PHYSICAL REVIEW D **80,** 023511 (2009)

Nonsingular cosmology with a scale-invariant spectrum of cosmological perturbations from Lee-Wick theory

Yi-Fu Cai,[1] Taotao Qiu,[1] Robert Brandenberger,[1,2,3] and Xinmin Zhang[1,3]

[1]*Institute of High Energy Physics, Chinese Academy of Sciences, P.O. Box 918-4, Beijing 100049, People's Republic of China*
[2]*Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada*
[3]*Theoretical Physics Center for Science Facilities (TPCSF), Chinese Academy of Science, People's Republic of China*
(Received 9 November 2008; revised manuscript received 15 June 2009; published 15 July 2009)

We study the cosmology of a Lee-Wick type scalar field theory. First, we consider homogeneous and isotropic background solutions and find that they are nonsingular, leading to cosmological bounces. Next, we analyze the spectrum of cosmological perturbations which result from this model. Unless either the potential of the Lee-Wick theory or the initial conditions are finely tuned, it is impossible to obtain background solutions which have a sufficiently long period of inflation after the bounce. More interestingly, however, we find that in the generic noninflationary bouncing cosmology, perturbations created from quantum vacuum fluctuations in the contracting phase have the correct form to lead to a scale-invariant spectrum of metric inhomogeneities in the expanding phase. Since the background is nonsingular, the evolution of the fluctuations is defined unambiguously through the bounce. We also analyze the evolution of fluctuations which emerge from thermal initial conditions in the contracting phase. The spectrum of gravitational waves stemming from quantum vacuum fluctuations in the contracting phase is also scale-invariant, and the tensor to scalar ratio is not suppressed.

DOI: 10.1103/PhysRevD.80.023511 PACS numbers: 98.80.Cq

I. INTRODUCTION

Recently, ideas originally due to Lee and Wick [1] were used to propose [2] a "Lee-Wick standard model," a modification of the standard model of particle physics in which the Higgs mass is stabilized against quadratically divergent radiative corrections and which in this sense is an alternative to supersymmetry for solving the hierarchy problem. The Lagrangian includes new higher derivative operators for each field. These operators can be eliminated by introducing a set of auxiliary fields, one for each field of the original model. The higher derivative terms have opposite signs for both the kinetic and mass terms, which indicates how the quadratic divergences in the Higgs mass can be cancelled.

Fields with opposite signs of the kinetic term in the action have recently been invoked in cosmology to provide models for dark energy. Fields with negative kinetic energy but positive potential energy are called "phantom fields" [3] and were introduced to provide a possible mechanism for obtaining an equation of state of dark energy with an equation of state parameter $w < -1$, where $w = p/\rho$, p and ρ being pressure and energy density, respectively. In addition to the conceptual problems of having phantom fields (see, e.g., [4]), phantom dark energy models lead to future singularities. To avoid these problems, the "quintom model" [5] was introduced. This model contains two scalar fields, one of them with a regular sign kinetic term, the second with an opposite sign kinetic term. This model allows for a crossing of the "phantom divide," i.e., a transition of the equation of state from $w < -1$ to $w > -1$. When applied to early universe cosmology, quintom models can lead to nonsingular cosmological backgrounds which correspond to a bouncing universe [6,7].

The Higgs sector of the Lee-Wick standard model has similarities with the Lagrangian of a quintom model: the Higgs field has a regular sign kinetic term but the auxiliary field has a negative sign kinetic term. Thus, it is logical to expect that the Lee-Wick model might give rise to a cosmological bounce and thus solve the cosmological singularity problem, in addition to solving the hierarchy problem. In this article we show that this expectation is indeed realized.

Given that the Lee-Wick model leads to a cosmological bounce, the cosmology of the very early universe may be very different from what is obtained by studying the cosmology of the standard model. It is possible to introduce a potential for the scalar field in order to obtain a sufficiently long period of inflation after the bounce in order to solve the problems of standard big bang cosmology and to obtain a spectrum of nearly scale-invariant cosmological fluctuations. However, this requires fine-tuning of the potential. On the other hand, given a bouncing cosmology it is possible that the cosmological fluctuations originate in the contracting phase, as in the pre–big bang[10] or ekpyrotic [11] scenarios. In this article, we study the generation and evolution of fluctuations in our Lee-Wick type model. We consider both vacuum and thermal initial conditions for the fluctuations in the contracting phase and follow the perturbations through the bounce, a process which can be done unambiguously since the bounce is nonsingular.

We find that initial quantum vacuum fluctuations in the contracting phase have the right spectrum to develop into a scale-invariant spectrum in the expanding phase. What is responsible for this result is the fact that there is a coupling of the growing mode in the contracting phase to the dominant (constant in time) mode in the expanding phase, and that this coupling scales with comoving wave number as k^2. The Lee-Wick model thus leads to a concrete realization of the proposal of [12] (see also [8,9,13] and more recently [14]) to obtain a scale-invariant spectrum of fluctuations from a matter-dominated contracting phase (see also [15] for an analysis of gravitational wave evolution in this background).

The outline of this article is as follows. In the following section we introduce the Lee-Wick scalar field model which we will study in the rest of the article. In Sec. III we study the background solutions of this model, taking initial conditions in the contracting phase. We show that, at least at the level of homogeneous and isotropic cosmology, it is easy to obtain a bouncing cosmology. In Sec. IV we study how cosmological fluctuations set up in the initial contracting phase pass through the bounce. The evolution of the fluctuations is well behaved. Section V contains the computation of the spectrum of gravitational waves, starting from quantum vacuum fluctuations in the contracting phase. We end with a discussion of our results.

II. A LEE-WICK SCALAR FIELD MODEL

We will take our starting Lagrangian for the scalar field $\hat{\phi}$ to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu \hat{\phi}\partial^\mu \hat{\phi} - \frac{1}{2M^2}(\partial^2 \hat{\phi})^2 - \frac{1}{2}m^2\hat{\phi}^2 - V(\hat{\phi}), \quad (1)$$

where m is the mass of the scalar field and V is its interaction potential. The second term on the right-hand side is the higher derivative term, involving a new mass scale M.

As discussed in [2], by introducing an auxiliary field $\tilde{\phi}$ and redefining the "normal" scalar field as

$$\phi = \hat{\phi} + \tilde{\phi}, \quad (2)$$

the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi\partial^\mu \phi - \frac{1}{2}\partial_\mu \tilde{\phi}\partial^\mu \tilde{\phi} + \frac{1}{2}M^2\tilde{\phi}^2 - \frac{1}{2}m^2(\phi - \tilde{\phi})^2 - V(\phi - \tilde{\phi}). \quad (3)$$

We thus see that M is the mass of the new scalar degree of freedom, the "Lee-Wick scalar" which comes from the extra degrees of freedom of the higher derivative theory. Note that both the kinetic term and the mass term of the Lee-Wick scalar have the opposite sign compared the signs for a regular scalar field. One may worry that the theory is unstable because of the wrong sign of the kinetic term of the Lee-Wick scalar [16–18]. However, as was argued in [19], the perturbative expansion can be defined in a consistent way and the theory is unitary. Building on these works, a recent study shows that Lee-Wick electrodynamics can be defined consistently as a ghost-free, unitary and Lorentz invariant theory [20].

By rotating the field basis, the mass term can be diagonalized. However, the coupling between the two fields in the interaction term remains. To be specific, we consider a quartic interaction term. Thus, the Lagrangian we study is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi\partial^\mu \phi - \frac{1}{2}\partial_\mu \tilde{\phi}\partial^\mu \tilde{\phi} + \frac{1}{2}M^2\tilde{\phi}^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}(\phi - \tilde{\phi})^4. \quad (4)$$

III. BACKGROUND COSMOLOGY

In this section we study the background cosmological equations which follow from coupling the matter Lagrangian (4) to Einstein gravity. For a homogeneous, isotropic and spatially flat universe the metric of space-time is

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2, \quad (5)$$

where t is physical time, and $\mathbf{x}$ denote the comoving spatial coordinates. The system of equations of motion consists of the Klein-Gordon equations

$$\begin{aligned}\ddot{\phi} + 3H\dot{\phi} + m^2\phi &= -\lambda(\phi - \tilde{\phi})^3\\ \ddot{\tilde{\phi}} + 3H\dot{\tilde{\phi}} + M^2\tilde{\phi} &= -\lambda(\phi - \tilde{\phi})^3\end{aligned} \quad (6)$$

for the two scalar fields and the Einstein expansion equation

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\tilde{\phi}}^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\tilde{\phi}^2 + \frac{\lambda}{4}(\phi - \tilde{\phi})^4\right], \quad (7)$$

where $H = \dot{a}/a$ is the Hubble expansion rate and G is Newton's gravitational constant. An overdot denotes the derivative with respect to t. Combining these equations leads to the following expression for the change in the Hubble expansion rate:

$$\dot{H} = -4\pi G(\dot{\phi}^2 - \dot{\tilde{\phi}}^2), \quad (8)$$

from which we immediately see that it is possible for the background cosmology to cross the "phantom divide" $\dot{H} = 0$.

Let us take a first look at how it is possible to obtain a bouncing cosmology in our model. We assume that the universe starts in a contracting phase and that the contribution of ϕ in the equations of motion dominates over that of the Lee-Wick scalar. This will typically be the case at low-energy densities and curvatures. As the universe contracts and the energy density increases, the relative importance of $\tilde{\phi}$ compared to ϕ will grow. From (7) it follows that there will be a time when $H = 0$–this is a necessary condition for the bounce point. From (8) it follows that at the bounce point $\dot{H} > 0$. Hence, we indeed have a transition from a contracting phase to an expanding phase, i.e., a cosmological bounce.

Let us now consider the above argument in a bit more detail. For the moment we will set the interaction Lagrangian to zero, i.e., we will assume $\lambda = 0$. We begin the evolution during the contracting phase when the energy density is sufficiently low so that we expect the contribution of the Lee-Wick scalar to the total energy density to be small. For these initial conditions, both matter fields will be oscillating, and the equation of state will hence be that of a matter-dominated universe. In fact, as follows from the Klein-Gordon Eqs. (6) which in this case reduce to

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \qquad \ddot{\tilde{\phi}} + 3H\dot{\tilde{\phi}} + M^2\tilde{\phi} = 0 \quad (9)$$

both scalar fields will be performing oscillations with amplitudes $\mathcal{A}(t)$ and $\tilde{\mathcal{A}}(t)$ which are blue-shifting (i.e., increasing) at the same rate

$$\mathcal{A}(t) \sim \tilde{\mathcal{A}}(t) \sim a(t)^{-3/2}. \quad (10)$$

Eventually, the oscillations of the field ϕ will freeze out. From studies of chaotic inflation [21] it is well known that this happens when the amplitude $\mathcal{A}$ becomes of the order of the Planck mass $m_{\rm pl}$, more specifically when

$$\mathcal{A} = (12\pi)^{-1/2} m_{\rm pl}. \quad (11)$$

After freeze-out, ϕ will slowly roll up the potential and the equation of state will shift from $w = 0$ to $w \simeq -1$ (where $w = p/\rho$, p and ρ denoting pressure and energy density, respectively) leading to a deflationary phase during which the scale factor is decreasing almost exponentially. This phase is the time reversal of a period of slow-roll inflation. However, during this period the Lee-Wick field $\tilde{\phi}$ is still oscillating with rapidly increasing amplitude. Hence, its contribution to the energy density will rapidly catch up to that of ϕ.

Let us give a rough estimate of the duration of the deflationary phase. It will depend crucially on the initial ratio of the energy density of the Lee-Wick scalar $\tilde{\phi}$ to that of the regular scalar ϕ. Let us denote this ratio by $\mathcal{F}$. In the absence of coupling between the two scalar fields, i.e., for $\lambda = 0$, the ratio will be unchanged during the period of matter domination when both fields are oscillating. However, once ϕ enters the slow-rolling phase, the amplitude $\tilde{\mathcal{A}}$ will increase exponentially according to (10) while that of ϕ will remain virtually unchanged. Thus, the condition on the duration Δt of the deflationary phase is

$$|H|\Delta t \equiv N = \frac{1}{3}\log(|\mathcal{F}|^{-1}). \quad (12)$$

Thus, to obtain a deflationary phase with $N > 50$ (which in the expanding phase will correspond to a period of inflation of sufficient length to solve the cosmological problems of the standard big bang model) required severe fine-tuning of the initial conditions. As we will discuss below, this problem may be even worse if coupling between the two fields is allowed.

Once the contribution of the Lee-Wick scalar to the energy density catches up to that of the original scalar field, the deflationary phase will end and a cosmological bounce will occur. Note that once $H = 0$, the Lee-Wick scalar is still oscillating whereas ϕ is slowly rolling. Thus, $\dot{H} > 0$ and we indeed have a transition from contraction to expansion. This is a behavior which is not possible for Einstein gravity coupled to matter satisfying the weak energy condition. However, due to the negative sign of the kinetic term in the Lagrangian, the weak energy condition is violated in our model. Note that in bouncing cosmologies obtained in higher derivative gravity models such as [22], it is the higher derivative gravitational terms which, when interpreted as matter, lead to a violation of the weak energy condition.

The duration of the bounce can be estimated as follows: The maximal amplitude H_m of $|H|$ before and after the bounce is set by

$$H_m \sim m, \quad (13)$$

since it is determined by the potential energy at the field value where the slow-rolling of ϕ begins. The amplitude of $\dot{H}$ at the bounce, denoted by $\dot{H}_b$, can in turn be estimated by

$$\dot{H}_b \sim 4\pi G\dot{\tilde{\phi}}^2 \sim 4\pi G m^2 m_{\rm pl}^2 \sim m^2, \quad (14)$$

where in the first step we have used the fact that the kinetic energy of ϕ is negligible at the bounce, and in the second step the fact that the bounce is determined by having the same absolute value of energy densities of ϕ and $\tilde{\phi}$, and that the field value of ϕ at the bounce is about $m_{\rm pl}$. The bounce time Δt_b can now be determined via

$$\dot{H}_b \Delta t_b = 2H_m. \quad (15)$$

This gives

$$\Delta t_b \sim m^{-1}. \quad (16)$$

Note from the above that the value of H_m is set by the mass of ϕ, not the mass of $\tilde{\phi}$. Similarly, the bounce time is determined by the mass of the original scalar field and not of its Lee-Wick partner.

After the bounce, the amplitude of the oscillations of the Lee-Wick scalar exponentially decreases while ϕ is now slowly rolling down the potential. This is a phase of inflation which is time-symmetric to the phase of deflation before the bounce. As we have seen, without fine-tuning of the initial contribution of the Lee-Wick scalar to the energy density, the period of inflation will be too small for inflation to solve the various problems of standard cosmology which inflation was invented to solve [23] (see also [24]) (such as the horizon and flatness problems).

Let us add some comments on the effects of allowing a coupling between the two scalar fields. We expect that this will lead to a gradual flow of energy between the regular scalar and the Lee-Wick scalar such that at an energy density corresponding to the scale of the Lee-Wick scalar, the energy density in the Lee-Wick scalar will begin to dominate. Thus, allowing for $\lambda \neq 0$ will lead to a shorter deflationary phase and may completely eliminate the period of deflation. Complete elimination of the deflationary phase will occur if the energy density in ϕ is larger than M^4 at $\phi = (12\pi)^{-1/2} m_{\rm pl}$, where $G \equiv m_{\rm pl}^{-2}$. This is the case if (making use of (11))

$$M < ((12\pi)^{-1/2} m m_{\rm pl})^{1/2}. \tag{17}$$

The approximate analytical analysis summarized above is supported by exact numerical results. We have solved the coupled equations of motion for the scale factor and the two scalar fields ϕ and $\tilde{\phi}$ numerically. Figure 1 presents the results in the base of a noninteracting model. We plot the time evolution of the scalar field ϕ (denoted ϕ_1 in the figure), its Lee-Wick partner $\tilde{\phi}$ (denoted by ϕ_2) and the equation of state parameter w. As is evident, there is a nonsingular cosmological bounce, there is no deflationary phase, but the equation of state parameter w crosses the phantom divide around the bounce point. Note that the scalar field ϕ stops oscillating near the bounce, whereas the Lee-Wick scalar continues to oscillate and therefore increases in magnitude by a large factor during the latter stages of the contracting phase (which is why we have plotted the time evolution of $\tilde{\phi}$ on two different scales).

Note that in the noninteracting model, the bounce is symmetric. In Fig. 2 we present the corresponding figure in the case of an interacting model with the value of λ chosen to be $\lambda = 1.64 \times 10^{-15}$. In this case, the bounce is clearly asymmetric. As a second major difference compared to the simulation of Fig. 1, the ratio of masses was chosen to be almost 100 in this case as opposed to only 2 in the first simulation. Because of the large ratio of the masses (and the corresponding initial conditions for which the energy in ϕ greatly dominates over than in $\tilde{\phi}$), the background evolution enters a brief deflationary phase at the end of the contraction phase. However, due to the presence of interactions, the energy density in ϕ does not come to dominate again right after the bounce and hence the period of inflation which would be the time reversal of the phase of delation is absent.

Finally, in Fig. 3 we plot the number of e-foldings of the deflationary phase as a function of the ratio of ρ_ϕ to $\rho_{\tilde{\phi}}$, in

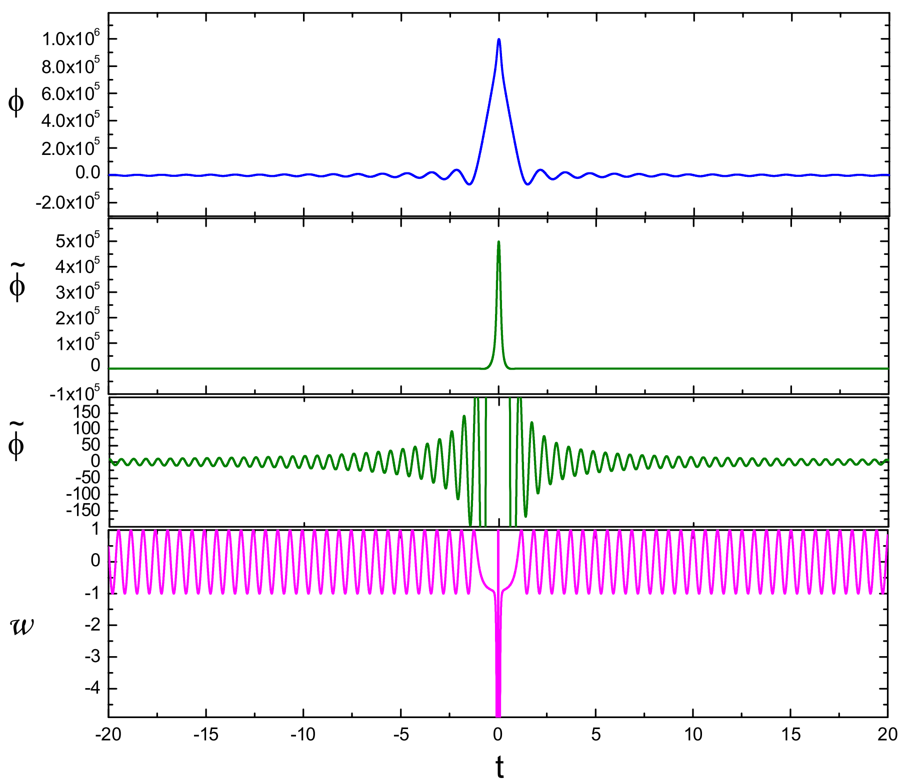

FIG. 1 (color online). Evolution of the background fields ϕ, $\tilde{\phi}$ and of the background equation of state parameter w in a noninteracting model as a function of cosmic time (horizontal axis). The background fields are plotted in dimensionless units by normalizing by the mass $M_{\rm rec} = 10^{-6} m_{\rm pl}$. Similarly, the time axis is displayed in units of $M_{\rm rec}^{-1}$. The mass parameters m and M were chosen to be $m = 5M_{\rm rec}$ and $M = 10M_{\rm rec}$. The initial conditions were $\phi_i = 1.74 \times 10^3 M_{\rm rec}$, $\dot{\phi}_i = 1.44 \times 10^4 M_{\rm rec}^2$, $\tilde{\phi}_i = 8.98 M_{\rm rec}$, $\dot{\tilde{\phi}}_i = -14.08 M_{\rm rec}^2$.

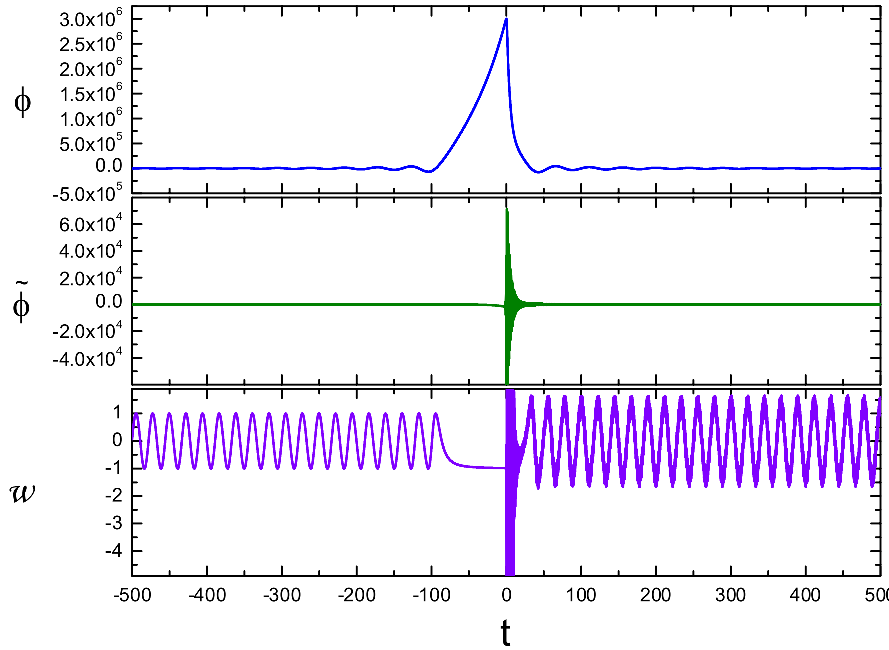

FIG. 2 (color online). Evolution of the background fields ϕ, $\tilde{\phi}$ and of the background equation of state parameter w in a noninteracting model as a function of cosmic time (horizontal axis). The background fields are plotted in dimensionless units by normalizing by the mass $M_{\rm rec} = 10^{-6} m_{\rm pl}$. Similarly, the time axis is displayed in units of $M_{\rm rec}^{-1}$. The mass parameters m and M were chosen to be $m = 1.4 \times 10^{-1} M_{\rm rec}$ and $M = 10 M_{\rm rec}$. The initial conditions were $\phi_i = -3.57 \times 10^3 M_{\rm rec}$, $\dot{\phi}_i = 5.56 \times 10^2 M_{\rm rec}^2$, $\tilde{\phi}_i = 2.98 \times 10_{-6} M_{\rm rec}$, $\dot{\tilde{\phi}}_i = -1.39 \times 10^{-6} M_{\rm rec}^2$.

the model without interactions between the two scalar fields. As predicted by our analytical approximations, the scaling of the period of deflation as a function of the ratio is roughly logarithmic.

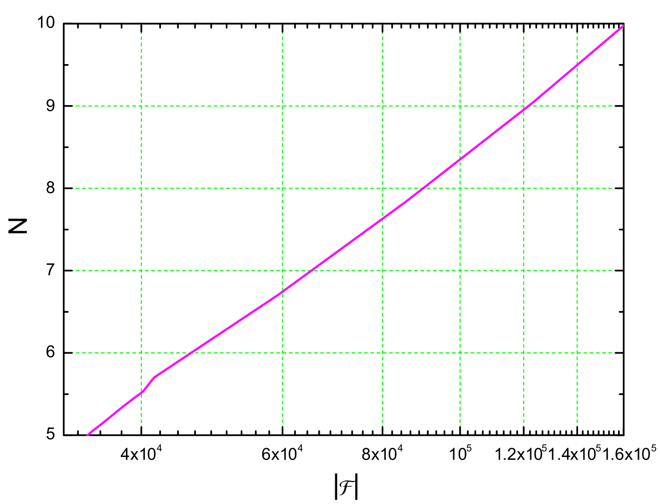

FIG. 3 (color online). Plot of the duration of the deflationary phase as a function of the ratio of energy densities of ϕ and $\tilde{\phi}$ (horizontal axis). The duration (vertical axis) is shown in terms of the e-folding number of deflation.

IV. COSMOLOGICAL FLUCTUATIONS

A. General considerations

It is useful to first consider the space-time sketch (4) of our nonsingular bouncing cosmology. We choose the bounce time to correspond to $t = 0$. Long before the bounce, the equation of state is that of matter. During this period, the Hubble radius is decreasing linearly and $\dot{H} < 1$. At a time denoted $-t_R$ (in analogy with the notation in inflationary cosmology) there is a transition to a period of deflation during which the Hubble radius $|H|^{-1}$ is constant. However, as argued in the previous section, this period will be of short duration and ends at a time $-t_i$ when a brief bouncing phase covering the time interval $-t_i < t < t_i$ begins. During this period $\dot{H} > 0$. After the bouncing phase there is a short period of inflation lasting from t_i to t_R, after which the universe enters a matter-dominated expansion phase with $\dot{H} < 0$.

In Fig. 4 we also plot the evolution of the physical length corresponding to a fixed comoving scale. This scale is the wavelength of the fluctuation mode k (k standing for the comoving wave number) which we want to follow. The wavelength begins in the matter-dominated phase of contraction on sub-Hubble scale, exits the Hubble radius during this phase at a time which we denote $-t_H(k)$, and reenters the Hubble radius during the matter-dominated phase at the time $t_H(k)$.

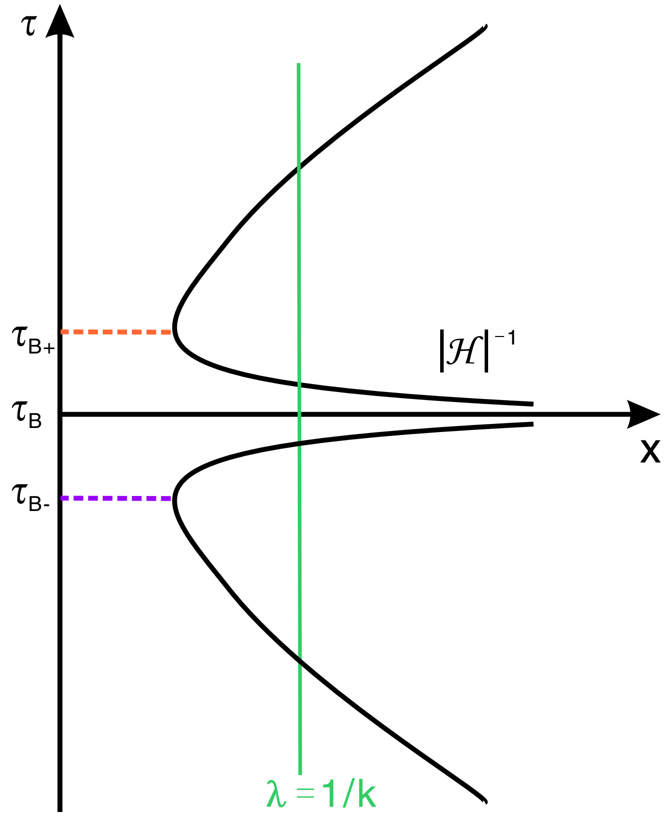

FIG. 4 (color online). A sketch of the evolution of scales in a bouncing universe. The horizontal axis is a comoving spatial coordinate, the vertical axis is conformal time. Plotted are the Hubble radius $|\mathcal{H}|^{-1}$ and the wavelength λ of a fluctuations with comoving wave number k.

Note that if the energy density at the bounce point is given by the scale η of Grand Unification ($\eta \sim 10^{16}$ GeV), then the physical wavelength of a perturbation mode corresponding to the current Hubble radius is of the order of 1 mm, i.e., in the far infrared. In this sense, the evolution of fluctuations in this bouncing cosmology is free of the trans-Planckian problem [25,26] which effects the evolution of fluctuations in all inflationary models in which the period of inflation lasts more than about 70 e-foldings (this number assumes that the scale of inflation is of the order of Grand Unification).

Since our bounce is nonsingular, the computation of the evolution of fluctuations is free of the matching condition ambiguities which affect the study of fluctuations in singular bouncing cosmologies such as the epyrotic scenario (see [27–32] for some early papers on the problem of matching fluctuations through the bounce in ekpyrotic cosmology).

There is another important difference in the study of cosmological fluctuations between nonsingular bouncing cosmologies and the inflationary scenario. It is usually argued that the exponential expansion of space during inflation red-shifts any preexisting matter and the related matter fluctuations, leaving behind a vacuum matter state. Thus, perturbations in this setup are quantum vacuum fluctuations [33]. On the other hand, in a bouncing cosmology the fluctuations are set up at low densities and temperatures in the contracting phase. There is no mechanism that red-shifts initial classical fluctuations. Thus, there is no reason to prefer vacuum over thermal initial perturbations. In the following, we will consider both choices.

B. Equations for cosmological perturbations

We begin by writing the metric including cosmological fluctuations in longitudinal gauge, assuming that there is no anisotropic stress (see [34] for a comprehensive discussion of the theory of cosmological perturbations and [35] for a briefer survey)

$$ds^2 = a(\eta)^2[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)d\mathbf{x}^2], \tag{18}$$

where $\Phi(\mathbf{x}, t)$ is the generalized Newtonian gravitational potential which represents the metric fluctuations. It is convenient to write the equations in terms of conformal time η defined via $dt = a(t)d\eta$.

The Einstein equations linearly expanded in Φ lead to the following equation of motion for the Fourier mode of Φ with comoving wave number k:

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi + k^2\Phi = 8\pi G\left(2\mathcal{H} + \frac{\phi''}{\phi'}\right)\tilde{\phi}'\delta\tilde{\phi} \tag{19}$$

where the derivative with respect to conformal time is denoted by a prime, $\mathcal{H} \equiv a'/a$, and $\delta\tilde{\phi}$ is the fluctuation in $\tilde{\phi}$. In deriving this equation, we have assumed that the background is dominated by the field ϕ. This will be the case except at the bounce.

In inflationary cosmology, it has proven to be convenient to use the variable ζ, the curvature fluctuation in comoving gauge, which in terms of Φ is given by

$$\zeta = \Phi + \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}(\Phi' + \mathcal{H}\Phi). \tag{20}$$

In any eternally expanding universe in which $1 + w \neq 0$, the variable ζ is conserved on super-Hubble scales in the absence of entropy fluctuations [36–38], since—neglecting terms of the order k^2—the equation of motion (19) for Φ is equivalent to

$$(1 + w)\dot{\zeta} = 0. \tag{21}$$

When considering the quantum theory of cosmological perturbations, it is important to identify the fluctuation variable which has a canonical kinetic term. It is with respect to this variable, commonly denoted by v, that the canonical commutation relations must be imposed (see [39,40] for the quantum theory of cosmological perturbations). It turns out that the variable is simply related to ζ

$$v = z\zeta, \tag{22}$$

where the background variable z is the following combination of the background metric and the background matter field ϕ (for simplicity we are assuming here only one matter field):

$$z = \frac{a\phi'}{\mathcal{H}}. \tag{23}$$

If the equation of state is constant in time, then $z(\eta)$ is proportional to $a(\eta)$.

The equation of motion for v is

$$v'' + \left[k^2 - \frac{z''}{z}\right]v = 0. \tag{24}$$

On sub-Hubble scales, it follows from (24) that v is performing harmonic oscillations as a function of conformal time. On the other hand, on super-Hubble scales v is frozen in and $v(\eta) \sim z(\eta)$.

In terms of the variable z, the relationship between the metric fluctuation Φ and the canonical field v takes the form [34]

$$\Phi = \frac{4\pi G}{k^2}\frac{\phi'^2}{\mathcal{H}}\left(\frac{v}{z}\right)'. \tag{25}$$

The variable ζ has proven to be a convenient variable to use in inflationary cosmology. It was therefore taken for granted that it would also be a useful variable in bouncing cosmologies, and that it would remain conserved between when the mode k exits the Hubble radius during the period of contraction at the time $-t_H(k)$ and the time $t_H(k)$ of reentry in the expanding phase. In the context of singular bouncing cosmologies, the Hwang-Vishniac [41] (Deruelle-Mukhanov [42]) matching conditions for fluctuations across the singularity lead to the conclusion that ζ should be conserved. However, as pointed out in [32], the applicability of these matching conditions is questionable since the matching conditions are not satisfied by the background.

Nonsingular bouncing cosmologies do not require ad-hoc matching conditions—the fluctuations can be followed through the bounce (as long as their amplitude remains sufficiently small such that linear perturbation theory does not break down). As has recently been shown in several examples of nonsingular bounces, the equation of motion for ζ develops singularities around the bounce point [43–45], whereas the equation of motion for Φ remains well defined. One of the reasons for the singularities in the equation of motion for ζ is that the comoving gauge has a singularity at a cosmological bounce. Thus, in the following we will follow the evolution of the fluctuations in terms of Φ.

If the initial fluctuations in the contracting phase are due to thermal matter, then the initial values of Φ and its derivative follow from the perturbations in the energy density of matter. If, on the other hand, we assume vacuum initial fluctuations, then the initial inhomogeneities are given in terms of the canonical variable v,and the initial values of Φ and $\dot{\Phi}$ must be induced from v via the relation (25).

C. General solutions

Let us briefly review the general solution of the equation of motion (19) for Φ on super-Hubble scales. We will keep the discussion quite general in this subsection and assume that the equation of state parameter is given by some w. In this case, $a(t)$ scales as

$$a(t) \sim t^p \tag{26}$$

with

$$p = \frac{2}{3(1+w)}. \tag{27}$$

From the definition of conformal time η it then follows that

$$\eta \sim t^{1-p}. \tag{28}$$

The condition for the Hubble radius crossing time $t_H(k)$ for a mode with comoving wave number k is

$$a(t_H(k))k^{-1} = H^{-1}(t_H(k)) \sim t_H(k). \tag{29}$$

Hence

$$\eta_H(k) \sim k^{-1}. \tag{30}$$

As is well known, one of the two modes of Φ on super-Hubble scales is constant, whereas the other is decaying in an expanding universe and growing in a contracting one. Specifically we have (see, e.g., [12])

$$\Phi(k, \eta) = D(k) + S(k)\eta^{-2\nu}, \tag{31}$$

where

$$2\nu = \frac{5+3w}{1+3w}, \tag{32}$$

and where $D(k)$ and $S(k)$ are independent of time and carry the information about the spectra of the two modes. In the following, we will determine the spectra of these two modes for various thermal and vacuum initial conditions.

D. Thermal initial conditions

Here we assume that the initial fluctuations are given by thermal matter perturbations. As was done in the case of string gas cosmology [46,47] (see [48] for a recent comprehensive review), we follow the matter perturbations up to Hubble radius crossing and then convert to metric fluctuations by making use of the perturbed Einstein constraint equation (the time-time component of the perturbed Einstein equations) which reads

$$-3\mathcal{H}(\mathcal{H}\Phi + \Phi') + \nabla^2\Phi = 4\pi G a^2 \delta T^0_0. \tag{33}$$

In the above, δT^0_0 is the fluctuation in the energy density, and ∇ is the comoving spatial gradient. At Hubble radius crossing all three terms on the left-hand side of the above equation are of the same order of magnitude. Hence, modulo a constant of the order 1, the Fourier space correlation function of Φ becomes

$$\langle|\Phi(k)|^2\rangle = 16\pi^2 G^2 k^{-4} a^4 \langle|\delta T^0_0(k)|^2\rangle, \qquad (34)$$

where the pointed brackets indicate ensemble averaging.

The energy density fluctuations are determined by thermodynamics. First, we express the momentum space energy density correlation function for comoving wave number k in terms of the rms position space mass fluctuation

$$\delta M(R)^2 = R^3 \langle|\delta T^0_0(k)|^2\rangle, \qquad (35)$$

where $R = ak^{-1}$ is the physical radius of the region corresponding to the wave number k. The mass fluctuations are determined by the specific heat capacity C_V

$$\delta M(R)^2 = T^2 C_V(R), \qquad (36)$$

where T is the temperature of the system. For a gas of point particles, the heat capacity is proportional to R^3, i.e.,

$$C_V(R) = c_V T^3 R^3, \qquad (37)$$

where c_V is a constant. Note that this result is in agreement with the intuition that on scales larger than the thermal correlation length T^{-1}, the heat capacity scales as a random walk.

Inserting (35)–(37) into (34) we obtain the following power spectrum for Φ:

$$k^3\langle|\Phi(k)|^2\rangle = 16\pi^2 G^2 k^{-1} T^5 c_V. \qquad (38)$$

We need to evaluate this expression at Hubble radius crossing $t_H(k)$ since we will be using the corresponding value as the initial condition for the evolution of Φ on super-Hubble scales:

$$k^3\langle|\Phi(k)|^2\rangle(t_H(k)) = 16\pi^2 G^2 k^{-1} T^5(t_H(k)) c_V. \qquad (39)$$

We now use (39) to infer the power spectra of the S and D modes in the case of thermal matter initial conditions. We are assuming that the initial value of Φ at Hubble radius crossing gets distributed equally among the two modes The power spectrum $P_D(k)$ of the constant mode D is the same as that of Φ at Hubble radius crossing

$$P_D(k) \sim k^{-1+(5p/(1-p))} \qquad (40)$$

where the second exponent comes from making use of

$$T(t_H(k)) \sim a^{-1}(t_H(k)) \sim t_H(k)^{-p} \sim k^{p/(1-p)}. \qquad (41)$$

The power spectrum of S is the spectrum of Φ at Hubble radius crossing modulated by the factor $\eta_H(k)^{2\nu}$:

$$P_S(k) \sim k^{-1+(5p/(1-p))-4\nu}. \qquad (42)$$

In the example we are interested in, fluctuations leave the Hubble radius in the matter epoch and hence $p = 2/3$ and $w = 0$. Thus, from the above we see that the spectra of D and S scale as

$$P_D(k) \sim k^9 \qquad P_S(k) \sim k^{-1}. \qquad (43)$$

The spectrum of the D mode is extremely blue. The blue tilt is due to the thermal suppression of the spectrum at large wavelengths. It is also easy to understand why the spectrum of the S mode is less blue than that of the D mode: the S mode grows on super-Hubble scales, and large wavelength modes experience the growth for a longer period of time. Our calculation shows that the difference in growth on super-Hubble scales dominates over the thermal suppression of long wavelength modes.

E. Vacuum initial conditions

Vacuum initial conditions are given in terms of the canonically normalized variable v_k being in its quantum vacuum state [34]

$$v_k(\eta) \sim k^{-1/2} e^{i\eta k}. \qquad (44)$$

Inserting this into (25) and making use of the fact that on sub-Hubble scales the derivative of the oscillating factor dominates over the derivative of other terms leads to the following initial conditions in terms of Φ:

$$\Phi_k(\eta) \sim i\frac{4\pi G}{k^{3/2}}\frac{\phi l^2}{z\mathcal{H}}, \qquad (45)$$

i.e., a spectrum which is proportional to $k^{-3/2}$. The same conclusion can be reached [28] by starting with vacuum fluctuations in the matter field ϕ and inserting the result into the equations expressing Φ and $\dot{\Phi}$ in terms of the matter field. Making use of (23) to eliminate z and of the background Friedmann equation to eliminate $\dot{\phi}$ in favor of H, we find the following result for the power spectrum of Φ:

$$P_\Phi(k, \eta) \equiv \frac{1}{2\pi^2} k^3 |\Phi_k(\eta)|^2 \simeq \frac{3}{\pi}\left(\frac{H(\eta)}{m_{\rm pl}}\right)^2. \qquad (46)$$

Making use of the definition of z from (23) it follows that the time-dependent terms in (45) scale as H^{-1}. Thus,

$$\Phi_k(t) \sim k^{-3/2} t^{-1}, \qquad (47)$$

which allows us to evaluate the result at Hubble radius crossing

$$\Phi_k(t_H(k)) \sim k^{-(3/2)+(1/(1-p))}. \qquad (48)$$

The above result (48) allows us to compute the spectra of both D and S modes of Φ on super-Hubble scales, assuming—as we did in the previous subsection—that $\Phi_k(t_H(k))$ sources both modes equally:

$$\Phi_D(k) \sim k^{-(3/2)+(1/(1-p))} \qquad (49)$$

$$\Phi_S(k) \sim k^{-(3/2)+(1/(1-p))-2\nu}. \quad (50)$$

In the case we are interested in $p = 2/3$, $w = 0$ and $2\nu = 5$ we obtain

$$\Phi_D(k) \sim k^{3/2} \quad (51)$$

$$\Phi_S(k) \sim k^{-7/2}. \quad (52)$$

Note that, as pointed out in [12], the S mode leads to a scale-invariant spectrum of fluctuations of ζ in the contracting phase.

This compares to the results obtained for ekpyrotic type contraction [28] where $p = 0$, $w = \infty$ and $2\nu = 1$ and therefore

$$\Phi_D(k) \sim k^{-1/2} \quad (53)$$

$$\Phi_S(k) \sim k^{-3/2}, \quad (54)$$

which leads to a scale-invariant spectrum for the S mode which is growing in the phase of contraction.

As we expect from the Hwang-Vishniac (Deruelle-Mukhanov) matching conditions, the S mode in the contracting phase will couple with a k^2 suppression to the dominant mode in the expanding phase. If this is realized, we will obtain a scale-invariant spectrum of curvature fluctuations in the expanding phase. In the following subsection we will evolve the fluctuations through the nonsingular bounce and infer the spectrum at late times. We indeed find a late-time scale-invariant spectrum.

F. Evolution of the fluctuations through the bounce

Let us step back and write down the equation of motion for Φ in a slightly modified form (which is equivalent to (19) except that we allow for a general speed of sound c_s which is equal to 1 in our scalar field model)

$$\Phi_k'' + 2\sigma\mathcal{H}\Phi_k' + k^2 c_s^2 \Phi_k = 0, \quad (55)$$

where

$$\sigma \equiv -\frac{\ddot{H}}{2H\dot{H}}. \quad (56)$$

1. Contracting phase

In the contracting phase Eq. (55) takes the form

$$\Phi_k'' + \frac{1+2\nu_c}{\eta - \tilde{\eta}_{B-}}\Phi_k' + k^2 c_s^2 \Phi_k = 0, \quad (57)$$

with

$$\nu_c \equiv \frac{5+3w_c}{2(1+3w_c)}, \quad (58)$$

where the subscript "c" indicates that we are discussing the contracting phase. The general analytical solution is

$$\Phi_k = (\eta - \tilde{\eta}_{B-})^{-\nu_c}\{k^{-\nu_c} D_- J_{\nu_c}[c_s k(\eta - \tilde{\eta}_{B-})] + k^{\nu_c} S_- J_{-\nu_c}[c_s k(\eta - \tilde{\eta}_{B-})]\}, \quad (59)$$

where the coefficients D_- and S_- can be determined by the initial condition of the gravitational potential as described in the two previous subsections for different sets of initial conditions. In the above, η_{B-} is a fixed time that corresponds to when the singular bounce would occur if the universe were to remain matter-dominated.

Note that, when the wavelength of the perturbation is larger than the Hubble radius with $k \ll |\mathcal{H}|$, the asymptotical form of Φ_k can be written as

$$\Phi_k^c = \bar{D}_- + \frac{\bar{S}_-}{(\eta - \tilde{\eta}_{B-})^{2\nu_c}}, \quad (60)$$

where we define

$$\bar{D}_- \equiv \frac{c_s^{\nu_c} D_-}{2^{\nu_c}\Gamma(1+\nu_c)}, \qquad \bar{S}_- \equiv \frac{2^{\nu_c} S_-}{c_s^{\nu_c}\Gamma(1-\nu_c)}. \quad (61)$$

As discussed in previous subsections, the $\bar{D}_-$ mode is constant and the $\bar{S}_-$ mode is growing in a contracting universe. Note from the definition of ζ_k we have to leading order in k

$$\zeta_k^c = \frac{5+3w_c}{3(1+w_c)}\bar{D}_-. \quad (62)$$

Thus, to this order, in the contracting phase ζ_k is determined by the constant mode of Φ_k which is subdominant. As discussed in detail in [28] the S mode does affect ζ when k^2 corrections to the solutions are taken into account.

If we match the asymptotic form for Φ (60) to the initial power spectrum of Φ (see (46)) at the Hubble radius crossing and assume that the initial power is equally distributed into the two modes, we obtain

$$\frac{1}{2\pi^2}k^3|S_-(k)|^2 \simeq \frac{3}{\pi}m_{\rm pl}^2 t_H(k)^{-2}\eta_H(k)^{4\nu_c}, \quad (63)$$

where the subscript H stands for the time of Hubble radius crossing.

2. Bouncing phase

As we have shown in the section on the background dynamics, the contribution of the higher derivative terms in the Lee-Wick model becomes more and more important as the universe contracts and will lead to a nonsingular bounce. Thus, the universe will exit from the phase of matter-dominated contraction at some time t_{B-}, and then the equation of state (EoS) of the universe will cross -1 and fall to negative infinity rapidly. Correspondingly, the Hubble parameter reaches zero and leads to a bounce of the universe at the time η_B. After the bounce, the Lee-Wick field will recover its normal state with the higher derivative terms rapidly decreasing in importance.

It is rather complicated to solve the perturbation equation directly from Eq. (55). In order to solve the equation analytically, we need to make some approximations to simplify it. Our approximation consists of choosing a convenient modelling of the Hubble parameter near the bounce of the form

$$H = \alpha t \tag{64}$$

with some positive constant α which has dimensions of k^2 and whose magnitude is set by the microphysics of the bounce, in our case by the mass M of the Lee-Wick scalar. The time of the bounce was chosen to be $t = 0$. In this case, we can obtain an analytical form for the comoving Hubble parameter in the bouncing phase:

$$\mathcal{H} = \frac{\frac{y}{3}(\eta - \eta_B)}{1 - \frac{y}{6}(\eta - \eta_B)^2}, \qquad y = \frac{12}{\pi}\alpha a_B^2, \tag{65}$$

where a_B denotes the value of the scale factor at the bounce point η_B.

Since the above parametrization should be valid only in the neighborhood of the bounce point, the quadratic and higher order terms of $|\eta - \eta_B|$ can be neglected. Consequently, the perturbation equation takes the following form:

$$\Phi_k'' + 2y(\eta - \eta_B)\Phi_k' + \left(c_s^2 k^2 + \frac{2}{3}y\right)\Phi_k = 0. \tag{66}$$

The solution of this equation can be written as

$$\Phi_k = \left\{E_k H_l[\sqrt{y}(\eta - \eta_B)] + F_{k1}F_1\left[-\frac{l}{2}, \frac{1}{2}, y(\eta - \eta_B)^2\right]\right\} \times \exp[-y(\eta - \eta_B)^2], \tag{67}$$

which is constructed from the l-th Hermite polynomial and a confluent hypergeometric function with

$$l \equiv -\frac{2}{3} + \frac{c_s^2 k^2}{2y} \tag{68}$$

and two undetermined coefficients E_k, F_k. These two functions are linearly independent, and their asymptotical behaviors are mainly determined by the parameter l.

When $c_s^2 k^2 \gg y$, i.e., the wave number of the mode is larger than the mass scale of the bounce, then both functions are oscillating. This case was already studied in Ref. [45].

However, in the current article we are interested in the opposite limit, the limit in which the wavelength is much larger than the inverse mass scale of the bounce, i.e., the limit when

$$c_s^2 k^2 \ll y. \tag{69}$$

As we have argued in the section of our article on the background evolution, the bounce takes place very fast and thus the condition (69) will be satisfied for all wavelengths we are interested in. In this case, we can expand the solution of the perturbation equation in a power series in terms of $\sqrt{y}(\eta - \eta_B)$. Then the solution is given by

$$\Phi_k^b = \hat{F}_k + \hat{E}_k\sqrt{y}(\eta - \eta_B) + (-1 - l)\hat{F}_k y(\eta - \eta_B)^2 + O(y^{3/2}(\eta - \eta_B)^3), \tag{70}$$

with

$$\hat{E}_k \equiv -\frac{2^{1+l}\sqrt{\pi}}{\Gamma(-\frac{l}{2})}E_k, \tag{71}$$

$$\hat{F}_k \equiv \frac{2^l\sqrt{\pi}}{\Gamma(\frac{1-l}{2})}E_k + F_k, \tag{72}$$

and the subscript "b" represents the bouncing phase. In this case, we have

$$\zeta_k^b \simeq \hat{F}_k\left[1 + \frac{c_s^2 k^2}{2}(\eta - \eta_B)^2\right]. \tag{73}$$

Therefore, the conservation of ζ_k is realized by the mode $\hat{F}_k$ when the bounce is fast enough.

Now we study how to establish the coefficients $\hat{E}_k$ and $\hat{F}_k$. We need to use the Hwang-Vishniac [41] (Deruelle-Mukhanov [42]) matching condition to link the fluctuations in contracting phase with those in the bouncing phase at the momentum η_{B-}. Note that since we are matching two contracting universes across a nonsingular surface, the background satisfies the matching conditions, unlike the situation in the ekpyrotic scenario with a singular bounce, Thus, it is justified to apply the matching conditions [32].

The matching conditions say that both Φ_k and

$$\hat{\zeta}_k \equiv \zeta_k + \frac{c_s^2 k^2}{3}\frac{\Phi_k}{\mathcal{H}^2 - \mathcal{H}'} \tag{74}$$

are continuous on the matching surface of constant energy density. Taking use of matching conditions in the solutions (60) and (70), we can obtain the following relations:

$$\hat{E}_k\sqrt{y}(\eta_{B-} - \eta_B) = -\left(\frac{1}{3} + 2l\right)\Phi_k^c - \hat{\zeta}_k^c|_{B-},$$
$$\hat{F}_k = \left(\frac{4}{3} + 2l\right)\Phi_k^c + \hat{\zeta}_k^c|_{B-}. \tag{75}$$

These relations show that the constant and growing modes of gravitational potential get mixed during the bounce. However, if we consider large wavelengths compared to the duration of the bounce, the second relation shows that ζ_k is indeed conserved across the bounce.

3. Expanding phase

After the bounce, the higher derivative terms of the Lee-Wick field rapidly decay. Therefore, a phase of matter-dominated expansion starts at the time η_{B+}. In the absence of interactions between the two scalar fields, the background cosmology will be time-symmetric about the bounce point. During the period after η_{B+} the background evolution is the time reverse of the contracting phase. In the case $\lambda \neq 0$ an asymmetric bounce is possible. To render our analysis more general, we assume that the equation of state in the expanding phase is w_e which could be different from that in the contracting stage which is w_c.

The equation of motion for the gravitational potential is similar as Eq. (57) but with the indexes

$$\nu_e \equiv \frac{5+3w_e}{2(1+3w_e)} \tag{76}$$

and "$B+$" instead of ν_c and "$B-$". Then the solution on super-Hubble scales takes the form

$$\Phi_k^e = \bar{D}_+ + \frac{\bar{S}_+}{(\eta - \tilde{\eta}_{B+})^{2\nu_e}}, \tag{77}$$

with

$$\tilde{\eta}_{B+} \equiv \eta_{B+} - \frac{2}{1+3w_e}\frac{1}{\mathcal{H}_{B+}}. \tag{78}$$

The $\bar{D}_+$ mode of the gravitational potential is constant in time, as is the $\bar{D}_-$ mode in contracting phase. However, the role of the S mode is very different. In the expanding phase $\bar{S}_+$ is the subdominant decreasing mode, whereas in the contracting phase $\bar{S}_-$ is the dominant expanding mode. Therefore, the dominant mode of the curvature perturbation in the period of expansion is $\bar{D}_+$. As we will show in the following, it inherits contributions from both $\bar{D}_-$ and $\bar{S}_-$ since these modes mix during the bounce.

To determine the coefficients of the two modes in the expanding phase, we need to apply the matching condition again, this time at the surface η_{B+}. A straightforward calculation yields

$$\hat{E}_k\sqrt{y}(\eta_{B+} - \eta_B) = -\left(\frac{1}{3} + 2l\right)\Phi_k^e - \hat{\zeta}_k^e|_{B+},$$
$$\hat{F}_k = \left(\frac{4}{3} + 2l\right)\Phi_k^e + \hat{\zeta}_k^e|_{B+}. \tag{79}$$

Note again that it is justified to apply the matching conditions since the universe is expanding on both sides of the matching surface and thus the background also satisfies the matching conditions.

By combining Eqs. (75) and (79), we can establish the relation between the gravitational potentials in contracting and expanding phases. Since $\bar{S}_+$ is a decaying mode, we will not write down its expression and focus our attention instead on the dominant mode $\bar{D}_+$. In terms of the modes in the contracting phase, it is given by

$$\bar{D}_+ = \frac{(5+3w_c)(1+w_e)}{(1+w_c)(5+3w_e)}\bar{D}_- + \frac{3(1+w_e)}{(5+3w_e)}c_s^2k^2 \times\left\{\frac{\eta_{B+}-\eta_B}{\eta_{B-}-\eta_B}M_+\left(\frac{2\bar{D}_-}{3(1+w_c)} - \frac{\bar{S}_-}{(\eta_{B-}-\tilde{\eta}_{B-})^{2\nu_c}}\right) - \frac{5+3w_c}{3(1+w_c)}M_+\bar{D}_- + M_-\left(\bar{D}_- + \frac{\bar{S}_-}{(\eta_{B-}-\tilde{\eta}_{B-})^{2\nu_c}}\right)\right\} + O(k^4), \tag{80}$$

where we defined the parameters

$$M_\pm \equiv \frac{2}{9\mathcal{H}_{B\pm}^2(1+w_c^e)} + \frac{1}{y}, \tag{81}$$

which are independent of k.

From the above result we see that both the constant and growing modes of gravitational potential in the contracting phase affect the dominant mode after the bounce. However, the growing mode is suppressed by k^2 on large scales whereas the constant one transfers through the bounce without a change in the spectral index. These results agree with what is obtained using the matching conditions at a singular hypersurface between the contracting and the expanding phase, as shown in [29].

Inspecting our result (80), we see that there are two ways to obtain a scale-invariant spectrum of cosmological perturbations after the bounce. The first is to consider a model in which the $D-$ mode in the contracting phase has a scale-invariant spectrum, i.e., $D-(k) \sim k^{-3/2}$, the other is to take a scenario where $S-(k) \sim k^{-7/2}$. As follows from (49), the first possibility is realized if $p = \infty$, i.e., in an inflationary contracting phase. The second way is realized in the case of a matter-dominated contraction, a possibility already pointed out in [12] (see also [49]. The Lee-Wick model yields a natural realization of this way.

Let us now come back to our Lee-Wick background, and assume quantum vacuum fluctuations. We insert the values $w_c = w_e = 0$, $c_s = 1$ into (80) and assume a symmetric fast bounce. Thus

$$\bar{D}_+ = \bar{D}_- + \left[-\frac{4}{5}\bar{D}_- + \frac{3}{5}\frac{\mathcal{H}_{B-}^5}{2^4}\bar{S}_-\right]\frac{2k^2}{9\mathcal{H}_{B-}^2}. \tag{82}$$

As discussed in the subsection on vacuum initial conditions, then if the initial conditions are imposed at a time η sufficiently early compared to the transition point $B-$, we have $\Phi_k^{\rm ini} \propto k^{-(3/2)}$ and therefore obtain $\bar{D}_- \propto k^{3/2}$ and $\bar{S}_- \propto k^{-(7/2)}$. Substituting these relations into Eq. (82), one can see that whereas the contribution of $\bar{D}_-$ to the final spectrum of $D+$ vanishes on large scales, the contribution of $\bar{S}_-$ which starts out deep red is blue-tilted by exactly the right amount to yield a final spectrum proportional to $k^{-(3/2)}$ which is the scale-invariant form.

To compute the amplitude of the spectrum, we insert the values (81) and the expression (63) for the value of S_- into

(82) and use the background Friedmann equation to replace the Hubble parameter by the energy density. This yields

$$\bar{D}_{+} = -\frac{\sqrt{\rho_{B-}}}{10\sqrt{2}M_p^2}k^{-(3/2)}. \tag{83}$$

Therefore, the power spectrum of the gravitational potential in this case can be expressed as

$$P_\Phi \equiv \frac{k^3}{2\pi^2}|\bar{D}_+|^2 = \frac{\rho_{B-}}{(20\pi)^2 M_p^4}. \tag{84}$$

Note that as long as $M \ll M_{\rm pl}$, the power spectrum of metric fluctuations remains much smaller than 1, and thus linear perturbation theory is applicable throughout the bouncing phase.

G. Numerical analysis

Our analytical calculations involve approximations. Specifically, in the contracting phase the scalar fields and hence the equation of state are oscillating. But in our analytical analysis we have replaced the time-dependent equation of state parameter by its temporal average. It is thus important to confirm the results by numerical integration of the full equations, namely, Eq. (19) coupled to the equation for the scalar matter field fluctuation.

At first sight, it appears that Eq. (19) contains a singularity at all turnaround points of ϕ. Such singularities are known from the study of the evolution of Φ during reheating taking into account the oscillatory nature of the inflaton field [50,51]. However, this singularity is actually not present. Let us consider in addition to the dynamical perturbed Einstein Eq. (19) the perturbed Einstein constraint equation

$$\Phi' + \mathcal{H}\Phi = 4\pi G(\phi'\delta\phi - \tilde{\phi}'\delta\tilde{\phi}). \tag{85}$$

Inserting (85) into (19) yields

$$\Phi'' + 6\mathcal{H}\Phi' + 2(\mathcal{H}' + 2\mathcal{H}^2)\Phi + k^2\Phi = 8\pi G\left(2\mathcal{H} + \frac{\phi''}{\phi'}\right)\phi'\delta\phi, \tag{86}$$

from which it is clear that the singularity has disappeared. Thus, we numerically solve (86) coupled to the perturbed ϕ equation

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + (k^2 + a^2V_{\phi\phi})\delta\phi = 4\phi'\Phi' - 2a^2V_\phi\Phi, \tag{87}$$

where the subscripts on V indicate the variables with respect to which the potential is differentiated.

Figures 5 and 6 show the results of our numerical integration. The first figure shows the evolution in time of the metric fluctuation Φ as a function of physical time (left side) and conformal time (right side) for different values of the comoving wave number k. We have chosen the bounce point to correspond to physical and conformal time 0. The initial conditions for Φ were set at the initial time of the simulation according to the vacuum initial condition prescription discussed earlier. We see from this figure that before the bounce the perturbations are dominated by the growing mode. When the universe enters the bouncing phase, we see that the amplitude approaches a constant and passes smoothly through the bounce. The numerical evolution agrees well with the analytical solution (70) in the bouncing phase. After the bounce, the perturbations are dominated by the constant mode. The

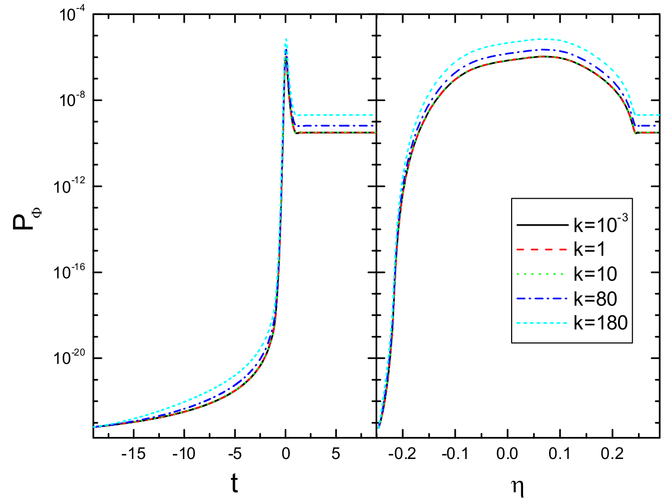

FIG. 5 (color online). Result of the numerical evolution of the curvature perturbations with different comoving wave numbers k in the Lee-Wick bounce. The horizontal axis in the left panel is cosmic time, and in the right panel it is comoving time. The initial values of the background parameters are the same as in Fig. 1. The units of the time axis are $M_{\rm rec}^{-1}$, the comoving wave number k is unity for $k = M_{\rm rec}$, as in Fig. 2.

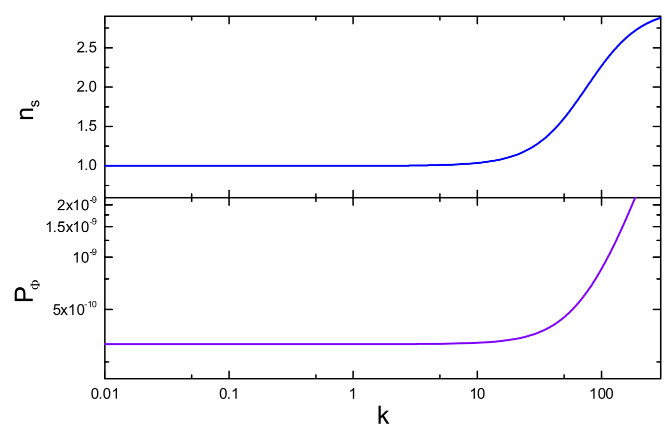

FIG. 6 (color online). Plot of the power spectrum of the curvature perturbation Φ (lower panel) and of the spectral index (upper panel) as functions of comoving wave numbers k in the Lee-Wick bounce. The initial values of the background parameters are the same as in Fig. 1.

numerical evolution demonstrates that this mode can be inherited from the growing mode in contracting phase.

Figure 6 shows the power spectrum of Φ (lower panel) and the spectral index n_s (upper panel) as a function of comoving wave number k. On large scales (small values of k), the power spectrum tends to a constant. The rise of the spectrum for large values of k is on scales which are comparable to maximal value of the Hubble rate, i.e., for modes which have not spent time outside of the Hubble radius.

One may worry that the existence of the perturbations of the Lee-Wick scalar could lead to an significant amplification on the physical modes of the density perturbations. However, in the case we considered, this amplification effect is secondary and so can be neglected roughly. The reason is as follows. In our background, the universe is usually dominated by the field with the lower mass. This is true in the periods far before and after the bounce. Thus, the contribution of the normal scalar ϕ dominates over that of the Lee-Wick scalar in the contracting and expanding phases. In these two phases, the perturbation of the ghost mode only appears as an entropy mode. Since its mass is heavy, the mode is hardly excited. Next, we argue that the contribution from the Lee-Wick ghost mode is also bounded in the bouncing phase. From the analysis in previous sections of the article, we have already learned that around the bounce the normal scalar has entered the slow-rolling region as in inflation while the Lee-Wick scalar still keeps oscillating. In this case we can use the approximations $\dot{\phi} \simeq -m^2\phi/3H$ and $\dot{\tilde{\phi}} \simeq M\tilde{\phi}$. Making use of these two relations and the equation of motion for ϕ, we can further express the right-hand side of Eq. (19) as follows:

$$\left(2\mathcal{H} + \frac{\phi''}{\phi'}\right)\tilde{\phi}'\delta\tilde{\phi} = -a^2m^2\phi\frac{\dot{\tilde{\phi}}}{\dot{\phi}}\delta\tilde{\phi} \simeq 3a^2HM\tilde{\phi}\delta\tilde{\phi}. \quad (88)$$

Since around the bounce the Hubble parameter goes through zero, the above term also becomes very small in this period. Therefore, we reach the conclusion that the modes of the Lee-Wick scalar only make a secondary contribution to the curvature perturbations in our model. A more detailed analysis is performed in the Appendix.

In order to support this argument, we have numerically calculated the perturbation $\delta\tilde{\phi}$ exactly, and we show its spectrum $P_{\delta\tilde{\phi}}$ in Fig. 7 which is normalized by the Planck mass. For comparison, in Fig. 8 we also plot the evolution of the ratio of theright-hand side to the term $k^2\Phi$ in Eq. (19). In both two figures we take $k = 10M_{\rm rec}$. The numerical results also show that the contribution of the Lee-Wick scalar is negligible.

As a side remark, it would be interesting to study the effect of the perturbations of the Lee-Wick mode in other Lee-Wick models. For example, if the Lee-Wick scalar would dominate for a while in the expanding phase, its perturbation might give rise to an amplification of the physical mode, and so could not be neglected as in the current article. We wish to give a complete analysis in future works.

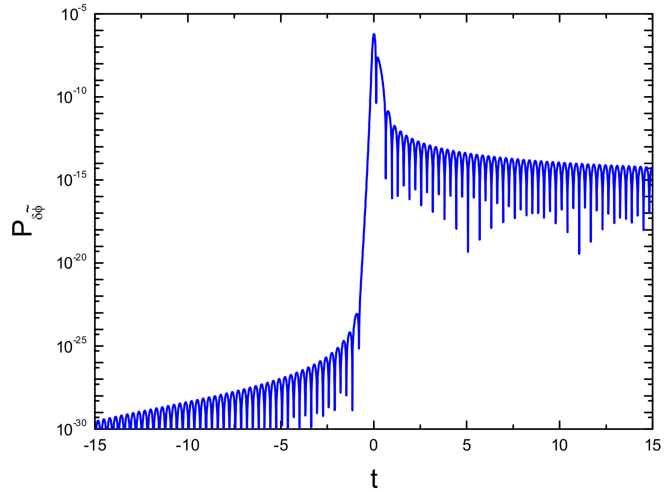

FIG. 7 (color online). Power spectrum of the Lee-Wick modes $\delta\tilde{\phi}$ as a function of cosmic time in the Lee-Wick bounce model (the blue dashed line). The comoving wave number is taken to be $k = 10$. The initial values of the background parameters are the same as in Fig. 1.

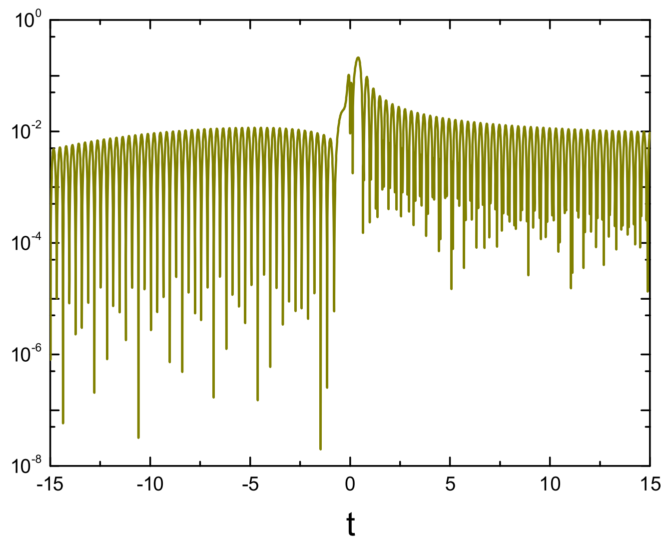

FIG. 8 (color online). Plot of the evolution of the absolute value of the ratio of the right-hand side to the $k^2\Phi$ term in Eq. (19). The comoving wave number is taken to be $k = 10$. The initial values of the background parameters are the same as in Fig. 1.

V. GRAVITATIONAL WAVES

Now we turn to consider the evolution of gravitational waves (tensor perturbations) in our background, assuming they start out in the vacuum state on sub-Hubble scales in the contracting phase. Since at the level of linear perturbation theory scalar metric fluctuations and gravitational waves decouple, we can focus on a metric containing only gravitational waves propagating in the background. The standard form of this metric in a spatially flat Friedmann-Robertson-Walker background is

$$ds^2 = a(\eta)^2[-d\eta^2 + (\delta_{ij} + \bar{h}_{ij})dx^i dx^j], \qquad (89)$$

where the Latin indexes run over the spatial coordinates, and the tensor perturbation $\bar{h}_{ij}$ is real, transverse and traceless, i.e.,

$$\bar{h}_{ij} = \bar{h}_{ji}; \qquad \bar{h}_{ii} = 0; \qquad \bar{h}_{ij,j} = 0. \qquad (90)$$

Because of these constraints, we only have 2 degrees of freedom in $\bar{h}_{ij}$ which correspond to two polarizations of gravitational waves. For each polarization state (labeled by r in the following), we can write $\bar{h}_{ij}(\eta, \mathbf{x})$ as a scalar field $h^r(\eta, \mathbf{x})$ multiplied by a polarization tensor e^r_{ij} which is constant in space and time.

If matter contains an anisotropic stress tensor σ_{ij}, there is a nonvanishing source term in the equation of motion for tensor perturbations, namely

$$\bar{h}''_{ij} + 2\frac{a'}{a}\bar{h}'_{ij} - \nabla^2\bar{h}_{ij} = 16\pi G a^2 \sigma_{ij}. \qquad (91)$$

If matter consists of a set of canonically normalized scalar fields or a set of perfect fluids, there is no anisotropic stress and thus no source term at linear order in perturbation theory for gravitational waves.

As usual, we go to Fourier space. The Fourier transformations of the tensor perturbations and anisotropic stress tensor are given by

$$\bar{h}_{ij}(\eta, \mathbf{x}) = \sum_{r=1}^{2}\int \frac{d^3k}{(2\pi)^{3/2}} h^r(\eta, \mathbf{k}) e^r_{ij} e^{i\mathbf{k}\mathbf{x}}. \qquad (92)$$

In order to canonically quantize the gravitational waves, it is important to identify the variable in terms of which the action has a canonical kinetic term. This variable turns out to be (see [52] for a derivation)

$$v^r_k = \sqrt{\frac{(e^r)^i_j (e^r)^j_i}{32\pi G}} a h^r_k \qquad (93)$$

(where h^r_k is a shorthand notation for $h^r(\eta, \mathbf{k})$) in terms of which the Einstein action expanded to second order in v^r becomes

$$S = \sum_{r=1}^{2}\frac{1}{2}\int [|(v^r_k)'|^2 - \left(k^2 - \frac{a''}{a}\right)|v^r_k|^2] d\eta d^3k. \qquad (94)$$

The resulting equation of motion for v^r is

$$(v^r)'' + \left(k^2 - \frac{a''}{a}\right)v^r = 0. \qquad (95)$$

We are interested in computing the power spectrum of the tensor modes. Making use of (93), it is related to the power spectrum of v (which is the same for each polarization state) via

$$P^T_k(h) = a^{-2} 32\pi G \sum_{r=1}^{2} P_k(v^r) = a^{-2} 64\pi G \frac{k^3}{2\pi^2}|v_k|^2. \qquad (96)$$

The tensor spectral index n_T is defined by

$$n_T \equiv \frac{d\ln P^T}{d\ln k}. \qquad (97)$$

The evolution of tensor perturbations is very similar to that of scalar perturbations. Initially the perturbations are inside the Hubble radius in the far past. Since the Hubble radius shrinks in the contracting phase, the modes with small comoving wave number exit the Hubble radius. After that the universe bounces to an expanding phase, so these Fourier modes will return into the Hubble radius.

In the current article we focus on a mode with small k so that it exits the Hubble radius in the contracting phase (rather than the bounce phase), then passes through the bounce and finally reenters the Hubble radius during the expanding phase.

We divide the time interval into three periods like we did for the analysis of scalar metric fluctuations. During the phase when the universe is contracting with an equation of state oscillating around $w = 0$, we have

$$v = (\eta - \tilde{\eta}_{B-})^{1/2}\{A^T_k J_{-(3/2)}[k(\eta - \tilde{\eta}_{B-})] + B^T_k J_{3/2}[k(\eta - \tilde{\eta}_{B-})]\}, \qquad (98)$$

where $\tilde{\eta}_{B-} = \eta_{B-} - 2/\mathcal{H}_{B-}$. Here, the parameters A^T_k and B^T_k can be determined by the initial condition for gravitational waves, which is taken as the Bunch-Davies vacuum

$$v \sim e^{-ik\eta}/\sqrt{2k}. \qquad (99)$$

So we have

$$A^T_k = i\frac{\sqrt{\pi}}{2} \quad \text{and} \quad B^T_k = -\frac{\sqrt{\pi}}{2}. \qquad (100)$$

Therefore, the asymptotic form of the solution to the tensor perturbation in the contracting phase is

$$v(k,\eta) = \begin{cases} -\frac{i}{\sqrt{2}}k^{-(3/2)}(\eta - \tilde{\eta}_{B-})^{-1}, & \text{outside Hubble radius} \\ \frac{1}{\sqrt{2k}}e^{-ik(\eta - \tilde{\eta}_{B-})}, & \text{inside Hubble radius} \end{cases}. \qquad (101)$$

During the bouncing phase, we have the approximate relation

$$\frac{a''}{a} \simeq \frac{4}{\pi}\alpha a_B^2 = \frac{y}{3}. \qquad (102)$$

Solving Eq. (95), we have

$$v(k,\eta)=\begin{cases} C_k^T\cos[l(\eta-\eta_B)]+D_k^T\sin[l(\eta-\eta_B)], & k^2\geq\frac{y}{3};\\ C_k^T e^{l(\eta-\eta_B)}+D_k^T e^{-l(\eta-\eta_B)}, & k^2<\frac{y}{3},\end{cases} \tag{103}$$

where we define $l^2=|k^2-\frac{y}{3}|$. Since the Hubble parameter approaches zero when the universe is transiting from the contracting to the expanding phase, all fluctuation modes return to the sub-Hubble region, but only for a very brief time. However, from the above solution we interestingly find that $k_{\rm ph}^2(\sim k^2/a_B^2)$ and $\dot{H}(\sim\alpha)$ are comparable.

After the bounce, an expanding phase with its EoS $w=0$ takes place. So the solution to the gravitational waves is given by

$$v=(\eta-\tilde{\eta}_{B+})^{1/2}\times\{E_k^T J_{-(3/2)}[k(\eta-\tilde{\eta}_{B+})] + F_k^T J_{3/2}[k(\eta-\tilde{\eta}_{B+})]\}, \tag{104}$$

where $\tilde{\tau}_{B+}=\tau_{B+}-2/\mathcal{H}_{B+}$. This solution takes on the asymptotic form,

$$v\simeq\sqrt{\frac{2}{\pi}}\frac{F_k^T}{3}k^{3/2}(\eta-\tilde{\eta}_{B+})^2, \tag{105}$$

when the modes are outside the Hubble radius.

Having obtained the solutions of the tensor perturbations in the different phases, now we need to match these solutions and determine the coefficients C_k^T, D_k^T, E_k^T and F_k^T respectively. This procedure is analogous to the matching process of scalar perturbations performed in the previous section. For a nonsingular bounce scenario such as the bounce we are considering, the continuity of the background evolution implies that both v and v' are able to pass through the bounce smoothly. So we match v and v' in (101) and (103) on the surface τ_{B-}, and those in (103) and (104) on the surface τ_{B+}. With these matching conditions, we can determine all the coefficients and finally get the solution for v at late times.

Since in the specific model we considered in this article, the evolution of the universe is symmetric with respective to the bounce point, we can simply take $\mathcal{H}_{B-}\simeq-\mathcal{H}_{B+}$. In addition, we have shown that the bounce takes place very fast on the time scale set by k^{-1}, so we have $l(\tau_{B+}-\tau_{B-})\gg 1$. Therefore, we eventually obtain the approximate result

$$F_k^T\simeq i\frac{\sqrt{\pi}\mathcal{H}_{B+}^3}{8k^3} \tag{106}$$

and the asymptotical form of v in the final stage can be expressed as

$$v^f\rightarrow i\frac{\sqrt{2}}{24}\frac{\mathcal{H}_{B+}^3}{k^{3/2}}(\eta-\tilde{\eta}_{B+})^2. \tag{107}$$

Now we are able to derive the power spectrum of primordial gravitational waves. From the definition of Eq. (96), the primordial power spectrum is given by

$$P_T(k)=G\frac{32k^3}{\pi}\left|\frac{v^f}{a}\right|^2=\frac{2\rho_{B+}}{27\pi^2M_p^4}. \tag{108}$$

From Eq. (108), we can read that the spectrum are scale-invariant on large scales (which is consistent with the result in Ref. [53]).

Comparing our result of the tensor power spectrum with the result (84) for the power spectrum of scalar metric fluctuations, we obtain a tensor to scalar ratio of the order of 30, which is in excess of the current observational bounds. The exact value of the ratio, however, will depend on the detailed modelling of the bounce phase [54]. However, the conclusion that the tensor to scalar ratio will be rather large will be robust, and also agrees with the analysis of [49] done in a different context.

VI. CONCLUSIONS AND DISCUSSION

Recently, the Lee-Wick standard model has been suggested as an extension of the standard model of particle physics providing an alternative to supersymmetry in terms of addressing the hierarchy problems.

In this article, we have considered the cosmology of the Higgs sector of the Lee-Wick standard model, an alternative to supersymmetry to solving the hierarchy problem. We have found that homogeneous and isotropic solutions are nonsingular. Thus, the Lee-Wick model provides a possible solution of the cosmological singularity problem.

We then considered the spectrum of cosmological perturbations and find that quantum vacuum fluctuations established in the contracting phase evolve into a scale-invariant spectrum in the expanding phase. Note that these results emerge without having to introduce any additional features into the model, unlike the situation in inflationary cosmology where the existence of a new scalar field satisfying slow-roll conditions must be assumed, or the situation inekpyrotic models where once again a scalar field with special features must be assumed.

Tuning the amplitude of the spectrum of scalar metric fluctuations to agree with the amplitude inferred from cosmic microwave background (CMB) observations [55], we can determine the scale M of the new physics which is present in the Lee-Wick model. The required value of M turns out to be about 10^{17} GeV.

We have also computed the spectrum of gravitational waves and also find a scale-invariant spectrum assuming that the fluctuations are quantum vacuum in nature. The tensor to scalar ratio may be in excess of the current observational bounces, but the exact value will depend on the detailed modelling of the bounce phase.

One of the main successes of cosmological inflation is the solution of the horizon, homogeneity, size and flatness problems of standard big bang cosmology which it provides. How does a bouncing cosmology such as our Lee-Wick model measure up against these successes? First of all, if the universe starts out large and cold, there are no horizon and size problems. If the spatial curvature at temperatures in the contracting phase comparable to the current temperature is not larger than the current spatial curvature, then there will be no flatness problem either because the deviation of Ω_K from 0 decreases in the contracting phase at the same rate that it increases in the expanding phase. The key challenge for any bouncing cosmology is to control the magnitude of the inhomogeneities and to provide a mechanism for preventing the universe from collapsing into a gas of black holes at the end of the phase of contraction. For an attempt to address this issue in the case of string gas cosmology see [56].

We would like to explain more about the relation between the "horizon" problem and the initial condition problem in the frame of bounce cosmology. Since for a bounce model the horizon can only be broken at an infinite time in the future if we start the cosmological evolution at the infinite past, we can conclude that the horizon problem does not exist in a bounce model, but it is transferred to another problem, namely, the choice of the initial condition as mentioned above. One may need to finely tune the initial state of a bouncing universe in order to avoid itself to collapse into a highly inhomogeneous one. Also we cannot neglect the vector perturbations if a gauge field exists. To alleviate this challenge, there have already been a few attempts; for example, we may need an ekpyrotic phase at the beginning of a collapsing universe to dilute classical perturbations, or introduce a Hagedorn string gas period near the bounce to wash out the classical instabilities. The works on this issue will be investigated in the near future.

We would like to conclude this article by putting our work in the context of previous work on perturbations in bouncing cosmologies. The issue of the mixing of the S and D modes at a cosmological bounce has been hotly debated in the literature since the ekpyrotic scenario was proposed. In the case of the ekpyrotic scenario, for vacuum initial conditions the S mode of Φ inherits a scale-invariant spectrum, whereas the D mode obtains a blue spectrum with index $n = 3$ [27–30] (see also the more recent analysis of [57] and the recent review of [58]). This is also the spectrum of ζ. According to the Hwang-Vishniac [41] (Deruelle-Mukhanov [42]) matching conditions applied at a hypersurface on which we glue the expanding to the contracting universe, the mixing between the $S-$ mode and the $D+$ mode is suppressed by a power of k^2 (see, e.g., [29] for a discussion of this point). Hence, the spectrum of metric fluctuations after the bounce is not scale-invariant. The pre–big bang scenario faces a similar problem [59]. These conclusions were confirmed in some specific models in which the bounce was smoothed out by making use of higher derivative gravity terms (see [60] in the case of the pre–big bang model and [61,62] in the case of the ekpyrotic scenario). However, the use of the matching conditions was challenged in [32] where it was pointed out that if the background solution does not satisfy the matching conditions at the bounce, there is no reason to expect the fluctuations to do so. In fact, in the case of the ekpyrotic scenario (which is intrinsically a higher-dimensional cosmology), computations done in the higher-dimensional framework yielded a successful transfer of the scale-invariant spectrum of metric fluctuations from the contracting to the expanding phase [63], a conclusion which was confirmed in [64] and, in a slightly different setting, in [65,66].

Calculations have also been done in some other nonsingular bouncing models [67]. For example, studies done in models in which the bounce is induced by a negative energy density scalar field found no unsuppressed matching between the growing perturbation mode in the contracting phase and the constant mode in the expanding phase [68,69], in contrast to what was obtained in some initial work [8,9]. Both studies in models in which a bounce was generated by a curvature term in the Einstein action [70] and analyses in some other bouncing models [71,72] yielded unsuppressed matching of the dominant modes of the contracting and expanding phases.

The upshot of these analyses is that the transfer of fluctuations through a cosmological bounce can depend quite sensitively on the physics of the bounce.

It was realized that the equation of motion for ζ has singularities in the case of a nonsingular bounce, thus casting doubt on the belief that in all cases ζ is conserved at a bounce. It was shown that the Φ equation is free of such singularities and is thus a safer equation to use [43,44]. In our previous work [45] it was shown in the case of the quintom bounce model that there is unsuppressed mixing between the $D+$ and $S-$ modes on length scales which are small compared to the duration of the bouncing period, whereas on longer length scales the mixing is suppressed (but not completely absent). In the present work, the bounce is short compared to the length scales we are interested in.

Our work shows that the evolution of fluctuations through the nonsingular bounce in the Lee-Wick model is rather standard. There is no unsuppressed coupling between the dominant modes of the contracting and expanding phases, and ζ is conserved at the bounce.

In the current article we have not considered radiation. Since the energy density in radiation increases faster than that in matter, radiation would dominate at early times. However, in the Lee-Wick standard model there is a Lee-Wick partner to each field. In particular, there is a Lee-Wick photon partner $\tilde{\gamma}$ of the radiation field γ. At high energy densities, then as a consequence of interactions between γ and $\tilde{\gamma}$, we expect that energy will flow from γ into $\tilde{\gamma}$, like it flows from ϕ to $\tilde{\phi}$ in our scalar field model. Then, a cosmological bounce would occur in a manner similar to how it occurs in our model. Adding intermediate phases of radiation between the bouncing phase and the contracting and expanding matter phases will not change our results concerning the spectrum of fluctuations for modes which exit the Hubble radius during the phase of matter domination, which are the modes we are interested in when trying to explain the large-scale structure of the universe and the CMB anisotropies. A study of these issues is left to a follow-up paper.

It would also be interesting to consider entropy fluctuations and non-Gaussian signatures of our scenario. We leave these topics for future research.

Note added: While this article was being prepared for submission, a preprint appeared [73] pointing out that the Lee-Wick model provides a realization of the quintom scenario and could be applied to study the current acceleration of the universe. We find it more natural to consider the corrections to the cosmological evolution which are obtained in the very early universe.

ACKNOWLEDGMENTS

We wish to thank Andy Cohen, Ben Grinstein, Hong Li, Jie Liu and Mark Wise for useful discussions. R. B. wishes to thank the Theory Division of the Institute of High Energy Physics (IHEP) for their wonderful hospitality and financial support. R. B. is also supported by an NSERC Discovery Grant and by the Canada Research Chairs Program. The research of X. Z., Y. C. and T. Q. is supported in part by the National Science Foundation of China under Grant Nos. 10533010 and 10675136, by the 973 Program No. 2007CB815401, and by the Chinese Academy of Sciences under Grant No. KJCX3-SYW-N2.

APPENDIX: ISOCURVATURE PERTURBATION FROM THE LEE-WICK SCALAR

Under the longitudinal (conformal-Newtonian) gauge, we recall the equation of motion for the Bardeen potential,

$$\Phi_k'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi_k' + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi_k + k^2\Phi_k = 8\pi G\left(2\mathcal{H} + \frac{\phi''}{\phi'}\right)\tilde{\phi}'\delta\tilde{\phi}_k. \quad \text{(A1)}$$

By varying the matter action with respect to the fluctuation of Lee-Wick scalar $\delta\tilde{\phi}$, we obtain the equation of motion,

$$\delta\tilde{\phi}_k'' + 2\mathcal{H}\delta\tilde{\phi}_k' + k^2\delta\tilde{\phi}_k + a^2M^2\delta\tilde{\phi}_k = -2a^2M^2\tilde{\phi}\Phi_k + 4\tilde{\phi}'\Phi_k', \quad \text{(A2)}$$

which combined with Eq. (A1) can describe all the perturbation modes consistently. Since the Lee-Wick scalar is almost a subdominant field during the evolution, its fluctuation seeds an isocurvature perturbation.

Note that we consider a model of Lee-Wick cosmology with the potential being a mass term for simplicity. Until now we have not used any approximations in deriving the above two perturbation equations. In the following calculation we will follow one mode of $\delta\tilde{\phi}$ along with the cosmological evolution.

1. Contracting phase

In the initial stage of the background evolution when $t \ll t_{B-}$, the average value of the EoS is $w_c = 0$. The universe behaves like a matter-dominated one, and the scale factor evolves as $a(t) \sim (t - \tilde{t}_{B-})^{2/3}$. So we have the expressions

$$\tilde{\phi} \simeq \frac{m_{\rm pl}}{\sqrt{3\pi}}\frac{\sin[M(t - \tilde{t}_{B-})]}{M(t - \tilde{t}_{B-})}, \quad \text{(A3)}$$

which is valid when $|H| \ll M$. As we have analyzed in Sec. III, the top value of the Hubble parameter $|H|$ is of order m which is much less than the mass of Lee-Wick scalar M. Therefore, this expression can be used in the contracting phase safely. Besides, one may notice that both the average values of $\tilde{\phi}$ and $\tilde{\phi}'$ are almost equal to 0. By taking the average of Eq. (A2), one can simplify the perturbation equation as follows:

$$\delta\tilde{\phi}_k'' + \frac{4}{\eta - \tilde{\eta}_{B-}}\delta\tilde{\phi}_k' + k^2\delta\tilde{\phi}_k + a^2M^2\delta\tilde{\phi}_k \simeq 0. \quad \text{(A4)}$$

We neglect the right-hand side of Eq. (A2) since a background parameter cannot be correlated with a perturbation variable and the background parameters are vanishing in average.

At the beginning with $\eta \to -\infty$ the scale factor is very large, and thus the mass term dominates over. One can impose an initial condition for $\delta\tilde{\phi}$ as follows:

$$a\delta\tilde{\phi}_k \simeq \frac{1}{\sqrt{2\omega(k,\eta)}}e^{i\int^{\eta}\omega(k,\eta')d\eta'}, \quad \text{(A5)}$$

where we define a frequency parameter

$$\omega(k,\eta)^2 \equiv k^2 + a(\eta)^2M^2. \quad \text{(A6)}$$

Along with the background contraction, the scale factor is decreasing and the comoving Hubble parameter is growing. A mode of Lee-Wick perturbation $\delta\tilde{\phi}_k$ is able to exit the Hubble radius when $k < |\mathcal{H}|$. However, as addressed previously, the largest value of the Hubble parameter $|H|$ is of order m which is still much smaller than M. Therefore, during the whole evolution, the perturbation equation is dominated by the mass term which appears as the last term of Eq. (A4). Consequently, we obtain an approximate solution to Eq. (A4) on super-Hubble scales,

$$\delta\tilde{\phi}_k \simeq \frac{1}{\sqrt{2M}a^{3/2}}e^{(2i/3)(M/H)}. \quad \text{(A7)}$$

From this solution, one can read that the Lee-Wick fluctuation is strongly oscillating and its amplitude is growing proportional to $a^{-(3/2)}$ in contracting phase. This is exactly consistent with the numerical result as shown in Fig. 7 when $t < 0$.

2. Bouncing phase

Since the contribution of the field $\tilde{\phi}$ becomes more and more important, the contraction will stop when $w = -1$ and then the universe will enter the bouncing phase at some moment t_{B-}. Correspondingly the EoS of the universe will fall to negative infinity rapidly. During this process, the Hubble parameter will shrink soon and arrive at zero at $t = 0$.

In order to get some analytic insight into the evolution of the Lee-Wick fluctuations in the bouncing phase, we use the parametrization $H = \alpha t$ which appeared in (64). As investigated at the end of Sec. IV G, its effects on the curvature perturbation Φ could be cancelled due to a vanishing Hubble parameter. However, the Lee-Wick fluctuation $\delta\tilde{\phi}$ could be affected by the curvature perturbation Φ conversely.

One can see that it is still the mass term dominated in the left-hand side of Eq. (A2) for the Lee-Wick fluctuation on large scales. The right-hand side plays a role of a source term, of which the first term dominates over. This is because around the bounce point, the Lee-Wick scalar $\tilde{\phi}$ reaches its maximal value $\tilde{\phi}_B \simeq \frac{m m_{\rm pl}}{\sqrt{12\pi}M}$ (we expect the energy densities of these two scalars can be cancelled at the bounce point which requires $m^2\phi^2 \simeq M^2\tilde{\phi}^2$ and at that moment the amplitude of the normal scalar ϕ is about $m_{\rm pl}/\sqrt{12\pi}$). This is consistent with the numerical calculation shown in Fig. 3. Therefore, we have an approximate solution

$$\delta\tilde{\phi}_k \simeq -2\tilde{\phi}\Phi_k \simeq -\frac{m m_{\rm pl}}{\sqrt{3\pi}M}\Phi_k, \tag{A8}$$

in the bouncing phase. This semianalytic solution indicates that the amplitude of $\delta\tilde{\phi}$ could be amplified in the bouncing phase.

One can use the approximate solution (A8) to make an estimate of the Lee-Wick fluctuation as follows:

$$P_{\delta\tilde{\phi}} = \frac{k^3}{2\pi^2}|\delta\tilde{\phi}_k|^2 \simeq \frac{m^2 m_{\rm pl}^2}{3\pi M^2}P_\Phi. \tag{A9}$$

From the numerical calculation shown in Fig. 5, one reads $P_\Phi \sim 10^{-5}$ at the bounce point when $t = 0$. Making use of the values of background parameter provided in Fig. 3, we obtain an approximate value of the amplitude to be $P_{\delta\tilde{\phi}} \sim 10^{-6} m_{\rm pl}$ which is scale-invariant. One can check that this estimate result is consistent with the numerical calculation shown in Fig. 7 very well.

3. Expanding phase

After the bounce, the universe starts to expand. The Lee-Wick scalar begins the oscillations again, and cannot dominate over at late times in the specific models we considered. The dynamics of its fluctuation is similar to that in the contracting phase of which the amplitude is proportional to $a^{-(3/2)}$. Therefore, the amplitude of $\delta\tilde{\phi}$ is decreasing after the bounce. However, as is shown previously, $\delta\tilde{\phi}$ could be amplified by the curvature perturbation during the bounce, and so its boundary value at the moment t_{B+} is much larger than the value at the moment t_{B-}. This implies the decreasing of $\delta\tilde{\phi}$ after the bounce is not symmetric to the growth before that. Since the Lee-Wick scalar is a subdominant field after the bounce, its fluctuation seeds a scale-invariant spectrum of isocurvature perturbation.

The numerical calculation is performed in Fig. 7. The amplitude of $\delta\tilde{\phi}$ initially grows in the contracting phase. However, the slope of the curve is different from that for the metric perturbation. Since in the bouncing phase the metric perturbation is able to amplify the perturbation of the subdominant field, $\delta\tilde{\phi}$ reaches its peak around the bounce point. After that, its amplitude decreases. The oscillation behavior exists anywhere along with the whole cosmological evolution.

As a side remark, the Lee-Wick scalar changes the dynamics of the cosmological background, but its fluctuation almost decouples from the scalar perturbation of the metric due to a large ghost mass. This decoupling actually depends on a critical comoving wave number of the perturbation, which is $k_M = aM$. For the modes with comoving wave numbers smaller than k_M, the decoupling could take place. Moreover, since in our model there is $k_M \gg \mathcal{H}$, all the possibly observable modes which are able to exit the Hubble radius in the contracting phase satisfy this decoupling condition.

[1] T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969); Phys. Rev. D **2**, 1033 (1970).
[2] B. Grinstein, D. O'Connell, and M. B. Wise, Phys. Rev. D **77**, 025012 (2008).
[3] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002).
[4] J. M. Cline, S. Jeon, and G. D. Moore, Phys. Rev. D **70**, 043543 (2004).
[5] B. Feng, X. L. Wang, and X. M. Zhang, Phys. Lett. B **607**, 35 (2005); B. Feng, M. Li, Y. S. Piao, and X. Zhang, Phys. Lett. B **634**, 101 (2006); Z. K. Guo, Y. S. Piao, X. M. Zhang, and Y. Z. Zhang, Phys. Lett. B **608**, 177 (2005).
[6] Y. F. Cai, T. Qiu, Y. S. Piao, M. Li, and X. Zhang, J. High Energy Phys. 10 (2007) 071.
[7] The use of a second scalar field to obtain a nonsingular bounce was already discussed in [8,9].
[8] F. Finelli, J. Cosmol. Astropart. Phys. 10 (2003) 011.

[9] J. Martin, P. Peter, N. Pinto Neto, and D. J. Schwarz, Phys. Rev. D **65**, 123513 (2002); P. Peter and N. Pinto-Neto, Phys. Rev. D **66**, 063509 (2002).
[10] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).
[11] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001).
[12] F. Finelli and R. Brandenberger, Phys. Rev. D **65**, 103522 (2002).
[13] D. Wands, Phys. Rev. D **60**, 023507 (1999).
[14] P. Peter and N. Pinto-Neto, Phys. Rev. D **78**, 063506 (2008).
[15] A. A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 719 (1979) [JETP Lett. **30**, 682 (1979)].
[16] D. G. Boulware and D. J. Gross, Nucl. Phys. **B233**, 1 (1984).
[17] N. Nakanishi, Phys. Rev. D **3**, 811 (1971); **3**, 3235 (1971).
[18] A. M. Gleeson, R. J. Moore, H. Rechenberg, and E. C. G. Sudarshan, Phys. Rev. D **4**, 2242 (1971).
[19] R. E. Cutkosky, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, Nucl. Phys. **B12**, 281 (1969).
[20] A. van Tonder, arXiv:0810.1928.
[21] A. D. Linde, Phys. Lett. B **129**, 177 (1983).
[22] T. Biswas, A. Mazumdar, and W. Siegel, J. Cosmol. Astropart. Phys. 03 (2006) 009; T. Biswas, R. Brandenberger, A. Mazumdar, and W. Siegel, J. Cosmol. Astropart. Phys. 12 (2007) 011.
[23] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[24] K. Sato, Mon. Not. R. Astron. Soc. **195**, 467 (1981).
[25] R. H. Brandenberger, arXiv:hep-ph/9910410.
[26] J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001); R. H. Brandenberger and J. Martin, Mod. Phys. Lett. A **16**, 999 (2001).
[27] D. H. Lyth, Phys. Lett. B **524**, 1 (2002); **526**, 173 (2002).
[28] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **66**, 046005 (2002).
[29] R. Brandenberger and F. Finelli, J. High Energy Phys. 11 (2001) 056.
[30] J. c. Hwang, Phys. Rev. D **65**, 063514 (2002).
[31] J. Martin, P. Peter, N. Pinto Neto, and D. J. Schwarz, Phys. Rev. D **65**, 123513 (2002).
[32] R. Durrer and F. Vernizzi, Phys. Rev. D **66**, 083503 (2002).
[33] In light of the trans-Planckian problem for inflationary fluctuations, one may view this prescription with some skepticism.
[34] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[35] R. H. Brandenberger, Lect. Notes Phys. **646**, 127 (2004).
[36] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[37] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[38] R. H. Brandenberger and R. Kahn, Phys. Rev. D **29**, 2172 (1984).
[39] M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986).
[40] V. F. Mukhanov, Zh. Eksp. Teor. Fiz. **94N7**, 1 (1988) [Sov. Phys. JETP **67**, 1297 (1988)].
[41] J. c. Hwang and E. T. Vishniac, Astrophys. J. **382**, 363 (1991).
[42] N. Deruelle and V. F. Mukhanov, Phys. Rev. D **52**, 5549 (1995).
[43] R. Brandenberger, H. Firouzjahi, and O. Saremi, J. Cosmol. Astropart. Phys. 11 (2007) 028.
[44] S. Alexander, T. Biswas, and R. H. Brandenberger, arXiv:0707.4679.
[45] Y. F. Cai, T. Qiu, R. Brandenberger, Y. S. Piao, and X. Zhang, J. Cosmol. Astropart. Phys. 03 (2008) 013.
[46] A. Nayeri, R. H. Brandenberger, and C. Vafa, Phys. Rev. Lett. **97**, 021302 (2006).
[47] R. H. Brandenberger, A. Nayeri, S. P. Patil, and C. Vafa, Int. J. Mod. Phys. A **22**, 3621 (2007).
[48] R. H. Brandenberger, arXiv:0808.0746.
[49] L. E. Allen and D. Wands, Phys. Rev. D **70**, 063515 (2004).
[50] F. Finelli and R. H. Brandenberger, Phys. Rev. Lett. **82**, 1362 (1999); Phys. Rev. D **62**, 083502 (2000).
[51] W. B. Lin, X. H. Meng, and X. M. Zhang, Phys. Rev. D **61**, 121301 (2000).
[52] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, UK, 2005), p. 421.
[53] Y. F. Cai and X. Zhang, J. Cosmol. Astropart. Phys. 06 (2009) 003.
[54] We have modeled the contracting and expanding phases with a background with a constant equation of state. In this case, the squeezing factor for scalar and tensor cosmological perturbations is the same. If the equation of state changes, then the amplitude of the scalar spectrum can be enhanced, as happens in inflationary cosmology at the time of reheating [37,38]. Since $1 + w$ does not change by orders of magnitude in the Lee-Wick bouncing cosmology—whereas it does change by orders of magnitude in the case of inflationary cosmology—we do not expect a dramatic enhancement of the scalar power spectrum. However, we do expect some increase in the amplitude. A detailed study of this issue is left to further work.
[55] G. F. Smoot *et al.*, Astrophys. J. **396**, L1 (1992).
[56] N. Lashkari and R. H. Brandenberger, J. High Energy Phys. 09 (2008) 082.
[57] P. Creminelli, A. Nicolis, and M. Zaldarriaga, Phys. Rev. D **71**, 063505 (2005).
[58] D. Wands, arXiv:0809.4556.
[59] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov, and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
[60] C. Cartier, J. c. Hwang, and E. J. Copeland, Phys. Rev. D **64**, 103504 (2001).
[61] S. Tsujikawa, R. Brandenberger, and F. Finelli, Phys. Rev. D **66**, 083513 (2002).
[62] A. Cardoso and D. Wands, Phys. Rev. D **77**, 123538 (2008).
[63] A. J. Tolley, N. Turok, and P. J. Steinhardt, Phys. Rev. D **69**, 106005 (2004).
[64] J. L. Lehners, P. McFadden, N. Turok, and P. J. Steinhardt, Phys. Rev. D **76**, 103501 (2007).
[65] T. J. Battefeld, S. P. Patil, and R. H. Brandenberger, Phys. Rev. D **73**, 086002 (2006).
[66] It would go beyond the scope of this discussion to review attempts to preserve a scale-invariant spectrum in pre–big bang and ekpyrotic scenarios via introducing entropy fluctuations.
[67] J. Martin and P. Peter, Phys. Rev. D **69**, 107301 (2004).
[68] V. Bozza and G. Veneziano, Phys. Lett. B **625**, 177 (2005).

[69] F. Finelli, P. Peter, and N. Pinto-Neto, Phys. Rev. D **77**, 103508 (2008).
[70] J. Martin and P. Peter, Phys. Rev. D **68**, 103517 (2003).
[71] L. R. Abramo and P. Peter, J. Cosmol. Astropart. Phys. 09 (2007) 001.
[72] P. Peter, N. Pinto-Neto, and D. A. Gonzalez, J. Cosmol. Astropart. Phys. 12 (2003) 003.
[73] S. Lee, arXiv:0810.1145.