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# Generalized DBI quintessence

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We investigate the phase space of a quintessence theory governed by a generalized version of the DBI action, using a combination of numeric and analytic methods. The additional degrees of freedom lead to a vastly richer phase-space structure, where the field covers the full equation of state parameter space: $-1 \leq \omega \leq 1$. We find many nontrivial solution curves to the equations of motion which indicate that DBI quintessence is an interesting candidate for a viable $k$-essence model.

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## I. INTRODUCTION

The dark energy problem continues to be a sticking point for theoretical physicists. The simplest solution to this problem is to postulate the existence of a vacuum energy or cosmological constant which agrees with all the current observational bounds [1–4]. However, we are then left with a secondary problem, namely, explaining why the vacuum energy is tuned to such a small value without some obvious symmetry to protect it. For many years we have hoped that UV complete theories of gravity would shed light on this issue, which is in effect an extremely embarrassing IR problem from this perspective. However, despite much effort, neither string theory nor loop quantum gravity has shed any compelling light on this issue—although there have been many interesting proposals.

An alternative approach is to assume that the cosmological constant is exactly zero, since supersymmetry can then be invoked as the regulating symmetry in this case. However, one then has to account for the fact that low energy supersymmetry must be broken and an alternative explanation for the current expansion, and for the vanishing of the cosmological constant, must be sought. One way to deal with the latter problem is to assume that the dark energy phase is driven by a dynamical field, implying that the equation of state is an explicit function of time [5,6]. Currently this cannot be ruled out by our best observations and therefore remains a possible solution to the dark energy problem. However, one cannot just consider *ad hoc* scalar fields coupling to gravity, since the low energy theory will still be sensitive to high energy physics. In particular, we must ensure that any additional scalars are neutral under all the standard model symmetries, and that they do not introduce additional fifth forces. Therefore, one must search for viable models of dynamical dark energy within UV sensitive theories.

Phenomenological models of our Universe have proven difficult to construct within string theory, due to technical difficulties arising from moduli stabilization, whereby we assume that the extra dimensions of the theory are compactified on manifolds with $SU(3) \times SU(3)$ structure (in the type IIB case) [7], and orientifolded to preserve the minimal amount of supersymmetry in four dimensions. Most of the work in this area assumes that the compact space is a Calabi-Yau threefold, which is a special limit of the $SU(3)$ structure manifold class.

As a result, embedding realistic cosmology into string theory has proven difficult. One area which has been well explored in recent years is inflation driven by the open string sector through dynamical $Dp$-branes. This is the so-called Dirac-Born-Infeld action (DBI) inflation [8,9]—which lies in a special class of $K$-inflation models. It was originally thought that such models yielded large levels of non-Gaussian perturbations which could be used as a falsifiable signature of string theory [10]. However, subsequent work has shown that this may not be the case, and that the simplest DBI models are essentially indistinguishable from standard field theoretic slow roll models [11–13].[1] The problem is that the WMAP 5 yr data set [2] imposes very tight constraints on the allowed tuning of the free parameters in the theory. We are then left with the choice of either having large non-Gaussianities but with vanishing tensors, or assume that the tensor spectrum will be visible—in which case there is no non-Gaussian signature. The models are only falsifiable once these conditions are relaxed. One can get around these conditions by considering more complicated models such as multiple fields [15,16], multiple branes [17–19], wrapped branes [20], or

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[1]Note, however, that the models proposed in [13,14] evade such problems.

 

monodromies [21]—but even here there are still problems with fine-tuning, backreaction, and the apparent breakdown of perturbation theory in the inflationary regime [22].

In models of dynamical dark energy, on the other hand [5,6,23], the WMAP constraints can be relaxed and therefore DBI models may still have some use as an explanation for a dynamical equation of state. Moreover, this fits in nicely with several intuitive ideas from string theory. Namely, that inflation can still occur, albeit only through the closed string sector—where one (or more) of the geometric pseudomoduli are actually responsible for the initial inflationary epoch (see [24] for the phenomenologically most viable proposals). After inflation the universe lives on branes that wrap various cycles within the compact space and are extended along the large Minkowski directions. In this sense we see that a grand unified theory or electroweak (EW) phase transition can manifest through a geometric fashion—namely, the Higgsing of branes in the compact bulk space. This suggests that dark energy may well be a dynamical process, and moreover, in the light of these open string constructions, it retains a sense of being geometric in nature.

With this in mind, various authors have begun to explore the phase space of DBI-driven dark energy [25,26]. The initial works have dealt with the dynamics of a solitary $D3$-brane moving through a particular warped compactification of type IIB. In this paper we wish to generalize this further to a more phenomenological class of models that include multiple and partially wrapped branes. We believe that this may be a more generic situation to consider, since typically one should expect branes of varying degrees to be wrapped on nontrivial cycles of the compact space. Our work is a first step into considerations of a more general setup for quintessence in IIB (open) string theory, and we hope it will be a valuable starting point for further endeavours.

## II. DYNAMICS OF THE EFFECTIVE THEORY

To begin let us assume that the universe at such late times can be adequately described by a flat Friedmann-Robertson-Walker (FRW) metric and that the matter sector consists of a dynamical scalar field and a perfect fluid, which are both separately conserved. The usual cosmological equations of motion are therefore independent of any particular model and can be written as

$$H^2 = \frac{(\rho + \rho_\phi)}{3M_p^2}, \qquad \dot{\rho}_i = -3H(P_i + \rho_i), \tag{2.1}$$

where $i$ runs over the contributing components. The equation of state is given by $\omega_i = P_i/\rho_i$; however, if $\omega$ of the fluid component is assumed to be constant, then we can integrate the appropriate conservation equation exactly to obtain

$$\rho \propto a^{-3(1+\omega)}, \tag{2.2}$$

where the scale factor varies as a function of time such that $a(t) \sim t^{2/(3[1+\omega])}$.

The model dependence arises in the parametrization of the scalar field sector. In our case we are assuming that the dark energy is driven by open string modes, which at low energies are described by fluctuations of a $Dp$-brane whose dynamics are governed by the DBI—which is a generalization of nonlinear electrodynamics [8,9]. Typically one assumes that the standard model is localized on an intersecting brane stack, in one of the many warped throats that are attached to the internal space. For consistency reasons, in the simplest cases, these are taken to be either $D3$- or $D7$-branes. In this paper we will consider a bottom-up approach; therefore, we shall not worry too much about the geometric deformations of the compact space, nor about any constraints imposed by orientifolded $Op$-planes—aside from those that ensure that all tadpoles are consistently canceled so that we can trust the low energy supergravity theory.

The action we consider is a generalized form of the DBI one coupled to Einstein-Hilbert gravity, which can be embedded into this background and takes the following generalized form[2]:

$$S = -\int d^4x a^3(t)\bigg(T(\phi)W(\phi)\sqrt{1 - \frac{\dot{\phi}^2}{T(\phi)}} - T(\phi) + \tilde{V}(\phi)\bigg) + S_M, \tag{2.3}$$

where $T(\phi)$ is the warped tension of the brane and $S_M$ is the action for matter localized in the standard model (SM) sector. Thus our assumption here is that our dynamical open string sector is coupled only gravitationally to the SM sector, and so we do not have to worry about additional forces or particle production. There are two potential terms for the scalar field, which are denoted by $W(\phi)$ and $\tilde{V}(\phi)$. The first of these terms can arise in different places within the theory. First, if the brane is actually a non-Bogomol'nyi-Prasad-Sommerfield (BPS) one [28], then the scalar field mode is actually tachyonic and the potential is therefore of the usual runaway form. If there are $N$ multiple coincident branes, then the world-volume field theory is a $U(N)$ non-Abelian gauge theory and the potential term is simply a reflection of the additional degrees of freedom [29]. Through the dielectric effect, one can also see that this configuration is related to a $D5$-brane wrapping a two-cycle within the compact space and carrying a nonzero magnetic flux along this cycle. Both of these configurations lead to an additional potential multiplying the usual DBI kinetic term.

[2]We refer the more interested readers to [19] for more details on the precise structure and origin of this action. The important thing to note is that $\phi$ is a matrix valued field. For recent work in a related direction see [27].

The origin of the $\tilde{V}(\phi)$ term is less explicit—but is a sum of terms. One expects open or closed string interactions to generate a scalar potential $V(\phi)$; however, the precise form of such an interaction depends upon many factors such as the number of additional branes and geometric moduli, the number of nontrivial cycles in the compact space, and the choice of embedding for branes on these cycles. Typically one can only compute this in special cases in the full string theory. There are also additional terms coming from coupling of the brane to any background Ramond-Ramond form fields. The action above is assumed to be that of a $D3$-brane(s) filling the space-time directions, which naturally couples to the field $C^{(4)}$ through the Chern-Simons part of the action. However, for wrapped $D5$-branes there is also the possibility of a coupling $C^{(4)} \wedge F$, where $F$ is the magnetic field through the two-cycle. For example, in the warped deformed conifold, one can see that $dC^{(6)} = \star dC^{(2)}$, and therefore there is an additional term in the DBI action,

$$S \sim \int d^4 x a^3(t) g_s^{-1} M \alpha' T(\phi), \tag{2.4}$$

up to a normalization factor of order 1. Terms such as this have been added to the interaction potential to define the *full* scalar potential $\tilde{V}(\phi)$. Recent extensions to standard DBI inflation have included the contribution from higher dimensional bulk forms, with the remarkable result that they cancel one another up to third order in the action and therefore do not affect the leading order perturbations [30]. Extending this work to higher orders is therefore extremely interesting.

The corresponding equations for the energy density and pressure of the DBI can then be written succinctly as

$$\begin{aligned} P_\phi &= \frac{T(\phi)}{\gamma}[\gamma - W(\phi)] - \tilde{V}(\phi), \\ \rho_\phi &= T(\phi)[W(\phi)\gamma - 1] + \tilde{V}(\phi), \end{aligned} \tag{2.5}$$

where $\gamma = [1 - \dot{\phi}^2/T(\phi)]^{-1/2}$ is the usual generalization of the relativistic factor. The subscript $\phi$ denotes the scalar field component here. We can also immediately define the equation of state parameter for the quintessence field to be

$$\omega_\phi = \frac{T(\phi)[\gamma - W(\phi)] - \tilde{V}(\phi)\gamma}{T(\phi)\gamma[W(\phi)\gamma - 1] + \tilde{V}(\phi)\gamma}, \tag{2.6}$$

from which one clearly sees that it is dynamically sensitive and can take a wide range of values. For instance, we only recover $\omega_\phi \sim -1$ in the limit that the field is nonrelativistic and the entire solution is dominated by the $\tilde{V}(\phi)$ terms—which will clearly require large amounts of fine-tuning to accomplish. There are clearly several regions of parameter space that are of interest. First let us assume that the potential term is zero, either because it is suppressed or there is an unlikely cancellation between the contributing terms. The more general case with nonzero $\tilde{V}$ leads to a wide variety of complex behavior. We can therefore identify several limits of interest—focusing on the behavior of $W$:

(i) $W(\phi) = 1$—which reduces the action back to the usual DBI case which has $\omega_\phi = 1/\gamma$ as discussed in [25].

(ii) $W(\phi) = \alpha\gamma$—which leads to constant $\gamma$ if $\alpha$ is constant, since the two are related via $\gamma^2 \omega_\phi \alpha = 1 - \alpha + \omega_\phi$. Moreover, this again means that $\dot{\phi} \propto t^{-(1+\omega_\phi)/(1+\omega)}$, as in the case where $W = 1$.

(iii) $W(\phi) \rightarrow 0$—as could occur in the case of a tachyonic theory, which mimics a dark energy dominated phase with $\omega_\phi = -1$. However, one must be careful if this is to be representative of non-BPS $D$-brane actions, since the coupling to the form field is noncanonical in this instance. In fact, the coupling term will typically be of the form $d\phi \wedge C$. This means that there is no solitary $T(\phi)$ term in the action, and therefore the equation of state in this instance will vary like $-1/\gamma^2$.

(iv) $W(\phi) \gg \gamma$—which can occur in the multibrane/wrapped brane case and yields $\omega_\phi \sim -1/\gamma^2$.

Note that in all cases the equation of state parameter remains bounded in the range $-1 \leq \omega_\phi \leq 1$.

One can combine the expressions for the energy-momentum tensor components, and together with the continuity equation, we obtain the following equation of motion (assuming that the scalar field follows a monotonic path):

$$\ddot{\phi} + \frac{3H\dot{\phi}}{\gamma^2} + \frac{3T_\phi}{2\gamma^2} + \frac{1}{W\gamma^3}(\tilde{V}_\phi - T_\phi) - \frac{T_\phi}{2} + \frac{TW_\phi}{W\gamma^2} = 0, \tag{2.7}$$

which is a generalization of the Klein-Gordon equation for the DBI Lagrangian. The subscript $\phi$ of $T$, $W$, and $\tilde{V}$ denotes a derivative with respect to the field value. The other dynamical equation of motion for the Hubble parameter can be written as

$$\dot{H} = -\frac{1}{2M_p^2}[\rho(1+\omega) + \gamma W(\phi)\dot{\phi}^2], \tag{2.8}$$

where we have defined the pressure of the noninteracting barotropic fluid to be$P = \omega\rho$. We leave the interesting case of interacting pressure for future endeavours.

Let us consider, as an example solution, the case where there is a scaling solution with $W = 1$, which has been reviewed elsewhere [26]. We will find it convenient to define the quantity

$$X = \frac{1 + \omega_\phi}{1 + \omega}, \tag{2.9}$$

in which case we see that $\dot{\phi} \sim t^{-X}$. This allows us to reconstruct the tension of the brane as follows:

$$T(\phi) = \mathcal{M}^4 e^{-\lambda\phi}, \qquad X = 1,$$
$$T(\phi) = \mathcal{M}^{4+\alpha}\phi^{-\alpha}, \qquad X \neq 1, \tag{2.10}$$

where $\mathcal{M}$ is a dimensionful mass scale, $\lambda$ is a constant, and $\alpha = 2X/(1-X)$. Using the fact that $\omega_\phi = 1/\gamma$, we can then see that for $X \neq 1$ the solution is physically valid only when $\omega > 2/\alpha$ since we define $\gamma$ to be the positive root. Let us now consider the phase-space dynamics of the theory in more detail, following along the lines of [5]. It is initially convenient to define the following new variables:

$$x = \sqrt{\frac{T(\phi)W(\phi)\gamma}{3}}\frac{1}{HM_p}, \qquad \mu_1 = \frac{\sqrt{T}M_p\tilde{V}_\phi}{\tilde{V}^{3/2}},$$
$$y = \sqrt{W(\phi)\gamma}\frac{\dot{\phi}}{HM_p}, \qquad \mu_2 = -\frac{\sqrt{T}M_pT_\phi}{\tilde{V}^{3/2}}, \tag{2.11}$$
$$z = \sqrt{\frac{\tilde{V}}{3}}\frac{1}{HM_p}, \qquad \mu_3 = \frac{W_\phi M_p}{W^{3/2}\gamma^{5/2}},$$

in terms of which we can see that $\gamma = [1 - y^2/(3x^2)]^{-1/2}$, and the fluid density parameter can be written as

$$\Omega = 1 - \Omega_\phi = 1 - \left(z^2 + x^2\left[1 - \frac{1}{W(\phi)\gamma}\right]\right), \tag{2.12}$$

while the equation of state in dimensionless variables will become

$$\omega_\phi = \frac{1}{\gamma}\left(\frac{x^2[\gamma - W(\phi)] - z^2W(\phi)\gamma^2}{x^2[W(\phi)\gamma - 1] + z^2W(\phi)\gamma}\right). \tag{2.13}$$

As is customary we will now switch to dimensionless derivatives, denoted by a prime, replacing time derivatives by derivatives with respect to the e-folding number $\mathcal{N}$. Therefore we can easily determine

$$\frac{H'}{H} = -\frac{y^2}{2} - \frac{3(1+\omega)}{2}\left(1 - z^2 - x^2\left[1 - \frac{1}{W(\phi)\gamma}\right]\right). \tag{2.14}$$

A useful quantity to calculate is the variation of the kinetic function, which we can write in the following manner using the equation of motion,

$$\frac{\dot{\gamma}}{\gamma} = -\frac{3H\dot{\phi}^2}{T} - \frac{W_\phi\dot{\phi}}{W} - \frac{T_\phi\dot{\phi}}{T} - \frac{\dot{\phi}}{\gamma WT}(\tilde{V}_\phi - T_\phi). \tag{2.15}$$

We can then determine the dynamical equations for the dimensionless fields as derivatives with respect to $\mathcal{N}$,

$$x' = -\frac{1}{2}(\mu_1 + \mu_2)\frac{yz^3}{x^2} - \frac{y^2}{2x} - x\frac{H'}{H},$$
$$y' = -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3}{xy}[\mu_1 + \mu_2]\right) + \frac{3\mu_2z^3W}{\gamma x} - 3x^2\mu_3 - y\frac{H'}{H}, \tag{2.16}$$
$$z' = \frac{z^2y\mu_1}{2x} - z\frac{H'}{H},$$

and the remaining parametric solutions are

$$\mu_1' = \frac{\mu_1^2yz}{x}\left(-\frac{3}{2} + \frac{\tilde{V}_{\phi\phi}\tilde{V}}{\tilde{V}_\phi^2} + \frac{T_\phi\tilde{V}}{\tilde{V}\tilde{V}_\phi}\right),$$
$$\mu_2' = \frac{\mu_1\mu_2yz}{x}\left(-\frac{3}{2} + \frac{T_\phi\tilde{V}}{2T\tilde{V}_\phi} + \frac{T_{\phi\phi}\tilde{V}}{T_\phi\tilde{V}_\phi}\right),$$
$$\mu_3' = y\mu_3^2\gamma^{3/2}\left(1 + \frac{W_{\phi\phi}W}{W_\phi^2} + \frac{5T_\phi W}{2TW_\phi} + \frac{5}{2T\gamma W_\phi}[\tilde{V}_\phi - T_\phi]\right) + \frac{5\mu_3y^2}{2x^2}. \tag{2.17}$$

Note that if the $\mu_i$ are constants, then the previous three equations form an autonomous set and should uniquely specify the dynamics of the quintessence field. We will consider this case as the simplest (canonical) example. If we wish to appeal to string theoretic constructions, then we restrict the parameter space of solutions. It is more interesting to consider the above equations in the context of a phenomenological model and see what kind of functions yield the correct behavior. Explicit constructions of string backgrounds are typically difficult, and there are only a few well-known examples that are ritually invoked; however, if we take string theory seriously, then there are undoubtedly other nontrivial backgrounds that are cosmologically interesting but not yet constructed. Since an analytic analysis of this generalized system is highly complicated, it is convenient to use a combination of analytic and numerical methods to understand the dynamics of the system. For a numeric analysis it is necessary to rewrite the fluid equation in terms of more useful variables. It turns out that the simplest variables to use are the following:

$$\phi' = \Phi, \tag{2.18}$$

$$\Phi' = -\frac{3\Phi}{\gamma^2} + \frac{3M_pz^3}{x}\left(\frac{\sqrt{W\gamma}\mu_2}{2}\left[\frac{3}{\gamma^2} - 1\right] - \frac{(\mu_1 + \mu_2)}{\sqrt{W}\gamma^{5/2}}\right) - \frac{3M_px^2\mu_3}{\sqrt{W\gamma}} - \Phi\frac{H'}{H}, \tag{2.19}$$

which are easily derivable from the terms written above. Equations (2.14), (2.16), (2.17), (2.18), and (2.19), together with the barotropic fluid equation $\rho' = -3\rho(\mathcal{N})(1 + w)$, hence form a closed ten-dimensional autonomous system if

$T$, $W$, or $\tilde{V}$ is given as an explicit function of $\phi$ or as a constant.

### A. Case I

Let us take the canonical string theoretic example arising when the local geometry can be approximated by an anti-de Sitter space. This geometry typically arises in the near horizon limit of coincident $D3$-branes (or flux). In this case we see that (at leading order)

$$T(\phi) = \frac{\phi^4}{\lambda^4}, \qquad \tilde{V}(\phi) = \frac{m^2\phi^2}{2}, \qquad W(\phi) = W, \tag{2.20}$$

where we have also included an effective $\phi^2$ potential for the system. This means that $\mu_3 = 0$, and we also have a constant $\mu_1$ which allows us to write the remaining $\mu$ terms as

$$\mu_1 = \frac{2\sqrt{2}M_p}{m\lambda^2}, \qquad \mu_2 = -\frac{2x^2\mu_1}{W\gamma z^2}, \tag{2.21}$$

and therefore the dynamical equations reduce to

$$\begin{aligned} x' &= -\frac{\mu_1 y z^3}{2x^2}\left(1 - \frac{2x^2}{W\gamma z^2}\right) - \frac{y^2}{2x} - \frac{xH'}{H}, \\ y' &= -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3\mu_1}{xy}\left[1 - \frac{2x^2}{W\gamma z^2}\right]\right) \\ &\quad - \frac{6\mu_1 z x}{\gamma^2} - \frac{yH'}{H}, \\ z' &= \frac{z^2 y \mu_1}{2x} - \frac{zH'}{H}. \end{aligned} \tag{2.22}$$

The simplest way to proceed with the analysis is to consider the final equation above, since this splits the solution space neatly into two components. Thus we search for solutions where either $z = 0$ or $z = (2x/y\mu_1)H'/H$ as initial conditions.

The first subset of solutions admits (0, 0, 0) as a (trivial) fixed point, which is a fluid dominated solution since $\Omega = 1$ in this instance. Let us remark here that this fixed point solution will occur for *all* the cases we consider; however, since this implies a vanishing of the action, causality implies that this fixed point must be unstable—i.e. phase-space trajectories will flow away from it. By making this field a phantom scalar, one can evade this causal bound and the point can become a stable fixed point. This behavior arises in many places in the literature, so we will not discuss it further here.

There is also a critical point at $(1, \sqrt{3}, 0)$ which is a kinetic dominated solution. This solution actually exists as solutions to the quadratic expression $y^2 = 3x^2$ which corresponds to the limit $\gamma \to \infty$. In terms of the density parameter, a quick calculation shows that along the general curve (parametrized by $y_0$ and $x_0$), we find $\Omega = 1 - x_0^2$. Thus at the trivial fixed point we see $\Omega \to 1$; however, for $x_0 \to 1$ we see that $\Omega \to 0$, corresponding to nonrelativistic matter, i.e. dust. In this instance we also find $a(t) \sim t^{2/3}$ as expected from the cosmological evolution equations. Again due to the special algebraic properties of the DBI action, we anticipate that this solution will also be found for the other cases of interest.

The second subset of solutions is more interesting, as initially one can solve the system by slicing the phase space at $y = 0$.[3] One can use the condition on $H'$ to fix $z$ through $z^2 = 1 - x^2(W - 1)/W$. Combining this with the equations of motion gives us the following fixed point (taking positive signs of all roots for simplicity):

$$x = \sqrt{\frac{W}{1 - W}}, \qquad y = 0, \qquad z = 1, \tag{2.23}$$

which is valid for all $W < 1$ in order for these points to be real, and at finite distance in phase space. If we then compute the density of the fluid, we find $\Omega = 0$ since $\Omega_\phi = 1$, which corresponds to a purely dustlike solution. Note that this class of solutions does not exist for the simple $D3$-brane analysis as in [26], since it arises from additional degrees of freedom which are neglected in these models. The remaining solutions in this subset are difficult to find analytically.

More generally, we can see that the above solution is a special case of the more general case I behavior, which we parametrize by

$$T(\phi) = \frac{\phi^\alpha}{\lambda^\alpha}, \qquad \tilde{V}(\phi) = \frac{m^\beta\phi^\beta}{\beta}, \qquad W(\phi) = W, \tag{2.24}$$

where we can then explicitly write

$$\mu_1 = A\left(\frac{x}{z\gamma^{1/2}}\right)^{(\alpha-\beta-2)/(\alpha-\beta)}, \qquad \mu_2 = -\frac{\alpha}{\beta}\frac{\mu_1}{W\gamma}\frac{x^2}{z^2}, \qquad \mu_3 = 0, \tag{2.25}$$

where $A$ is a (real, positive) constant provided that $\beta > 0$,

$$A = \frac{M_p\beta^{3/2}}{\lambda^{\alpha/2}m^{\beta/2}}\left(\frac{\lambda^\alpha m^\beta}{\beta W}\right)^{(\alpha-\beta-2)/2(\alpha-\beta)}, \tag{2.26}$$

but which simplifies in the limit $\alpha = \beta + 2$. As before, the solution space splits into two disconnected subsets; therefore, in the first instance where we take slices through $z = 0$, we find the following bound:

$$\frac{2}{(\alpha - \beta)} > 0, \tag{2.27}$$

which implies that $\alpha > \beta$ and so the brane tension should dominate the dynamics (in the large field regime). Let us therefore assume that $\alpha$, $\beta$ are chosen such that this

[3] Note that one cannot do this for $x = 0$ since the action becomes singular and ill defined.

condition is satisfied—then we find the solution branch is governed again by the relation $y^2 = 3x^2$ as expected—which contains the solution (0, 0, 0) as a special case. Moreover, this is valid for all values of $\alpha$, $\beta$ satisfying the above constraint. The secondary solution branch occurs when we find solutions to

$$\frac{zy\mu_1}{2x} = \frac{H'}{H}, \tag{2.28}$$

which is generally very complicated. A simple set of solutions does arise when we consider slices at $y = 0$, since the fixed points are localized along the curve

$$x = \pm\sqrt{\frac{\beta W}{(\alpha - \beta)(1 - W)}}, \qquad y = 0, \qquad z = \pm\sqrt{\frac{\alpha}{(\alpha - \beta)}}, \tag{2.29}$$

which corresponds to a dustlike solution: $\Omega = 0 \ \forall \ \alpha, \beta$. The reality constraint here demands that $\alpha > \beta$, which in turn fixes $W < 1$. However there are also additional solutions where $\beta < 0$ and $\alpha$ is positive—provided that $W > 1$. Explicit realizations of this scenario within a string theory context can arise through potentials arising from brane/antibrane interactions and it is therefore a nontrivial and interesting solution.

Figures 1 and 2 show the numerical solutions in phase space. For the $W = 1$ case, the numerical constants are given as $M_p = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case). Other parameters are $\alpha = 4$, $\beta = 2$, and $A = 2\sqrt{2}$. As expected, the (five) fixed points all lie along the curve $y^2 = 3x^2$. We also plot the evolution of each parameter $(x, y, z)$ as a function of the e-folding number in Fig. 3, where each of the coordinates tends to its critical value. As expected the phase-space dynamics are $\mathbb{Z}_2$ symmetric about the origin. Note that in the case of $y(N)$ one can keep $y$ suppressed for a few e-foldings with enough tuning, before it eventually evolves towards the points $\pm\sqrt{3}$ at late times. The full numerical solution of the case $W < 1$ is illustrated in Fig. 4 where $W = 0.95$, which uniquely fixes the critical points to be $x = \pm\sqrt{20}$, $y = 0$, $z = 1$. As one can see from the resulting plot, this is an unstable node because the general behavior is divergent. Note that $x \longrightarrow \infty$ in this regime effectively solves all the dynamical equations trivially.

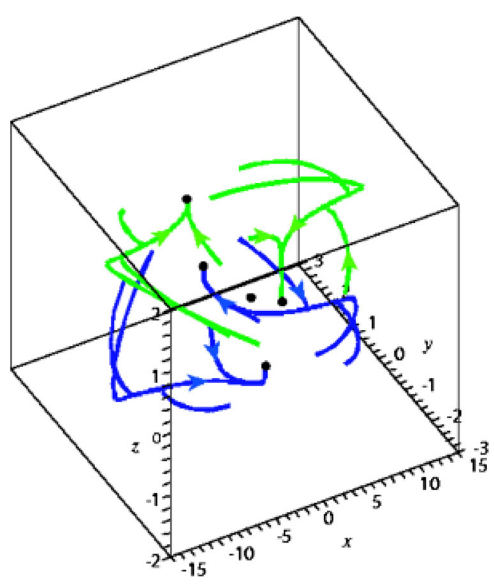


FIG. 1 (color online). Case I: 3D $xyz$ phase-space trajectories for $T(\phi) = \phi^4/\lambda^4$, $\tilde{V}(\phi) = m^2\phi^2/2$, and $W(\phi) = W$. Here we have set $M_p = 1$, $W = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case).

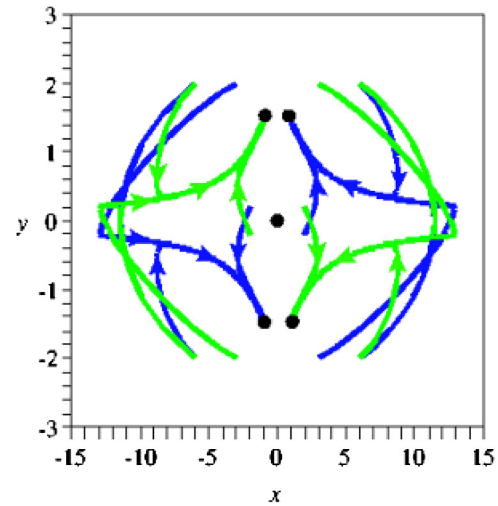


FIG. 2 (color online). Case I: Phase-space trajectories in the $xy$ plane. Four attractors $(\pm 1, \pm\sqrt{3}, 0)$ and one unstable node (0, 0, 0) can be seen here. $z$ is bounded within the $(-1, 1)$ range.

### B. Case II

Analogous to the first case, let us now consider another branch of solutions where this time the tension of the brane is taken to be constant. This dramatically alters the relativistic rolling of the scalar field since the $\gamma$ factor is no longer warped. Initially, let us consider the ansatz

$$\tilde{V}(\phi) = \frac{m^2\phi^2}{2}, \qquad T(\phi) = T, \qquad W(\phi) = \frac{\phi^4}{\lambda^4}, \tag{2.30}$$

which implies that

$$\mu_1 = \left(\frac{4\sqrt{2T^3}M_p}{\lambda^4 m^3}\right)\frac{z^2\gamma}{x^2}, \qquad \mu_2 = 0, \qquad \mu_3 = \frac{2z\mu_1}{\gamma^2 x}, \tag{2.31}$$

and the corresponding field equations become

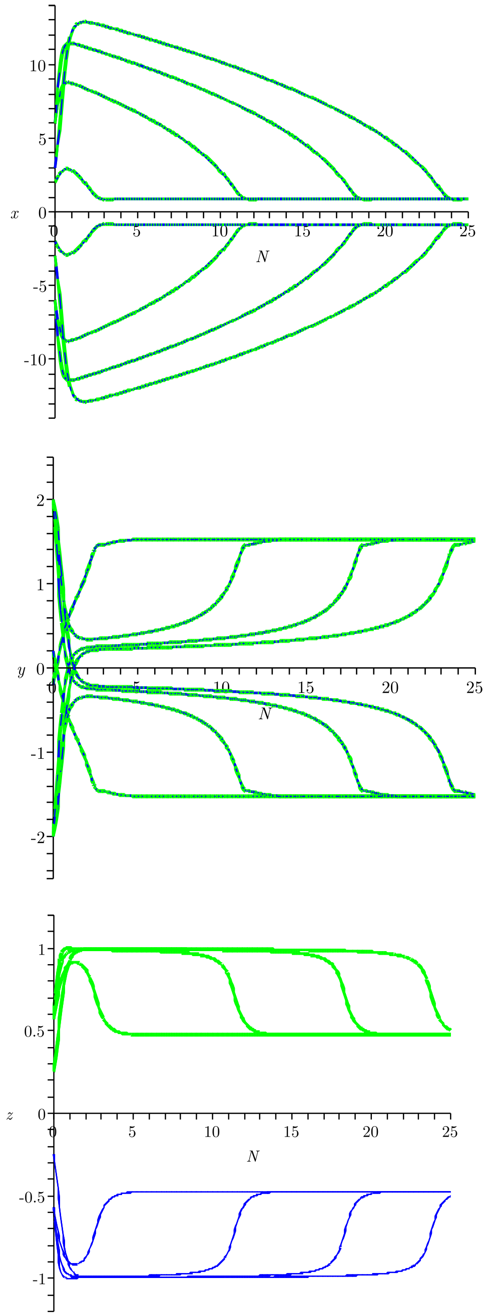


FIG. 3 (color online). Case I: Evolution of $x$, $y$, $z$ versus the e-folding number, setting $W = 1$.

$$x' = -\frac{\alpha\gamma y z^5}{x^4} - \frac{y^2}{2x} - \frac{xH'}{H},$$
$$y' = -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{\alpha\gamma z^5}{2x^3 y}\right) - \frac{6\alpha z^3}{\gamma x} - \frac{yH'}{H}, \quad (2.32)$$
$$z' = \frac{\alpha\gamma z^4}{2x^3} - \frac{zH'}{H},$$

where we have defined $\alpha$ as the constant prefactor in the definition of $\mu_1$.

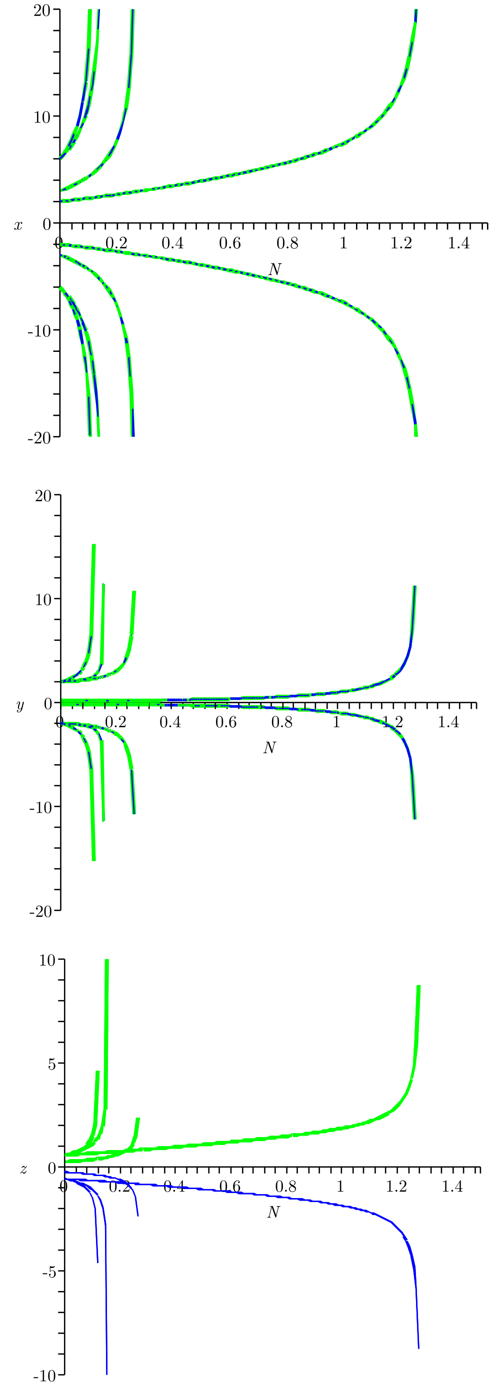


FIG. 4 (color online). Case I: Evolution of $x$, $y$, $z$ versus the e-folding number, setting $W = 0.95$. All solutions diverge from the origin.

As before we separate the solution space into two—first finding solutions to $z = 0$ and then solutions to $H'/H = \alpha\gamma z^3/(2x^3)$. In the first case it is straightforward to see that there are the usual fixed point solutions at (0, 0, 0) and $(1, \sqrt{3}, 0)$ (with their respective partner solutions) respectively coming from the usual condition that $y^2 = 3x^2$. The secondary branch of solutions also admits fixed points when $y = 0$; however, the condition on $z$ is that $z = 0, -4x^2$. Since we want real solutions we are forced to set $z = 0$ as a secondary constraint. This forces $W$ to diverge, and therefore in the limit that $z \to 0$, we find that $x^2 \to \pm 1$ which is a unique solution. Again the density parameter vanishes identically in this limit as one would expect. The remaining solutions are actually extremely difficult to solve analytically as they correspond to high order polynomials. As a result we are forced to sketch their behavior numerically.

Phenomenologically, we see that the ansatz presented above is a special class of the more general solution

$$T(\phi) = T, \qquad \tilde{V}(\phi) = \frac{m^\beta \phi^\beta}{\beta}, \qquad W(\phi) = \frac{\phi^\alpha}{\lambda^\alpha}, \tag{2.33}$$

which has the parametrization constraints

$$\mu_1 = A\left(\frac{z\gamma^{1/2}}{x}\right)^{(2+\beta)/(\alpha-\beta)}, \qquad \mu_2 = 0, \qquad \mu_3 = B\gamma^{(2-4\alpha+5\beta)/2(\alpha-\beta)}\left(\frac{z}{x}\right)^{(2+\alpha)/(\alpha-\beta)}, \tag{2.34}$$

where $A$, $B$ are both constants. One can see from the dynamical equations that fixed points with $z = 0$ can only occur when the following condition is met:

$$\frac{2(1+\alpha)-\beta}{\alpha-\beta} > 0, \tag{2.35}$$

which is trivially satisfied for cases where $\alpha > \beta$ (which we assume as an initial constraint).

More generically we see that, provided $\alpha > -2$, we recover the usual fixed point equation $y^2 = 3x^2$. However, we need to be careful here because if this condition is satisfied, then $W$ becomes undefined. Since this is the overall prefactor multiplying the DBI action, the action is undefined in this limit and it should therefore correspond to a point of instability in the phase space. In the limit where $\alpha = -2$, which implies that $\beta > -2$, the fixed point solution now lives on the zeros of the polynomial

$$3x^4 B\gamma^{-5(2+\beta)/2(1+\beta)} + 3x^2 y - y^3 = 0, \tag{2.36}$$

which can be used to fix $x = x(y)$ or vice versa. This solution is actually indicative of a more general branch of physical solutions where we take $\beta > 2(1+\alpha)$. The resulting fixed point equation (provided $\alpha \neq 2$) is trivially calculated to be $y^2 = 3x^2$ as before, but now we see that $W$ vanishes identically. In turn this means that the kinetic terms also vanish and the solution is dominated solely by the potential interaction. One could imagine a situation such as this occurring in the condensation of an open string tachyon mode on a non-BPS brane, where the vanishing of $W$ indicates that we are living in the closed string vacuum. For dynamic solutions it seems reasonable to consider this particular case as the late-time attractor for the solution $z \to 0$.

The second subset of solutions is again complicated, but again we can analytically understand the plane at $y = 0$, which gives us the fixed point solutions

$$x = \pm\sqrt{-\frac{\beta z^2}{\alpha}}, \qquad y = 0, \qquad z = \pm\left(1 - \frac{\beta}{\alpha}\left(1 - \left[-\frac{\alpha T}{\lambda^\alpha m^\beta}\right]^{\alpha/(\alpha-\beta)}\right)\right)^{-1/2}. \tag{2.37}$$

Clearly, for the solution to be real we require that $\alpha$, $\beta$ have opposite signs. This satisfies our primary constraint, and therefore is a physical possibility. Moreover, in the limit where we set $\beta = -\alpha$, we find that $\Omega = 0$, which is again the dust solution. Illustrations of numerical solutions for case II are shown in Figs. 5–7. Constants are set as $M_p = 1$, $T = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case). Other parameters are $\alpha = 4$, $\beta = 2$. From the numerical analysis one sees that there are six saddle nodes, only two attractors, and one repulsive point which is the origin (0, 0, 0) as expected. The dynamical trajectories are particularly interesting due to their apparent lack of monotonicity as a function of e-fold number. The $z$ term, in particular, appears to have a large variation in trajectory, diverging in some instances while rapidly reaching zero in other instances. Conversely, the $y$ variable displays very uniform (physical) trajectory behavior, with several curves almost on top of one another at $y = 0$ and the remainder smoothly driven to the (unstable) critical point $y_c \sim 1.8$ in the example given.

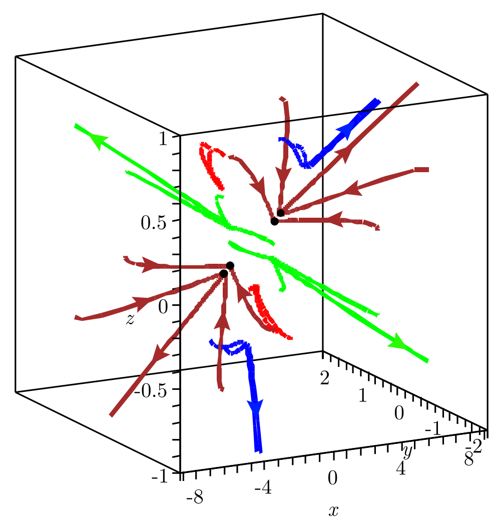


FIG. 5 (color online). Case II: 3D $xyz$ phase-space trajectories for $T(\phi) = T$, $\tilde{V}(\phi) = m^2\phi^2/2$, and $W(\phi) = \phi^4/\lambda^4$. Here we have set $M_p = 1$, $T = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case).

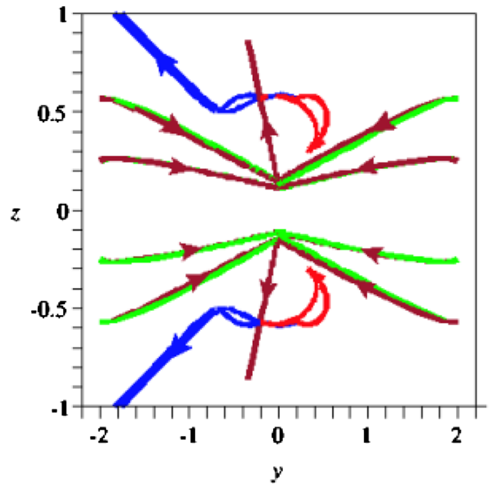


FIG. 6 (color online). Case II: Trajectory slice through the $yz$ plane.

### C. Case III

Let us now consider a new case where only $W = W(\phi)$, with all the other terms being constant. We will take $W = \phi^\alpha/\lambda^\alpha$ for generality—which in turn should impose a constraint on the allowed values of $\alpha$. In this case we see that

$$\mu_1 = 0, \qquad \mu_2 = 0, \qquad \mu_3 = \alpha A\left(\frac{z}{x}\right)^{(\alpha+2)/\alpha} \frac{1}{\gamma^{(2\alpha-1)/\alpha}}, \tag{2.38}$$

where $A$ is a function of the constant parameters $A = M_p/\lambda(T/\tilde{V})^{(\alpha+2)/2\alpha}$. Because only $\mu_3$ is nonzero, the resulting dynamical expressions are considerably easy to work with,

$$\begin{aligned} x' &= -\frac{y^2}{2x} - \frac{xH'}{H}, \\ y' &= -3y\left(1 - \frac{y^2}{6x^2}\right) - 3\alpha A z^{(\alpha+2)/\alpha} x^{(\alpha-2)/\alpha} - \frac{yH'}{H}, \\ z' &= -\frac{zH'}{H}. \end{aligned} \tag{2.39}$$

Considering the slice again through $z = 0$, we see that the solutions split into two types depending upon the integer $\alpha$. We recover the usual $y^2 = 3x^2$ curve only when $\alpha > 0$ or when $\alpha < -2$. If $\alpha = -2$ then the corresponding polynomial equation becomes

$$y\gamma^{9/2} = 2Ax^2 \tag{2.40}$$

which is difficult to solve analytically due to the dependence of $\gamma$ on both $x$, $y$. This expression does not admit anything but the trivial solution if we set $y$ to zero.[4] Again we see that there is a potential problem here since the potential $W$ goes like $1/z^2$, and is therefore divergent in this limit. Solutions to this expression are possible, but complicated. Interestingly there does exist a solution curve given by

$$y^2 = ax_c^2, \qquad x_c = \frac{81}{2A}\frac{\sqrt{3a}}{(9-3a)^{9/4}} \tag{2.41}$$

where the parameter $a$ factor must satisfy $0 \le a < 3$ for this solution to be physical. Since $a$ need not be integer,

[4]By trivial we mean the point (0, 0, 0).

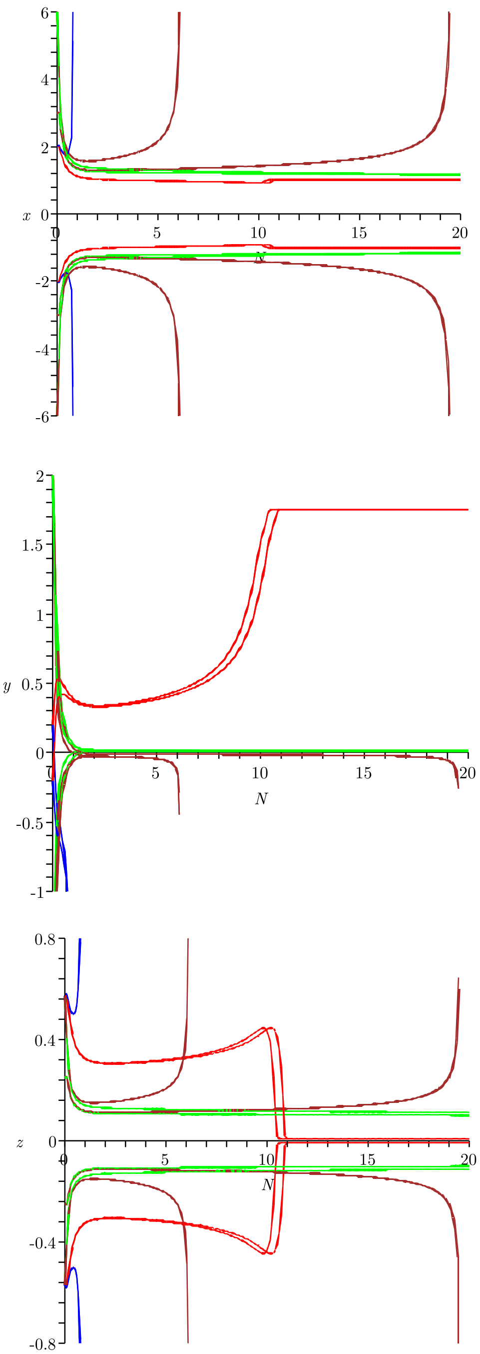


FIG. 7 (color online). Case II: Evolution of $x$, $y$, $z$ versus the e-folding number.

there is essentially a continuum of curves giving rise to fixed points in this theory.

The secondary branch of solutions again admits fixed point behavior for $y = 0$; however, things are more complicated since the fixed points are now obtained by solving more nonlinear expressions. There are two cases of immediate interest though. First, if we have $\alpha = 2$ then we see that $z^2 = -1/(2A)$ which is only real when $A$ is negative. Since we have chosen our parametrization such that this quantity is positive, this particular branch of solutions is ruled out. Interestingly, when $\alpha = -2$ there is a unique fixed point located at

$$x = \pm\frac{1}{2A}, \qquad y = 0, \qquad z = \pm\frac{1}{\sqrt{T/\tilde{V} - 1}}\sqrt{\frac{1}{2A} - 1}, \tag{2.42}$$

which corresponds to a positive definite equation of state parameter

$$\Omega = \frac{2T^2A(A-1) + \tilde{V}^2(T/\tilde{V} - 1)}{AT\tilde{V}(T/\tilde{V} - 1)(2A - 1)}. \tag{2.43}$$

Note that we must require $T > \tilde{V}$ for this solution to be nonsingular, which means (again) that the tension term dominates the energetics of the theory. What is also obvious is that demanding $A = 1/2$ leads to a novel fixed point at $(\pm 1, 0, 0)$ regardless of the ratio $T/\tilde{V}$. Using the definition of $A$, this fixes $\lambda = 2M_p$, and therefore $W$ is vanishingly small unless the scalar is trans-Planckian. This is manifest in a divergence in the equation of state parameter and is therefore unphysical. Therefore we must ensure that $A < 1/2$, implying that $\lambda > 2M_p$. Since this is the largest scale in our theory, one again expects this to be unphysical.

The more general solutions can be found numerically and correspond to $x_0^2 = 1 + z_0^2(T/\tilde{V} - 1)$, where $z_0^2$ are the characteristic solutions to the nonlinear equation

$$1 + \alpha A z^{(2+\alpha)/2}(1 + z^2(T/\tilde{V} - 1))^{(\alpha-2)/2\alpha} = 0. \tag{2.44}$$

In this more general case we can set $T = \tilde{V}$ without the solution diverging, and we therefore find the corresponding fixed point solution is thus given by

$$x = \pm 1, \qquad y = 0, \qquad z = \left(-\frac{1}{A\alpha}\right)^{2/(2+\alpha)} \tag{2.45}$$

which implies that $\alpha$ is negative. Moreover we see that $\Omega$ is again zero here for all physical values of $\alpha$, although there is no additional constraint upon the magnitude of $A$. Now, we see numerical solutions in Figs. 8–10. Constants are set as $M_p = 1$, $T = 1$, $\tilde{V} = 1$, $\lambda = 1$, and $w = 0$ (dust case). Other parameters are $\alpha = 1$ and $A = 1$.

## D. Case IV

Following on from the previous class of models, we can find solutions where the scalar potential is now constant,

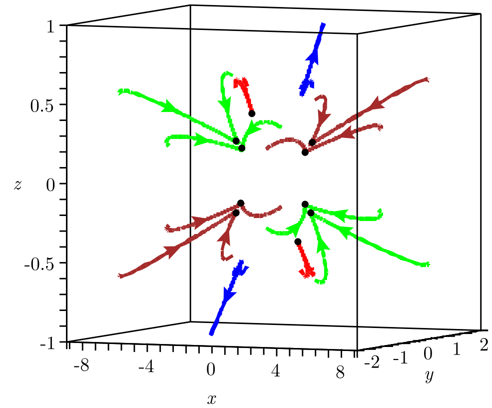


FIG. 8 (color online). Case III: 3D $xyz$ phase-space trajectories for $T(\phi) = T$, $\tilde{V}(\phi) = V$, and $W(\phi) = \phi^\alpha/\lambda^\alpha$. Here we set $M_p = 1$, $T = V = 1$, $m = 1$, $\lambda = 1$, and $w = 0$ (dust case), $\alpha = 1$.

using the ansatz

$$\tilde{V} = V, \qquad T(\phi) = \left(\frac{\phi}{\lambda}\right)^\alpha, \qquad W(\phi) = \left(\frac{\phi}{\delta}\right)^\beta \tag{2.46}$$

where $\lambda$, $\delta$ are terms of the requisite dimensionality. From this expression we see that $\mu_1$ is identically zero. It will be convenient to define the following function, $Q = V\lambda^\alpha\delta^\beta$, which in turn can be used in the definitions of the remaining $\mu_i$ functions

$$\mu_2 = -\frac{\alpha M_p}{\lambda^{\alpha/2}V}\left(\frac{Qx^2}{\gamma z^2}\right)^{n_1},$$
$$\mu_3 = \frac{\beta M_p \delta^{\beta/2}}{\gamma^{4/2}}\left(-\frac{\mu_2\lambda^{\alpha/2}V^{3/2}}{\alpha M_p}\right)^{-n_2}, \tag{2.47}$$
$$n_1 = \frac{3\alpha - 2}{2(\alpha + \beta)}, \qquad n_2 = \frac{1 + \beta}{3\alpha - 2},$$

and now the dynamical equations simplify to become

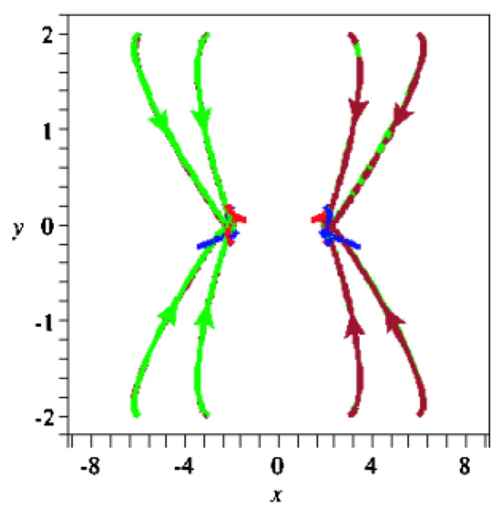


FIG. 9 (color online). Case III: Phase-space trajectories in the $xy$ plane.

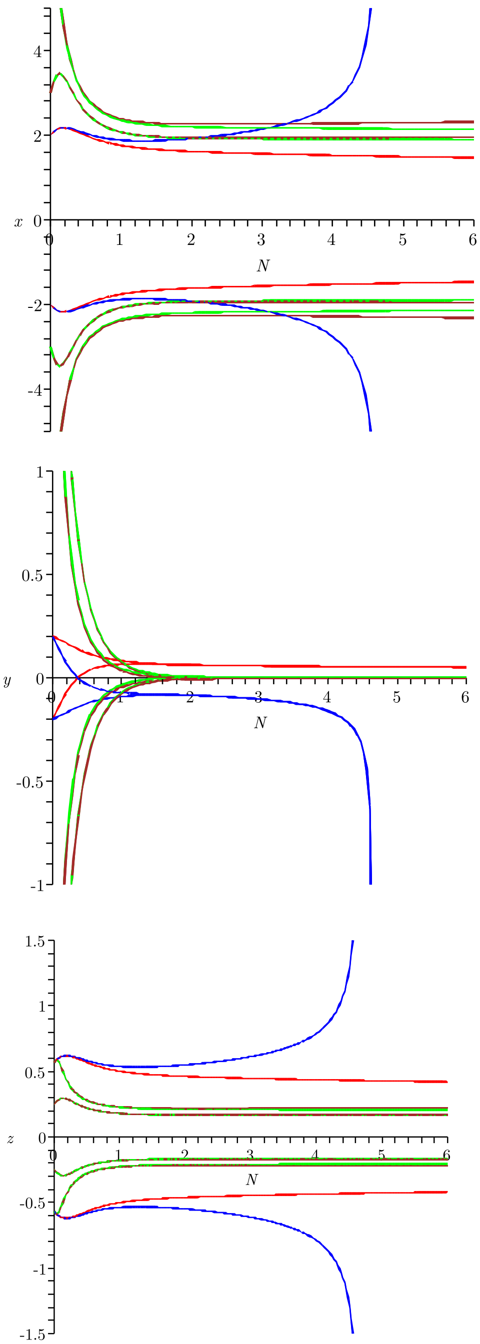


FIG. 10 (color online). Case III: Evolution of $x$, $y$, $z$ versus the e-folding number.

$$x' = -\frac{\mu_2 y z^3}{2x^2} - \frac{y^2}{2x} - \frac{xH'}{H},$$
$$y' = -3y\left(1 - \frac{y^2}{6x^2}\right)\left(1 + \frac{z^3\mu_2}{xy}\right) + 3\mu_2\left(\frac{z^{3\alpha+\beta}}{x^{\beta-\alpha}y^{2\beta+\alpha}}\left[\frac{Q}{\delta}\right]^{\beta}\right)^{1/(\alpha+\beta)} - 3x^2\mu_3 - \frac{yH'}{H},$$
$$z' = -\frac{zH'}{H}. \tag{2.48}$$

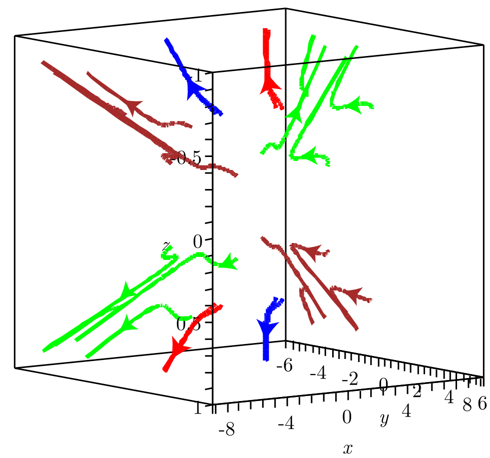


FIG. 11 (color online). Case IV: 3D $xyz$ phase-space trajectories for $T(\phi) = (\phi/\lambda)^\alpha$, $\tilde{V}(\phi) = V$, and $W(\phi) = (\phi/\delta)^\beta$. Here, $M_p = 1$, $V = 1$, $m = 1$, $\lambda = 1$, $\alpha = 1$, $\beta = 1$, $\delta = 1$, and $w = 0$ (dust case).

The resulting analysis is far more complicated than in the previous cases. Let us again start with the simplest solution slices at $z = 0$. The expressions for $x'$ and $z'$ readily simplify in this instance; however, the equation for $y'$ requires us to be more careful. We see that in order for the $z^3\mu_2$ term to vanish in this limit, we require $(2 + 3\beta)/(\alpha + \beta) > 0$. The remaining $\mu_2$ term only vanishes if this condition is tightened to $(2 + \beta)/(\alpha + \beta) > 0$, and the term coming from $\mu_3$ only vanishes if $(1 + \beta)/(\alpha + \beta) > 0$. If these inequalities are reversed, for example, then these terms diverge in the $z \to 0$ limit. If we restrict ourselves to well-behaved solutions such that $\alpha$, $\beta$ satisfy the above bounds (either by both $\alpha, \beta \geq 0$ or by $\alpha \geq 0$, $\beta \leq 0$ with $|\beta| > |\alpha|$), then we obtain the solution curve $y^2 = 3x^2$ as usual. If the parameters $\alpha$, $\beta$ do not satisfy at

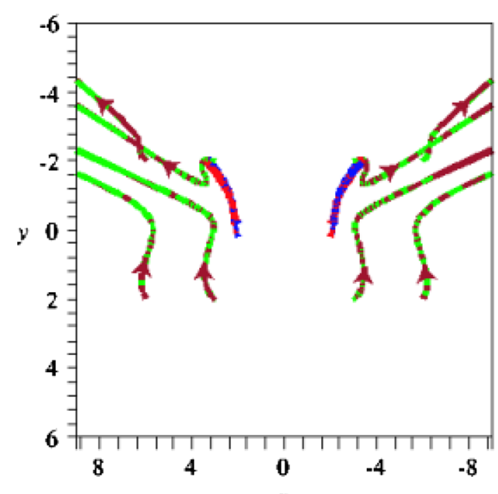


FIG. 12 (color online). Case IV: Phase-space trajectories in the $xy$ plane.

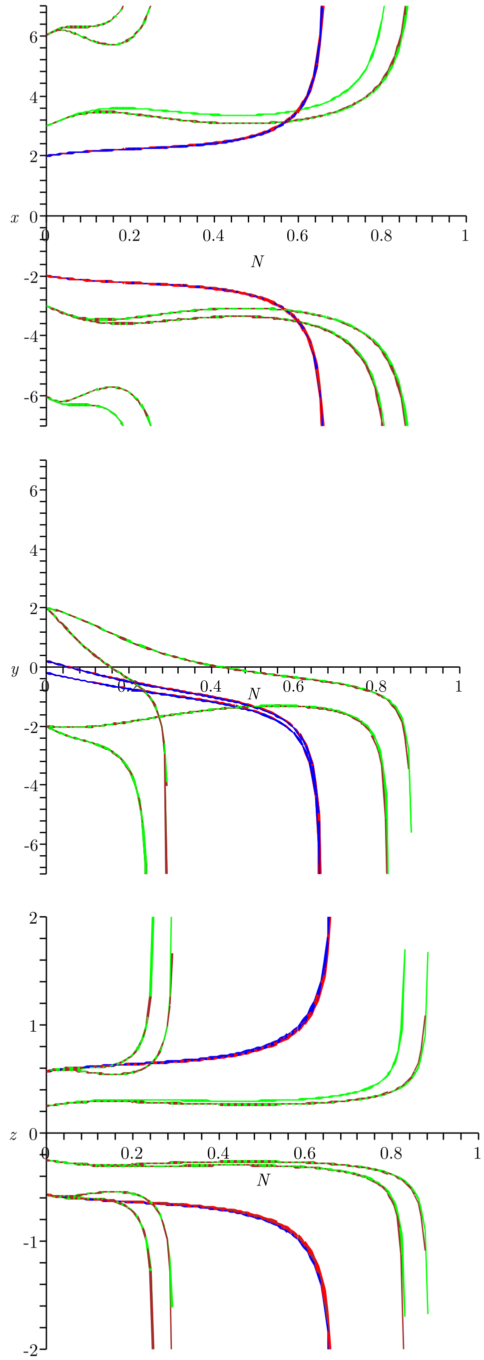

FIG. 13 (color online). Case IV: Evolution of $x$, $y$, $z$ versus the e-folding number.

least the minimal bound, then one can only solve these expressions numerically.

The only other solution branch occurs when $H'/H = 0$. This is again a complicated solution; however, things simplify somewhat when we slice through $y = 0$, but also tune the solution such that $\alpha = \beta$, which gives us

$$z = \frac{x\delta^{\alpha}}{2\sqrt{Q}}\left(1 \pm \sqrt{1 - \frac{4Q(x^2 - 1)}{x^2\delta^{2\alpha}}}\right), \qquad (2.49)$$

and therefore the fixed point solution in this instance is given by solutions of the polynomial

$$x\sqrt{\frac{Q}{\delta}} + \left(\frac{\sqrt{Q}}{z}\right)^{1/(2\alpha)} \frac{x^{(1+8\alpha)/(4\alpha)}}{\lambda^{\alpha/2}\delta^{\alpha/2}} = 1. \qquad (2.50)$$

This can actually be solved exactly when $\alpha = -1$, but numerically for more general $\alpha$. The exact case gives us the following solution:

$$x_0 = \frac{Q\delta - 2\sqrt{Q} + \delta^2 \pm \delta\sqrt{F(\lambda, \delta)}}{2\lambda\delta^4},$$

$$F(\lambda, \delta) = Q^2 + \delta^2 - 4\sqrt{Q^3\delta} + 6Q\delta - 4\sqrt{Q\delta^3} + 16\sqrt{Q^3\delta^7} - 4Q^3\delta^2 + 16\sqrt{Q^{5/2}\delta^{5/2}} - 24Q^2\delta^3 - 4Q\delta^4 + 4\delta^6\lambda, \qquad (2.51)$$

where $z_0$ is given by the term written above. This is a highly complicated solution, but one sees that, in principle, there are many fixed points along the plane $(x_0, 0, z_0)$ depending on the constants $\lambda$, $\delta$. One also sees that there is a simple solution when $x = 1$, since this implies that $z_0 = \delta^{\alpha}/\sqrt{Q}$ or $z_0 = 0$, the latter again giving rise to the point (1, 0, 0) which corresponds to the nonpropagating end point of the brane dynamics as shown in Figs. 11–13.

### E. Case V

Finally let us comment on perhaps the most general form of the solution one could obtain from this model, namely, that corresponding to turning on all the relevant degrees of freedom. One can therefore see that cases I–IV are actually slices through the full phase space described in this section. We will take the following parametrization for simplicity:

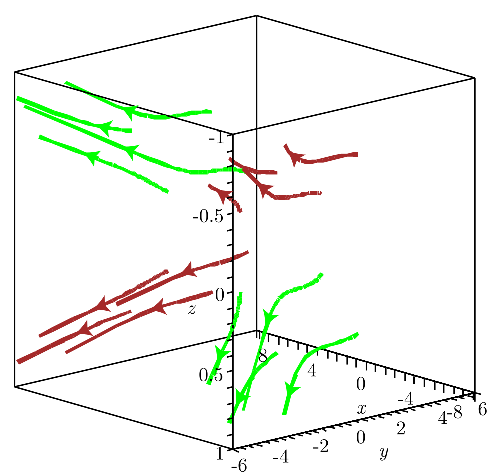

FIG. 14 (color online). Case V: 3D $xyz$ phase-space trajectories for $T(\phi) = (\phi/\lambda)^{\alpha}$, $\tilde{V}(\phi) = (m\phi)^{\xi}/\xi$, and $W(\phi) = (\phi/\delta)^{\beta}$. Here, $M_p = 1$, $V = 1$, $m = 1$, $\lambda = 1$, $\alpha = 1$, $\beta = 1$, $\delta = 1$, $\xi = 2$, and $w = 0$ (dust case).

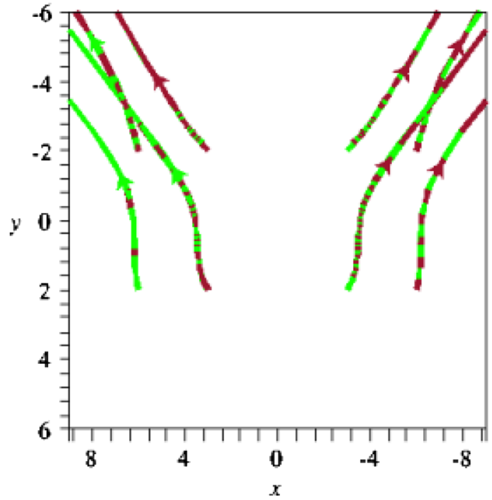


FIG. 15 (color online). Case V: Phase-space trajectories in the $xy$ plane.

$$T = \left(\frac{\phi}{\lambda}\right)^{\alpha}, \qquad W = \left(\frac{\phi}{\delta}\right)^{\beta}, \qquad \tilde{V} = \frac{m^{\xi}\phi^{\xi}}{\xi}. \tag{2.52}$$

In this case all three $\mu_i$ will be nonzero, which complicates the analysis somewhat, and reality again imposes the condition that $\xi > 0$. Let us initially search for the fixed points around $z = 0$. The primary constraint equation for this becomes

$$\frac{\alpha - \xi + 2(1 - \beta)}{(\alpha + \beta - \xi)} > 0. \tag{2.53}$$

Let us initially assume that the denominator is positive definite. Going through the same analysis as before yields the usual solution curve $y^2 = 3x^2$, provided that we tune $\beta > 0$ and $\alpha + \beta > \xi$. However, with reference to the action, we see that this situation leads to both $W$, $T$ diverging, and therefore we should be wary of this part of the solution. Returning to the constraint equation, let us therefore assume that $\xi > \alpha + \beta$ and redo the analysis. We then find that the $y^2 = 3x^2$ is perfectly valid, and moreover the parameters $W$, $T$ are not divergent, provided that the parameters satisfy $\alpha + \beta - \xi < -(2 + \beta)$. Furthermore, we also see that $\beta$ is bounded from above such that $\beta < -2/3$—thus severely restricting the form of the variable phase space.

If we search for solutions along the $y = 0$ slicing, things are again complicated. However, we can simplify things by identifying $\alpha = \xi$, since we can then solve explicitly for $x$ via

$$x^2 = 1 + z^2\left(\frac{\xi^2}{\lambda^{\xi}m^{\xi}} - 1\right). \tag{2.54}$$

The remaining equation coming from $y' = 0$ has several solutions. The simplest ones are $z^2 = 0$, $(\lambda^{-\xi}m^{-\xi}\xi^2 - 1)^{-1}$, which give rise to the points

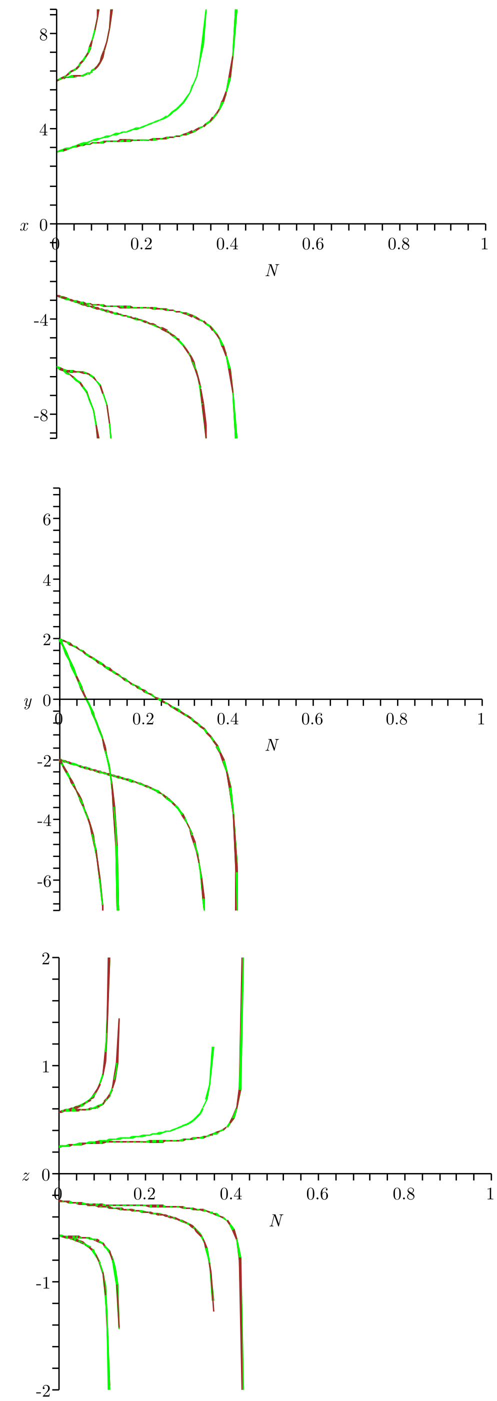


FIG. 16 (color online). Case V: Evolution of $x$, $y$, $z$ versus the e-folding number.

$$x_0 = \pm\sqrt{2}, \qquad y_0 = 0, \qquad z_0 = \frac{1}{\sqrt{\lambda^{-\xi}m^{-\xi}\xi^2 - 1}},$$
$$x_0 = \pm 1, \qquad y_0 = 0, \qquad z_0 = 0. \tag{2.55}$$

However, the first of these conditions also requires that $\xi^{2/\xi} > \lambda m$ for the solution to be real. The maximal value of $\xi^{2/\xi}$ is actually given by $\xi = e^1$, which imposes a tight constraint on the background parameters which can only be satisfied through substantial fine-tuning. Again, more general solutions are only available through numeric methods as shown in Figs. 14–16.

## III. PERTURBATIONS AND FIXED POINT STABILITY

We now need to evaluate the stability of these fixed point solutions. Clearly one may anticipate that solutions such as (0, 0, 0) may well be unstable. We must perturb the field equations about small values; therefore we need

$$x \to x_0 + \delta x, \qquad y \to y_0 + \delta y, \qquad z \to z_0 + \delta z. \tag{3.1}$$

Now the analysis is more complicated than in standard models due to the complexity of the DBI action and the general (unknown) phase-space dependence of the variables $T$, $W$, $\tilde{V}$. Since $\gamma$ is independent of any particular parametrization, we can calculate the general result.

$$\gamma \to \gamma\left(1 + \frac{\gamma^2 y_0 \delta y}{3x_0^2} - \frac{\gamma^2 y_0^2 \delta x}{3x_0^3} + \ldots\right). \tag{3.2}$$

Using this we can write the perturbation in $H'/H$. In general, we can Taylor expand the function $W$ such that we have $W(x^i + \epsilon^i) \sim W(x_0^i) + \partial_i W \epsilon^i$, and therefore the general result is true:

$$\delta\left(\frac{H'}{H}\right) = -y_0\delta y - \frac{3(1+\omega)}{2} \times \left(-2z_0\delta z - 2x_0\delta x\left[1 - \frac{1}{W\gamma}\right] - \frac{x_0^2}{\gamma W}\left\{-\frac{\gamma^2 y_0\delta y}{3x_0^2} + \frac{\gamma^2 y_0^2\delta x}{3x_0^3} - \frac{\partial_i W\epsilon^i}{W}\right\}\right),$$

where all terms such as $\gamma$, $W$ are evaluated on the classical solution and there is a summation over Latin indices.

The general equations, even for the linear perturbation, are shown below for case V—which encompasses all the other solutions in the relevant limit:

$$\begin{aligned}
\delta x' &= -\frac{yz^3}{2x^2}(\mu_1+\mu_2)\left(\frac{\delta y}{y} + 3\frac{\delta z}{z} - 2\frac{\delta x}{x}\right) - \frac{yz^3}{2x}(\mu_1\delta\mu_1 + \mu_2\delta\mu_2) - \frac{y^2}{2x}\left(2\frac{\delta y}{y} - \frac{\delta x}{x}\right) - \delta x\frac{H_0'}{H_0} - x\delta\left(\frac{H'}{H}\right),\\
\delta y' &= -\frac{3z^3}{x}\left\{\mu_1\delta\mu_1 + \mu_2\delta\mu_2 + [\mu_1+\mu_2]\left(3\frac{\delta z}{z} - \frac{\delta x}{x} - \frac{\delta y}{y}\right)\right\} - 3x^2\mu_3\left(2\frac{\delta x}{x} + \delta\mu_3\right) + \left(1 + \frac{z^3}{xy}[\mu_1+\mu_2]\right)\\
&\quad\times\left(\frac{y^3}{x^2}\left[\frac{\delta y}{y} - \frac{\delta x}{x}\right] - 3\delta y\right) - \delta y\frac{H_0'}{H_0} - y\delta\left(\frac{H'}{H}\right)\\
&= \frac{3z^3\mu_2 W}{\gamma x}\left(\delta\mu_2 + 3\frac{\delta z}{z}\left(1 - \frac{2\beta}{3n}\right) - \frac{\gamma^2 y\delta y}{3x^2}\left(1 + \frac{\beta}{n} + \frac{\delta x}{x}\left\{\frac{2\beta}{n} - 1 + \frac{\gamma^2y^2}{3x^2}\left(1 + \frac{\beta}{n}\right)\right\}\right)\right),\\
\delta z' &= \frac{z^2y\mu_1}{2x}\left(2\frac{\delta z}{z} + \frac{\delta y}{y} - \frac{\delta x}{x} + \delta\mu_1\right) - \delta z\frac{H_0'}{H_0} - z\delta\left(\frac{H'}{H}\right),
\end{aligned} \tag{3.3}$$

where we have defined $n = \alpha + \beta - \rho$ for simplicity and also the following terms:

$$\begin{aligned}
\delta\mu_1 &= -\frac{2(\alpha - 2\rho)}{n}\frac{\delta z}{z} - \frac{(\alpha - 2 - \rho)}{2n}\frac{\gamma^2 y\delta y}{3x^2} + \frac{2(\alpha - 2 - \rho)}{n}\frac{\delta x}{x}\left(1 + \frac{\gamma^2y^2}{12x^2}\right),\\
\delta\mu_2 &= -\frac{4(\alpha - 1 - \rho)}{n}\frac{\delta z}{z} - \frac{(3\alpha - 3\rho - 2)}{n}\frac{\gamma^2 y\delta y}{6x^2} + \frac{\delta x}{nx}\left(4(\alpha - 1 - \rho) + \frac{(3\alpha - 3\rho - 2)\gamma^2y^2}{6x^2}\right),\\
\delta\mu_3 &= \frac{2(\alpha + 2 + 3\rho + 2\beta)}{n}\frac{\delta z}{z} + \frac{(2\alpha - 2 - 10\rho + \beta)\gamma^2 y\delta y}{6nx^2} + \frac{\delta x}{nx}\frac{(-2\alpha + 2 + 10\rho - \beta)\gamma^2y^2}{6x^2}\\
&\quad - \frac{\delta x}{x}\frac{(2\alpha + 4 + 6\rho + 4\beta)}{n}.
\end{aligned} \tag{3.4}$$

We will work through an explicit example to illustrate the formalism, namely, the case I solutions. First, we can calculate the following expression,

$$\delta\left(\frac{H'}{H}\right) \sim -y_0\delta y + \frac{3(1+\omega)}{2}\left(2z_0\delta z + 2x_0\delta x\left[1 - \frac{1}{W\gamma}\right] + \frac{x_0^2\gamma y_0}{3Wx_0^2}\left(\delta y - \frac{y\delta x}{x_0}\right)\right), \tag{3.5}$$

which will allow us to calculate the perturbed phase-space variables. The perturbed dynamic expressions then take the following form:

$$\begin{aligned}
\delta x' &= \frac{yz^3\mu_1}{2x^2}\left(\frac{\alpha x^2}{\beta W\gamma^2}\left[\frac{2\delta x}{x}\left(1+\frac{\gamma^2 y^2}{6x^3}\right)-2\frac{\delta z}{z}-\frac{\gamma^2 y\delta y}{3x^2}\right]\right)-\frac{y^2}{2x}\left(\frac{2\delta y}{y}-\frac{\delta x}{x}\right)-\frac{yz^3\mu_1}{2x^2}\left(1-\frac{\alpha x^2}{\beta W\gamma^2}\right)\\
&\quad\times\left(\frac{\delta y}{y}\left(1-\frac{n\gamma^2 y}{6x^2}\right)+(3-n)\frac{\delta z}{z}+\frac{\delta x}{x}\left(n-2+\frac{n\gamma^2 y^2}{6x^2}\right)\right)-\delta x\frac{H_0'}{H_0}-x\delta\left(\frac{H'}{H}\right),\\
\delta y' &= 3z^3\mu_1\left(1-\frac{y^2}{6x^2}\right)\frac{\alpha x}{\beta W\gamma^2}\left[2\frac{\delta x}{x}\left(1+\frac{\gamma^2 y^2}{6x^2}\right)-2\frac{\delta z}{z}-\frac{\gamma^2 y\delta y}{3x^2}\right]-\frac{3z^3\mu_1}{2x}\left(1-\frac{\alpha x^2}{\beta W\gamma^2}\right)\left(1-\frac{y^2}{6x^2}\right)\\
&\quad\times\left((3-n)\frac{\delta z}{z}+\frac{\delta x}{x}\left(n-1+\frac{n\gamma^2 y^2}{6x^2}\right)-\frac{\delta y}{y}\left(1+\frac{n\gamma^2 y^2}{6x^2}\right)\right)+\left(1+\frac{z^3\mu_1}{xy}\left[1-\frac{\alpha x^2}{\beta W\gamma^2}\right]\right)\\
&\quad\times\left\{\frac{y^3}{x^2}\left[\frac{\delta y}{y}-\frac{\delta x}{x}\right]-3\delta y\left(1-\frac{y^2}{6x^2}\right)\right\}-\frac{3xz\alpha\mu_1}{\beta\gamma^2}\left(\frac{\delta x}{x}\left[1+n+\frac{2\gamma^2 y^2}{3x^2}\left(1+\frac{n}{4}\right)\right]+(1-n)\frac{\delta z}{z}-\frac{\delta y}{y}\frac{2\gamma^2 y^2}{3x^2}\left(1+\frac{n}{4}\right)\right)\\
&\quad-y\delta\left(\frac{H'}{H}\right)-\delta y\frac{H_0'}{H_0},\\
\delta z' &= \frac{z^2 y\mu_1}{2x}\left((2-n)\frac{\delta z}{z}+\frac{\delta x}{x}\left[n-1+\frac{n\gamma^2 y^2}{6x^2}\right]+\frac{\delta y}{y}\left[1-\frac{n\gamma^2 y^2}{6x^2}\right]\right)-\delta z\frac{H_0'}{H_0}-z\delta\left(\frac{H'}{H}\right),
\end{aligned}\tag{3.6}$$

where the notation $H_0'/H_0$ implies that we take this function evaluated at the critical points, and we have defined $n = (\alpha - \beta - 2)/(\alpha - \beta)$ for simplicity. Note that these are the leading order solutions only, and that all terms proportional to $\delta^2$ have been neglected.

The stability of the fixed point solutions is therefore determined by the eigenvalues of the resulting perturbation matrix. A lengthy calculation which we will omit here shows that the point (0, 0, 0) leads to the eigenvalues

$$\lambda_1 = \frac{3(\omega-1)}{2}, \qquad \lambda_2 = \frac{3(1+\omega)}{2}, \qquad \lambda_3 = \frac{3(1+\omega)}{2}, \tag{3.7}$$

which indicates that this is never a point of stability for the theory unless the equation of state is phantom, i.e. $\omega < -1$. In fact, this statement will be true for all the various cases we have considered in the physical limit, since the dynamical equations of motion all reduce to the exact same form in this instance.

Another relatively simple case to consider is that in case III. For slices through the $(x, y)$ plane at $z = 0$, we find the eigenvalues

$$\begin{aligned}
&\lambda = \frac{1}{2}(x^2+y^2)+\frac{3}{2}(\omega(x^2-1)-1), \qquad \lambda_\pm = \frac{1}{4x^2}(-6x^4(1+\omega)-6x^2+2y^2x^2+5y^2 \pm F(x,y)),\\
&F(x,y) = \sqrt{12y^2x^2\omega+96y^2x^4\omega-48y^2x^6\omega-8y^4x^2+48y^2x^4+16y^4x^4-48y^2x^6+36\omega^2x^4+17y^4}.
\end{aligned}\tag{3.8}$$

If one now slices this through $y = 0$, we see that we are left with the same situation discussed above (as expected), indicative of a phantom equation of state.

On the other hand, through the $y = 0$ plane we see that the eigenvalues become

$$\begin{aligned}
\lambda &= \frac{3}{2}(1+\omega)\left(1-z^2-x^2\left(1-\frac{Qz^2}{x^2}\right)\right),\\
\lambda_\pm &= -\frac{3}{2x}(-Qz^2-x+2xz^2+x^3-xz^2Q+x^2\\
&\quad\pm F(x,y)),
\end{aligned}\tag{3.9}$$

where $F$ is another polynomial in $x$, $z$, and we have defined $Q = T/\tilde{V}$ for simplicity. In the limit that $z \to 0$, we find that these simplify to yield

$$\begin{aligned}
\lambda &\to \tfrac{3}{2}(1+\omega)(1-x^2),\\
\lambda_\pm &\to \tfrac{3}{2}(1+\omega)(x^2-1-x^2(1\pm 1)).
\end{aligned}\tag{3.10}$$

Note that two of the eigenvalues are therefore degenerate as before, requiring a phantom equation of state; however, the final eigenvalue has the opposite sign, and therefore this fixed point is always unstable.

The remaining fixed points can be analyzed in precisely the same manner, although the analysis is somewhat awkward. We will postpone the relevant discussion and return to it in a follow-up publication.

## IV. DISCUSSION

We have initiated an alternate approach to the problem of $k$-essence, or DBI quintessence [25], using a more generalized form of the DBI action. Since this has more degrees of freedom, the resulting analysis is typically complicated, but the phase-space structure is far richer. We have attempted to make some headway by restricting the phase-space volume to various two-dimensional slices, and by attempting to identify the relevant solution curves upon which the fixed points may lie. Our ansatz for each of the unknown functions is also potentially restrictive; however, we are confident that it represents the leading semiclassical contributions which may (or may not) be derivable from a full string theory embedding of our model.

What is clear is that the ratio of the (warped) brane tension to the potential is an important factor in the dynamics of the theory, where we found $T \geq \tilde{V}$ in several cases. Moreover, the additional multiplicative factor $W(\phi)$ plays a crucial role, even when it is a constant, since it comes into the field equations nontrivially in the expression for $H'/H$. In the usual DBI analysis, $W = 1$ and the tension is the sole term responsible for the interesting quintessence behavior. In some string compactifications, where the warp factor has no cutoff at small distances, we typically find $W$ is constant and greater than unity. However, there may be entire classes of solutions where $W \leq 1$, which can lead to novel phase-space trajectories. Since our approach has been phenomenological, and since there may be additional string backgrounds of interest that have yet to be fully explored, we cannot rule out $W < 1$—which is vital for obtaining fixed point solutions in case I, for example.

Our numerical results have shown that there is indeed a rich phase-space structure present due to the increased number of degrees of freedom. We expect many of these to yield highly nontrivial stable fixed points in the full analysis, which is beyond the scope of the current paper. We have classified the nature of as many of the fixed points as is feasible within the current analysis. Ultimately we hope that this will lead to a renewed interest in dynamical dark energy models driven by a more generalized approach to $D3$-brane dynamics.

In light of the recent developments in holographic dark energy [31,32] and the apparent relation to agegraphic [33,34] dark energy, we hope that it may be possible to reconstruct the various potentials in our generalized model along the lines of [35].

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