 + 3k)(1 + k)}.$$
(3.26)

By using Eq. (3.14), the radial velocity v^r can be written as

$$v^r = -\frac{k^{1/3}(1 + 3k)}{2b^{1/3}(1 + k)^{1/3}}, \qquad b = \frac{F_3}{B^{3/2}}.$$
(3.27)

Then, using Eqs. (3.8), (3.9), and (3.10), the equation determining k is given by

$$\frac{dk}{d\zeta} = \frac{k\{1 - (v^r)^2\}}{v^r}.$$
(3.28)

Imposing the boundary condition $k = 0$ at $\zeta = 0$, we obtain the solution $k = k(\zeta)$ containing the parameter b. For $n = 3$ case, we also have $|v^r| \to 1$ in the limit $\zeta \to \infty$, and the value of k remains finite at $\zeta = \infty$. It is clear from Eq. (3.26) that Ω_{asym} decreases monotonically as k increases. Let us denote the values of k and Ω_{asym} in the limit $\zeta \to \infty$ as k_∞ and Ω_∞, respectively. These limiting values depend on the value of the parameter b. The profile of Ω_{asym} as a function of ζ is shown in Fig. 2.

Although $\Omega_{\text{asym}} = 4/9$ at the center $\zeta = 0$ irrespective of the parameter b, the value Ω_∞ decreases as b increases. For $b \gg 1$, we have $k_\infty \simeq 2\sqrt{b}/3$ and $\Omega_\infty \simeq 1/3b$. Recall that $b > b_c$ gives the condition for the naked singularity formation. The numerical calculation shows that this inequality corresponds to $\Omega_\infty < \Omega_c \simeq 0.03$. Using the central value $\Omega_0 = 4/9$ of Ω_{asym}, the density contrast defined by the ratio $\delta = \Omega_0/\Omega_\infty$ has a critical value $\delta_c : \delta < \delta_c$ corresponds to the black hole formation and $\delta > \delta_c$ corresponds to the naked singularity formation. For $\delta \gg \delta_c$, the density contrast becomes very large and this leads to the naked singularity formation. This behavior is consistent with the asymptotic profile of Ω discussed in the case $n = 2$.[3]

We finally comment on the self-similar evolution in dust collapse in terms of the Hubble normalized variables. The spherically symmetric self-similar solutions are defined as solutions of the Einstein equations reduced to a set of the ordinary differential equations with respect to a dimensionless variable (see [22] for its details). For $n = 3$, the Hubble normalized density Ω, as well as the other Hubble normalized variables $(v^r, \lambda^r, A^r, E_r^{\hat{r}}, E_\theta^{\hat{\theta}})$ is given as the function of the single coordinate r in the limit $t \to \infty$. Although we have derived the asymptotic behavior of v^r

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[3]Although the naked singularity formation occurs both for $n = 2$ and for $n = 3$ with the parameter $b > b_c$, its strength is known to be different in the sense of Tipler [21]. Namely, the arising central singularity is called gravitationally weak for the former case but strong for the latter one [2]. This difference of the strength can also be interpreted from the viewpoint of the asymptotic behavior of Ω, which claims that the "vacuum-dominated" evolution under the condition $\Omega \to 0$ around the central region results in the formation of "weak" naked singularity.

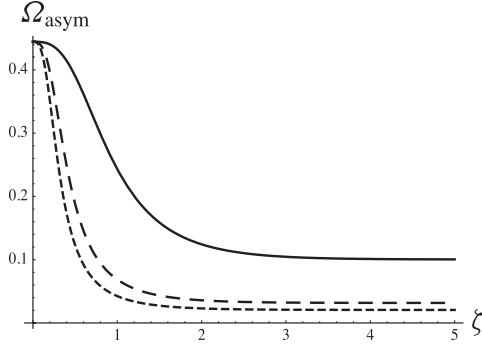FIG. 2. The profile of Ω_{asym} as a function of the spatial coordinate ζ. Each lines correspond to $b < b_c$ (the solid line), $b = b_c$ (the long-dotted line) and $b > b_c$ (the short-dotted line). As the value of b increases, the density contrast $\delta = \Omega_0/\Omega_\infty$ increases. For $\delta > \delta_c \approx 15$, the resulting singularity becomes naked.

and Ω through the coordinate transformations (3.8), (3.9), and (3.10), it is possible to obtain all the asymptotic forms of these variables directly from an autonomous system of the evolution equations (2.25), (2.26), (2.27), (2.28), (2.29), and (2.30) and the constraints (2.31), (2.32), and (2.33) for $\gamma = 1$. As Σ^{rr} and q approach zero in the limit $t \to \infty$, if these two variables in addition to variables with partial derivatives with respect to t are neglected in the equations, we arrive at a closed system of the ordinary differential equations for $(\Omega, v^r, \lambda^r, A^r, E_\theta{}^\theta)$ using the dimensionless coordinate ζ(see the next section for the details). This closed system will be equivalent to the Einstein equations with the requirement of self-similarity.[4] The result obtained in this subsection means that such a self-similar behavior should develop at the final stage $t \to \infty$, even if the initial distribution function $F(\rho)$ given by Eq. (3.5) contains the higher order inhomogeneous terms with $\rho^{n'}$ for $n' \geq 4$ in addition to the term $F_3\rho^3$.

D. Condition for the naked singularity formation

Combining the result of our analysis for the asymptotic behavior of the Hubble normalized density function Ω in the LTB solution, we propose the following conjecture for the condition of the naked singularity formation:

The development of the large spatial gradient or density contrast in the asymptotic profile of the Hubble normalized density parameter Ω in the separable volume gauge gives a sufficient condition of the naked singularity formation.

We have confirmed this conjecture holds for the dust case. In the next section, we examine whether this conjecture can be extended to the gravitational collapse of perfect fluid with pressure.

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[4]In fact, we can check that the function Ω_{asym} given by Eq. (3.26) has the same form as Eq. (3.13) in [18] except for the numerical factor.

IV. EFFECT OF PRESSURE ON THE HUBBLE NORMALIZED DENSITY

In the previous section, we have discussed the asymptotic behavior of the Hubble normalized variables through the coordinate transformation from the LTB solution of dust collapse. Unfortunately, there does not exist such a generic analytical solution of perfect fluid collapse for the equation of state (2.6) with $1 < \gamma \leq 2$. We therefore have to analyze directly the asymptotic solutions of an autonomous system of the evolution equations (2.25), (2.26), (2.27), (2.28), (2.29), and (2.30) and the constraints (2.31), (2.32), and (2.23).

As was schematically shown in Fig. 1, under the separable volume gauge (2.24) for the line element (2.21), we can assume the timelike congruence parametrized by the coordinates t and r converges to the central singularity arising at $t = \infty$ and the possible asymptotic time dependence can be assumed to be $I \sim J \sim R \sim \exp(-t)$. The Hubble normalized variables q and $\Sigma_r{}^r$ then go to zero in the limit $t \to \infty$, while the other variables $(E_r{}^{\hat{r}}, E_\theta{}^{\hat{\theta}}, A^r, \lambda^r, \Omega, v^r)$ may become finite dependent only on r. We express their asymptotic forms as $(E, A, \lambda, \Omega, v)$ without the indices r and θ, and also introduce the coordinate ζ given by Eq. (3.17). It is easy to obtain $\lambda = -3\gamma\Omega v/2G$ from Eq. (2.32) and $dE/d\zeta = -(A + \lambda)E$ from Eq. (2.33). The assumptions for the timelike congruence and the possible asymptotic time dependence of (I, J, R) allow us to arrive at the closed set of equations for (A, Ω, v) as follows

$$\{(\gamma - 1)v^2 + 1\}\{v^2 - (\gamma - 1)\}\frac{d\Omega}{d\zeta}$$
$$= 2\gamma\Omega v^2 A\{2\gamma - 3 - (\gamma - 1)v^2\}$$
$$\quad - \frac{3}{2}\gamma(\gamma - 2)G^{-1}\Omega^2 v\{1 - (\gamma - 1)v^2\}(1 - v^2)$$
$$\quad - 2\Omega v\{(3\gamma - 4)(\gamma - 1) - (2\gamma^2 - 5\gamma + 4)v^2\},$$
$$(4.1)$$

$$\frac{\gamma\{(\gamma - 1)v^2 + 1\}\{v^2 - (\gamma - 1)\}}{1 - v^2}\frac{dv}{d\zeta}$$
$$= 2\gamma(\gamma - 1)GvA + \frac{3}{2}\gamma^2(\gamma - 2)\Omega v^2$$
$$\quad - G\{(\gamma - 1)(3\gamma - 2) + (\gamma - 2)v^2\}, \quad (4.2)$$

$$\frac{dA}{d\zeta} = 3\gamma G^{-1}\Omega vA - \frac{3}{2}G^{-1}(\gamma - 1 + v^2)\Omega$$
$$\quad - \frac{3}{2}\Omega - A^2 + 1. \quad (4.3)$$

From Eq. (4.2), we can find that there exists the null boundary $|v| = 1$ of the timelike congruence in the limit $\zeta \to \infty$ just in the same way as dust collapse.

It should be noted that this set of equations for (A, Ω, v) is derived under the assumption $\Omega \neq 0$. If we consider the

case $\Omega \to 0$ in the limit $t \to \infty$, we must use the equation

$$E^2 - A^2 + 1 = 0 \qquad (4.4)$$

instead of Eqs. (4.1) and (4.2). The similar equation can be obtained for the case $n = 2$ of the dust collapse. In fact, under the requirement $\Omega = \lambda = 0$, we have the solution $E = -1/\sinh\zeta$, $A = \coth\zeta$ which corresponds to the flat metric given by (3.22). It was shown in the previous section that this "vacuum-dominated" evolution should break down in the vicinity of the regular center due to the large spatial gradient of $\Omega_{\rm asym}$. For the perfect fluid collapse, the effect of pressure should become important at least in the subsonic region $v^2 < \gamma - 1$ near the center and works to suppress the development of the density contrast. Nevertheless, it is plausible that the "vacuum-dominated" evolution leading to the decay $\Omega \to 0$ is allowed in the supersonic region $v^2 > \gamma - 1$, where pressure becomes ineffective. This means that a large gradient of $\Omega_{\rm asym}$ generated in the supersonic region is in favor of the naked singularity formation. Unfortunately, it is very difficult to study in detail the perfect fluid collapse allowing the decay of Ω with the lapse of time in the supersonic region and to check the validity of our expectation. Therefore, in the following, our discussion is restricted to the case that Ω remains nonzero in the limit $t \to \infty$ in $0 \leq \zeta < \infty$, and Eqs. (4.1), (4.2), and (4.3) can be applied.

The set of the ordinary differential equations (4.1), (4.2), and (4.3) with respect to the nondimensional coordinate ζ describes the self-similar behavior of the Hubble normalized variables such as Ω. Although in this paper, we assume that they become asymptotically dominant equations as a result of gravitational collapse started from general non self-similar initial conditions, they are equivalent to the exact self-similar Einstein equations which have been extensively studied in the comoving coordinate system. We can see the existence of a singular point of the differential equations at $v^2 = \gamma - 1$ where the coefficients of the derivatives $d\Omega/d\zeta$ and $dv/d\zeta$ in Eqs. (4.1) and (4.2) vanish. This corresponds to the sonic point where the fluid velocity relative to a tangent surface of a homothetic Killing vector equals to the sound speed [17,23], and the self-similar solutions may become singular.

We consider the self-similar solutions regular in $0 \leq \zeta < \infty$ including the sonic point. The simplest example is the so-called flat Friedmann solution with $\gamma > 1$. It is well known that the arising singularity becomes spacelike in the homogeneous perfect fluid collapse. For this solution, the asymptotic density function is given by

$$\Omega_{\rm asym} = \frac{4}{9\gamma^2}\left[\frac{1 + \gamma\sinh^2\{(3\gamma - 2)\zeta/3\gamma\}}{1 + \sinh^2\{(3\gamma - 2)\zeta/3\gamma\}}\right]. \qquad (4.5)$$

The ζ dependence of this function is shown in Fig. 3. This form can be obtained through the coordinate transformation from the comoving coordinate system to the coordinate system $\{t, r, \theta, \phi\}$ as was performed in Sec. III. Of
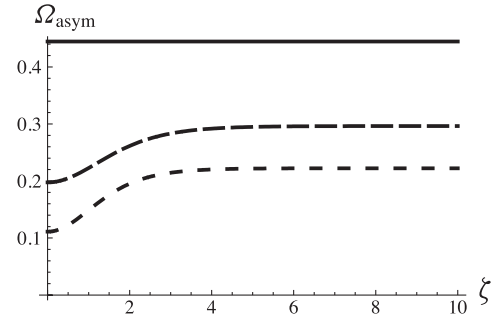

FIG. 3. The spatial profile of $\Omega_{\rm asym}$ for the flat Friedmann solution with perfect fluid. Each lines correspond to $\gamma = 1$ (the solid line), $\gamma = 3/2$ (the dotted-line) and $\gamma = 2$ (the short-dotted line).

course, it is also possible to derive this form directly from Eqs. (4.1), (4.2), and (4.3) with $v^r = -\tanh\{(3\gamma - 2)\zeta/3\gamma\}$. This solution can be interpreted as an extension of the case $n \geq 4$ of the dust collapse, for which we have $\Omega_{\rm asym} = 4/9$. By virtue of the effect of pressure, we have the density contrast $\delta = \Omega_0/\Omega_\infty = 1/\gamma < 1$ in this homogeneous perfect fluid collapse. As the density contrast or the spatial gradient is not so large, the evolution cannot become "vacuum-dominated" and the resulting singularity is not naked. This example supports our conjecture.

The effect of pressure becomes clearer if we consider the general relativistic Larson-Penston (GRLP) solution [17]. This is the numerically obtained self-similar solution which describes monotonic collapse of an inhomogeneous perfect fluid with the regularity imposed at the sonic point. This self-similar solution may correspond to the case $n = 3$ of the dust collapse provided that γ is treated as a free parameter instead of b. The central singularity of this solution becomes naked for $\gamma < \gamma_c \simeq 1.0105$ (weak pressure), while it is hidden behind a horizon for $\gamma > \gamma_c$ (strong pressure).

To obtain $\Omega_{\rm asym}$ corresponding to the GRLP solution, we numerically solve the ordinary differential equations (4.1),
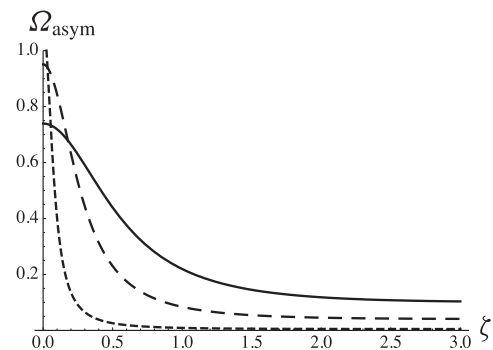

FIG. 4. The profile of the Hubble normalized density $\Omega_{\rm asym}$ corresponding to the GRLP solution. Each lines correspond to $\gamma > \gamma_c$ (the solid line), $\gamma = \gamma_c$ (the long-dotted line), $\gamma < \gamma_c$ (the short-dotted line). As the value of γ decreases, the density contrast $\delta = \Omega_0/\Omega_\infty$ increases. For $\delta > \delta_c \approx 24$, the resulting singularity becomes naked.

(4.2), and (4.3) by requiring the regularity of the solution both at the center $\zeta = 0$ and the sonic point $v^2 = \gamma - 1$. The result is shown in Fig. 4.

It is clear from Fig. 4 that $\Omega_{\rm asym}$ decreases monotonically as ζ increases, just in the same way as the case $n = 3$ of the dust model (see Fig. 2). Further, we note that the density contrast $\delta = \Omega_0/\Omega_\infty$ in this self-similar perfect fluid collapse increases with the decrease of γ. In particular, the critical value of γ for the naked singularity formation is $\gamma = \gamma_c \simeq 1.0105$ and using this value, we can roughly estimate the corresponding critical density contrast to be $\delta_c = \Omega_0/\Omega_\infty \simeq 24 \gg 1$ ($\Omega_\infty \simeq 0.04$). Recall that for the case $n = 3$ of dust collapse, we have the critical value $\delta_c = \Omega_0/\Omega_\infty \simeq 15 \gg 1$. Although the critical value of the density contrast will slightly depend on the property of collapsing matter, the asymptotic analysis of the perfect fluid collapse also supports our conjecture that the very steep change in the asymptotic spatial profile of Ω characterizes the type of the arising singularity.

The asymptotic behavior of the Hubble normalized variable with the separable volume gauge becomes self-similar provided that the asymptotic value of Ω is not zero. This may be interesting in relation to the so-called self-similar hypothesis [24] which asserts that a self-similar behavior should be dominant near the dense central region at the final stage of collapse starting from general initial conditions.

V. SUMMARY AND DISCUSSION

We have studied the asymptotic dynamics of the naked singularity formation in spherically symmetric gravitational collapse of perfect fluid. We first have examined inhomogeneous dust gravitational collapse described by the marginally bound LTB solution. As our main result, we have revealed the different asymptotic behavior of the Hubble normalized density Ω depending on the type of the initial inhomogeneity. By comparing to the known causal structure of singularity arising in the dust collapse, we arrive at the important conjecture that the very steep decrease in the asymptotic spatial profile of Ω is the characteristic of the naked singularity formation. The validity of this conjecture has been also supported in the perfect fluid collapse with pressure.

Let us remark on the "vacuum-dominated" case of the dust collapse (the case $n = 2$), for which Ω goes to zero in the limit $t \rightarrow \infty$ and the arising singularity becomes naked. Because the central value Ω_0 should remain nonzero, the vacuum-dominated evolution leading to the naked singularity formation has been interpreted as the development of the steep spatial gradient in the profile of Ω with the lapse of time. For the dust collapse, as was mentioned in Sec. III, the asymptotic behavior $\Omega \rightarrow 0$ with the naked singularity formation is rather generic. Unfortunately, for perfect fluid collapse with pressure, the dynamical evolution from an initial state to the vacuum-dominated state $\Omega \rightarrow 0$ is not confirmed. One may find that Ω approaches zero only in the outer supersonic region $|v^r| > \sqrt{\gamma - 1}$. The self-similar behavior may be then dominant in the inner subsonic region. To reveal the existence of the vacuum-dominated evolution and the structural change from the inner region to the outer one remains as an important problem to be investigated, especially, in relation to the criterion for the naked singularity formation.

If no symmetry is assumed in gravitational collapse, one can assert that the generic singularity becomes spacelike from the so-called BKL conjecture [25,26] that states the Einstein equation becomes local near the singularity. In terms of the $1 + 3$ orthonormal frame formalism considered here, the BKL conjecture has been reformulated in [16] and applied to various types of collapsing matter (e.g., the perfect fluid with $\gamma = 2$ [27]). Although the BKL conjecture has been supported even in the vacuum-dominated case, the arising singularity is known to be null (see [1,28]) for spherically symmetric dust collapse if $\Omega \rightarrow 0$. In this paper, we have claimed that the naked singularity is originated from the steep gradient in the profile of Ω. For nonspherically symmetric collapse, of course, the freedom of gravitational waves becomes important in the vacuum-dominated case and the long-wavelength modes will be crucial to suppress such a steep density gradient and avoid the naked singularity formation. The BKL conjecture on the causal structure of singularity is based on the assumption that the spatial gradient of dynamical variables becomes negligible at the final stage of collapse. This may not be valid for some models of gravitational collapse including spherically symmetric models.

[1] T. P. Singh and P. S. Joshi, Classical Quantum Gravity **13,** 559 (1996).
[2] S. Jhingan, P. S. Joshi, and T. P. Singh, Classical Quantum Gravity **13**, 3057 (1996).
[3] T. Harada, H. Iguchi, and K. Nakao, Prog. Theor. Phys. **107**, 449 (2002).
[4] T. Harada, Pramana **63**, 741 (2004).
[5] S. S. Deshingkar, arXiv:0710.1866.
[6] R. Giambò, F. Giannoni, and G. Magli, J. Math. Phys. (N.Y.) **49**, 042504 (2008).
[7] T. Harada, Phys. Rev. D **58**, 104015 (1998).
[8] S. Jhingan and G. Magli, Phys. Rev. D **61**, 124006 (2000).
[9] S. M. C. V. Gonçalves, S. Jhingan, and G. Magli, Phys. Rev. D **65**, 064011 (2002).

[10] P. S. Joshi and R. Goswami, Phys. Rev. D **69**, 064027 (2004).

[11] A. Mahajan, R. Goswami, and P. S. Joshi, Classical Quantum Gravity **22**, 271 (2005).

[12] P. S. Joshi, N. Dadhich, and R. Maartens, Phys. Rev. D **65**, 101501 (2002).

[13] P. S. Joshi, R. Goswami, and N. Dadhich, Phys. Rev. D **70**, 087502 (2004).

[14] F. C. Mena, B. C. Nolan, and R. Tavakol, Phys. Rev. D **70**, 084030 (2004).

[15] H. van Elst and C. Uggla, Classical Quantum Gravity **14**, 2673 (1997).

[16] C. Uggla, H. van Elst, J. Wainwright, and G. F. R. Ellis, Phys. Rev. D **68**, 103502 (2003).

[17] A. Ori and T. Piran, Phys. Rev. D **42**, 1068 (1990).

[18] B. J. Carr, Phys. Rev. D **62**, 044022 (2000).

[19] B. J. Carr and C. Gundlach, Phys. Rev. D **67**, 024035 (2003).

[20] R. A. Sussman, Classical Quantum Gravity **25**, 015012 (2008).

[21] F. J. Tipler, Phys. Lett. A **64**, 8 (1977).

[22] B. J. Carr and A. A. Coley, Classical Quantum Gravity **16**, R31 (1999).

[23] T. Foglizzo and R. N. Henriksen, Phys. Rev. D **48**, 4645 (1993).

[24] B. J. Carr and A. A. Coley, Gen. Relativ. Gravit. **37**, 2165 (2005).

[25] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).

[26] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **31**, 639 (1982).

[27] J. Curtis and D. Garfinkle, Phys. Rev. D **72**, 064003 (2005).

[28] D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984).