PHYSICAL REVIEW D **80,** 016005 (2009)

Longitudinal rescaling and high-energy effective actions

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(Received 21 January 2009; published 10 July 2009)

Under a longitudinal rescaling of coordinates $x^{0,3} \to \lambda x^{0,3}$, $\lambda \ll 1$, the classical QCD action simplifies dramatically. This is the high-energy limit, as $\lambda \sim s^{-1/2}$, where s is the center-of-mass energy squared of a hadronic collision. We find the quantum corrections to the rescaled action at one loop, in particular, finding the anomalous powers of λ in this action, with $\lambda < 1$. The method is an integration over high-momentum components of the gauge field. This is a Wilsonian renormalization procedure, and counterterms are needed to make the sharp-momentum cutoff gauge invariant. Our result for the quantum action is found, assuming $|\ln\lambda| \ll 1$, which is essential for the validity of perturbation theory. If λ is sufficiently small (so that $|\ln\lambda| \gg 1$), then the perturbative renormalization group breaks down. This is due to uncontrollable fluctuations of the longitudinal chromomagnetic field.

DOI: 10.1103/PhysRevD.80.016005 PACS numbers: 11.10.Jj, 11.15.Ha, 11.15.Tk

I. INTRODUCTION

Effective gauge-theory descriptions are a promising approach to high-energy proton-proton collisions [1–5], and nuclear collisions [6–10].

The approximation of Verlinde and Verlinde [3] was to eliminate some gauge-theory degrees of freedom through a longitudinal rescaling. These authors argued that this rescaling yields the Balitsky-Fadin-Kuraev-Lipatov (BFKL) theory [11]. In particular, they were able to rederive the BFKL vertex and argued that gluon Reggeization occurs. A similar idea was incorporated by McLerran and Venugopalan [6] into a picture which came to be known as the color-glass condensate [7]. Longitudinal rescaling in Ref. [3] was done purely *classically*, by a simple change of variables in the action. After the rescaling, quark and gluon matter travels primarily longitudinally. Most of the energy is in the transverse color field strength, just as in a Weizsacker-Williams shock wave. Effective actions incorporating such shock waves have been extensively discussed by Lipatov [12] and Balitsky [13]

In Ref. [5], the cutoff rescaled theory was shown to be completely integrable, massive and confining, in the high-energy limit. Our interest here is to see whether this limit is justified.

In this paper, we determine how the quantum action changes under longitudinal rescaling. We will only consider the gluon field in our calculation. Quarks will be included in a later publication.

The explicit rescaling of coordinates and gauge fields is $x^{0,3} \to \lambda x^{0,3}$, $x^{1,2} \to x^{1,2}$, $A_{0,3} \to \lambda^{-1} A_{0,3}$, $A_{1,2} \to A_{1,2}$, where $A_\mu = A_\mu^a t_a$, $a = 1, \ldots, N^2 - 1$ are SU(N) Yang-Mills field. Sometimes we shall use L as an abbreviation for the longitudinal Lorentz indices 0, 3 and $\perp$ as an abbreviation for the transverse Lorentz indices 1, 2. We normalize $\mathrm{Tr} t_a t_b = \delta_{ab}$ and define $\mathrm{i} f_{ab}^{c} t_c = [t_a, t_b]$. Since momentum components transform as $p_L \to \lambda^{-1} p_L$, $p_\perp \to p_\perp$, we can think of the rescaling factor as $\lambda = \sqrt{s'/s}$, where s' and s are the center-of-mass energies squared, before and after the rescaling, respectively. To describe extremely high energies, we would, in principle, take $\lambda \ll 1$ [3].

Perhaps a better motivation for this rescaling is that transverse transport of glue is suppressed and longitudinal transport is enhanced. This can be seen by perusing the Hamiltonian. If the scale factor λ is small, but not zero, the resulting Hamiltonian has one extremely small coupling and one extremely large coupling. The classically rescaled action is

$$S = \frac{1}{2g_0^2} \int d^4x \, \mathrm{Tr}\left(\lambda^{-2} F_{03}^2 + \sum_{j=1}^{2} F_{0j}^2 - \sum_{j=1}^{2} F_{j3}^2 - \lambda^2 F_{12}^2\right), \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]$. The Hamiltonian in $A_0 = 0$ gauge is therefore

$$H = \int d^3x \left[\frac{g_0^2}{2} \mathcal{E}_\perp^2 + \frac{1}{2g_0^2} \mathcal{B}_\perp^2 + \lambda^2 \left(\frac{g_0^2}{2} \mathcal{E}_3^2 + \frac{1}{2g_0^2} \mathcal{B}_3^2\right)\right], \tag{1.2}$$

where the electric and magnetic fields are $\mathcal{E}_i = -\mathrm{i}\delta/\delta A_i$ and $\mathcal{B}_i = \epsilon^{ijk}(\partial_j A_k + A_j \times A_k)$, respectively, and $(A_j \times A_k)^a = f_{bc}^a A_j^b A_k^c$. Physical states Ψ must satisfy Gauss's law

$$(\partial_\perp \cdot \mathcal{E}_\perp + \partial_3 \mathcal{E}_3 - \rho)\Psi = 0, \tag{1.3}$$

where ρ is the quark color-charge density. If the term of order λ^2 is neglected, all the energy is contained in the transverse electric and magnetic fields. Chromo-electromagnetic waves can only move longitudinally.

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This is most easily seen in an axial gauge $A_3 = 0$, in which case the $\lambda = 0$ Hamiltonian contains no transverse derivatives [5].

As we mentioned above, the longitudinal rescaling considered above is classical. In a fully quantized Yang-Mills theory, the rescaled action is not as simple as (1.1). In the quantum case, all the coefficients of the field strength-squared terms must be rescaled. Furthermore, these coefficients are not simply multiplied by integer powers of λ; anomalous dimensions are present.

The rescaling is done for the quantized Yang-Mills theory in two steps. First a Wilson-style renormalization [14] from an isotropic to an anisotropic cutoff is performed. Second, the longitudinal rescaling discussed above is done to restore the isotropy of the cutoff. One way to visualize this procedure is to imagine a lattice ultraviolet cutoff, with lattice spacing a. The lattice rescaling procedure is illustrated in Fig. 1. Degrees of freedom are thinned out by a Kadanoff or ‘‘block-spin’’ transformation, which changes the lattice spacing in the longitudinal directions to a/λ, while leaving the lattice spacing in the transverse directions unchanged. After this reduction of degrees of freedom, the entire lattice is rescaled longitudinally, so that the lattice spacing in the direction of any coordinate axis has the original value a.

Some papers on anisotropic renormalization were written [15], not long after Refs. [3,6] appeared. Perturbative renormalization of the Yang-Mills field is not performed in these papers.

The Wick-rotated Yang-Mills theory is defined by introducing the functional integral $\int \exp -S$, where S is the action, with an ultraviolet cutoff Λ on the variables of integration, namely, the gauge field $A_\mu(x)$ (we do not include quark fields in this paper). The cutoff is introduced by requiring that the Fourier components of these fields, which are functions of Euclidean four-momentum p, vanish for $p^2 > \Lambda^2$. This sharp momentum cutoff breaks gauge invariance, meaning that counterterms restoring this invariance are necessary. We denote the two components of longitudinal momenta by $p_L = (p_0, p_3)$ and the two components of transverse momenta by $p_\perp = (p_1, p_2)$.

We first isolate the degrees of freedom depending on momenta satisfying $\tilde{b} p_L^2 + p_\perp^2 > \tilde{\Lambda}^2$, where $\Lambda \geq \tilde{\Lambda}$, $\tilde{b} \geq 1$, and integrate these out of the functional integral. This yields a new functional integral whose action has new couplings, but with an ellipsoidal cutoff, with the remaining degrees of freedom vanishing unless $\tilde{b} p_L^2 + p_\perp^2 < \tilde{\Lambda}^2$. The different coefficients of the field-strength-squared component are rescaled differently. Finally, we rescale $p_L \to \lambda^{-1} p_L$ and $p_\perp \to p_\perp$. We identify $\lambda^{-2} = \tilde{b}$. The ultraviolet regularization is once again isotropic, with components vanishing unless $p^2 = p_L^2 + p_\perp^2 < \tilde{\Lambda}^2$. As a result, the different coefficients of the field-strength-squared components are rescaled again, yielding the final form of the action.

It is possible to assume that $\tilde{\Lambda} = \Lambda$. In that case, we integrate out all the degrees of freedom in the original momentum sphere, except for those in an ellipsoid, whose two major axes are equal to the diameter of the sphere. It is illustrative, however, to consider the more general case of $\tilde{\Lambda} \leq \Lambda$.

The plan of the paper is as follows. In the next section, we discuss generally how the Wilson renormalization for an SU(N) Yang-Mills theory is carried out. The isotropic case is briefly reviewed in Sec. III. The integration from a spherical cutoff to an ellipsoidal cutoff is explained in Sec. IV. This result is then used to find the effect of a longitudinal rescaling on the Yang-Mills action in Sec. V. We touch upon the utility of effective actions for high-energy collisions, in the light of our results, in Sec. VI. In the last section, we mention some calculations which should be done, in the near future.

II. RENORMALIZATION OF QCD WITH A MOMENTUM CUTOFF: GENERAL CONSIDERATIONS

In this section, we review how the QCD action changes if we integrate, to one loop, from one sharp momentum cutoff to a smaller sharp momentum cutoff. For readers not already familiar with this method, a discussion can be found in Ref. [16]. The techniques do not differ appreci-

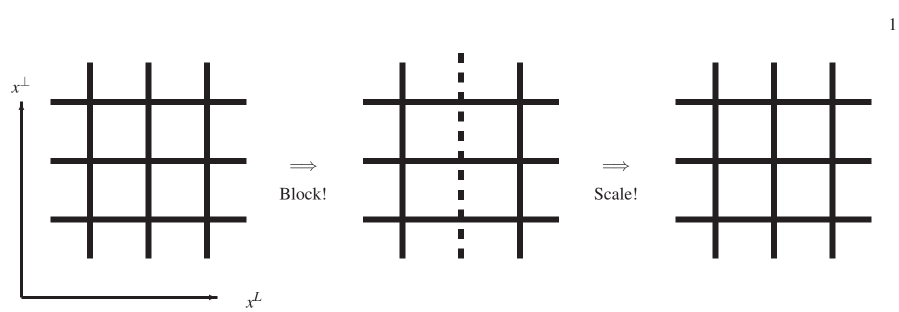

FIG. 1. Rescaling of field theory on a lattice with $\lambda = 1/2$. First, a Kadanoff transformation increases the longitudinal lattice spacing. The spacing is then restored to its original value by a longitudinal rescaling.

ably from those for the background-field calculation of the effective action.

First we Wick rotate the Yang-Mills theory to obtain the standard Euclidean metric. We choose Λ and $\tilde{\Lambda}$ to be real positive numbers with units of cm^{-1} and b and $\tilde{b}$ to be two dimensionless real numbers, such that $b \geq 1$ and $\tilde{b} \geq 1$. We require furthermore that $\Lambda > \tilde{\Lambda}$ and that $\Lambda^2/b \geq \tilde{\Lambda}^2/\tilde{b}$. We define the region of momentum space $\mathbb{P}$ to be the set of points p, such that $bp_L^2 + p_\perp^2 < \Lambda^2$. We define the region $\tilde{\mathbb{P}}$ to be the set of points p, such that $\tilde{b}p_L^2 + p_\perp^2 < \tilde{\Lambda}^2$. Finally, we define $\mathbb{S}$ to be the Wilsonian "onion skin" $\mathbb{S} = \mathbb{P} - \tilde{\mathbb{P}}$.

The functional integral we consider is

$$Z_\Lambda = \int \left[\prod_{p\in\mathbb{P}} dA(p) \right] \exp -S, \qquad S = \int d^4x \frac{1}{4g_0^2} \mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + S_{\mathrm{c.t.},\Lambda,b}, \tag{2.1}$$

where $S_{\mathrm{c.t.},\Lambda,b}$ contains counterterms, needed to maintain gauge invariance with the sharp-momentum cutoff Λ, and anisotropy parameter b.

The restriction on the measure of integration in (2.1) means that the gauge field has the Fourier transform

$$A_\mu(x) = \int_{\mathbb{P}} \frac{d^4p}{(2\pi)^4} A_\mu(p) e^{-\mathrm{i}p\cdot x}.$$

We split the field A_μ into slow parts $\tilde{A}_\mu$, and fast parts a_μ, defined by

$$\tilde{A}_\mu(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} A_\mu(p) e^{-\mathrm{i}p\cdot x}, \qquad a_\mu(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4} A_\mu(p) e^{-\mathrm{i}p\cdot x},$$

so that $A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x)$. This can also be written in momentum space: $A_\mu(p) = \tilde{A}_\mu(p) + a_\mu(p)$, by defining

$$\tilde{A}_\mu(p) = \begin{cases} A_\mu(p), & p \in \tilde{\mathbb{P}}, \\ 0, & p \in \mathbb{S} \end{cases}, \qquad a_\mu(p) = \begin{cases} 0, & p \in \tilde{\mathbb{P}}, \\ A_\mu(p), & p \in \mathbb{S} \end{cases}. \tag{2.2}$$

We shall integrate out the fast components a_μ, of the field to obtain

$$Z_\Lambda = e^{-f} Z_{\tilde{\Lambda}}, \qquad Z_{\tilde{\Lambda}} = \int \left[\prod_{p\in\tilde{\mathbb{P}}} dA(p) \right] \exp -\tilde{S}, \qquad \tilde{S} = \int d^4x \frac{1}{4\tilde{g}_0^2} \mathrm{Tr}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + S_{\mathrm{c.t.},\tilde{\Lambda},\tilde{b}}, \tag{2.3}$$

where f is an unimportant ground-state-energy renormalization, $\tilde{g}_0$ is the coupling at the new cutoff $\tilde{\Lambda}$, $\tilde{b}$, $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - \mathrm{i}[\tilde{A}_\mu, \tilde{A}_\nu]$, and $S_{\mathrm{c.t.},\tilde{\Lambda},\tilde{b}}$ contains the counterterms needed to restore gauge invariance with the new cutoff.

To integrate out the fast gauge field, yielding the new action in (2.3), we expand the original action in terms of this field to quadratic order:

$$S = \frac{1}{4g_0^2} \int d^4x \,\mathrm{Tr}\{\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - 4[\tilde{D}_\mu, \tilde{F}^{\mu\nu}]a_\nu + ([\tilde{D}_\mu, a_\nu] - [\tilde{D}_\nu, a_\mu])([\tilde{D}^\mu, a^\nu] - [\tilde{D}^\nu, a^\mu]) - 2\mathrm{i}\tilde{F}^{\mu\nu}[a_\mu, a_\nu]\}, \tag{2.4}$$

where $\tilde{D}_\mu = \partial_\mu - \mathrm{i}\tilde{A}_\mu$ is the covariant derivative determined by the slow gauge field.

The action is invariant under the gauge transformation of the fast field:

$$\tilde{A}_\mu \to \tilde{A}_\mu, \qquad a_\mu \to a_\mu + [\tilde{D}_\mu - \mathrm{i}a_\mu, \omega].$$

Variations δa_μ orthogonal to these gauge transformation satisfy $[\tilde{D}_\mu, \delta a_\mu] = 0$. We can add with impunity the term $\frac{1}{2g_0^2} \int d^4x \,\mathrm{Tr}[\tilde{D}_\mu, a_\mu]^2$ to the action.

Notice that there is a linear term in a_μ in the action (2.4). Once we integrate out the fast field, the only result of this term will be to induce terms of order $[\tilde{D}_\mu, \tilde{F}^{\mu\nu}]^2$ in $\tilde{S}$. These terms will be of dimension greater than four or nonlocal, so we ignore them, as they will be irrelevant. We can thereby replace (2.4) with

$$S = \frac{1}{4g_0^2} \int d^4x \,\mathrm{Tr}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2g_0^2} \int d^4x ([\tilde{D}_\mu, a_\nu][\tilde{D}^\mu, a^\nu] - 2\mathrm{i}\tilde{F}^{\mu\nu}[a_\mu, a_\nu]),$$

In terms of coefficients of the generators t_b, $b = 1, \ldots, N^2 - 1$, this expression may be written as

$$S = \frac{1}{4g_0^2} \int d^4x \tilde{F}^b_{\mu\nu}\tilde{F}^{\mu\nu}_b + S_{\mathrm{O}} + S_{\mathrm{I}} + S_{\mathrm{II}},$$

where

$$S_{\mathrm{O}} = \frac{1}{2g_0^2} \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} q^2 a^b_\mu(-q) a^\mu_b(q), \tag{2.5}$$

$$S_{\mathrm{I}} = \frac{\mathrm{i}}{g_0^2} \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} q^\mu f_{bcd} a^b_\nu(q) \tilde{A}^c_\mu(p) a^d_\nu(-q-p) + \frac{1}{2g_0^2} \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4l}{(2\pi)^4} f_{bcd} f_{bfg} a^d_\nu(q) \times \tilde{A}^c_\mu(p) \tilde{A}^f_\mu(l) a^g_\nu(-q-p), \tag{2.6}$$

and

$$S_{\mathrm{II}} = \frac{1}{2g_0^2} \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} f_{bcd} a^b_\mu(q) \tilde{F}^c_{\mu\nu}(p) a^d_\nu(-p-q). \tag{2.7}$$

The gluon propagator is given by the expression for S_{O} in

(2.5) as

$$\langle a^b_\mu(q) a^c_\nu(p)\rangle = g_0^2 \delta^{bc}\delta_{\mu\nu}\delta^4(q+p)q^{-2}. \tag{2.8}$$

We define the meaning of brackets $\langle W\rangle$, around any quantity W to be the expectation value of W with respect to the measure $\mathcal{N}\exp - S_{\rm O}$, where $\mathcal{N}$ is chosen so that $\langle 1\rangle = 1$.

One more term must be included in the action, which depends on the anticommuting ghost fields $G^b_\mu(x)$, $H^b_\mu(x)$, associated with the gauge fixing of $a^b_\mu(x)$. The ghost action is

$$S_{\rm ghost} = \frac{\rm i}{g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4} q^\mu f_{bcd} G^b(q)\tilde{A}^c_\mu(p) \times H^d(-q-p) + \frac{1}{2g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4} \times \int_{\tilde{\mathbb{P}}}\frac{d^4l}{(2\pi)^4} f_{bcd}f_{bfg}G^d(q)\tilde{A}^c_\mu(p)\tilde{A}^f_\mu(l) \times H^g(-q-p),$$

which is similar to $S_{\rm I}$, except that the fast vector gauge field has been replaced by the scalar ghost fields. Integration over the ghost fields eliminates two of the four spin degrees of freedom of the fast gauge field.

To integrate out the fast gauge field and its associated ghost fields, we use the connected-graph expansion for the expectation value of the exponential of minus a quantity R:

$$\langle e^{-R}\rangle = \exp\left[-\langle R\rangle + \frac{1}{2!}(\langle R^2\rangle - \langle R\rangle^2) - \frac{1}{3!}(\langle R^3\rangle - 3\langle R^3\rangle\langle R\rangle + 2\langle R\rangle^3) + \cdots\right].$$

Applying this expansion to second order, we find

$$\exp - \tilde{S} = \exp\left(-\frac{1}{4g_0^2}\int d^4x \tilde{F}^b_{\mu\nu}\tilde{F}^{\mu\nu}_b\right) \times \left\langle \exp\left(-\frac{1}{2}S_{\rm I} - S_{\rm II}\right)\right\rangle \approx \exp\left[-\frac{1}{4g_0^2}\int d^4x \tilde{F}^b_{\mu\nu}\tilde{F}^{\mu\nu}_b\right]\exp\left[-\frac{1}{2}\langle S_{\rm I}\rangle + \frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) + \frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2)\right]. \tag{2.9}$$

We remark briefly on the coefficients in the last exponential in (2.9). The coefficient of $\langle S_{\rm I}\rangle$ has a contribution -1 from a fast gluon loop and $1/2$ from a fast ghost loop. The coefficient of $\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2$ has a contribution $1/2$ from a fast gluon loop and $-1/4$ from a fast ghost loop. The coefficient of $\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2$ has no ghost contribution. Other terms in the exponential of the same order vanish upon contraction of group indices.

The terms in the new action (2.9) are given by

$$\frac{1}{2}\langle S_{\rm I}\rangle - \frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) = \frac{C_N}{4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)P_{\mu\nu}(p),$$

$$P_{\mu\nu}(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[-\frac{q_\mu(p_\nu + 2q_\nu)}{4q^2(q+p)^2} + \frac{\delta_{\mu\nu}}{4q^2}\right], \tag{2.10}$$

where C_N is the Casimir of SU(N), defined by $f^{bcd}f^{hcd} = C_N\delta^{bh}$, and

$$-\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) = -\frac{C_N}{2}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p) \times \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2(p+q)^2}. \tag{2.11}$$

The remaining work to be done is to evaluate integrals in (2.10) and (2.11).

Notice that if the integral $I(p)$ is defined by

$$I(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{p_\alpha + 2q_\alpha}{q^2(q+p)^2},$$

then $I(p) + I(-p) = 0$. We can see this by changing the sign of q in the integration. Hence we can replace the polarization tensor $P_{\mu\nu}(p)$ in (2.10) by the manifestly symmetric form $\Pi_{\mu\nu}(p)$:

$$\frac{1}{2}\langle S_{\rm I}\rangle - \frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) = C_N\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\Pi_{\mu\nu}(p),$$

$$\Pi_{\mu\nu}(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[-\frac{(p_\mu + 2q_\mu)(p_\nu + 2q_\nu)}{8q^2(q+p)^2} + \frac{\delta_{\mu\nu}}{4q^2}\right]. \tag{2.12}$$

As it is now defined, the polarization tensor is symmetric, but breaks gauge invariance. This is because at this order in the loop expansion, $p_\mu\Pi_{\mu\nu}(p)\neq 0$. The reason for this is clear; gauge symmetry is explicitly broken by sharp-momentum cutoffs. The purpose of the counterterms $S_{{\rm c.t.},\Lambda,b}$ and $S_{{\rm c.t.},\tilde{\Lambda},\tilde{b}}$ in (2.1) and (2.3), respectively, is to restore this symmetry.

III. RENORMALIZATION OF QCD WITH A MOMENTUM CUTOFF: THE SPHERICAL CASE

Next we present the results of the one-loop calculation presented in the last section for spherical cutoffs, i.e. $b = \tilde{b} = 1$. Absolutely nothing new will be found in this section. Our only reason for discussing the spherical case is that it is a serviceable template for the more complicated ellipsoidal case.

Let us first evaluate $\Pi_{\mu\nu}(p)$ in (2.12), segregating it into a gauge-invariant part and a non-gauge-invariant part. At

$p = 0$,

$$\Pi_{\mu\nu}(0) = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\left[-\frac{q_\mu q_\nu}{2(q^2)^2} + \frac{\delta_{\mu\nu}}{4q^2}\right].$$

If we change the sign of one component only of q, e.g. $q_0 \to -q_0$, $q_{\rm i} \to q_{\rm i}$,i $= 1, 2, 3$, the first term of the integrand changes sign for $\mu = 0$ and $\nu = {\rm i}$. Hence $\Pi_{\mu\nu}(0)$ vanishes unless $\mu = \nu$. Thus

$$\Pi_{\mu\nu}(0) = \frac{1}{8}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{\delta_{\mu\nu}}{q^2} = \frac{1}{128\pi^2}(\Lambda^2 - \tilde{\Lambda}^2)\delta_{\mu\nu}.$$

Writing $\Pi_{\mu\nu}(p) = \hat{\Pi}_{\mu\nu}(p) + \Pi_{\mu\nu}(0)$, we find

$$\hat{\Pi}_{\mu\nu}(p) = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\left[-\frac{(p_\mu + 2q_\mu)(p_\nu + 2q_\nu)}{8q^2(q+p)^2} + \frac{\delta_{\mu\nu}}{8q^2}\right].$$

If we subtract the polarization tensor at zero momentum by a counterterms of identical form at each scale, or in other words

$$S_{{\rm c.t.},\Lambda} = -\frac{\Lambda^2}{128\pi^2}\int d^4x A^2, \qquad S_{{\rm c.t.},\tilde{\Lambda}} = -\frac{\tilde{\Lambda}^2}{128\pi^2}\int d^4x \tilde{A}^2, \tag{3.1}$$

the result is gauge invariant, as we show below.

Next we expand the polarization tensor $\hat{\Pi}_{\mu\nu}(p)$ in powers of p. The terms which are more than quadratic order in p have canonical dimension greater than four, so can be ignored in the new action. To this order,

$$\hat{\Pi}_{\mu\nu}(p) = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\left[\frac{p_\mu p_\nu + \delta_{\mu\nu}p^2}{8(q^2)^2} - \frac{2p_\alpha p_\beta q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4}\right] + \cdots \tag{3.2}$$

The right-hand side of (3.2) is readily evaluated using Euclidean O(4) symmetry: we emphasize this point, because in the aspherical case, we will not have invariance under O(4), but under its subgroup O(2) × O(2). Exploiting this symmetry, we write the nontrivial tensor integral in (3.2) in terms of a scalar integral:

$$\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4} = \frac{1}{24}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\nu}\delta_{\mu\beta} + \delta_{\alpha\mu}\delta_{\beta\nu}).$$

The polarization tensor is therefore

$$\hat{\Pi}_{\mu\nu}(p) = \frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}(\delta_{\mu\nu} - p_\mu p_\nu) + \cdots. \tag{3.3}$$

Gauge invariance is satisfied to this order of p, i.e. $p^\mu \hat{\Pi}_{\mu\nu}(p) = 0$.

Next we turn to (2.11). As before, the terms of dimension higher than four can be dropped, by expanding the integral over $\mathbb{S}$ in powers of p:

$$\begin{aligned} -\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) &= -\frac{C_N}{2}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2} + \cdots \\ &= -\frac{C_N}{16\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p) + \cdots. \end{aligned} \tag{3.4}$$

Putting together (2.12), (3.1), (3.3), and (3.4) gives the standard result for the new coupling $\tilde{g}_0$ in (2.3):

$$\frac{1}{\tilde{g}_0^2} = \frac{1}{g_0^2} - \frac{C_N}{8\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{12}\frac{C_N}{8\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} = \frac{1}{g_0^2} - \frac{11C_N}{96\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}. \tag{3.5}$$

IV. RENORMALIZATION OF QCD WITH A MOMENTUM CUTOFF: THE ELLIPSOIDAL CASE

In the general case of ellipsoidal cutoffs, integration over the region $\mathbb{S}$ is done by the change of variables, from q_μ to two angles θ and ϕ, and two variables with dimensions of momentum squared, u and w. The relation between the old and new variables is

$$q_1 = \sqrt{u}\cos\theta, \qquad q_2 = \sqrt{u}\sin\theta, \qquad q_3 = \sqrt{w-u}\cos\phi, \qquad q_0 = \sqrt{w-u}\sin\phi \tag{4.1}$$

(note that $u = q_\perp^2$ and $w - u = q_L^2$), which gives

$$\int_{\mathbb{S}} d^4q = \frac{1}{4}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi\left[\int_0^{\tilde{\Lambda}^2} du \int_{\tilde{b}^{-1}\tilde{\Lambda}^2 + (1-\tilde{b}^{-1})u}^{b^{-1}\Lambda^2 + (1-b^{-1})u} dw + \int_{\tilde{\Lambda}^2}^{\Lambda^2} du \int_u^{b^{-1}\Lambda^2 + (1-b^{-1})u} dw\right]. \tag{4.2}$$

The O(2) × O(2) symmetry group is generated by translations of the angles $\theta \to \theta + d\theta$ and $\phi \to \phi + d\phi$.

The polarization tensor $\Pi_{\mu\nu}(p)$ in (2.12), expanded to second order in p_α may be written as the sum of six terms:

$$\Pi_{\mu\nu}(p) = \Pi^1_{\mu\nu}(p) + \Pi^2_{\mu\nu}(p) + \Pi^3_{\mu\nu}(p) + \Pi^4_{\mu\nu}(p) + \Pi^5_{\mu\nu}(p) + \Pi^6_{\mu\nu}(p),$$

where

$$\Pi^1_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{4}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}, \qquad \Pi^2_{\mu\nu}(p) = -\frac{1}{2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{q_\mu q_\nu}{(q^2)^2},$$
$$\Pi^3_{\mu\nu}(p) = \frac{p_\mu p_\alpha}{2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{q_\nu q_\alpha}{(q^2)^3} + \frac{p_\nu p_\alpha}{2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{q_\mu q_\alpha}{(q^2)^3}, \qquad \Pi^4_{\mu\nu}(p) = -\frac{p_\mu p_\nu}{8}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2}, \tag{4.3}$$
$$\Pi^5_{\mu\nu}(p) = \frac{p^2}{2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{q_\mu q_\nu}{(q^2)^3}, \qquad \Pi^6_{\mu\nu}(p) = -2p_\alpha p_\beta I^6_{\alpha\beta\mu\nu}(p), \quad \text{where } I^6_{\alpha\beta\mu\nu}(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4}.$$

We next evaluate each of the six terms of the polarization tensor (4.3). This is done using the integration (4.2) over the variables (4.1) which is tedious, though not difficult. Since the integrals are invariant under O(2) × O(2), but not O(4), we introduce a bit of notation. We assume the indices C and D take only the values 1 and 2, and the indices Ω and Ξ take only the values 3 and 0. As usual, the indices μ, ν, etc., can take any of the four values 1, 2, 3 and 0. The results are

$$\Pi^1_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{64\pi^2}\left(\frac{\Lambda^2\ln b}{b-1} - \frac{\tilde{\Lambda}^2\ln\tilde{b}}{\tilde{b}-1}\right), \tag{4.4}$$

$$\Pi^2_{CD}(p) = -\frac{\Lambda^2\delta_{CD}}{64\pi^2}\left[1 + \frac{b}{(b-1)^2}(1-b+\ln b)\right] + \frac{\tilde{\Lambda}^2\delta_{CD}}{64\pi^2}\left[1 + \frac{\tilde{b}}{(\tilde{b}-1)^2}(1-\tilde{b}+\ln\tilde{b})\right],$$
$$\Pi^2_{\Omega\Xi}(p) = -\frac{\Lambda^2\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{1}{b-1} - \frac{\ln b}{(b-1)^2}\right] + \frac{\tilde{\Lambda}^2\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{1}{\tilde{b}-1} - \frac{\ln\tilde{b}}{(\tilde{b}-1)^2}\right], \qquad \Pi^2_{C\Omega}(p) = \Pi^2_{\Omega C}(p) = 0, \tag{4.5}$$

$$\Pi^3_{CD}(p) = \frac{p_C p_D}{32\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p_C p_D}{64\pi^2}\left[\frac{b\ln b}{(b-1)^2} - \frac{b}{b-1}\right] + \frac{p_C p_D}{64\pi^2}\left[\frac{\tilde{b}\ln\tilde{b}}{(\tilde{b}-1)^2} - \frac{\tilde{b}}{\tilde{b}-1}\right],$$
$$\Pi^3_{\Omega\Xi}(p) = \frac{p_\Omega p_\Xi}{32\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p_\Omega p_\Xi}{64\pi^2}\left[\frac{2b\ln b}{b-1} - \frac{b\ln b}{(b-1)^2} + \frac{b}{b-1}\right] + \frac{p_\Omega p_\Xi}{64\pi^2}\left[\frac{2\tilde{b}\ln\tilde{b}}{\tilde{b}-1} - \frac{\tilde{b}\ln\tilde{b}}{(\tilde{b}-1)^2} + \frac{\tilde{b}}{\tilde{b}-1}\right], \tag{4.6}$$
$$\Pi^3_{C\Omega}(p) = \Pi^3_{\Omega C}(p) = \frac{p_C p_\Omega}{32\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p_C p_\Omega}{64\pi^2}\frac{b\ln b}{b-1} + \frac{p_C p_\Omega}{64\pi^2}\frac{\tilde{b}\ln\tilde{b}}{\tilde{b}-1},$$

$$\Pi^4_{\mu\nu}(p) = -\frac{p_\mu p_\nu}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{p_\mu p_\nu}{128\pi^2}\left(\frac{b\ln b}{b-1} - \frac{\tilde{b}\ln\tilde{b}}{\tilde{b}-1}\right), \tag{4.7}$$

$$\Pi^5_{CD}(p) = \frac{p^2\delta_{CD}}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p^2\delta_{CD}}{128\pi^2}\left[\frac{b\ln b}{(b-1)^2} - \frac{b}{b-1}\right] + \frac{p^2\delta_{CD}}{128\pi^2}\left[\frac{\tilde{b}\ln\tilde{b}}{(\tilde{b}-1)^2} - \frac{\tilde{b}}{\tilde{b}-1}\right],$$
$$\Pi^5_{\Omega\Xi}(p) = \frac{p^2\delta_{\Omega\Xi}}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p^2\delta_{\Omega\Xi}}{128\pi^2}\left[\frac{b(2b-3)\ln b}{(b-1)^2} + \frac{b}{b-1}\right] + \frac{p^2\delta_{\Omega\Xi}}{128\pi^2}\left[\frac{\tilde{b}(2\tilde{b}-3)\ln\tilde{b}}{(\tilde{b}-1)^2} + \frac{\tilde{b}}{\tilde{b}-1}\right], \tag{4.8}$$
$$\Pi^5_{C\Omega}(p) = \Pi^5_{\Omega C}(p) = 0,$$

and finally, we present the components of the tensor $I^6_{\alpha\beta\mu\nu}(p)$ (from which the components of $\Pi^6_{\mu\nu}(p)$ can be obtained)

$$I^6_{CCCC}(p) = \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{b^3}{128\pi^2(b-1)^3}\left[\ln b - \frac{2(b-1)}{b} + \frac{b^2-1}{2b^2}\right] + \frac{\tilde{b}^3}{128\pi^2(\tilde{b}-1)^3}\left[\ln\tilde{b} - \frac{2(\tilde{b}-1)}{\tilde{b}} + \frac{\tilde{b}^2-1}{2\tilde{b}^2}\right],$$
$$I^6_{1122}(p) = \frac{1}{3}I^6_{CCCC}(p),$$
$$I^6_{\Omega\Omega\Omega\Omega}(p) = \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2(b-1)^3}\left[\ln b - 2(b-1) + \frac{b^2-1}{2}\right] + \frac{1}{64\pi^2(\tilde{b}-1)^3}\left[\ln\tilde{b} - 2(\tilde{b}-1) + \frac{\tilde{b}^2-1}{2}\right],$$
$$I^6_{0033}(p) = \frac{1}{3}I^6_{\Omega\Omega\Omega\Omega}(p),$$
$$I^6_{CC\Omega\Omega} = \frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{384\pi^2}\left[\frac{3b(2b-3)\ln b}{(b-1)^2} - \frac{2b^3\ln b}{(b-1)^3} + \frac{3b}{b-1} + \frac{2b-1}{b} + \frac{b^2-1}{2b^2}\right]$$
$$+ \frac{1}{384\pi^2}\left[\frac{3\tilde{b}(2\tilde{b}-3)\ln\tilde{b}}{(\tilde{b}-1)^2} - \frac{2\tilde{b}^3\ln\tilde{b}}{(\tilde{b}-1)^3} + \frac{3\tilde{b}}{\tilde{b}-1} + \frac{2\tilde{b}-1}{\tilde{b}} + \frac{\tilde{b}^2-1}{2\tilde{b}^2}\right]. \tag{4.9}$$

All other nonvanishing components of $I^6_{\alpha\beta\mu\nu}(p)$ can be obtained by permuting indices of those shown in (4.9).

Notice that $\Pi^j_{\mu\nu}(p)$, $j = 1, \ldots, 6$ each change sign under the interchange of Λ and b with $\tilde{\Lambda}$ and $\tilde{b}$, respectively. We can eliminate $\Pi^1_{\mu\nu}(p)$ and $\Pi^2_{\mu\nu}(p)$ by a mass counterterm. The sum of the other pieces of the polarization tensor, $\sum_{j=3}^6 \Pi^j_{\mu\nu}(p)$, reduces to the expression in (3.3) if $b = \tilde{b}$; integrating degrees of freedom with momenta between two similar ellipsoids yields the same result as integrating degrees of freedom with momenta between two spheres.

Next we set $b = 1$ and expand $\tilde{b} = 1 + \ln\tilde{b} + \cdots$. We drop the part of the polarization tensor of order $(\ln\tilde{b})^2$. We write the polarization tensor as matrix whose rows and columns are ordered by 1, 2, 3, 0. After some work, we obtain

$$\sum_{j=3}^{6}\Pi^j(p) = \frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}(\mathbb{1} - pp^T) + \frac{\ln\tilde{b}}{64\pi^2}$$
$$\times\begin{pmatrix} -\frac{3}{4}p_1^2 - \frac{1}{6}p_2^2 - \frac{13}{12}p_L^2 & -\frac{7}{12}p_1p_2 & -\frac{7}{4}p_1p_3 & -\frac{7}{4}p_1p_0 \\ -\frac{7}{12}p_1p_2 & -\frac{3}{4}p_2^2 - \frac{1}{6}p_1^2 - \frac{13}{12}p_L^2 & -\frac{7}{4}p_2p_3 & -\frac{7}{4}p_2p_0 \\ -\frac{7}{4}p_1p_3 & -\frac{7}{4}p_2p_3 & \frac{7}{4}p_3^2 + \frac{2}{3}p_0^2 + \frac{1}{3}p_\perp^2 & \frac{13}{12}p_3p_0 \\ -\frac{7}{4}p_1p_0 & -\frac{7}{4}p_2p_0 & \frac{13}{12}p_3p_0 & \frac{2}{3}p_3^2 + \frac{7}{4}p_0^2 + \frac{1}{3}p_\perp^2 \end{pmatrix}, \tag{4.10}$$

where $\mathbb{1}$ is the four-by-four identity matrix and the superscript T denotes the transpose. The first term on the right-hand side of (4.10) is the polarization tensor found in the previous section (3.3). The second term does not depend on Λ or $\tilde{\Lambda}$. Had we taken $b > 1$, and expanded $b = 1 + \ln b + \cdots$, the quantity $\ln\tilde{b}$ in (4.10) would have been $\ln(\tilde{b}/b)$.

Notice that the second term on the right-hand side of (4.10) violates gauge invariance (multiplying the vector p by the matrix in this term does not yield zero). Therefore, an additional counterterm is necessary. The most general local action of dimension 4, which is quadratic in $\tilde{A}_\mu$ and which does not change under $O(2)\times O(2)$ transformations and is gauge invariant to linear order is

$$S_{\text{quad}} = \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\,\text{Tr}\tilde{A}(-p)^T[a_1M_1(p) + a_2M_2(p) + a_3M_3(p)]\tilde{A}(p),$$

where a_1, a_2 and a_3 are real coefficients and

$$M_1(p) = \begin{pmatrix} p_2^2 & -p_1p_2 & 0 & 0 \\ -p_1p_2 & p_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$M_2(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_3^2 & -p_3p_0 \\ 0 & 0 & -p_3p_0 & p_0^2 \end{pmatrix},$$
$$M_3(p) = \begin{pmatrix} p_L^2 & 0 & -p_1p_3 & -p_1p_0 \\ 0 & p_L^2 & -p_2p_3 & -p_2p_0 \\ -p_1p_3 & -p_2p_3 & p_\perp^2 & 0 \\ -p_1p_0 & -p_2p_0 & 0 & p_\perp^2 \end{pmatrix}.$$

We next determine a_1, a_2 and a_3 such that the difference

$$S_{\text{diff}} = \int_{\tilde{\mathbb{P}}}\frac{d^4q}{(2\pi)^4}\,\text{Tr}\tilde{A}(-p)^TM_{\text{diff}}(p)\tilde{A}(p) = \int_{\tilde{\mathbb{P}}}\frac{d^4q}{(2\pi)^4}\,\text{Tr}\tilde{A}(-p)^T\sum_{j=3}^{6}\Pi^j(p)\tilde{A}(p) - S_{\text{quad}} \tag{4.11}$$

is maximally non-gauge invariant. By this we mean that the projection of tensor $M_{\rm diff}(p)$ to a gauge-invariant expression:

$$\Big(\mathbb{1}-\frac{pp^T}{p^Tp}\Big)M_{\rm diff}(p)\Big(\mathbb{1}-\frac{pp^T}{p^Tp}\Big),$$

has no local part. This gives a precise determination of $S_{\rm diff}$, which is proportional to the counterterm to be subtracted. To carry this procedure out, we break up the second term of (4.10) into a linear combination of M_1, M_2 and M_3 and a diagonal matrix:

$$\sum_{j=3}^{6}\Pi^j(p)=\frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}(\mathbb{1}-pp^T)+\frac{\ln\tilde{b}}{64\pi^2}\Big[\frac{7}{12}M_1(p)-\frac{13}{12}M_2(p)+\frac{7}{4}M_3(p)\Big]$$
$$+\frac{\ln\tilde{b}}{64\pi^2}\begin{pmatrix}-\frac{3}{4}p_\perp^2-\frac{17}{6}p_L^2 & 0 & 0 & 0\\ 0 & -\frac{3}{4}p_\perp^2-\frac{17}{6}p_L^2 & 0 & 0\\ 0 & 0 & -\frac{17}{12}p_\perp^2+\frac{7}{4}p_L^2 & 0\\ 0 & 0 & 0 & -\frac{17}{12}p_\perp^2+\frac{7}{4}p_L^2\end{pmatrix}. \tag{4.12}$$

The diagonal matrix is maximally non-gauge-invariant. It is local, $O(2)\times O(2)$ invariant and of dimension four; we remove it with local counterterms, rendering our ellipsoidal cutoffs gauge invariant, to one loop. Therefore

$$a_1=\frac{\ln\tilde{b}}{64\pi^2}\cdot\frac{7}{12},\qquad a_2=-\frac{\ln\tilde{b}}{64\pi^2}\cdot\frac{13}{12},\qquad a_3=\frac{\ln\tilde{b}}{64\pi^2}\cdot\frac{7}{4}.$$

Removing the last term from (4.12) leaves us with our final result for the polarization tensor

$$\hat{\Pi}(p)=\sum_{j=3}^{6}\Pi^j(p)-\frac{\ln\tilde{b}}{64\pi^2}\begin{pmatrix}-\frac{3}{4}p_\perp^2-\frac{17}{6}p_L^2 & 0 & 0 & 0\\ 0 & -\frac{3}{4}p_\perp^2-\frac{17}{6}p_L^2 & 0 & 0\\ 0 & 0 & -\frac{17}{12}p_\perp^2+\frac{7}{4}p_L^2 & 0\\ 0 & 0 & 0 & -\frac{17}{12}p_\perp^2+\frac{7}{4}p_L^2\end{pmatrix}$$
$$=\frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}(\mathbb{1}-pp^T)+\frac{\ln\tilde{b}}{64\pi^2}\Big[\frac{7}{12}M_1(p)-\frac{13}{12}M_2(p)+\frac{7}{4}M_3(p)\Big].$$

One of the terms to be induced in the renormalized action by integrating out fast degrees of freedom is thereby

$$\frac{1}{2}\langle S_{\rm I}\rangle-\frac{1}{4}(\langle S_{\rm I}^2\rangle-\langle S_{\rm I}\rangle^2)=C_N\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\hat{\Pi}_{\mu\nu}(p)$$
$$=C_N\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\Big\{\frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}(\mathbb{1}-pp^T)+\frac{\ln\tilde{b}}{64\pi^2}\Big[\frac{7}{12}M_1(p)-\frac{13}{12}M_2(p)+\frac{7}{4}M_3(p)\Big]\Big\}. \tag{4.13}$$

The other term induced by this integration, namely $-(\langle S_{\rm II}^2\rangle-\langle S_{\rm II}\rangle^2)/2$, will be discussed next.

We showed in Sec. II that the term $-(\langle S_{\rm II}^2\rangle-\langle S_{\rm II}\rangle^2)/2$ is given by (2.11). This term may be expanded in powers of p as we did for the spherical case in (3.4). The result is

$$-\frac{1}{2}(\langle S_{\rm II}^2\rangle-\langle S_{\rm II}\rangle^2)=-\frac{C_N}{2}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2}+\cdots$$
$$=-C_N\Big[\frac{1}{16\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}-\frac{b\ln b}{32\pi^2(b-1)}+\frac{\tilde{b}\ln\tilde{b}}{32\pi^2(\tilde{b}-1)}\Big]\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)+\cdots. \tag{4.14}$$

For $b = 1$ and to leading order in $\ln\tilde{b}$, (4.14) becomes

$$-\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) = -C_N\left(\frac{1}{16\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{\ln\tilde{b}}{32\pi^2}\right) \times \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p) + \cdots. \tag{4.15}$$

Putting together (4.13) and (4.15) gives the following expression for the new action $\tilde{S} = \int d^4x\tilde{\mathcal{L}}$:

$$\tilde{\mathcal{L}} = \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{C_N\ln\tilde{b}}{64\pi^2}\right)(\tilde{F}^2_{01} + \tilde{F}^2_{02} + \tilde{F}^2_{13} + \tilde{F}^2_{23}) + \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{37C_N\ln\tilde{b}}{192\pi^2}\right)\tilde{F}^2_{03} + \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{17C_N\ln\tilde{b}}{192\pi^2}\right)\tilde{F}^2_{12} + \cdots. \tag{4.16}$$

V. THE LONGITUDINALLY RESCALED YANG-MILLS ACTION

The main result of Sec. IV, Eq. (4.16), tells what happens after aspherically integrating out degrees of freedom. We will write this in a form which allows comparison with standard renormalization with an isotropic cutoff, i.e. (3.5). We define $\tilde{g}_0$ using (3.5). To leading order in $\ln\tilde{b}$, the effective coupling in the first term of (4.16) is given by

$$\frac{1}{g^2_{\rm eff}} = \frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{C_N\ln\tilde{b}}{64\pi^2} = \frac{1}{\tilde{g}_0^2}\tilde{b}^{-(C_N/64\pi^2)\tilde{g}_0^2} + \cdots.$$

After we set $\tilde{b} = \lambda^{-2}$, we find to leading order in $\ln\lambda$

$$g^2_{\rm eff} = \tilde{g}_0^2\lambda^{-(C_N/32\pi^2)\tilde{g}_0^2}. \tag{5.1}$$

and

$$\tilde{\mathcal{L}} = \frac{1}{4g^2_{\rm eff}}{\rm Tr}(\tilde{F}^2_{01} + \tilde{F}^2_{02} + \tilde{F}^2_{13} + \tilde{F}^2_{23} + \lambda^{(17C_N/48\pi^2)\tilde{g}_0^2}\tilde{F}^2_{03} + \lambda^{(7C_N/48\pi^2)\tilde{g}_0^2}\tilde{F}^2_{12}) + \cdots,$$

where the corrections are of order $(\ln\lambda)^2$. We perform the rescaling of longitudinal coordinates, $x^L \to \lambda x^L$, drop the tildes on the fields, and Wick-rotate back to Minkowski signature, to find the longitudinally rescaled effective Lagrangian

$$\mathcal{L}_{\rm eff} = \frac{1}{4g^2_{\rm eff}}{\rm Tr}(F^2_{01} + F^2_{02} - F^2_{13} - F^2_{23} + \lambda^{-2+(17C_N/48\pi^2)\tilde{g}_0^2}F^2_{03} - \lambda^{2+(7C_N/48\pi^2)\tilde{g}_0^2}F^2_{12}) + \cdots, \tag{5.2}$$

where again the corrections are of order $(\ln\lambda)^2$. Comparing this with the classically rescaled action (1.1) we see that the field-strength-squared terms are anomalously rescaled.

If we naively consider the limit as $\lambda \to 0$ of (5.2), all couplings become zero or infinite, except $g_{\rm eff}$ [3]. For very high energy, that is for small λ, this effective coupling becomes strong, as can immediately be seen from (5.1). We are fortunate, however, that this energy is enormous. If we take $\tilde{g}_0$ of order one, then

$$g^2_{\rm eff} \sim \lambda^{-(1/100)}. \tag{5.3}$$

This tells us that $g^2_{\rm eff}$ is less than a number of order ten, unless λ is roughly less than an inverse googol, $\lambda \sim 10^{-100}$. Thus the experimentally accessible value of $g_{\rm eff}$ is small. Even so, there is the concern that the coefficient of F^2_{12} in the effective Lagrangian is very small as $\lambda \to 0$. This is also true for the classically rescaled theory (1.1) [5]. This means that there is very little energy in longitudinal magnetic flux. Hence the longitudinal magnetic flux fluctuates wildly. If we call the coefficient of this term in the Lagrangian $1/(4g_L^2)$, then

$$g_L^2 = g^2_{\rm eff}\lambda^{-2-(7C_N/48\pi^2)\tilde{g}_0^2}, \tag{5.4}$$

which explodes for small λ, even if $g_{\rm eff}$ is small.

VI. IMPLICATIONS FOR EFFECTIVE HIGH-ENERGY THEORIES

We have determined how a quantized non-Abelian gauge action changes under a longitudinal rescaling $\lambda < 1$, but $\lambda \approx 1$. Though our analysis suggests the form of the effective action for the high-energy limit, $\lambda \ll 1$, we cannot prove that this form is correct. The main problem concerns how the Yang-Mills action changes as λ is decreased. The coefficient of the longitudinal magnetic field squared, in the action, decreases, as λ is decreased. Eventually, we can no longer compute how couplings will run.

Our difficulty is very similar to that of finding the spectrum of a non-Abelian gauge theory. Assuming that there is no infrared-stable fixed point at nonzero bare coupling, a guess for the long-distance effective theory is a strongly coupled cutoff action. The regulator can be a lattice, for example. One can then use strong-coupling expansions to find the spectrum. The problem is that no one knows how to specify the true cutoff theory (which presumably has many terms, produced by integrating over all the short-distance degrees of freedom). The best we can do is guess the regularized strongly coupled action. Such strong-coupling theories are not (yet) derivable from QCD, but are best thought of as models of the strong interaction at large distances.

Similarly, we believe that (1.1) for $\lambda \ll 1$, and variants we discuss below, cannot be proved to describe the strong interaction at high energies. Thus it appears that the same statement applies to the BFKL/BK theory (designed to describe the region where Mandelstam variables satisfy $s \gg t \gg \Lambda_{\rm QCD}$) [11,17]. Two closely related problems in this theory are lack of unitarity and infrared diffusion of gluon virtualities. These problems indicate that the BFKL theory breaks down at large length scales. There is numerical evidence [18] that unitarizing using the BK evolution equation [17] suppresses diffusion into the infrared and leads to saturation, at least for fixed small impact parameters. This BK equation is a nonlinear generalization of the BFKL evolution equation. The nonlinearity only becomes important at small x, at large longitudinal distances, where perturbation theory is not trustworthy.

In the color-glass-condensate picture [6,7], the Yang-Mills action with $\ln\lambda = 0$ is coupled to sources. The classical field strength is purely transverse. If this action is quantized, however, this is no longer the case. The fluctuations of the longitudinal magnetic field $\mathcal{B}_3$ will become extremely large [this can be seen by inspecting (1.1) and (1.2)]. In principle, we would hope to derive the color-glass condensate by a longitudinal renormalization-group transformation, with background sources. The obstacle to doing this is precisely the problem of large fluctuations of $\mathcal{B}_3$. This does not suggest any inconsistency of the color-glass-condensate idea itself, but indicates how difficult it may be to establish the color-glass condensate directly in QCD.

Finally we wish to comment on an approach to soft-scattering and total cross sections. In Ref. [5] an effective lattice SU(N) gauge theory was proposed. This gauge theory is a regularization of (1.2) and (1.3). This gauge theory can be formulated as coupled (1 + 1)-dimensional SU(N) $\times$ SU(N) nonlinear sigma models and reduces to a lattice Yang-Mills theory at $\lambda = 1$ (in which case, it is equivalent to the light-cone lattice theory of Bardeen *et al.* [19]). The nonlinear sigma model is asymptotically free and has a mass gap. These facts together with the assumption that the terms proportional to λ^2 are a weak perturbation leads to confinement and diffraction in the gauge theory. Similar gauge models in (2 + 1) dimensions were proposed as laboratories of color confinement [20], and string tensions for different representations [21], the low-lying glueball spectrum [22], and corrections of higher order in order λ to the string tension [23] were found (these calculations were performed using the exact S-matrix [24] and form factors [25] of the (1 + 1)-dimensional nonlinear sigma model). In such theories (whether in (2 + 1) or (3 + 1) dimensions), transverse electric flux is built of massive partons (made entirely of glue, but not conventional gluons). These partons move (and scatter) only longitudinally, to leading order in λ. The behavior of such gauge-theory models is very close to the picture of the forward-scattering amplitude suggested by Kovner [26].

The effective gauge theory of Ref. [5] has a small value of $g_{\rm eff}$, as well as a small value of λ, in the Hamiltonian (1.2). We have found in Sec. V that $g_{\rm eff}$ grows extremely slowly, as the energy is increased. If we can naively extrapolate our results to extremely high energies, this effective gauge theory appears correct. We should not, however, regard this as proof that the effective theory is valid, since the perturbative calculation of Sec. V breaks down at such energies.

VII. DISCUSSION

In this paper, we found how the action of an SU(N) gauge changes under longitudinal rescaling at one loop. This was done by a Wilsonian renormalization procedure. As the energy increases, the coefficient of F_{12}^2 in the action eventually becomes too small to trust the method further. Therefore, neither classical nor perturbative methods may be entirely trusted beyond a certain energy. The breakdown of these methods at high energies is similar to the breakdown of perturbation theory to compute the force between charges at large distances, in an asymptotically-free theory. Nonetheless, high-energy effective theories, inspired by the longitudinal rescaling idea, may be phenomenologically useful.

The next step is to repeat our calculation including fermions. Aside from the importance of considering QCD with quarks, it would be interesting to study how longitudinal rescaling affects the QED action.

We should point out that another way to derive our effective Lagrangian (5.2) and investigate anomalous dimensions of other operators would be to carefully study Green's functions of the operator

$$\mathcal{D}(x) = x^0 \mathcal{T}_{00}(x) + x^3 \mathcal{T}_{03}(x), \tag{7.1}$$

where $\mathcal{T}_{\mu\nu}(x)$ is the stress-energy-momentum tensor. The spacial integral of this operator generates longitudinal rescalings on states. Correlators of products of $\mathcal{D}(x)$ and other operators could be studied with simpler regularization methods (such as dimensional regularization) instead of our sharp momentum cutoff. The commutator of $\mathcal{D}(x)$ and an operator $\mathcal{O}(y)$ will reveal how $\mathcal{O}(y)$ behaves under longitudinal rescaling. Such an analysis should be easier than the method used in this paper, especially beyond one loop.

ACKNOWLEDGMENTS

P. O. had early conversations about this work with Poul Henrik Damgaard at the Niels Bohr International Academy and Gordon Semenoff at the University of British Columbia. We would also like to thank Adrian Dumitru and Jamal Jalilian-Marian for extensive discussions. Finally, we are grateful to Rob Pisarski for making a helpful technical suggestion. This research was supported in part by the National Science Foundation, under Grant No. PHY0653435 and by a grant from the PSC-CUNY.

[1] L. N. Lipatov, Nucl. Phys. **B365**, 614 (1991).
[2] Yu. V. Kovchegov, Phys. Rev. D **54**, 5463 (1996); **55**, 5445 (1997).
[3] H. Verlinde and E. Verlinde, Princeton University report, PUPT-1319, 1993.
[4] R. Kirschner, L. N. Lipatov, and L. Szymanowski, Phys. Lett. B **425**, 597 (1994); Phys. Rev. D **51**, 838 (1995).
[5] P. Orland, Phys. Rev. D **77**, 056004 (2008).
[6] L. McLerran and R. Venugopalan, Phys. Rev. D **49**, 2233 (1994); **49**, 3352 (1994); **50**, 2225 (1994); **59**, 094002 (1999).
[7] E. Iancu, A. Leonidov, and L. D. McLerran, Nucl. Phys. **A692**, 583 (2001); Phys. Lett. B **510**, 133 (2001).
[8] J. Jalilian-Marian, A. Kovner, L. McLerran, and H. Weigert, Phys. Rev. D **55**, 5414 (1997); J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, Nucl. Phys. **B504**, 415 (1997); Phys. Rev. D **59**, 014014 (1998); **59**, 034007 (1999); J. Jalilian-Marian, A. Kovner, and H. Weigert, Phys. Rev. D **59**, 014015 (1998); J. Jalilian-Marian and X. N. Wang, Phys. Rev. D **60**, 054016 (1999); A. Kovner and J. G. Milhano, Phys. Rev. D **61**, 014012 (1999).
[9] J. Jalilian-Marian, S. Jeon, and R. Venugopalan, Phys. Rev. D **63**, 036004 (2001).
[10] Y. Hatta, E. Iancu, L. McLerran, A. Stasto, and D. N. Triantafyllopoulos, Nucl. Phys. **A764**, 423 (2006).
[11] V. Fadin, E. Kuraev, and L. N. Lipatov, Sov. Phys. JETP **44**, 443 (1976); I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28**, 822 (1978).
[12] L. N. Lipatov, Nucl. Phys. **B452**, 369 (1995).
[13] I. Balitsky, Phys. Rev. D **72**, 074027 (2005).
[14] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
[15] I. Ya. Aref'eva and I. V. Volovich, Steklov Math. Inst. report, SMI-15-94, 1994; V. Periwal, Phys. Rev. D **52**, 7328 (1995).
[16] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, Chur Switzerland, 1987).
[17] I. Balitsky, Nucl. Phys. **B463**, 99 (1996); Phys. Rev. Lett. **81**, 2024 (1998); Phys. Rev. D **60**, 014020 (1999); Phys. Lett. B **518**, 235 (2001); Yu. V. Kovchegov, Phys. Rev. D **60**, 034008 (1999).
[18] K. Golec-Biernat, L. Motyka, and A. M. Stasto, Phys. Rev. D **65**, 074037 (2002); K. Golec-Biernat and A. M. Stasto, Nucl. Phys. **B668**, 345 (2003).
[19] W. A. Bardeen and R. B. Pearson, Phys. Rev. D **14**, 547 (1976); W. A. Bardeen, R. B. Pearson, and E. Rabinovici, Phys. Rev. D **21**, 1037 (1980).
[20] P. Orland, Phys. Rev. D **71**, 054503 (2005).
[21] P. Orland, Phys. Rev. D **75**, 025001 (2007).
[22] P. Orland, Phys. Rev. D **75**, 101702(R) (2007).
[23] P. Orland, Phys. Rev. D **74**, 085001 (2006); Phys. Rev. D **77**, 025035 (2008).
[24] E. Abdalla, M. C. B. Abdalla, and A. Lima-Santos, Phys. Lett. B **140**, 71 (1984); P. B. Wiegmann, Phys. Lett. B **142**, 173 (1984); A. M. Polyakov and P. B. Wiegmann, Phys. Lett. B **131**, 121 (1983); P. B. Wiegmann, Phys. Lett. B **141**, 217 (1984).
[25] M. Karowski and P. Weisz, Nucl. Phys. **B139**, 455 (1978).
[26] A. Kovner, Acta Phys. Pol. B **36**, 3551 (2005).