Renormalization of quark bilinear operators in a momentum-subtraction scheme with a nonexceptional subtraction point

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We extend the Rome-Southampton regularization independent momentum-subtraction renormalization scheme (RI/MOM) for bilinear operators to one with a nonexceptional, symmetric subtraction point. Two-point Green's functions with the insertion of quark bilinear operators are computed with scalar, pseudoscalar, vector, axial-vector and tensor operators at one-loop order in perturbative QCD. We call this new scheme RI/SMOM, where the S stands for "symmetric." Conversion factors are derived, which connect the RI/SMOM scheme and the $\overline{\mathrm{MS}}$ scheme and can be used to convert results obtained in lattice calculations into the $\overline{\mathrm{MS}}$ scheme. Such a symmetric subtraction point involves nonexceptional momenta implying a lattice calculation with substantially suppressed contamination from infrared effects. Further, we find that the size of the one-loop corrections for these infrared improved kinematics is substantially decreased in the case of the pseudoscalar and scalar operator, suggesting a much better behaved perturbative series. Therefore it should allow us to reduce the error in the determination of the quark mass appreciably.

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I. INTRODUCTION

Lattice simulations in quantum chromodynamics (QCD) allow for *ab initio* nonperturbative determinations of operator matrix elements and physical quantities such as quark masses and the strong coupling constant. One starts with a direct computation of the bare quantities with the lattice spacing acting as the ultraviolet cutoff in some particular discretization of QCD. Providing that the lattice spacing is sufficiently small, it is in principle possible to obtain the corresponding renormalized quantities using perturbation theory. However, the coefficients in lattice perturbation theory frequently prove to be large and for this reason techniques using nonperturbative renormalization (NPR) have been developed and are being successfully implemented. With these techniques lattice perturbation theory is avoided entirely, and one obtains renormalized quantities in some appropriate renormalization scheme such as the regularization independent momentum-subtraction (RI/MOM) scheme [1].

On the other hand perturbative calculations in continuum QCD are conventionally and conveniently performed using dimensional regularization [2] and the $\overline{\mathrm{MS}}$ renormalization scheme [3,4] which is not directly amenable to the NPR procedure. The continuum perturbation theory is therefore used to match the quantities computed in the RI/MOM and $\overline{\mathrm{MS}}$ schemes. For example, the computation of the mass conversion factor $C_m^{\mathrm{RI/MOM}}$, which converts a quark mass renormalized in the RI/MOM scheme into the $\overline{\mathrm{MS}}$ scheme or the conversion factor $C_q^{\mathrm{RI/MOM}}$, which performs the corresponding conversion of the quark fields, are both known up to three-loop order in perturbative QCD [1,5,6]. Another scheme, which is useful in lattice simulations is the $\mathrm{RI}'/\mathrm{MOM}$ scheme in which these conversion factors are also known up to three-loop order [6,7]. A more detailed definition of these schemes will be discussed in Sec. II. The conversion factors C_m and C_q in both schemes can be obtained through the evaluation of self-energy diagrams. Not only quark masses, but also the strong coupling constant α_s has been studied in MOM schemes [8–13].

With regard to the vertex diagrams one has many choices of defining the subtraction point at which the renormalization constants are fixed through different momentum configurations. In this paper we determine the one-loop matching coefficients for a generalization of the RI/MOM scheme in which there are no channels with exceptional momenta and which we propose to use in our numerical simulations. Because the kinematic configuration in this scheme is symmetrical in the three channels, we call it the RI/SMOM scheme. In the following we define the symmetric and asymmetric Minkowski momentum configurations by

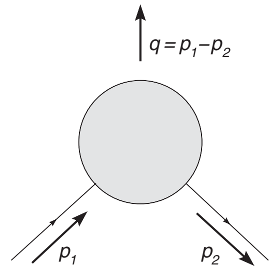


FIG. 1. Momentum flow of a generic diagram required for the renormalization procedure with nonexceptional momenta. The gray bubble stands for an operator insertion and higher order corrections.

(i) symmetric or nonexceptional momentum configuration:

$$p_1^2 = p_2^2 = q^2 = -\mu^2, \qquad \mu^2 > 0,$$
$$q = p_1 - p_2;$$

(ii) asymmetric or exceptional momentum configuration:

$$p_1^2 = p_2^2 = -\mu^2, \qquad \mu^2 > 0,$$
$$p_1 = p_2, \qquad q = 0,$$

where the momentum flow is shown diagrammatically in Fig. 1.

In Ref. [14] quark masses were determined through lattice simulations using nonperturbative renormalization [1] in the RI/MOM scheme and subsequently converted to the $\overline{\text{MS}}$ scheme. In order to renormalize the bare quark masses in the lattice simulation, the renormalization constants need to be computed on the lattice. In regularization and renormalization schemes which preserve flavor and chiral symmetries in the limit of vanishing quark masses, the perturbative renormalization constants of the axial-vector and vector operators as well as the ones for the pseudoscalar and scalar operators need to be equal. In the standard RI/MOM and RI′/MOM schemes the normalization conditions for quark bilinear operators are imposed on Green's functions with the operator inserted between equal incoming and outgoing momenta, say, p, and $-p^2 \equiv \mu^2$ is the renormalization scale. The momentum q inserted at the operator is therefore 0 so that there is an *exceptional* channel, i.e. one in which the square of the momentum is much smaller than the typical large scale (μ^2). For the asymmetric subtraction point effects of chiral symmetry breaking vanish only slowly like $1/p^2$ for large external momenta p^2. In Ref. [15] it was proposed instead to use a similar renormalization procedure but with the incoming and outgoing quarks having different momenta, p_1 and p_2, respectively, with $p_1^2 = p_2^2 = (p_1 - p_2)^2 \equiv p^2$. There are now no exceptional channels and we explain below that this decreases chiral symmetry breaking and other unwanted infrared effects. The choice of such a symmetric subtraction point is very convenient; the renormalized quantities depend also only on a single scale p^2. When the renormalization constants of quark bilinear operators are fixed at a symmetric subtraction point (chosen to have nonexceptional kinematics) chiral symmetry breaking and other unwanted infrared effects are better behaved and vanish with larger asymptotic powers of the order $1/p^6$. This behavior has been derived in Ref. [15] as a consequence of Weinberg's theorem [16] and demonstrated by explicitly computing the renormalization constants on the lattice. Hence these RI/SMOM kinematics suppress infrared effects much more strongly than the usual exceptional configuration for large external momenta. The symmetric momentum configuration is thus much more favorable. However, in order to be able to use it to evaluate the matrix elements of quark bilinear operators and the quark mass, the matching factors need to be determined perturbatively for this new, symmetric choice of momenta. A nonperturbative test of the RI/SMOM scheme for the quark mass renormalization can be found in Ref. [17].

Another drawback in the case of the exceptional momenta is that the perturbative expansion of the usual conversion factor $C_m^{\text{RI/MOM}}$ shows poor convergence and makes a significant contribution to the systematic uncertainty in the quark masses obtained from the lattice studies. In fact, in Ref. [14] the error ($\approx 11\%$) in the quark masses arising from the truncation of the perturbative series in the matching factor amounts to around 60% of the total error. Therefore determining the conversion factor for a symmetric momentum configuration will also allow us to see if the convergence will be better behaved. If it is better behaved, then the symmetric configuration would be preferred for both of these reasons. Motivated by these considerations we study in this work the renormalization of quark bilinear nonsinglet operators of the form $\hat{O} = \bar{u}\Gamma d$ for a symmetric subtraction point, where Γ represents a Dirac matrix and $\bar{u}$ and d are fermion quark fields.

Even with the use of the symmetric, nonexceptional kinematics, the renormalization prescription is not unique and the chiral Ward-Takahashi identities can be satisfied using a variety of procedures. In the following sections we study a specific scheme which we consider to be convenient and practicable for the nonperturbative renormalization of lattice quark bilinear operators. In order to preserve the Ward-Takahashi identity, the definitions of the vertex and wave function renormalizations are related as we explain in the following section.

The outline of this paper is as follows: In Sec. II we define our notation and conventions and introduce the framework required for performing renormalization of the quark bilinear operators with a symmetric subtraction point. Subsequently we present in Sec. III two methods for the extraction of the conversion factor C_m in the RI/SMOM scheme, apply the concepts of Sec. II to calculate the vector, axial-vector, pseudoscalar, scalar and tensor operators between two off-shell quark states at one-loop order in perturbative QCD for the nonexceptional momentum configuration and determine the matching factors. Finally we close with a brief summary and our conclusions in Sec. IV. Even with the symmetric nonexceptional kinematics the choice of renormalization conditions is not unique. In Appendix A we therefore present the one-loop perturbative results in a form which can be used to calculate the conversion factors from a general scheme with a symmetric subtraction point to the $\overline{\mathrm{MS}}$ scheme. For illustration we study one alternative scheme called the RI/SMOM$_{\gamma_\mu}$ scheme, in which the vertex renormalization condition is the same as in the RI/MOM scheme, but with nonexceptional kinematics and with a different wave function renormalization. We also provide the results for the conversion factors and, in Appendix B, the corresponding two-loop anomalous dimensions.

II. CONCEPTS AND FRAMEWORK OF THE RI/SMOM SCHEME

We will begin with a bare, continuum theory of QCD which has been regulated using a scheme which guarantees that Green's functions involving the quark field and quark field bilinears obey the usual chiral and flavor symmetries of QCD. Dimensional regularization is an example of such a scheme.

Let us consider the nonamputated Green's function $G_{\hat{O}}$ of an operator $\hat{O}$ computed between two external off-shell quark lines in a fixed gauge. The corresponding diagrams up to one-loop order in perturbative QCD are shown in Fig. 2.

The amputated Green's function is defined by

$$\Lambda_{\hat{O}} = S^{-1}(p_2) G_{\hat{O}} S^{-1}(p_1), \tag{1}$$

where $S(p)$ is given by the quark propagator

$$\begin{aligned} -iS(p) &= \int dx e^{ipx} \langle T[\Psi(x)\bar{\Psi}(0)]\rangle \\ &= \frac{i}{\not{p} - m + i\epsilon - \Sigma(p)}, \end{aligned} \tag{2}$$

where $\Sigma(p)$ contains the higher order corrections and can, in perturbation theory, be decomposed into its Lorentz structure: $\Sigma(p) = \not{p}\Sigma_V(p^2) + m\Sigma_S(p^2)$. The lowest order and one-loop diagrams contributing to $\Sigma(p)$ are shown in Fig. 3.

In the following we will consider quark bilinear operators $\hat{O} = \bar{u}\Gamma d$ with scalar ($\Gamma = \mathbb{1}$), pseudoscalar ($\Gamma = i\gamma_5$), vector ($\Gamma = \gamma^\mu$), axial-vector ($\Gamma = \gamma^\mu\gamma_5$) and tensor ($\Gamma = \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$) kernels. We will distinguish between bare and renormalized quantities by assigning the index B to a bare quantity and the index R to a renormalized one. In the case of renormalized quantities an additional quantifier specifying the scheme is attached. Renormalized and bare quantities are related through the renormalization constants Z:

$$\Psi_R = Z_q^{1/2}\Psi_B, \qquad m_R = Z_m m_B, \qquad \hat{O}_R = Z_{\hat{O}}\hat{O}_B. \tag{3}$$

The renormalization constants of the scalar ($\hat{O} = S$), pseudoscalar ($\hat{O} = P$), vector ($\hat{O} = V$), axial-vector ($\hat{O} = A$) and tensor ($\hat{O} = T$) operator will be denoted as Z_S, Z_P, Z_V, Z_A and Z_T, respectively. In the RI/MOM scheme the renormalization conditions which fix the renormalization constants Z_m and Z_q are given by

$$\begin{aligned} &\lim_{m_R\to 0} \frac{1}{12 m_R} \mathrm{Tr}[S_R^{-1}(p)]|_{p^2=-\mu^2} = 1 \quad \text{and} \\ &\lim_{m_R\to 0} \frac{1}{48} \mathrm{Tr}\left[\gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu}\right]|_{p^2=-\mu^2} = -1, \end{aligned} \tag{4}$$

where the symbol "Tr" denotes the trace over color and spins. The second equation determines $Z_q^{\mathrm{RI/MOM}}$ and subsequently the first one can be used to extract $Z_m^{\mathrm{RI/MOM}}$. Now in the RI$'$/MOM scheme the second condition of Eqs. (4) is replaced by

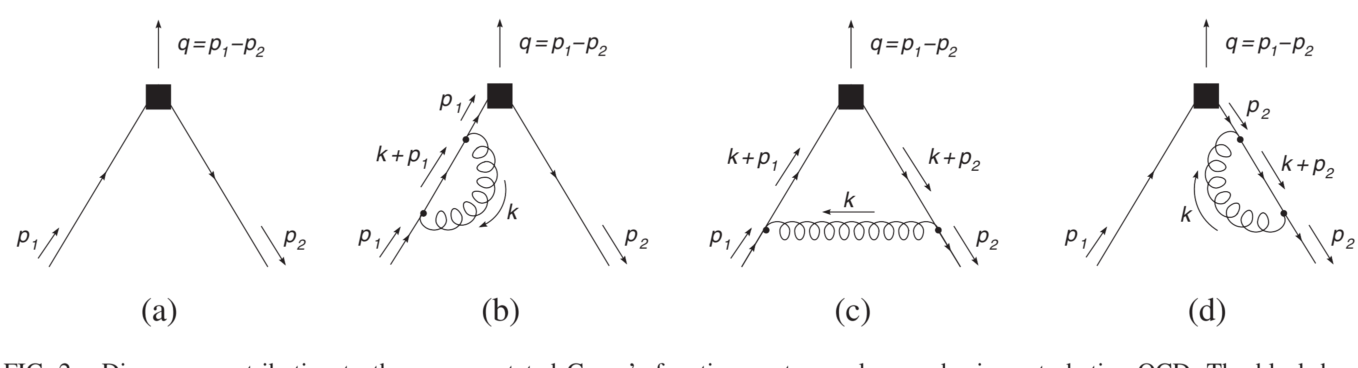
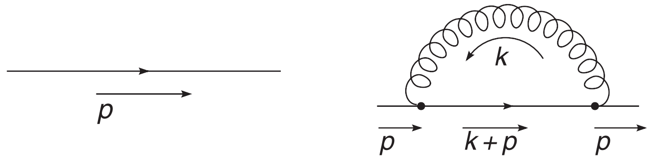


FIG. 2. Diagrams contributing to the nonamputated Green's function up to one-loop order in perturbative QCD. The black box indicates the inserted operator. Spiral lines denote gluons and solid lines fermions.



(a) (b)

FIG. 3. Propagator-type diagrams up to one-loop order in QCD.

$$\lim_{m_R\to 0}\frac{1}{12p^2}\,\mathrm{Tr}[S_R^{-1}(p)\not{p}]|_{p^2\to-\mu^2} = -1. \tag{5}$$

The quark propagator in the RI′/MOM scheme is fixed to its lowest order value at the point $p^2 = -\mu^2$, where p^2 is the squared, external, Minkowski momentum and μ is the renormalization scale.

The propagator and vertex diagrams (Fig. 2) for the vector and axial-vector operators are related through the vector Ward-Takahashi identity for degenerate masses $m_u = m_d = m$

$$q_\mu \Lambda^\mu_{V,B}(p_1, p_2) = S_B^{-1}(p_2) - S_B^{-1}(p_1) \tag{6}$$

and the axial-vector Ward-Takahashi identity

$$-iq_\mu\Lambda^\mu_{A,B}(p_1, p_2) = 2m_B\Lambda_{P,B}(p_1, p_2) - i\gamma_5 S_B^{-1}(p_1) - S_B^{-1}(p_2)i\gamma_5, \tag{7}$$

with the momentum transfer $q = p_1 - p_2$. The renormalized and bare amputated Green's functions are connected by

$$S_R(p) = Z_q S_B(p), \qquad \Lambda_{\hat{O},R}(p_1, p_2) = \frac{Z_{\hat{O}}}{Z_q}\Lambda_{\hat{O},B}(p_1, p_2). \tag{8}$$

In the following we want to renormalize the quark bilinear operators using a symmetric subtraction point. For functions f, which are restricted to the symmetric momentum configuration we use the shorthand $f(p_1^2, p_2^2, q^2)|_{p_1^2=p_2^2=q^2=-\mu^2} \equiv f(p_1^2, p_2^2, q^2)|_{\mathrm{sym}}$ and for the asymmetric subtraction point we introduce the abbreviation $f(p_1^2, p_2^2, q^2)|_{q=0, p_1^2=-\mu^2=p_2^2} \equiv f(p_1^2, p_2^2, q^2)|_{\mathrm{asym}}$.

We perform the quark mass and wave function renormalization by imposing on the two-point function $S(p)$ the condition of Eq. (5) and

$$\lim_{m_R\to 0}\frac{1}{12m_R}\Big\{\mathrm{Tr}[S_R^{-1}(p)]|_{p^2=-\mu^2} - \frac{1}{2}\,\mathrm{Tr}[q_\mu\Lambda^\mu_{A,R}(p_1, p_2)\gamma_5]|_{\mathrm{sym}}\Big\} = 1. \tag{9}$$

The second term in the curly brackets on the left-hand side of Eq. (9) starts at $\mathcal{O}(\alpha_s)$ and is absent in the RI/MOM and RI′/MOM schemes. This term is needed to maintain the Ward-Takahashi identities for renormalized quantities, as we will see below. For the vector and axial-vector quark bilinear operators we impose the conditions

$$\begin{aligned}\lim_{m_R\to 0}\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu\Lambda^\mu_{V,R}(p_1, p_2)\not{q}]|_{\mathrm{sym}} &= 1,\\ \lim_{m_R\to 0}\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu\Lambda^\mu_{A,R}(p_1, p_2)\gamma_5\not{q}]|_{\mathrm{sym}} &= 1.\end{aligned} \tag{10}$$

The projectors for the amputated Green's functions in Eqs. (10) are different from those used in the RI/MOM scheme (see Table I). Using instead these original RI/MOM projectors leads to a different wave function renormalization and will be discussed in Appendix A. For the pseudoscalar and scalar amputated Green's functions we use the renormalization conditions

$$\begin{aligned}\lim_{m_R\to 0}\frac{1}{12i}\,\mathrm{Tr}[\Lambda_{P,R}(p_1, p_2)\gamma_5]|_{\mathrm{sym}} &= 1,\\ \lim_{m_R\to 0}\frac{1}{12}\,\mathrm{Tr}[\Lambda_{S,R}(p_1, p_2)\mathbb{1}]|_{\mathrm{sym}} &= 1,\end{aligned} \tag{11}$$

and for the tensor operator the condition

$$\lim_{m_R\to 0}\frac{1}{144}\,\mathrm{Tr}[\Lambda^{\mu\nu}_{T,R}\sigma_{\mu\nu}]|_{\mathrm{sym}} = 1. \tag{12}$$

Note that all of the renormalization schemes being considered in this paper are mass-independent. Thus, each condition is imposed at fixed external momentum and vanishing quark mass. The renormalization conditions of the RI/MOM and RI/SMOM schemes are summarized in Table I.

In the remainder of this section we will show that if the normalization conditions in Eqs. (5) and (9)–(11) of this

TABLE I. The renormalization conditions for the RI/MOM and RI/SMOM schemes.

RI/MOM	$\lim_{m_R\to 0}\frac{1}{48}\,\mathrm{Tr}[\gamma^\mu\frac{\partial S_R^{-1}(p)}{\partial p^\mu}]|_{p^2=-\mu^2} = -1,$	$\lim_{m_R\to 0}\frac{1}{12m_R}\,\mathrm{Tr}[S_R^{-1}(p)]|_{p^2=-\mu^2} = 1,$
	$\lim_{m_R\to 0}\frac{1}{48}\,\mathrm{Tr}[\Lambda^\mu_{V,R}(p_1, p_2)\gamma_\mu]|_{\mathrm{asym}} = 1,$	$\lim_{m_R\to 0}\frac{1}{48}\,\mathrm{Tr}[\Lambda^\mu_{A,R}(p_1, p_2)\gamma_5\gamma_\mu]|_{\mathrm{asym}} = 1,$
	$\lim_{m_R\to 0}\frac{1}{12}\,\mathrm{Tr}[\Lambda_{S,R}(p_1, p_2)\mathbb{1}]|_{\mathrm{asym}} = 1,$	$\lim_{m_R\to 0}\frac{1}{12i}\,\mathrm{Tr}[\Lambda_{P,R}(p_1, p_2)\gamma_5]|_{\mathrm{asym}} = 1.$
RI/SMOM	$\lim_{m_R\to 0}\frac{1}{12p^2}\,\mathrm{Tr}[S_R^{-1}(p)\not{p}]|_{p^2=-\mu^2} = -1,$	
	$\lim_{m_R\to 0}\frac{1}{12m_R}\{\mathrm{Tr}[S_R^{-1}(p)]|_{p^2=-\mu^2} - \frac{1}{2}\,\mathrm{Tr}[q_\mu\Lambda^\mu_{A,R}(p_1, p_2)\gamma_5]|_{\mathrm{sym}}\} = 1,$	
	$\lim_{m_R\to 0}\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu\Lambda^\mu_{V,R}(p_1, p_2)\not{q}]|_{\mathrm{sym}} = 1,$ $\lim_{m_R\to 0}\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu\Lambda^\mu_{A,R}(p_1, p_2)\gamma_5\not{q}]|_{\mathrm{sym}} = 1,$	
	$\lim_{m_R\to 0}\frac{1}{12}\,\mathrm{Tr}[\Lambda_{S,R}(p_1, p_2)\mathbb{1}]|_{\mathrm{sym}} = 1,$ $\lim_{m_R\to 0}\frac{1}{12i}\,\mathrm{Tr}[\Lambda_{P,R}(p_1, p_2)\gamma_5]|_{\mathrm{sym}} = 1.$	

RI/SMOM scheme are imposed on the quark bilinear operators, the Ward-Takahashi identities of Eqs. (6) and (7) are also obeyed for the resulting renormalized quantities and the properties $Z_V = 1 = Z_A$, $Z_P = 1/Z_m$ and $Z_S = Z_P$ are preserved, as they are in the $\overline{\rm MS}$, RI′/MOM and RI/MOM schemes (see e.g. Ref. [18,19]). Some of these properties hold nonperturbatively while the others are proven only in the perturbation theory as we will see below.

Let us start by considering the object $\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu \Lambda^\mu_{V,B} \not{q}]|_{\rm sym}$ and insert the vector Ward-Takahashi identity of Eq. (6):

$$\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu \Lambda^\mu_{V,B}\not{q}]|_{\rm sym} = \frac{1}{12q^2}\{\mathrm{Tr}[S_B^{-1}(p_2)\not{q}] - \mathrm{Tr}[S_B^{-1}(p_1)\not{q}]\}|_{\rm sym} = -\frac{1}{12q^2}\,\mathrm{Tr}[S_B^{-1}(q)\not{q}]|_{\rm sym}. \quad (13)$$

Expressing bare quantities in terms of renormalized ones using Eq. (8) and imposing the condition in Eq. (5) and the one on the left in Eq. (10) leads to $Z_V^{\rm RI/SMOM} = 1$. Similarly one obtains $Z_V^{\rm RI/SMOM} = Z_A^{\rm RI/SMOM}$ by inserting Eq. (7) into $\frac{1}{12q^2}\,\mathrm{Tr}[q_\mu \Lambda^\mu_{A,B}\gamma_5 \not{q}]|_{\rm sym}$, combining it with Eqs. (13) and imposing the conditions of Eq. (10) for the renormalized quantities in the massless limit. Note that the above derivation of $Z_A^{\rm RI/SMOM} = Z_V^{\rm RI/SMOM}$ is independent of the choice of the renormalization point μ. This is in contrast to the RI/MOM scheme for which the Ward-Takahashi identity for the axial current only holds at large μ^2. The renormalized vector current satisfies the Ward-Takahashi identity in both the RI/MOM and the RI/SMOM schemes even in the low energy region. However, the relation $Z_A = Z_V = 1$ implies that the axial vertex function given in Eq. (10) remains exactly equal to one in the limit of vanishing quark mass even when evaluated in the infrared region of QCD where large vacuum chiral symmetry breaking might have been expected to introduce large asymmetries between such vector and axial-vector correlation functions.

From $Z_V^{\rm RI/SMOM} = 1 = Z_A^{\rm RI/SMOM}$ it follows that the renormalization constant $Z_q^{\rm RI/SMOM}$ can be extracted from Eqs. (10). However, since Eq. (5) determines Z_q in both the RI/SMOM and RI′/MOM schemes, $Z_q^{\rm RI/SMOM} = Z_q^{\rm RI'/MOM}$, whose value is known up to order α_s^3 in Refs. [6,7]. Nevertheless, in Sec. III A, we will renormalize the vector and axial-vector operators for the symmetric momentum configuration in the RI/SMOM scheme using the conditions in Eqs. (10) in order to demonstrate that the value of $Z_q^{\rm RI/SMOM}$ obtained from Eq. (10) is in fact equal to the value for $Z_q^{\rm RI'/MOM}$ obtained from Eq. (5) by explicit calculation up to one-loop order.

From the axial Ward-Takahashi identity it follows that the renormalization constant for the pseudoscalar operator $Z_P^{\rm RI/SMOM}$ and the mass renormalization constant $Z_m^{\rm RI/SMOM}$ are related. If one multiplies Eq. (7) by $(-i\gamma_5)$, takes the trace of both sides over spin and color and restricts it to the symmetric momentum configuration, one obtains

$$-\frac{1}{12}\,\mathrm{Tr}[q_\mu \Lambda^\mu_{A,B}\gamma_5]|_{\rm sym} = 2m_B \frac{1}{12i}\,\mathrm{Tr}[\Lambda_{P,B}\gamma_5]|_{\rm sym} - \frac{1}{6}\,\mathrm{Tr}[S_B(p)^{-1}]|_{p^2=-\mu^2}. \quad (14)$$

Taking the zero-mass limit, expressing again the bare equation with the help of Eqs. (3) and (8) in terms of renormalized quantities and imposing the conditions in Eqs. (9) and (11) for the RI/SMOM scheme leads to $Z_P^{\rm RI/SMOM} = 1/Z_m^{\rm RI/SMOM}$.

The conditions in Eq. (11) for the pseudoscalar and scalar operator can be expressed in terms of the bare Green's function and the renormalization constants. The traces over the two bare Green's functions become equal in the massless limit in perturbation theory, which leads to $Z_S = Z_P$.

In the above discussion the renormalization constants relate the bilinear operators renormalized in the RI/SMOM scheme to those in the bare theory which we had assumed to satisfy the Ward-Takahashi identities (6) and (7). Since many lattice formulations of QCD break the chiral or flavor symmetries, in general Eqs. (6) and (7) do not hold in these (bare) theories. Nevertheless, our renormalization scheme is indeed *regularization independent* and the Ward-Takahashi identities hold for the RI/SMOM renormalized quantities. The renormalization constants relating the renormalized and bare lattice operators depend on the regularization of course, so that, for example, Z_V and Z_A will typically be different from 1 in such cases.

III. CONVERSION FACTORS: RESULTS OF THE NEXT-TO-LEADING ORDER CALCULATION

The properties discussed in Sec. II can be used to convert quark masses determined through lattice simulations in the RI/SMOM scheme into the $\overline{\rm MS}$ scheme by computing the matching factor $C_m^{\rm RI/SMOM} = Z_m^{\overline{\rm MS}}/Z_m^{\rm RI/SMOM}$ with $m_R^{\overline{\rm MS}} = C_m^{\rm RI/SMOM} m_R^{\rm RI/SMOM}$. The explicit calculation to determine this conversion factor at one-loop order in perturbative QCD will be performed in the next subsections using two different methods, which allows us to cross-check our results.

First, the matching factor $C_m^{\rm RI/SMOM}$ can be obtained with the help of Eq. (9) through

$$(C_m^{\mathrm{RI/SMOM}})^{-1} = (C_m^{\mathrm{RI'/MOM}})^{-1} - \frac{1}{2} C_q^{\mathrm{RI/SMOM}} \lim_{m_R \to 0} \frac{1}{12 m_R^{\overline{\mathrm{MS}}}} \times \mathrm{Tr}[q_\mu \Lambda_{A,R}^{\mu,\overline{\mathrm{MS}}} \gamma_5]|_{\mathrm{sym}}, \quad (15)$$

which will be evaluated in Sec. III A. In analogy to $C_m^{\mathrm{RI/SMOM}}$ we define here the conversion factor $C_q^{\mathrm{RI/SMOM}} = Z_q^{\overline{\mathrm{MS}}}/Z_q^{\mathrm{RI/SMOM}}$ for the fermion fields.

Second, the conversion factor can be related to the renormalization constants of the pseudoscalar operator

$$C_m^{\mathrm{RI/SMOM}} = \frac{Z_m^{\overline{\mathrm{MS}}}}{Z_m^{\mathrm{RI/SMOM}}} = \frac{Z_P^{\mathrm{RI/SMOM}}}{Z_P^{\overline{\mathrm{MS}}}} \equiv \frac{1}{C_P^{\mathrm{RI/SMOM}}} \quad (16)$$

and hence

$$m_R^{\overline{\mathrm{MS}}} = \frac{1}{C_P^{\mathrm{RI/SMOM}}} m_R^{\mathrm{RI/SMOM}} = \frac{1}{C_P^{\mathrm{RI/SMOM}}} \frac{1}{Z_{P,\mathrm{latt.}}^{\mathrm{RI/SMOM}}} m_{B,\mathrm{latt.}}. \quad (17)$$

In particular in Sec. III B we will evaluate the conversion factor $C_P^{\mathrm{RI/SMOM}}$, which converts the pseudoscalar operator from the RI/SMOM scheme to the $\overline{\mathrm{MS}}$ scheme. The matching factor $C_P^{\mathrm{RI/SMOM}}$ is in general gauge dependent; however, this gauge dependence will cancel out with the corresponding gauge dependence in the factor $Z_{P,\mathrm{latt.}}^{\mathrm{RI/SMOM}}$ determined in the lattice calculation. In the following we will perform the computation in the general covariant gauge using the tree level gluon propagator

$$\frac{i\delta^{ab}}{q^2 + i\epsilon}\left(-g^{\mu\nu} + (1-\xi)\frac{q^\mu q^\nu}{q^2 + i\epsilon}\right) \quad (18)$$

and we will restrict ourselves to the Landau gauge ($\xi = 0$) at the end of the calculation. We choose the renormalization scales of both schemes to be equal $\mu^{\overline{\mathrm{MS}}} = \mu^{\mathrm{RI/SMOM}}$. The conversion factors $C_x^{\mathrm{RI/SMOM}}$ with $x \in \{m, q, S, P, V, A, T\}$ denote always the conversion from the RI/SMOM to the $\overline{\mathrm{MS}}$ scheme.

A. The vector and axial-vector operator

In this section we want to use the vector and axial-vector operator separately to extract the matching factor $C_q^{\mathrm{RI/SMOM}}$ for the quark field for the symmetric subtraction point. This result is then used in the next step to compute $C_m^{\mathrm{RI/SMOM}}$ with the help of Eq. (15).

Our perturbative computation is performed in dimensional regularization with the space-time dimension $d = 4 - 2\varepsilon$. For the vector operator case $C_q^{\mathrm{RI/SMOM}}$ can be obtained by

$$(C_q^{\mathrm{RI/SMOM}})^{-1} = \lim_{m_R \to 0} \frac{1}{12 q^2} \mathrm{Tr}[q_\mu \Lambda_{V,R}^{\mu,\overline{\mathrm{MS}}} \not{q}]|_{\mathrm{sym}}. \quad (19)$$

The calculation of the one-loop QCD corrections to the vector operator, computed between two off-shell quark lines, is straightforward and leads to

$$C_q^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F \xi + \mathcal{O}(\alpha_s^2). \quad (20)$$

The symbol C_F denotes the Casimir operator of the SU(3) group in the fundamental representation; $C_F = 4/3$. As expected Eq. (20) agrees with the result in Refs. [6,7], since $C_q^{\mathrm{RI/SMOM}} = C_q^{\mathrm{RI'/MOM}}$, as shown in Sec. II.

Similarly one can also derive this result from the axial-vector operator by using

$$(C_q^{\mathrm{RI/SMOM}})^{-1} = \lim_{m_R \to 0} \frac{1}{12 q^2} \mathrm{Tr}[q_\mu \Lambda_{A,R}^{\mu,\overline{\mathrm{MS}}} \gamma_5 \not{q}]|_{\mathrm{sym}}. \quad (21)$$

For the treatment of γ_5 in dimensional regularization [2,20] we use a naive anticommuting definition of γ_5 for evaluating the loop integrals, which obeys the equations $\{\gamma_5, \gamma^\mu\} = 0$ and $\gamma_5^2 = 1$. This is a self-consistent prescription for the flavor nonsinglet contributions considered in this work [21,22]. On the other hand one can use Eq. (5) in order to determine $Z_q^{\mathrm{RI/SMOM}}$ and then extract $Z_V^{\mathrm{RI/SMOM}}$ and $Z_A^{\mathrm{RI/SMOM}}$ from Eqs. (10). For both we explicitly confirm that at one-loop order $Z_V^{\mathrm{RI/SMOM}} = Z_A^{\mathrm{RI/SMOM}} = 1$ as expected.

The conversion factor $C_m^{\mathrm{RI/SMOM}}$ can now be computed from the axial-vector operator with the help of Eq. (15) by determining

$$\lim_{m_R \to 0} \frac{1}{12 m_R^{\overline{\mathrm{MS}}}} \mathrm{Tr}[q_\mu \Lambda_{A,R}^{\mu,\overline{\mathrm{MS}}} \gamma_5]|_{\mathrm{sym}} = \frac{\alpha_s}{4\pi} C_F (3+\xi) C_0 + \mathcal{O}(\alpha_s^2), \quad (22)$$

with

$$C_0 = \tfrac{2}{3}\Psi'(\tfrac{1}{3}) - (\tfrac{2}{3}\pi)^2, \quad (23)$$

where $\Psi(x)$ is the digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$.[1] The matching factor $C_m^{\mathrm{RI'/MOM}}$ can be taken from Refs. [6,7] and $C_q^{\mathrm{RI/SMOM}}$ from Eq. (20). This leads to

$$C_m^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F \left(4 + \xi - (3+\xi)\frac{1}{2} C_0\right) + \mathcal{O}(\alpha_s^2). \quad (24)$$

B. The pseudoscalar and scalar operator

In this section we determine the conversion factor $C_m^{\mathrm{RI/SMOM}} = (C_P^{\mathrm{RI/SMOM}})^{-1} = Z_P^{\mathrm{RI/SMOM}}/Z_P^{\overline{\mathrm{MS}}}$ through the calculation of the pseudoscalar operator. At one-loop order in perturbative QCD its computation leads to the decomposition

[1] The prime denotes here the derivative.

$$\Lambda_{P,B} = A_{P,B} i\gamma_5 + B_{P,B} i\gamma_5 \frac{m\not{q}}{q^2} + C_{P,B} i\gamma_5 \frac{[\not{p}_1, \not{p}_2]}{q^2}, \quad (25)$$

with

$$A_{P,B} = 1 + \frac{\alpha_s}{4\pi} a_{p,1} + \cdots, \qquad B_{P,B} = \frac{\alpha_s}{4\pi} b_{p,1} + \cdots,$$
$$C_{P,B} = \frac{\alpha_s}{4\pi} c_{p,1} + \cdots, \quad (26)$$

where the dots stand for higher order corrections and where we have set $p_1^2 = q^2 = p_2^2$. The quantities $b_{p,1}$ and $c_{p,1}$ are finite, whereas $a_{p,1}$ contains $1/\varepsilon$ poles. In the limit of massless fermions, considered here, we obtain $b_{p,1} = 0$. The matching factor $C_P^{\mathrm{RI/SMOM}}$ can be obtained from Eq. (11) by evaluating

$$C_P^{\mathrm{RI/SMOM}} = C_q^{\mathrm{RI/SMOM}} \lim_{m_R \to 0} \frac{1}{12i} \mathrm{Tr}[\Lambda_{P,R}^{\overline{\mathrm{MS}}} \gamma_5]|_{\mathrm{sym}}. \quad (27)$$

The fermion field conversion factor $C_q^{\mathrm{RI/SMOM}}$ is known and has been discussed in the previous Sec. III A. Since the amplitudes $B_{P,B}$ and $C_{P,B}$ in Eq. (25) do not contribute to the trace in Eq. (11), this condition depends only on the pseudoscalar amplitude $A_{P,B}$, which is fixed to its lowest order value at the symmetric subtraction point. The explicit calculation yields

$$C_P^{\mathrm{RI/SMOM}} = C_q^{\mathrm{RI/SMOM}} \Big\{ 1 + \frac{\alpha_s}{4\pi} C_F \Big[4 + 2\xi + \Big(1 + \frac{\xi}{3}\Big) \times \Big(\frac{2}{3}\pi^2 - \Psi'\Big(\frac{1}{3}\Big)\Big)\Big] + \mathcal{O}(\alpha_s^2)\Big\}. \quad (28)$$

Inserting $C_q^{\mathrm{RI/SMOM}}$ from Eq. (20) and exploiting $C_m^{\mathrm{RI/SMOM}} = (C_P^{\mathrm{RI/SMOM}})^{-1}$ leads to the same result as given in Eq. (24). Numerical evaluation in the Landau gauge leads to

$$C_P^{\mathrm{RI/SMOM}} = (C_m^{\mathrm{RI/SMOM}})^{-1} = 1 + \frac{\alpha_s}{4\pi} C_F 0.484\,139\,1\ldots + \mathcal{O}(\alpha_s^2). \quad (29)$$

Comparing Eq. (29) to the RI′/MOM scheme with $C_m^{\mathrm{RI'/MOM}} = 1 - \frac{\alpha_s}{4\pi} C_F 4 + \cdots$ or the RI/MOM scheme (which is in the Landau gauge at one-loop order equal to the RI′/MOM scheme), we see that the result in Eq. (29) has a smaller one-loop coefficient by almost a factor of 10.

In order to study the conversion factor $C_m^{\mathrm{RI/SMOM}}$ for different subtraction points, we introduce the parameter ω and fix our renormalization condition for the subtraction "point" $p_1^2 = p_2^2 = -\mu^2$ and $q^2 = -\omega\mu^2$. This allows us also to study the limit $\omega \to 0$, which results in an exceptional momentum configuration, whereas the limit $\omega \to 1$ gives the symmetric one. The result depending on ω is given by

$$C_m^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F \Big[4 + \xi - (3 + \xi) \frac{\omega}{2} C_0(\omega) \Big] + \mathcal{O}(\alpha_s^2), \quad (30)$$

where the function $C_0(\omega)$ for $\omega \in [0, 4]$ is given by

$$C_0(\omega) = -\mu^2 \int \frac{d^4k}{i\pi^2} \frac{1}{(k + p_1)^2 (k + p_2)^2 k^2} \Big|_{p_1^2 = p_2^2 = -\mu^2, q^2 = -\omega\mu^2}$$
$$= \frac{2i}{\sqrt{4 - \omega}\sqrt{\omega}} \Big[\mathrm{Li}_2\Big(\frac{-\sqrt{4-\omega} + i\sqrt{\omega}}{-\sqrt{4-\omega} - i\sqrt{\omega}}\Big) - \mathrm{Li}_2\Big(\frac{-\sqrt{4-\omega} - i\sqrt{\omega}}{-\sqrt{4-\omega} + i\sqrt{\omega}}\Big)\Big], \quad (31)$$

and $\mathrm{Li}_2(z)$ is the dilogarithm function. In the case $\omega = 1$ one obtains the result of Eq. (24) with $C_0(\omega = 1) = C_0$. In order to display the dependence of this result for $C_m^{\mathrm{RI/SMOM}}$ on the gauge parameter ξ, we introduce the one-loop coefficient function $c_m^{(1),\mathrm{RI/SMOM}}(\omega, \xi)$ extracted from Eq. (30) using the definition $C_m^{\mathrm{RI/SMOM}} = 1 + \frac{\alpha_s}{4\pi} C_F c_m^{(1),\mathrm{RI/SMOM}}(\omega, \xi)$. The coefficient $c_m^{(1),\mathrm{RI/SMOM}}(\omega, \xi)$ is plotted as a function of ω in the interval $\omega \in [0, 4]$ for different gauges in Fig. 4(a).

Going from the exceptional ($\omega = 0$) to the nonexceptional ($\omega = 1$) momentum configuration leads to a smaller one-loop coefficient in the Landau gauge; however even for almost all other gauges the one-loop coefficient becomes smaller as well, except for gauges in the small interval $\xi \in (\frac{4}{C_0 - 4} - 3, -3)$, which is shown in Fig. 4(b). The smaller coefficient might indicate that the symmetric configuration is less disposed to infrared effects.

In analogy to the pseudoscalar operator the computation of the scalar operator leads to the conversion factor $C_S^{\mathrm{RI/SMOM}}$ by employing the renormalization condition in Eq. (11). As expected the one-loop result for the matching factor $C_m^{\mathrm{RI/SMOM}} = (C_S^{\mathrm{RI/SMOM}})^{-1}$, extracted from the scalar operator, is equal to the result obtained from the pseudoscalar one.

C. The tensor operator

The matching factor converting the Green's function of the tensor operator from the RI/SMOM to the $\overline{\mathrm{MS}}$ scheme can be obtained from Eq. (12) in complete analogy to the other operators. It is given by

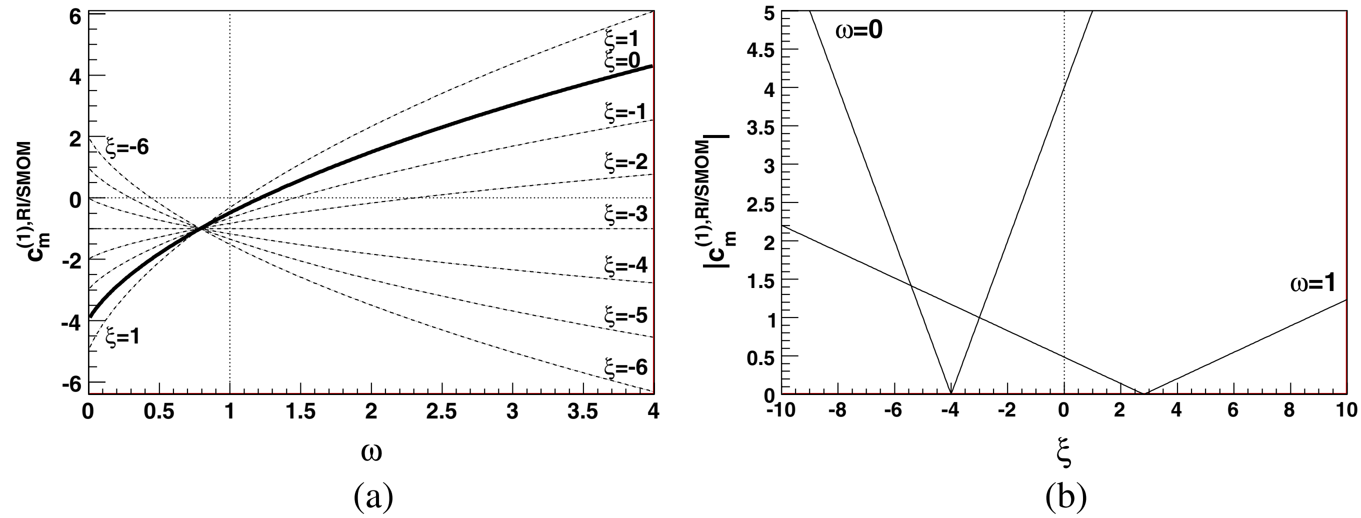


FIG. 4 (color online). (a) shows the one-loop coefficient $c_m^{(1),\mathrm{RI/SMOM}}$ of the matching factor $C_m^{\mathrm{RI/SMOM}}$ as a function of ω for different gauges. The value at $\omega = 0$ is the result for the exceptional momentum configuration. The nonexceptional configuration is indicated through the vertical line at $\omega = 1$. The bold line indicates the Landau gauge ($\xi = 0$), which is usually adopted in lattice calculations. (b) shows the exceptional ($\omega = 0$) and nonexceptional ($\omega = 1$) configurations as a function of the gauge parameter ξ.

TABLE II. Summary of the matching factors for the mass and fermion field conversion ($C_m^{\mathrm{RI/SMOM}}$ and $C_q^{\mathrm{RI/SMOM}}$) as well as for the conversion of the tensor operator ($C_T^{\mathrm{RI/SMOM}}$) in the Landau gauge.

$C_m^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F 0.484\,139\,1\ldots + \mathcal{O}(\alpha_s^2)$
$C_q^{\mathrm{RI/SMOM}} = 1 + \mathcal{O}(\alpha_s^2)$
$C_T^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F 0.161\,379\,7\ldots + \mathcal{O}(\alpha_s^2)$

$$C_T^{\mathrm{RI/SMOM}} = 1 + \frac{\alpha_s}{4\pi} C_F \left[(1-\xi)\left(\frac{C_0}{2} - \frac{4}{3}\right) - \xi \right], \qquad (32)$$

where we have defined $C_T^{\mathrm{RI/SMOM}} = Z_T^{\overline{\mathrm{MS}}}/Z_T^{\mathrm{RI/SMOM}}$. The numerical evaluation in the Landau gauge leads to

$$C_T^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F 0.161\,379\,7\ldots. \qquad (33)$$

For the RI′/MOM scheme the tensor operator has been evaluated up to three-loop order in Ref. [7]. The conversion factor at one-loop order is found to be proportional to the gauge parameters ξ. This contribution is therefore zero in the Landau gauge.

All conversion factors discussed in Sec. III are summarized in Table II. The matching factors for the scalar and pseudoscalar operator are equal to the inverse of the mass conversion factor $1/C_m^{\mathrm{RI/SMOM}} = C_S^{\mathrm{RI/SMOM}} = C_P^{\mathrm{RI/SMOM}}$.

IV. SUMMARY AND CONCLUSION

We provide the framework and concepts for renormalizing the quark bilinear operators in a MOM scheme (RI/SMOM) with a symmetric subtraction point which has no channels with exceptional momenta. This generally suppresses the infrared chiral symmetry breaking effects compared to the standard RI/MOM (or RI′/MOM) scheme in which there is an exceptional channel (with zero momentum). An exception is the vector current for which the RI/MOM scheme satisfies the Ward-Takahashi identity also at low values of p^2. We demonstrate that the chiral Ward-Takahashi identities (for degenerate masses) are satisfied nonperturbatively, and thus $Z_V = 1 = Z_A$ for all values of p^2, in the RI/SMOM scheme. We calculate the matching factors relating operators renormalized in this scheme and the $\overline{\mathrm{MS}}$ schemes at one-loop order in perturbation theory. The one-loop coefficients are given in Table II and we note that they are small. In particular, for the quark mass the coefficient is much smaller than that between the RI/MOM (RI′/MOM) and $\overline{\mathrm{MS}}$ schemes which, if confirmed at higher orders, would lead to a significant reduction in the uncertainty on the calculated value of the quark mass.

Nonperturbative renormalization of operators in lattice QCD using the RI/MOM (or RI′/MOM) scheme has been successfully implemented for many years. The evaluation of matrix elements in the RI/SMOM renormalization scheme in lattice simulations is equally practicable and in view of the advantages explained above we strongly advocate its use.

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APPENDIX A: ALTERNATIVE PROJECTORS FOR THE VECTOR AND AXIAL-VECTOR OPERATOR GREEN'S FUNCTIONS

In general one can also use other projectors than those of the RI/SMOM scheme as defined in Sec. II in order to define a scheme with a symmetric subtraction point. The general structure before taking the trace with projectors of the one-loop corrected amputated Green's functions of the operators in the $\overline{\mathrm{MS}}$ scheme for massless quarks with the momenta $p_1^2 = p_2^2 = q^2$ are given by

$$\bar{\Lambda}_S = \left(\bar{A}_S \mathbb{1} + \bar{C}_S \frac{[\not{p}_1, \not{p}_2]}{q^2}\right)\delta_{ij}, \tag{A1}$$

$$\bar{\Lambda}_P = \left(\bar{A}_P i\gamma_5 + \bar{C}_P i\gamma_5 \frac{[\not{p}_1, \not{p}_2]}{q^2}\right)\delta_{ij}, \tag{A2}$$

$$\bar{\Lambda}_V^\mu = \left(\bar{A}_V \gamma^\mu + \bar{B}_V \frac{\not{p}_1\gamma^\mu\not{p}_1 + \not{p}_2\gamma^\mu\not{p}_2}{q^2} + \bar{C}_V \frac{\not{p}_1\gamma^\mu\not{p}_2}{q^2} + \bar{D}_V \frac{\not{p}_2\gamma^\mu\not{p}_1}{q^2}\right)\delta_{ij}, \tag{A3}$$

$$\bar{\Lambda}_A^\mu = \left(\bar{A}_A \gamma^\mu\gamma_5 - \bar{B}_A \frac{\not{p}_1\gamma^\mu\gamma_5\not{p}_1 + \not{p}_2\gamma^\mu\gamma_5\not{p}_2}{q^2} - \bar{C}_A \frac{\not{p}_1\gamma^\mu\gamma_5\not{p}_2}{q^2} - \bar{D}_A \frac{\not{p}_2\gamma^\mu\gamma_5\not{p}_1}{q^2}\right)\delta_{ij}, \tag{A4}$$

$$\bar{\Lambda}_T^{\mu\nu} = \left(\bar{A}_T \sigma^{\mu\nu} + \bar{B}_T \frac{\sigma^{\mu\nu}\not{p}_2\not{p}_1 - \not{p}_1\not{p}_2\sigma^{\mu\nu}}{q^2} + \bar{C}_T \frac{\not{p}_1\not{p}_2\sigma^{\mu\nu}\not{p}_1\not{p}_2}{q^4}\right)\delta_{ij}, \tag{A5}$$

where the indices i and j denote color indices and the coefficient functions read

$$\bar{A}_{S,P} = 1 + \frac{\alpha_s}{4\pi} C_F\left[4 + 3\log\left(\frac{\mu^2}{-q^2}\right) - \frac{3}{2}C_0 + \xi\left(2 + \log\left(\frac{\mu^2}{-q^2}\right) - \frac{C_0}{2}\right)\right], \tag{A6}$$

$$\bar{C}_{S,P} = \frac{\alpha_s}{4\pi} C_F (1-\xi)\frac{C_0}{6}, \tag{A7}$$

$$\bar{A}_{V,A} = 1 + \frac{\alpha_s}{4\pi} C_F\left[-\frac{C_0}{3} + \xi\left(1 + \log\left(\frac{\mu^2}{-q^2}\right) - \frac{C_0}{3}\right)\right], \tag{A8}$$

$$\bar{B}_{V,A} = -\frac{\alpha_s}{4\pi}\frac{C_F}{3}[(1-\xi)(C_0 - 2) - 2], \tag{A9}$$

$$\bar{C}_{V,A} = -\frac{\alpha_s}{4\pi}\frac{C_F}{3}[C_0 + 2 + \xi(C_0 - 1)], \tag{A10}$$

$$\bar{D}_{V,A} = \frac{\alpha_s}{4\pi}\frac{C_F}{3}[(1-\xi)(C_0 - 1) - 1], \tag{A11}$$

$$\bar{A}_T = 1 - \frac{\alpha_s}{4\pi} C_F(1-\xi)\left[\frac{5}{3} + \log\left(\frac{\mu^2}{-q^2}\right) - \frac{2}{3}C_0\right], \tag{A12}$$

$$\bar{B}_T = -\frac{\alpha_s}{4\pi}\frac{C_F}{3}[2C_0 - (1-\xi)], \tag{A13}$$

$$\bar{C}_T = \frac{\alpha_s}{4\pi}\frac{C_F}{3}(1-\xi)[2 - C_0]. \tag{A14}$$

An example of a second possible choice for the projectors is the use of the projectors of the RI/MOM scheme for the amputated Green's function of the vector and axial-vector operator

$$\lim_{m_R\to 0}\tfrac{1}{48}\mathrm{Tr}[\Lambda_{V,R}^\mu(p_1, p_2)\gamma_\mu]|_{\mathrm{sym}} = 1, \quad \lim_{m_R\to 0}\tfrac{1}{48}\mathrm{Tr}[\Lambda_{A,R}^\mu(p_1, p_2)\gamma_5\gamma_\mu]|_{\mathrm{sym}} = 1, \tag{A15}$$

in the renormalization conditions with a symmetric subtraction point together with the conditions of Eqs. (11). One also has to modify the conditions of Eqs. (4)

$$\lim_{m_R\to 0}\frac{1}{48}\left\{\mathrm{Tr}\left[\gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu}\right]\Bigg|_{p^2=-\mu^2} + \mathrm{Tr}\left[q_\mu\gamma^\alpha\frac{\partial}{\partial q^\alpha}\Lambda_{V,R}^\mu\right]\Bigg|_{\mathrm{sym}}\right\} = -1, \tag{A16}$$

$$\lim_{m_R\to 0}\frac{1}{12m_R}\left\{\mathrm{Tr}[S_R^{-1}(p)]|_{p^2=-\mu^2} - \frac{1}{2}\mathrm{Tr}[q_\mu\Lambda_{A,R}^\mu(p_1, p_2)\gamma_5]|_{\mathrm{sym}}\right\} = 1, \tag{A17}$$

to maintain the Ward-Takahashi identities of Eqs. (6) and (7) for renormalized quantities. This leads to a wave function renormalization factor Z_q which is different from the one of the RI/MOM or RI$'$/MOM scheme. For this reason the projectors used in Eqs. (10) of Sec. II have the advantage to produce the same well-known renormalization constant Z_q like in the RI′/MOM scheme. With the conditions of Eqs. (11) and (A15)–(A17) one obtains in this $\mathrm{RI/SMOM}_{\gamma_\mu}$ scheme the following conversion factors:

$$C_q^{\mathrm{RI/SMOM}_{\gamma_\mu}} = 1 - \frac{\alpha_s}{4\pi} C_F \left[-1 + \frac{\xi}{2} \left(3 - \frac{2}{3} \Psi'\left(\frac{1}{3}\right) + \left(\frac{2}{3}\pi\right)^2 \right) \right] + \mathcal{O}(\alpha_s^2), \tag{A18}$$

$$C_P^{\mathrm{RI/SMOM}_{\gamma_\mu}} = 1 + \frac{\alpha_s}{4\pi} C_F \left[5 + \frac{2}{3}\pi^2 - \Psi'\left(\frac{1}{3}\right) + \frac{\xi}{2} \right] + \mathcal{O}(\alpha_s^2). \tag{A19}$$

The numerical evaluation of the resulting mass conversion factor in the Landau gauge reads

$$C_m^{\mathrm{RI/SMOM}_{\gamma_\mu}} = 1 - \frac{\alpha_s}{4\pi} C_F 1.484\,139\,1\ldots. \tag{A20}$$

APPENDIX B: ANOMALOUS DIMENSIONS

In order to evaluate the mass in the RI/SMOM scheme at different scales the mass anomalous dimension $\gamma_m^{\mathrm{RI/SMOM}}$ is required. It is defined by

$$\gamma_m = \frac{d\log m(\mu)}{d\log(\mu^2)} = -\gamma_m^{(0)}\frac{\alpha_s}{\pi} - \gamma_m^{(1)}\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3). \tag{B1}$$

The result up to order α_s^2 in the RI/SMOM scheme reads in the Landau gauge

$$\begin{aligned} \gamma_m^{(0),\mathrm{RI/SMOM}} &= \gamma_m^{(0),\overline{\mathrm{MS}}}, \\ \gamma_m^{(1),\mathrm{RI/SMOM}} &= \gamma_m^{(1),\overline{\mathrm{MS}}} - \frac{\beta^{(0)}}{4} C_F c_m^{(1),\mathrm{RI/SMOM}}(1,0), \end{aligned} \tag{B2}$$

with $c_m^{(1),\mathrm{RI/SMOM}}(1,0) = -0.484\,139\,1\ldots$ and the β function defined through

$$\beta = \frac{d\alpha_s(\mu)/\pi}{d\log(\mu^2)} = -\beta^{(0)}\left(\frac{\alpha_s}{\pi}\right)^2 - \beta^{(1)}\left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4). \tag{B3}$$

The first expansion coefficients for the mass anomalous dimension in the $\overline{\mathrm{MS}}$ scheme and the $\overline{\mathrm{MS}}$ β function are given by

$$\begin{aligned} \gamma_m^{(0),\overline{\mathrm{MS}}} &= \tfrac{3}{4} C_F, \\ \gamma_m^{(1),\overline{\mathrm{MS}}} &= \tfrac{1}{16}\left(\tfrac{3}{2}C_F^2 + \tfrac{97}{6}C_F C_A - \tfrac{10}{3} C_F T_F n_f\right), \\ \beta^{(0)} &= \tfrac{1}{4}\left(\tfrac{11}{3}C_A - \tfrac{4}{3}T_F n_f\right), \end{aligned}$$

where C_A is the Casimir operator in the adjoint representation of SU(3) and n_f is the number of active fermions. The symbol T_F denotes the normalization of the trace of the SU(3) generators in the fundamental representation, conventionally chosen as 1/2. For the $\mathrm{RI/SMOM}_{\gamma_\mu}$ scheme, defined in Appendix A, the two-loop mass anomalous dimension is given by

$$\gamma_m^{(1),\mathrm{RI/SMOM}_{\gamma_\mu}} = \gamma_m^{(1),\overline{\mathrm{MS}}} - \frac{\beta^{(0)}}{4} C_F c_m^{(1),\mathrm{RI/SMOM}_{\gamma_\mu}}, \tag{B4}$$

with $c_m^{(1),\mathrm{RI/SMOM}_{\gamma_\mu}} = -1.484\,139\,1\ldots$ as given in Eq. (A20).

Similarly the anomalous dimension $\gamma_q^{\mathrm{RI/SMOM}} = 2\frac{d\log\Psi}{d\log(\mu^2)}$ can be defined, which is equal to the one in the RI′/MOM scheme, $\gamma_q^{\mathrm{RI/SMOM}} = \gamma_q^{\mathrm{RI'/MOM}}$, and can be found in Refs. [6,7] up to order α_s^3. For completeness we give here the result up to $\mathcal{O}(\alpha_s^2)$ which in the Landau gauge is the same as in the $\overline{\mathrm{MS}}$ scheme

$$\gamma_q^{\mathrm{RI/SMOM}} = \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{3}{32}C_F^2 - \frac{25}{64}C_F C_A + \frac{1}{8}C_F T_F n_f\right) + \mathcal{O}(\alpha_s^3). \tag{B5}$$

In the $\mathrm{RI/SMOM}_{\gamma_\mu}$ scheme, defined in Appendix A, the order α_s^2 coefficient of the anomalous dimension $\gamma_q^{\mathrm{RI/SMOM}_{\gamma_\mu}}$ in the Landau gauge reads

$$\gamma_q = \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{3}{32}C_F^2 - \frac{31}{192}C_F C_A + \frac{1}{24}C_F T_F n_f\right) + \mathcal{O}(\alpha_s^3). \tag{B6}$$

We define the anomalous dimension γ_T of the tensor operator by

$$\gamma_T^{\mathrm{RI/SMOM}} = \frac{d\log Z_T}{d\log(\mu^2)} = -\gamma_T^{(0)}\left(\frac{\alpha_s}{\pi}\right) - \gamma_T^{(1)}\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3). \tag{B7}$$

In the RI/SMOM scheme it is given in the Landau gauge by

$$\begin{aligned} \gamma_T^{(0),\mathrm{RI/SMOM}} &= \gamma_T^{(0),\overline{\mathrm{MS}}}, \\ \gamma_T^{(1),\mathrm{RI/SMOM}} &= \gamma_T^{(1),\overline{\mathrm{MS}}} - \frac{\beta^{(0)}}{4} C_F c_T^{(1),\mathrm{RI/SMOM}}(0), \end{aligned} \tag{B8}$$

where we have introduced the one-loop coefficient function $c_T^{(1),\mathrm{RI/SMOM}}(\xi)$ extracted from Eq. (32) using the definition $C_T^{\mathrm{RI/SMOM}} = 1 + \frac{\alpha_s}{4\pi} C_F c_T^{(1),\mathrm{RI/SMOM}}(\xi)$, with $c_T^{(1),\mathrm{RI/SMOM}}(0) = -0.161\,379\,7\ldots$. The $\overline{\mathrm{MS}}$ anomalous dimension $\gamma_T^{\overline{\mathrm{MS}}}$ is known from Refs. [7,23,24] and reads

$$\begin{aligned} \gamma_T^{(0),\overline{\mathrm{MS}}} &= \tfrac{1}{4} C_F, \\ \gamma_T^{(1),\overline{\mathrm{MS}}} &= \tfrac{1}{16}\left(-\tfrac{19}{2}C_F^2 + \tfrac{257}{18}C_F C_A - \tfrac{26}{9} C_F T_F n_f\right). \end{aligned} \tag{B9}$$

Since the renormalization constants of the pseudoscalar and scalar operators are related to the mass renormalization constant, the anomalous dimensions of the pseudoscalar and scalar operators follow from the mass anomalous dimension in Eq. (B2).

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